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ABSTRACT. Let G denote a connected reductive algebraic group over an algebraically closed field k and let X denote a projective $G \times G$-equivariant embedding of G. The large Schubert varieties in X are the closures of the double cosets BgB, where B denotes a Borel subgroup of G, and $g \in G$. We prove that these varieties are globally F-regular in positive characteristic, resp. of globally F-regular type in characteristic 0. As a consequence, the large Schubert varieties are normal and Cohen-Macaulay.

1. INTRODUCTION

The class of globally F-regular varieties was introduced by Smith in [20] ; these are projective algebraic varieties in positive characteristics such that all the ideals in their homogeneous coordinate rings are tightly closed. The globally F-regular varieties (and their analogues in characteristic 0, the varieties of globally F-regular type) have remarkable properties, e.g., they are normal and Cohen-Macaulay, and the higher cohomology groups of all nef invertible sheaves are trivial.

Examples of globally F-regular varieties include the projective toric varieties (Prop.6.4 in [20]). In this note, we obtain the global F-regularity of a wider class of varieties with algebraic group action: the $G \times G$-equivariant projective embeddings of any connected reductive group G, and the closures in any such embedding of the double cosets BgB, where B denotes a Borel subgroup of G, and $g \in G$ is arbitrary. By the Bruhat decomposition, these "large Schubert varieties" are parametrized by the Weyl group of G; examples are the closures of parabolic subgroups. We also show that the large Schubert varieties are of globally F-regular type in characteristic 0.

For this, we exploit the close relation between global F-regularity and Frobenius splitting established in [20], and the Frobenius splitting properties of the large Schubert varieties, proved in [2] and [18]. Another key ingredient is the global F-regularity of the flag varieties and their Schubert varieties ([12]). Note that, unlike for Schubert varieties, no desingularization of large Schubert varieties is known in general; this makes our arguments somewhat indirect.

As a consequence of our result, the large Schubert varieties in any equivariant embedding of G are normal and Cohen-Macaulay. This was first proved in the case of the canonical compactification of a semisimple adjoint group, by Frobenius splitting methods ([2]). Then Rittatore showed that all the equivariant embeddings of connected reductive groups are Cohen-Macaulay, again by Frobenius splitting methods ([18]). On the other hand, the Cohen-Macaulayness of large Schubert varieties in the space of $n \times n$ matrices (regarded as an equivariant embedding of the general linear group GL_n) was established by Knutson and Miller via a degeneration argument, see [10].

The group G, regarded as a homogeneous space under $G \times G$, is an example of a spherical homogeneous space, i.e., it contains only finitely many orbits of the Borel subgroup $B \times B$ of $G \times G$. More generally, one may consider an equivariant embedding X of a spherical homogeneous G/H, and ask whether the closures in X of the B-orbits in G/H are globally F-regular. The answer is generally negative as some of these closures have bad singularities, see Ex.6 in [3]. However, the question makes sense for the class of multiplicity-free orbit closures introduced in [3], since these are normal and Cohen-Macaulay (see [3] in characteristic 0, and [4] in arbitrary characteristic). In fact, the class of multiplicity-free orbit closures includes the large Schubert varieties in toroidal embeddings of G, i.e., those which dominate the canonical compactification of the associated adjoint semisimple group.

2. Strong F-regularity

In this section k denotes an algebraically closed field of characteristic $p > 0$ and R denotes a commutative k-algebra which is essentially of finite type, i.e., is isomorphic to a localization of a finitely generated k-algebra.

Composing the R-module structure on an R-module M with the Frobenius map $F : R \to R$, $r \mapsto r^p$, defines a new R-module which we denote by F_*M. The module defined by iterating this procedure n times will be denoted by $F_*^n M$. In particular, this defines an R-module $F_*^n R$ for each positive integer n which as an abelian group coincides with R but where the R-module structure is twisted by the n-th iterated Frobenius morphism $r \mapsto r^{p^n}$.

When $s \in R$ and n is a positive integer we may define an R-module map by

$$\begin{aligned} F_s^n : R &\to F_*^n R, \\ r &\mapsto r^{p^n} s. \end{aligned}$$

A *splitting* of F_s^n is a R-module map $\phi : F_*^n R \to R$ such that the composed map $\phi \circ F_s^n$ coincides with the identity map on R.

Definition. ([8]) The ring R is *strongly F-regular* if for each $s \in R$, not contained in a minimal prime of R, there exists a positive integer n such the map F_s^n is split. The affine scheme $\mathrm{Spec}(R)$ is said to be strongly F-regular if R is strongly F-regular.

It is known (see [8]) that strongly F-regular rings are reduced, normal, Cohen-Macaulay, and F-rational. Moreover, strongly F-regular rings are weakly F-regular, i.e. all ideals are tightly closed. Being strongly F-regular is a local condition in the sense that R is strongly F-regular if and only if all its local rings are strongly F-regular.

3. Global F-regularity

In this section X will denote a projective variety over an algebraically closed field k of characteristic $p > 0$. (By a variety, we mean a separated integral scheme of finite type over k; in particular, varieties are irreducible). When $\mathcal{L}$ is an ample invertible sheaf on X we define the associated *section ring* to be

$$R = R(X, \mathcal{L}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^n).$$

This ring R is a positively graded, finitely generated k-algebra. We may now state

Definition. ([20]) The projective variety X is *globally F-regular* if the ring $R(X, \mathcal{L})$ is strongly F-regular for some ample invertible sheaf $\mathcal{L}$ on X.

It can be shown (see Thm.3.10 in [20]) that if X is globally F-regular then the section ring associated to any ample invertible sheaf is strongly F-regular. Moreover, if X is globally F-regular then all its local rings are strongly F-regular. In particular,

Corollary 3.1. *If the projective variety X is globally F-regular then X is normal and Cohen-Macaulay. Furthermore, the section ring $R(X, \mathcal{L})$ of any ample invertible sheaf $\mathcal{L}$ on X is normal and Cohen-Macaulay.*

3.1. **Frobenius splitting.** The absolute Frobenius morphism $F : Y \to Y$ on a scheme Y of finite type over k, is the map which is the identity on the set of points and where the associated map of structure sheafs $F^\sharp : \mathcal{O}_Y \to F_* \mathcal{O}_Y$ is the p-th power map. Following [13] we say that Y is *Frobenius split* if $F^\sharp$ splits as a map of $\mathcal{O}_Y$-modules, i.e. if there exists an $\mathcal{O}_Y$-linear map $\phi : F_*\mathcal{O}_Y \to \mathcal{O}_Y$ such that the composed map $\phi \circ F^\sharp$ is the identity. The map ϕ is in this case called a *Frobenius splitting* of Y. If ϕ is a (Frobenius) splitting of Y and Z is a closed subscheme of Y with associated ideal sheaf $\mathcal{I}$, we say that Z is *compatibly split* (by ϕ) if $\phi(\mathcal{I}) \subseteq \mathcal{I}$. In this case ϕ induces a splitting of Z. It is easily seen that globally F-regular varieties are Frobenius split but the converse is in general not true.

Let D denote an effective Cartier divisor on Y and let s denote the canonical section of the associated invertible sheaf $\mathcal{O}_Y(D)$. When n is a positive integer we let $F^n(D)$ denote the $\mathcal{O}_Y$-linear map

$$F^n(D) : \mathcal{O}_Y \to F^n_* \mathcal{O}_Y(D),$$
$$t \mapsto t^{p^n} s.$$

We say that Y is *stably Frobenius split along D* if $F^n(D)$ is split, as a map of $\mathcal{O}_Y$-modules, for some n. In this case Y is Frobenius split as well; the *induced Frobenius splitting* is given by composing the splitting of $F^n(D)$ with the map $F_*\mathcal{O}_Y \to F^n_*\mathcal{O}_Y(D)$, $t \mapsto t^{p^{n-1}} s$. In case $F^1(D)$ is split we simply say that Y is *Frobenius split along D*.

In the following lemma we will, for later use, collect a number of standard facts about the concepts introduced above.

Lemma 3.1. *Let Y be a scheme of finite type over over k.*

1. *If ϕ is a Frobenius splitting of Y which compatibly splits closed subschemes Z_1 and Z_2 then the scheme theoretic intersection $Z_1 \cap Z_2$ is also compatibly split by ϕ.*
2. *If Z is compatibly Frobenius split in Y by ϕ then every irreducible component of Z is compatibly split by ϕ.*
3. *Assume that Y is Frobenius split along D and that the induced splitting compatibly splits a closed subscheme Z. Assume further that none of the irreducible components of Z is contained in the support of D. Then Z is Frobenius split along $D \cap Z$, where $D \cap Z$ denotes the restriction of D to Z.*
4. *Let $D' \leq D$ be effective Cartier divisors on Y. Then every (stable) splitting of Y along D induces a (stable) splitting along D'. Moreover, the induced Frobenius splittings of Y, defined by these two splittings, coincide.*

(5) *Let D and D' denote effective Cartier divisors on Y. If Y is stably Frobenius split along both D and D' then Y is also stably split along the sum $D + D'$. Furthermore, the stable splitting along $D + D'$ may be chosen such that the induced Frobenius splitting of Y coincides with the one induced by the stable splitting along D'.*

(6) *If Y is Frobenius split along $(p-1)D$ for some effective Cartier divisor D, then D (regarded as the zero subscheme of its canonical section) is compatibly split by the induced Frobenius splitting.*

Proof. For the proof of (1), (2) and (3) see Prop.1.9 in [16]. For the proof of (4) let D'' denote the effective Cartier divisor $D - D'$ and let s_D, $s_{D'}$ and $s_{D''}$ denote the canonical sections of $\mathcal{O}_Y(D)$, $\mathcal{O}_Y(D')$ and $\mathcal{O}_Y(D'')$ respectively. Let

$$\phi : F_*^n \mathcal{O}_Y(D) \to \mathcal{O}_Y$$

denote the splitting of the map $F^n(D)$ which exists by assumption. Consider now the diagram

$$\begin{tikzcd} \mathcal{O}_Y \arrow[r, "(F^n)^\sharp"] \arrow[dr, "F^n(D)"'] & F_*^n \mathcal{O}_Y \arrow[r, "s_{D'}"] \arrow[d, "s_D"] & F_*^n \mathcal{O}_Y(D') \arrow[dl, "s_{D''}"'] \arrow[d, "\phi'"] \\ & F_*^n \mathcal{O}_Y(D) \arrow[r, "\phi"'] & \mathcal{O}_Y \end{tikzcd}$$

where ϕ' is defined such that the diagram is commutative. It follows that ϕ' defines a stable splitting along D' and that it induces the same Frobenius splitting

$$F_* \mathcal{O}_Y \to \mathcal{O}_Y$$

as the splitting ϕ along D.

Let now D and D' be as described in (5) and let $\phi : F_*^n \mathcal{O}_Y(D) \to \mathcal{O}_Y$ and $\phi' : F_*^m \mathcal{O}_Y(D') \to \mathcal{O}_Y$ denote the associated stable splittings. By applying the projection formula and the isomorphism $F^* \mathcal{L} \simeq \mathcal{L}^p$, when $\mathcal{L}$ is any invertible sheaf on Y, we may define the map

$$\eta : F_*^{n+m} \mathcal{O}_Y(D + p^n D') \simeq F_*^m(\mathcal{O}_Y(D') \otimes F_*^n \mathcal{O}_Y(D)) \to F_*^m \mathcal{O}_Y(D'),$$

where the latter map is induced by tensoring ϕ with $\mathcal{O}_Y(D')$ and applying the functor F_*^m. The composition of η with ϕ' then defines a stable Frobenius splitting along $D + p^n D'$, and it is easily checked that the induced Frobenius splitting coincides with the one induced by ϕ'. The statement now follows from (4).

Next we prove (6). Consider an effective Cartier divisor D and a Frobenius splitting of Y along $(p-1)D$, defined by the morphism

$$\phi : F_* \mathcal{O}_Y((p-1)D) \to \mathcal{O}_Y.$$

As the statement of (6) is a local condition we may assume that Y is affine and that $\mathcal{O}_Y(D) \simeq \mathcal{O}_Y$. Let s be the regular function on Y associated, under the latter isomorphism, to the canonical section of $\mathcal{O}_Y(D)$. Then ϕ is identified with an $\mathcal{O}_Y$-linear morphism $\tilde{\phi} : F_* \mathcal{O}_Y \to \mathcal{O}_Y$, and the induced Frobenius splitting is defined by

$$\eta : F_* \mathcal{O}_Y \to \mathcal{O}_Y, \quad t \mapsto \tilde{\phi}(t s^{p-1}).$$

In particular, for $t \in \mathcal{O}_Y(Y)$ it follows that $\eta(ts) = s\tilde{\phi}(t) \in (s)$. Hence the zero subscheme of the canonical section of D, whose ideal is generated by s, is compatibly Frobenius split. □

We also record the following result.

Lemma 3.2. *Let $f : \tilde{X} \to X$ be a morphism of projective varieties. Let $\tilde{Y}$ be a closed subvariety of $\tilde{X}$ and put $Y := f(\tilde{Y})$. Assume that $\tilde{X}$ is stably Frobenius split along an ample effective divisor $\tilde{D}$ not containing $\tilde{Y}$, and that the induced Frobenius splitting of $\tilde{X}$ compatibly splits $\tilde{Y}$. If the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{\tilde{X}}$ is an isomorphism, then the map $\mathcal{O}_Y \to f_*\mathcal{O}_{\tilde{Y}}$ is an isomorphism as well.*

Proof. It suffices to show that the composition $\mathcal{O}_X \to f_*\mathcal{O}_{\tilde{X}} \to f_*\mathcal{O}_{\tilde{Y}}$ is surjective. This will follow if the map

$$\Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{L} \otimes f_*\mathcal{O}_{\tilde{Y}})$$

is surjective for any very ample invertible sheaf $\mathcal{L}$ on X. By the projection formula, this amounts to the surjectivity of the restriction map

$$\Gamma(\tilde{X}, f^*\mathcal{L}) \to \Gamma(\tilde{Y}, f^*\mathcal{L}).$$

The latter map is part of a commutative diagram

$$\begin{tikzcd}
\Gamma(\tilde{X}, f^*\mathcal{L}) \arrow[r] \arrow[d, shift right] & \Gamma(\tilde{Y}, f^*\mathcal{L}) \arrow[d, shift right] \\
\Gamma(\tilde{X}, F^n_*(\mathcal{O}_{\tilde{X}}(\tilde{D}) \otimes f^*\mathcal{L}^{p^n})) \arrow[r] \arrow[u, shift right] & \Gamma(\tilde{Y}, F^n_*(\mathcal{O}_{\tilde{X}}(\tilde{D}) \otimes f^*\mathcal{L}^{p^n})) \arrow[u, shift right]
\end{tikzcd}$$

where the split vertical maps are induced from the stable Frobenius splitting of $\tilde{X}$ (resp. $\tilde{Y}$) along $\tilde{D}$ (resp. $\tilde{D} \cap \tilde{Y}$ using Lemma 3.1(3)), and where the horizontal maps are restriction maps. By Prop.3 in [13] the lower horizontal map is surjective. Hence, as the splittings of the vertical maps are compatible, we conclude that the upper horizontal map is also surjective. □

Assume now that Y is a nonsingular variety and let ω_Y denote its dualizing sheaf. By duality for the finite morphism F it follows that

$$\mathcal{H}om_{\mathcal{O}_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y) \simeq F_*(\omega_Y^{1-p}),$$

which means that a Frobenius splitting of Y is the same as a global section of ω_Y^{1-p} with certain properties. More precisely, let

$$C : F_*(\omega_Y^{1-p}) \to \mathcal{O}_Y,$$
$$s \mapsto s(1),$$

be the morphism defined by the isomorphism above. Then a global section s of ω_Y^{1-p} defines a Frobenius splitting if and only if $C(s)$ coincides with the constant function 1 on Y. Assume that s is a global section of ω_Y^{1-p} which defines a Frobenius splitting, and let D denote the divisor of zeroes of s. Then, by the discussion above, the composed map $C \circ F^1(D)$ is the identity map on $\mathcal{O}_Y$ and hence C defines a Frobenius splitting of Y along D.

3.2. **A criterion for global F-regularity.** The following important result by Smith (see Thm.3.10 in [20]) connects global F-regularity, Frobenius splitting and strong F-regularity.

Theorem 3.1. *If X is a projective variety over k then the following are equivalent*

(1) *X is globally F-regular.*
(2) *X is stably Frobenius split along an ample effective Cartier divisor D and the (affine) complement $X \setminus D$ is strongly F-regular.*
(3) *X is stably Frobenius split along every effective Cartier divisor.*

The connection between (1) and (3) in this theorem leads to the following result which can be found in [12].

Corollary 3.2. *Let $f : \tilde{X} \to X$ be a morphism of projective varieties. If the map $f^\sharp : \mathcal{O}_X \to f_* \mathcal{O}_{\tilde{X}}$ is an isomorphism and $\tilde{X}$ is globally F-regular then X is also globally F-regular.*

4. Equivariant embeddings of reductive groups

In this section G will denote a connected reductive algebraic group over an algebraically closed field k of arbitrary characteristic. We will fix a Borel subgroup B and a maximal torus $T \subseteq B$ of G. The Weyl group $N_G(T)/T$ will be denoted by W. For any $w \in W$ we denote by $\dot{w}$ a representative in $N_G(T)$. The set of roots defined by T will be denoted by Φ. To each root α is associated a reflection s_α in W. We choose the set of positive roots Φ^+ to consist of the roots in Φ defined by B, i.e. Φ^+ consists of the T-weights of the Lie algebra of the unipotent radical of B. The set of positive simple roots will be denoted by Δ and the associated simple reflections will be denoted by $s_1, \ldots, s_\ell$. Each element w in W is a product of simple reflections and the least number of factors needed in such a product will be denoted by $\ell(w)$ and will be called the length of w. The unique element in W of maximal length is denoted by w_0.

We will denote by Λ the character group of T and by Λ^+ the subset of dominant weights (i.e., those characters having nonnegative scalar product with all the simple coroots). We have a partial ordering $\leq$ on the group Λ, where $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a linear combination of the simple roots with nonnegative integer coefficients.

For any $w \in W$, the double coset $B\dot{w}B$ is a locally closed subvariety of G which only depends on w; we will denote this subvariety by BwB. By the Bruhat decomposition the group G is the disjoint union of the double cosets BwB, $w \in W$. Moreover, $\dim(BwB) = \ell(w) + \dim(B) = \ell(w) + \ell(w_0) + \ell$. The closure in G of any BwB is the union of the BvB, where $v \in W$ and $v \leq w$ for the Bruhat ordering of W.

An *equivariant embedding* of G is a normal $G \times G$-variety X containing G as an open subset and where the induced $G \times G$-action on G is given by left and right translation. (In other words, X is an equivariant embedding of the homogeneous space $G \times G / \operatorname{diag} G \simeq G$.)

The *boundary* of the equivariant embedding X is the closed $G \times G$-stable subset $X \setminus G$, denoted by ∂X. Its irreducible components $D_1, \ldots, D_n$ are the *boundary divisors*; they are indeed of codimension 1, as the open subset G is affine (see Prop.II.3.1 in [7]).

When X is an equivariant embedding of G we denote by $X(w)$, $w \in W$, the closure in X of the double coset BwB (in particular, $X(w_0) = X$). In this section we want to study the geometry of these *large Schubert varieties* $X(w)$. Those of codimension 1 are the $X(w_0 s_i)$, $i = 1, \ldots, \ell$; they will be denoted by $X_1, \ldots, X_l$.

4.1. **Two preliminary geometric results.** A key ingredient in our study is the following

Proposition 4.1. *Let X be a projective embedding of G, with boundary divisors $D_1, \ldots, D_n$.*

(1) *There exists a very ample $G \times G$-linearized invertible sheaf $\mathcal{L}$ over X such that* $\operatorname{Spec} R(X, \mathcal{L})$ *is an affine embedding of the group $G \times \mathbb{G}_m$, where action of the multiplicative group $\mathbb{G}_m$ on* $\operatorname{Spec} R(X, \mathcal{L})$ *corresponds to the grading of $R(X, \mathcal{L})$.*
(2) *There exist positive integers $a_1, \ldots, a_n$ such that the invertible sheaf*

$$\mathcal{O}_X\Big(\sum_{i=1}^{n} a_i D_i\Big)$$

is ample.

Proof. (1) We may find a very ample $G \times G$-linearized invertible sheaf $\mathcal{L}$ on X, see e.g. Cor.1.6 in [14]. Then the pull-back of $\mathcal{L}$ to the open orbit $G \simeq G \times G/\operatorname{diag} G$ is the linearized invertible sheaf associated with a character of the isotropy group $\operatorname{diag} G$. Such a character extends to a character of $G \times G$, so that (changing the linearization) we may assume that the pull-back of $\mathcal{L}$ to G is trivial as a linearized invertible sheaf.

Replacing $\mathcal{L}$ with some positive power, we may also assume that the ring $R(X, \mathcal{L})$ is normal. Then $\hat{X} := \operatorname{Spec} R(X, \mathcal{L})$ is a normal affine variety endowed with an action of $G \times G \times \mathbb{G}_m$, where $\mathbb{G}_m$ acts via the grading of $R(X, \mathcal{L})$. By our assumptions on $\mathcal{L}$, the affine cone $\hat{X}$ is an affine embedding of the group $G \times \mathbb{G}_m =: \hat{G}$.

(2) Let $\hat{X}$, $\hat{G}$ as above. Then $\hat{X}$ is a linear algebraic monoid with unit group $\hat{G}$, by Prop.1 in [17]. So, by Thm.3.15 in [15], $\hat{X}$ admits an embedding into some matrix ring $M_n(k)$ as a closed submonoid (with respect to the multiplication of matrices). We claim that $\hat{G}$ identifies with $\hat{X} \cap \mathrm{GL}_n(k)$ under this embedding. Indeed, the inclusion $\hat{G} \subseteq \hat{X} \cap \mathrm{GL}_n(k)$ is clear. Conversely, if $\gamma \in \hat{X} \cap \mathrm{GL}_n(k)$ then the images $\gamma^i \hat{X}$ form a decreasing sequence of closed subsets of $\hat{X}$. Thus, $\gamma^i \hat{X} = \gamma^{i+1} \hat{X}$ for $i \gg 0$. It follows that $\hat{X} = \gamma \hat{X}$, whence γ has a right inverse. Likewise, γ has a left inverse, which completes the proof of the claim.

Let s be the regular function on $\hat{X}$ given by the restriction of the determinant function on $M_n(k)$. By the claim, the zero set of s is precisely the boundary $\partial \hat{X}$. Further, s is an eigenvector of $\mathbb{G}_m$, by the multiplicative property of the determinant. So s is a section of a positive power of $\mathcal{L}$, with zero set being ∂X. $\square$

We also recall the following result which is known under a stronger form (Prop.3 in [18], see also Prop.6.2.5 in [5]).

Lemma 4.1. *For any equivariant embedding X of G, there exists a nonsingular equivariant embedding $\tilde{X}$ and a projective morphism*

$$f : \tilde{X} \to X$$

which induces the identity on G.

We will refer to f as an *equivariant resolution* of X. Note that f is $G \times G$-equivariant and birational. Since X is assumed to be normal, it follows that $f_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. Further, $\tilde{X}$ is projective if X is. Also, note that f restricts to a birational morphism $\tilde{X}(w) \to X(w)$, for any $w \in W$. Together with Lemma 3.2, this will allow us to reduce questions on $X(w)$ to the case where X is nonsingular.

4.2. **Frobenius splitting of nonsingular embeddings.** Now fix a nonsingular equivariant embedding X of G. We assume from now on that the ground field k has characteristic $p > 0$. It is known (see [18], or Prop.6.2.6 in [5]) that the inverse of the dualizing sheaf on X equals

$$\omega_X^{-1} \simeq \mathcal{O}_X\Big(\partial X + \sum_{i=1}^{l}(X_i + \tilde{X}_i)\Big),$$

where $\tilde{X}_i = (\dot{w}_0, \dot{w}_0)X_i$, and that the $(p-1)$-th power s_X^{p-1} of the canonical section s_X of the right hand side defines a Frobenius splitting of X. As noticed at the end of Section 3.1, this yields in fact a splitting of X along D, where

$$D := (p-1)\Big(\partial X + \sum_{i=1}^{l}(X_i + \tilde{X}_i)\Big).$$

This leads to the following result.

Proposition 4.2. *Let X denote a nonsingular equivariant embedding of G over a field of characteristic $p > 0$. Then X is Frobenius split along $(p-1)\partial X$, compatibly with the large Schubert subvarieties $X(w)$, $w \in W$.*

Proof. Denote by $\eta : F_*\mathcal{O}_X \to \mathcal{O}_X$ the underlying Frobenius splitting of X induced by s_X. By Lemma 3.1(2)(6), each X_i is compatibly Frobenius split by η. In other words, η is compatible with the $X(w)$, where $\ell(w) = \ell(w_0) - 1$. Now consider $w \in W$ such that $\ell(w) \leq \ell(w_0) - 2$. By Lem.10.3 in [1], there exist distinct w_1, w_2 in W such that $w < w_1$, $w < w_2$, and $\ell(w_1) = \ell(w_2) = \ell(w) + 1$. Then $X(w)$ is contained in $X(w_1) \cap X(w_2)$ as an irreducible component. Now Lemma 3.1(1)(2) implies by decreasing induction on $\ell(w)$ that $X(w)$ is compatibly Frobenius split by η. That η is induced by a Frobenius splitting of X along $(p-1)\partial X$ follows from Lemma 3.1(4). □

4.3. **The main results.** We still assume that k has characteristic $p > 0$.

Theorem 4.3. *Let X denote a projective equivariant embedding of G. Then each $X(w)$, $w \in W$, is globally F-regular.*

Proof. First we consider the case where X is nonsingular. Then, by Lemma 3.1(3) and Proposition 4.2 each $X(w)$ is Frobenius split along $(p-1)(\partial X \cap X(w))$. Together with Lemma 3.1(4)(5), it follows that $X(w)$ is stably Frobenius split along any divisor $\sum_{i=1}^{n} a_i(D_i \cap X(w))$, with $a_i > 0$. By Proposition 4.1, we may find such a divisor which is ample on X. Then the restriction

$$D = \sum_{i=1}^{n} a_i(D_i \cap X(w)),$$

is an effective ample Cartier divisor on $X(w)$ with support $\partial X \cap X(w)$. Hence, by Theorem 3.1 it is enough to prove that the open affine subset

$$G(w) = X(w) \setminus \partial X$$

is strongly F-regular.

Notice that the set $G(w)$ coincides with the closure of BwB in G. Hence, there is a surjective map

$$\pi(w) : G(w) \to S(w) \subseteq G/B,$$

onto the corresponding Schubert variety $S(w)$. By [12] $S(w)$ is globally F-regular and hence locally strongly F-regular. Moreover, by the Bruhat decomposition there exists a covering of $S(w)$ by open affine subsets U_i, $i \in I$, such that $\pi(w)^{-1}(U_i) \simeq U_i \times B$. As B is smooth and U_i is strongly F-regular it follows that $U_i \times B$ is strongly F-regular (Lem.4.1 in [11]). Hence, the affine variety $G(w)$ is also strongly F-regular. This completes the proof in the case of nonsingular X.

In the general case, we may choose an equivariant resolution $f : \tilde{X} \to X$ (Lemma 4.1). By the considerations above the equivariant embedding $\tilde{X}$ is stably Frobenius split along an ample Cartier divisor. Furthermore according to the last part of Lemma 3.1(4), this stable splitting may be chosen such that each $\tilde{X}(w)$ is compatibly Frobenius split. Then, by Lemma 3.2, the map $\mathcal{O}_{X(w)} \to f_* \mathcal{O}_{\tilde{X}(w)}$ is an isomorphism and the global F-regularity of $X(w)$ hence follows from Corollary 3.2. □

Corollary 4.1. *Let X denote an affine equivariant embedding of G. Then each $X(w)$, $w \in W$, is strongly F-regular.*

Proof. We may embed X as a closed $G \times G$-stable subvariety of a $G \times G$-module M. Let $\overline{X}$ be the normalization of the closure of X in the projectivization of $M \oplus k$. Then $\overline{X}$ is a projective equivariant embedding of G containing X as an open affine subset. By Theorem 4.3 each $\overline{X}(w)$ is globally F-regular. In particular, every local ring of $\overline{X}(w)$ is strongly F-regular. As $X(w)$ is an open subset of $\overline{X}(w)$ this implies that every local ring of the affine variety $X(w)$ is strongly F-regular. This proves the claim as the condition of being strongly F-regular is local. □

Corollary 4.2. *Let X denote any equivariant embedding of G. Then each $X(w)$, $w \in W$, is normal and Cohen-Macaulay.*

Proof. This follows from Theorem 4.3 by using that X has an open cover by equivariant embeddings which are also open subsets of projective equivariant embeddings, see [21, 22]. □

4.4. **From characteristic p to characteristic 0.** In this section, k is of characteristic 0. We will obtain versions of Theorem 4.3 and of Corollaries 4.1, 4.2, by using the notions of strongly (resp. globally) F-regular type ([20]) that we briefly review.

Let Y be a scheme of finite type over k. Then Y is defined over some finitely generated subring A of k. This yields a scheme Y_A which is flat and of finite type over $\mathrm{Spec}(A)$, such that Y is naturally identified with the scheme $Y_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$. On the other hand, the geometric fibers of Y_A at closed points of $\mathrm{Spec}(A)$ are schemes over algebraic closures of finite fields (of various characteristics).

Definition. ([20]) An affine (resp. projective) variety X is *of strongly (resp. globally) F-regular type* if X is defined over some finitely generated subring A of k such that the geometric fibers of X_A over a dense subset of closed points of $\mathrm{Spec}(A)$ are strongly (resp. globally) F-regular.

Remember that any strongly (resp. globally) F-regular variety is locally F-rational. It follows that any variety X of strongly (resp. globally) F-regular type is of F-rational type (this latter notion is defined similarly to the definition of strongly/globally F-regular type). Hence by Thm4.3. in [19] it follows that X has rational singularities, in particular, X is normal and Cohen-Macaulay.

Theorem 4.4. *Let X denote an affine (resp. projective) equivariant embedding of G over a field of characteristic 0. Then any $X(w)$, $w \in W$, is of strongly (resp. globally) F-regular type.*

Proof. By Proposition 4.1 (1), it suffices to treat the affine case. For this, we will recall the classification of affine equivariant embeddings, after [23] (generalized in [17] to arbitrary characteristic), and show that any such embedding X is defined and flat over $\mathrm{Spec}(\mathbb{Z})$.

Put $R := \Gamma(G, \mathcal{O}_G)$ and $S := \Gamma(X, \mathcal{O}_X)$, then S is a $G \times G$-stable subalgebra of R. Further, S is finitely generated and normal, with the same quotient field as R. Recall the isomorphism of $G \times G$-modules

$$R \cong \bigoplus_{\lambda \in \Lambda^+} \nabla(\lambda) \otimes \nabla(-w_0\lambda),$$

where $\nabla(\lambda)$ denotes the simple G-module with highest weight λ. It follows that

$$S \cong \bigoplus_{\lambda \in \mathcal{M}} \nabla(\lambda) \otimes \nabla(-w_0\lambda),$$

for some subset $\mathcal{M}$ of Λ^+. Thus, the weights of $T \times T$ in the invariant subring $S^{U \times U}$ are exactly the $(\lambda, -w_0\lambda)$, where $\lambda \in \mathcal{M}$; each such weight has multiplicity 1. Since $S^{U \times U}$ is a finitely generated, normal domain (see e.g. [6]), the corresponding affine variety is a toric variety for the left T-action. Thus, $\mathcal{M}$ is the intersection of Λ with a rational polyhedral convex cone of nonempty interior in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, contained in the positive chamber.

One may show that $\mathcal{M}$ satisfies the following saturation property: For any $\lambda \in \mathcal{M}$ and $\mu \in \Lambda^+$ such that $\mu \leq \lambda$, then $\mu \in \mathcal{M}$. Conversely, any $\mathcal{M}$ satisfying the preceding properties yields an affine embedding of G, see [23].

Next let $G_{\mathbb{Z}}$ be the split $\mathbb{Z}$-form of G, with affine coordinate ring $R_{\mathbb{Z}}$. For any ring A, this defines the ring $R_A := R_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$ and the corresponding group G_A. In particular, we obtain the $\mathbb{Q}$-form $R_{\mathbb{Q}}$ of R. Now the preceding decomposition of R is defined over $\mathbb{Q}$; further, the subspace

$$S_{\mathbb{Q}} := S \cap R_{\mathbb{Q}} = \bigoplus_{\lambda \in \mathcal{M}} \nabla_{\mathbb{Q}}(\lambda) \otimes \nabla_{\mathbb{Q}}(-w_0\lambda)$$

(with obvious notation) is a subalgebra of $R_{\mathbb{Q}}$, and a $\mathbb{Q}$-form of S. Put $S_{\mathbb{Z}} := S_{\mathbb{Q}} \cap R_{\mathbb{Z}}$ (then the quotient $R_{\mathbb{Z}}/S_{\mathbb{Z}}$ is torsion-free), and

$$R_p := R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}_p}, \quad S_p := S_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}_p},$$

where p is any prime number, $\mathbb{F}_p$ denotes the field with p elements, and $\overline{\mathbb{F}_p}$ denotes its algebraic closure. Define likewise the connected reductive group G_p over $\overline{\mathbb{F}_p}$ and its subgroups B_p, T_p, U_p. Then $R_p = \Gamma(G_p, \mathcal{O}_{G_p})$, and S_p is a $\overline{\mathbb{F}_p}$-subalgebra of R_p, stable under the action of $G_p \times G_p$. We will show that S_p is the coordinate ring of an affine equivariant embedding X_p of G_p.

By Prop.II.4.20 in [9] (see also Thm.4.2.5 in [5]), the $G_p \times G_p$-module R_p has an increasing filtration with subquotients being the $\nabla_p(\lambda) \otimes \nabla_p(-w_0\lambda)$ ($\lambda \in \Lambda^+$), where now $\nabla_p(\lambda)$ denotes the dual Weyl module of highest weight λ. Further, the proof of this result given in [5] also shows that the $G_p \times G_p$-module S_p has an increasing filtration with subquotients being the $\nabla_p(\lambda) \otimes \nabla_p(-w_0\lambda)$ ($\lambda \in \mathcal{M}$). In particular, this module has a good filtration. Using Lem.II.2.13 and Prop.II.4.16 in [9], it follows that the weights of $T_p \times T_p$ in the invariant subring $S_p^{U_p \times U_p}$ are again the $(\lambda, -w_0\lambda)$, where $\lambda \in \mathcal{M}$; each such weight has multiplicity 1. Therefore, the algebra $S_p^{U_p \times U_p}$ is finitely generated and normal. By [6], the algebra S_p is finitely generated and normal as well.

Put $X_p := \mathrm{Spec}(S_p)$, then X_p is a normal affine variety where $G_p \times G_p$ acts with a dense orbit. We now show that this orbit is isomorphic to $G_p \times G_p / \operatorname{diag} G_p$; equivalently, the morphism $G_p \to X_p$ associated with the inclusion $S_p \subseteq R_p$ is an open immersion. Since the corresponding morphism $G \to X$ is an open immersion, we may find $f \in S^{U \times U}$ with zero set the complement of the open $B \times B$-orbit Bw_0B. Replacing f with a scalar multiple, we may assume that $f \in S_{\mathbb{Z}}$ is a lift of a nonzero $f_p \in S_p^{U_p \times U_p}$. Then $R[f^{-1}] = S[f^{-1}] = \Gamma(Bw_0B, \mathcal{O}_{Bw_0B})$, so that $R_{\mathbb{Z}}[f^{-1}] = S_{\mathbb{Z}}[f^{-1}]$. Thus, $R_p[f_p^{-1}] = S_p[f_p^{-1}]$; equivalently, the morphism $(f_p \neq 0) = B_p w_0 B_p \to X_p$ is an open immersion. Since the $G_p \times G_p$-translates of $B_p w_0 B_p$ cover G_p, we have shown that the reduction X_p is an equivariant embedding of G_p. Further, since all the double classes BwB in G are defined over $\mathbb{Z}$, their closures $X(w)$ in X are also defined over $\mathbb{Z}$, with reductions $X(w)_p$. $\square$

By the argument of Corollary 4.2, this implies readily

Corollary 4.3. *Let X denote an equivariant embedding of G over a field of characteristic 0. Then each $X(w)$, $w \in W$, has rational singularities.*

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