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1. Introduction

Let G be a reductive algebraic group over a field k of characteristic $p \geq 0$. If $\mathcal{L}$ is a line bundle on the flag variety $\mathfrak{X}$ for G then the cohomology modules $H^i(\mathfrak{X}, \mathcal{L}),\ i \geq 0$ have a natural G-structure. The G-modules arising in this way play a prominent role in the representation theory of G. This is for instance illustrated by the following four results.

1.1. **The Chevalley classification of irreducible G-modules** [15]**.** This theorem says that all the finite dimensional irreducible G-modules occur as submodules in $H^0(\mathfrak{X}, \mathcal{L})$ with $\mathcal{L}$ running through the set of effective line bundles on $\mathfrak{X}$. Moreover, if $p = 0$ then any line bundle on $\mathfrak{X}$ has at most one non-vanishing cohomology module, and that one is irreducible. This last result is the Borel-Weil-Bott theorem [13].

1.2. **The strong linkage principle** [2]**.** The Borel-Weil-Bott theorem mentioned above fails badly when $p > 0$. As a weaker substitute for this result we have in positive characteristics the strong linkage principle. It says that the composition factors of a given cohomology module $H^i(\mathfrak{X}, \mathcal{L}),\ i \geq 0, \mathcal{L}$ a line bundle on $\mathfrak{X}$, have highest weights strongly linked (see loc. cit.) to the dominant weight which is conjugated under the Weyl group to the weight determining $\mathcal{L}$.

1.3. **The sum formula for Weyl modules** [24]**,** [5]**.** Via Kempf's vanishing theorem [25] (cf. also 2.2.e) below) one may identify Weyl modules with the top cohomology modules $H^N(\mathfrak{X}, \mathcal{L}),\ N = \dim \mathfrak{X},\ \mathcal{L}$ running through the antidominant line bundles on $\mathfrak{X}$. Using $\mathbb{Z}$-forms of G and $\mathfrak{X}$ one may for each such $\mathcal{L}$ define a filtration of $H^N(\mathfrak{X}, \mathcal{L})$ and compute the sum of its terms. In low rank this allows a calculation of all the irreducible characters for G.

1.4. **Restriction to Schubert varieties** [29]**,** [30]**,** [7]**.** Vanishing theorems for the higher cohomology of the restrictions of effective line bundles on $\mathfrak{X}$ to Schubert varieties have played a big role in establishing geometric properties like normality and Cohen Macaulayness for these. They have also lead to the Demazure character formula first formulated in [16].

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1.5. Let now U_q denote the quantum group corresponding to G. Here q is an arbitrary nonzero element of k and U_q is the specialization of the Lusztig integral form of the generic quantum group. Then we have quantized versions of the cohomology modules $H^i(\mathfrak{X}, \mathcal{L})$ constructed via induction from the quantized Borel subalgebra to U_q, cf. [11]. It turns out that there are analogues of all the above results. When q is not a root of unity then the Borel-Weil-Bott theorem carries over while the other statements above hold (with l replacing the role of p) when q is an l-th root of unity.

Despite the many efforts leading to the results we have mentioned in 1.1–4 (as well as to many further related theorems) there is still an abundance of open questions concerning the cohomology $H^i(\mathfrak{X}, \mathcal{L})$. In this note we have collected some of the known facts about these modules and at the same time we have tried to call attention to several such open problems. We do this in such a way that each result and question may easily be 'quantized'.

Related to the problem of describing the cohomology of line bundles on $\mathfrak{X}$ is the calculation of the Hochschild cohomology groups for B, $H^i(B, \lambda) = \mathrm{Ext}^i_B(k, k_\lambda), i \geq 0, \lambda$ a character of B (considered as a 1-dimensional B-module denoted either just λ or sometimes k_λ). We shall consider the quantized root of unity analogues of these computations and show that they are related to the cohomology of line bundles on the cotangent bundle for $\mathfrak{X}$. Even when $p = 0$ this latter cohomology is not known. We illustrate the complexity of this problem by giving some details of the computations when $G = SL_3$.

2. Vanishing behaviour

In this section we shall consider the vanishing behaviour of the cohomology of line bundles on $\mathfrak{X}$: For a given i we ask for all line bundles $\mathcal{L}$ such that $H^i(\mathfrak{X}, \mathcal{L}) \neq 0$. Likewise we can fix $\mathcal{L}$ and ask for the set of i such that $H^i(\mathfrak{X}, \mathcal{L}) \neq 0$.

2.1. **Notation.** Let k, p, G, and $\mathfrak{X}$ be as in the introduction. Choose a Borel subgroup B in G and identify $\mathfrak{X}$ with G/B. Let T be a maximal torus in B and set X equal to the character group of T (and of B). In the root system R for (G, T) we choose a set of simple roots S such that the corresponding positive roots are the roots of the Borel subgroup opposite to B. The dominant weights X^+ is the subset consisting of those $\lambda \in X$ for which $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. Here we have denoted by $\alpha^\vee$ the coroot of a root α.

When $p > 0$ we set for each $r > 0$

$$X_r = \{\lambda \in X^+ \mid \langle \lambda, \alpha^\vee \rangle < p^r, \quad \alpha \in S\}.$$

This is called the set of p^r-restricted weights. For each $\lambda \in X$ we then have a unique expansion $\lambda = \lambda_0 + p^r \lambda_1$ with $\lambda_0 \in X_r$ and $\lambda_1 \in X$.

The Weyl group W for (G, T) acts naturally on X. Moreover, we have the 'dot' action given by $w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \in W, \quad \lambda \in X$. Here $2\rho = \sum_{\alpha \in R^+} \alpha$.

We shall also use the dot notation for the action of p^r on X given by $p^r \cdot \lambda = p^r(\lambda + \rho) - \rho$. Note that this commutes with the W dot action, i.e we have $w \cdot p^r \cdot \lambda = p^r \cdot w \cdot \lambda$ for all $w \in W$.

All modules considered in this paper will be finite dimensional. For any B-module E we denote by $\mathcal{L}(E)$ the vector bundle on $\mathfrak{X}$ induced by E. We shall then write $H^i(E)$ short for $H^i(\mathfrak{X}, \mathcal{L}(E))$. In particular, we shall consider the weights $\lambda \in X$ as 1-dimensional B-modules and study the corresponding cohomology modules $H^i(\lambda)$.

2.2. Standard base change arguments show that the vanishing behaviour of $H^i(\lambda)$ only depends on p (not on k itself). Therefore we set for $i \geq 0$

$$D_p(i) = \{\lambda \in X \mid H^i(\lambda) \neq 0\}.$$

Then the problem we are studying in this section consists of describing the subsets $D_p(i)$. In general, this is a wide open problem. However, certain cases are known:

a) The Borel-Weil-Bott theorem (cf. 1.1) says that

$$D_0(i) = \bigcup_{w \in W,\, l(w)=i} w \cdot X^+.$$

b) It is well known [22] that independently of p we have

$$D_p(0) = X^+.$$

Serre duality says that for any finite dimensional B-module E we have isomorphisms of G-modules $H^i(E)^* \simeq H^{N-i}(E^* \otimes -2\rho)$ for all $i \geq 0$. Here $N = \dim \mathfrak{X} = |R^+|$ and * denotes dual module with contragredient action.

This gives for all p

c) $D_p(i) = -D_p(N-i) - 2\rho$.

In particular, b) is then equivalent to

d) $D_p(N) = -X^+ - 2\rho$.

Finally we mention that Kempf's vanishing theorem [25] says (for all p)

e) $D_p(i) \cap X^+ =$ for $i > 0$.

2.3. The case $i = 1$ is completely solved in [1]. In the notation from 2.2 the result can be stated[1]

$$D_p(1) = \bigcup_{r \geq 0} (p^r \cdot D_0(1) - X_r). \tag{1}$$

Then by Serre duality (2.2.c)) we get

$$D_p(N-1) = \bigcup_{r \geq 0} (-p^r \cdot D_0(1) + X_r - 2\rho) = \bigcup_{r \geq 0} (p^r \cdot D_0(N-1) + X_r). \tag{2}$$

Remark 2.1. The above results completely describe the vanishing behaviour for the cohomology of line bundles on the 3-dimensional flag variety (i.e $G = SL_3$). This case was first solved by W. Griffith [21] (by methods completely different from those used in [1]. Apart from the trivial case $G = SL_2$ (where $\mathfrak{X} = \mathbb{P}^1$) this is still the only flag variety for which a full solution of our problem is known.

[1]Please note that equation numbers restarts from (1) in each subsection. Unless otherwise stated, a cross reference to an equation number only refer to the current subsection.

2.4. We shall now recall a result whose main application is a short proof of the Kempf vanishing theorem. As we shall see it has also some further bearing on our problem.

Let $r \geq 0$ and set $St_r = L((p^r - 1)\rho)$. This is the r-th Steinberg module. Denote by $F_r : G \to G$ (as well as its restriction to B) the r-th Frobenius homomorphism. If M is a G- (or B-) module we denote by $M^{(r)}$ the same vector space but with G (or B) action given by $g \cdot m = F_r(g)m$, $g \in G$ (or B), $m \in M$.

Theorem 2.2. [4] *For any B-module E and any $i, r \geq 0$ there is a natural G-isomorphism $H^i(E)^{(r)} \otimes St_r \simeq H^i(E^{(r)} \otimes (p^r - 1)\rho)$.*

Note that for any two B-modules E_1 and E_2 we have 'cup product' maps $\cup_{i,j} : H^i(E_1) \otimes H^j(E_2) \to H^{i+j}(E_1 \otimes E_2)$, $i, j \geq 0$. The Frobenius homomorphism F_r gives rise to natural maps $F_r^* : H^i(E)^{(r)} \to H^i(E^{(r)})$, $i \geq 0$, E a B-module. The isomorphism in this theorem is the composite $\cup_{i,0} \circ (F_r^* \otimes 1)$. (Note that $St_r = H^0((p^r - 1)\rho)$ – an easy consequence of the strong linkage principle mentioned in the introduction.)

2.5.

Corollary 2.3. *Let $i \geq 0$. Then for all $r \geq 0$ we have*

$$p^r \cdot D_p(i) \subset D_p(i) \quad \text{and} \quad p^r D_p(i) \subset D_p(i).$$

Proof: The first inclusion is clear from Theorem 2.2 by taking $E = \lambda$. The theorem also implies (via the description of the isomorphism) that $F_r^* : H^i(E)^{(r)} \to H^i(E^{(r)})$ is injective for all i and all E. Taking again $E = \lambda$ we obtain the second inclusion.

Remark 2.4. The corollary contains in particular Kempf's vanishing theorem (cf. 2.2.e)): If $\lambda \in X^+$ then $\mathcal{L}(\lambda + \rho)$ is an ample line bundle on $\mathfrak{X}$ and therefore $H^i(p^r \cdot \lambda) = 0$ for $i > 0$ and $r \gg 0$. This means that for r large $p^r \cdot \lambda \notin D_p(i)$ for $i > 0$, and hence by the corollary $D_p(i) \cap X^+ = \emptyset$ for $i > 0$.

2.6. We can improve the result in Corollary 2.3 . Note that $p^r \cdot D_p(i) \subset p^r D_p(i) + X_r$ and $p^r \cdot D_0(i) - X_r = p^r D_0(i) + X_r$.

Proposition 2.5. [3] *For any $i, r \geq 0$ we have $p^r \cdot D_p(i) - X_r \subset D_p(i)$.*

Proof: Let $\lambda \in D_p(i)$, $\nu \in X_r$ and set $\nu' = (p^r - 1)\rho - \nu$. Then we have a commutative diagram

$$\begin{array}{ccc}
H^i(\lambda)^{(r)} \otimes H^0(\nu) \otimes H^0(\nu') & \xrightarrow{1 \otimes \cup_{0,0}} & H^i(\lambda)^{(r)} \otimes St_r \\
\Big\downarrow{\scriptstyle (\cup_{i,0} \circ (F_r^* \otimes 1)) \otimes 1} & & \Big\downarrow \\
H^i(p^r\lambda + \nu) \otimes H^0(\nu') & \xrightarrow[\cup_{i,0}]{} & H^i(p^r \cdot \lambda)
\end{array}$$

where the right vertical map is the isomorphism from Theorem 2.2 . Since St_r is irreducible the top horizontal map is surjective. Hence so is the bottom map and the proposition follows.

2.7. By semi-continuity we have $D_0(i) \subset D_p(i)$ for all i. Combining this with Serre duality and Proposition 2.5 we get

$$\begin{aligned} \lambda \in D_0(i) &\iff -\lambda - 2\rho \in D_0(N-i) \subset D_p(N-i) \\ &\implies -p^r\lambda - 2p^r\rho + X_r \subset D_p(N-i) \\ &\implies p^r\lambda + 2p^r\rho - X_r - 2\rho \subset D_p(i) \\ &\iff p^r \cdot \lambda + X_r \subset D_p(i). \end{aligned}$$

Hence $p^r \cdot D_0(i) + X_r \subset D_p(i)$. We therefore have

Corollary 2.6. *For every $i \in \mathbb{N}$ the following inclusion holds*

$$\bigcup_{r\geq 0} (p^r \cdot D_0(i) \pm X_r) \subset D_p(i).$$

Remark 2.7. a. For $i = 0, 1, N$ and $N-1$ we have equality in this corollary. For $i = 0, 1$ (respectively $i = N, N-1$) the '+'-sign (respectively '−'-sign) on the left is redundant.

b. Equality does not hold in general in Corollary 2.6. The first case where it fails is for type B_2 and $i = 2$, see [3]. In loc. cit. there are also some further study (in part based on the translation principle) of the sets $D_p(i)$.

c. The above results make it possible to find examples where $D_p(i) \cap D_p(j) \neq \emptyset$ for any $i, j > 0$: Suppose $0 < i < j$ and let $G = SL_{j+1}$. Choose $r > 0$ such that $p^r > j$. Let s_m denote the m-th simple reflection in W (in the standard enumeration) and set $\lambda = p^r(s_1 s_2 \cdots s_i \cdot 0)$. Then $\lambda \in p^r D_0(i) \subset D_p(i)$. On the other hand we have also $\lambda \in D_0(j)$ (there are exactly j positive roots α with $\langle \lambda + \rho, \alpha^\vee \rangle < 0$) so that also $\lambda \in D_p(j)$.

It is not clear whether in this case we have in fact $\lambda \in D_p(m)$ for all $i < m < j$.

3. G-structure

In this section we shall consider the G-module structure on $H^i(\lambda)$, $i \geq 0$, $\lambda \in X$. For instance, we can ask for the formal characters of these modules, their composition factors, submodule configurations etc. Even though all of these questions are wide open it is possible to prove some key facts which have non-trivial applications in the representation theory for G.

3.1. Recall that if M is a T-module and $\lambda \in X$ then the weight space M_λ is defined by $M_\lambda = \{m \in M \mid tm = \lambda(t)m,\ t \in T\}$. The character of M is defined by

$$\mathrm{ch}(M) = \sum_{\lambda \in X} (\dim M_\lambda) e^\lambda \ \in \mathbb{Z}[X].$$

For any B-module we denote the Euler character of the induced vector bundle $\mathcal{L}(E)$ by $\chi(E)$, i.e

$$\chi(E) = \sum_i (-1)^i \, \mathrm{ch}(H^i(E)).$$

Then χ is additive on short exact sequences of B-modules. This implies that $\chi(E) = \sum_\lambda \dim(E_\lambda)\chi(\lambda)$. Moreover, for any $\lambda \in X$ the Euler character $\chi(\lambda)$ is given by Weyl's character formula, see e.g [17]

$$\chi(\lambda) = \sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho)} \Big/ \sum_{w\in W} (-1)^{l(w)} e^{w(\rho)}.$$

This means in particular, that if all cohomology of $\mathcal{L}(\lambda)$ is concentrated in degree i then the character of $H^i(\lambda)$ is known: in that case $\mathrm{ch}(H^i(\lambda)) = (-1)^i\chi(\lambda)$. One such situation is when $\lambda \in X^+$ because then Kempf's vanishing theorem 2.2.e) gives the vanishing of all the higher cohomology such that

$$\mathrm{ch}(H^0(\lambda)) = \chi(\lambda) \text{ for all } \lambda \in X^+.$$

Remark 3.1.

a. We do not know $\mathrm{ch}(H^i(\lambda))$ in general. When λ is sufficiently generic (see [8]) in $D_0(i)$ one can prove that $H^i(\lambda)$ is in fact the only non-vanishing cohomology module. In this case the above applies.

b. Donkin [18], [19] has recently given an algorithm which computes $\mathrm{ch}(H^i(\lambda))$ in terms of smaller weights. To effectively apply this algorithm one has for some of these smaller weights to deal with vector bundles of rank more than 1.

3.2. Even though we cannot determine the vanishing behaviour nor find the G-structure of the individual cohomology modules $H^i(\lambda)$ we can still prove

Theorem 3.2. *(The Strong Linkage Principle) Let $\mu,\ \lambda+\rho \in X^+$. If $L(\mu)$ is a composition factor of $H^i(w\cdot\lambda)$ for some $w \in W,\ i \geq 0$ then μ is strongly linked to λ.*

Note that any $\lambda' \in X$ may be written in the form $\lambda' = w\cdot\lambda$ for some $w \in W,\ \lambda+\rho \in X^+$.

For the definition of the strong linkage relation and for the proof of this theorem we refer to [2]. See also [9] for the quantum case.

Corollary 3.3. *Let M be an indecomposable G-module and let $\lambda,\ \mu \in X^+$. If $L(\lambda)$ and $L(\mu)$ both are composition factors of M then $\lambda \in W_p\cdot\mu$.*

Here W_p denotes the affine Weyl group associated with G. This is the group generated by W together with all translations on X by elements of pR.

We sketch a proof (for details see e.g [9])

Proof: It is enough to check that if $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$ then $\lambda \in W_p\cdot\mu$. For this we may assume $\lambda \not> \mu$. In this case easy weight considerations show that $\mathrm{Ext}^1_G(L(\lambda), H^0(\mu)) = 0$. The exact sequence $0 \to L(\mu) \to H^0(\mu) \to H^0(\mu)/L(\mu) \to 0$ therefore gives that $L(\lambda)$ is a composition factor of $H^0(\mu)$ (in fact that it is a submodule of $H^0(\mu)/L(\mu)$). Now Theorem 3.2 gives the conclusion.

3.3. It is well-known (and a key ingredient in the Chevalley classification theorem mentioned in the introduction) that $H^0(\lambda)$ contains $L(\lambda)$ as its unique simple submodule for all $\lambda \in X^+$. It is also true that $H^1(\lambda)$ has simple socle for all $\lambda \in D_p(1)$:

Theorem 3.4. *Let $\lambda \in D_p(1)$. Then there exists a unique simple root α with $\langle \lambda+\rho, \alpha^\vee\rangle < 0$. If $\langle \lambda+\rho, \alpha^\vee\rangle = -ap^r$ for some $0 < a < p$ and $r \geq 0$ then the socle of $H^1(\lambda)$ is $L(s_\alpha \cdot \lambda)$. If $-(a+1)p^r < \langle \lambda+\rho, \alpha^\vee\rangle < -ap^r$ for some $0 < a < p$ and $r > 0$ then the socle of $H^1(\lambda)$ is $L(\lambda + ap^r\alpha)$.*

We refer to [1] for the proof.

Via Serre duality we deduce that $H^N(\lambda)$ and $H^{N-1}(\lambda)$ for $\lambda \in D_p(N)$, respectively $D_p(N-1)$, have simple heads.

Remark 3.5. As already observed it is not generally true that $H^i(\lambda)$ has simple socles (or heads). However, this is so for generic weights in $D_0(i)$, see [8].

3.4. Let $G_{\mathbb{Z}}$ denote the Chevalley group over $\mathbb{Z}$ corresponding to G. Then we have similar cohomology modules for $G_{\mathbb{Z}}$ which we denote $H^i_{\mathbb{Z}}(\lambda)$ and the universal coefficient theorem leads to the short exact sequences of G-modules

$$0 \to H^i_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k \to H^i(\lambda) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{i+1}_{\mathbb{Z}}(\lambda), k) \to 0. \tag{1}$$

For any simple root α with $\langle \lambda+\rho, \alpha^\vee\rangle > 0$ we have natural $G_{\mathbb{Z}}$-homomorphisms $H^{i+1}_{\mathbb{Z}}(s_\alpha \cdot \lambda) \to H^i_{\mathbb{Z}}(\lambda)$ and $H^i_{\mathbb{Z}}(\lambda) \to H^{i+1}_{\mathbb{Z}}(s_\alpha \cdot \lambda)$. Composing these homomorphisms for a sequence of simple roots corresponding to a reduced expression for the longest word w_0 in W we obtain a homomorphism $H^i_{\mathbb{Z}}(\lambda) \to H^{N-i}_{\mathbb{Z}}(w_0 \cdot \lambda)$. This allows us to define a Jantzen filtration of $H^i_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$. In 'good' situations this module coincides with $H^i(\lambda)$. It follows from (1) that this is always so when $i = N$ in which case we obtain the usual Jantzen filtration ([23] for the Weyl module $H^N(\lambda)$. The setup allows us also to derive sum formulae for these filtrations (in the Weyl module case they are the Jantzen sum formulae). We refer to [5] for details.

Remark 3.6. Some further results on the structure of $H^i(\lambda)$ may be found in [28].

4. Hochschild cohomology for B

In this section we demonstrate how to calculate some of the Hochschild cohomology of 1-dimensional B-modules. This cohomology which we denote $H^i(B,-)$ is the derived functors of the fixed point functor, i.e $H^i(B,-) = \mathrm{Ext}^i_B(k,-)$ where k denotes the trivial B-module. Rather little is known about this cohomology except for $i = 0$ (where it is of course zero unless $\lambda = 0$) and i $= 1$ (where only $\lambda = -p^r\alpha$ for some $r \geq 0$ and α a simple root give nonvanishing contributions, see [6]). The case $i = 2$ has recently been dealt with in [12].

To simplify things we shall here limit ourselves to consider the quantum case at a complex root of 1. In this way we avoid having to deal with higher powers of p which complicates the modular case.

4.1. **Notation.** Recall from the introduction that U_q is the quantum group corresponding to G. In the rest of this paper we assume – if not specifically said otherwise – that q is a primitive complex l-th root of 1. We shall assume that l is odd, larger than the Coxeter number for R, and prime to 3 if R contains a component of type G_2.

There is a Borel subalgebra B_q in U_q corresponding to B in G (i.e associated with the negative roots in R).

The standard generators of the quantum group are denoted E_i, F_i and $K_i^{\pm 1}$ and the divided powers by $E_i^{(n)}$ etc. The small quantum group u_q is the subalgebra of U_q generated by all $E_i, F_i, K_i^{\pm 1}$ modulo the ideal generated by the $K_i^l - 1$. We have similarly a small quantum Borel subalgebra b_q. We have a quantum Frobenius homomorphism $F_q : U_q \to U_{\mathbb{C}}$ with $U_{\mathbb{C}}$ denoting the enveloping algebra of the Lie algebra for the complex semisimple group $\bar{G}$ with root system R. We shall also denote the restriction of F_q to B_q by the same name. Note that the Frobenius homomorphism induces the trivial map on u_q and b_q.

Let M be a U_q-module. By this we shall always mean a finite dimensional module of type **1**. If M restricts to a trivial module for u_q then the Frobenius homomorphism produces from this a $U_{\mathbb{C}}$-module. This we identify with a module for $\bar{G}$ which we denote $M^{[-l]}$. On the other hand, if $\bar{M}$ is a $\bar{G}$-module then this gives via F_q rise to a U_q-module which we denote $\bar{M}^{[l]}$. Clearly u_q acts trivially on $\bar{M}^{[l]}$.

Similar conventions and notations apply to B_q- and $\bar{B}$-modules where $\bar{B}$ is the Borel subgroup in $\bar{G}$ corresponding to B.

4.2. Let M be a U_q-module. Then the modules $H^s(u_q, M)$ are naturally U_q-modules with trivial u_q-action, and we have the Lyndon-Hochschild-Serre spectral sequence

$$H^r(\bar{G}, H^s(u_q, M)^{[-l]}) \implies H^{r+s}(U_q, M). \tag{1}$$

Likewise, if M is a B_q-module we have the spectral sequence

$$H^r(\bar{B}, H^s(b_q, M)^{[-l]}) \implies H^{r+s}(B_q, M). \tag{2}$$

Note that in our situation the first sequence degenerates because $H^r(\bar{G}, -) = 0$ for $r > 0$ ($\bar{G}$ being reductive). This means that we have

$$\left(H^s(u_q, M)^{[-l]}\right)^{\bar{G}} \simeq H^s(U_q, M). \tag{3}$$

On the other hand, the spectral sequence (2) does not degenerate (as we shall see below).

4.3. The cohomology $H^s(b_q, \lambda)$ is completely known for all $\lambda \in X$. In fact, we have (see [26] and compare [10], [20] for the corresponding modular case)

$$H^s(b_q, \lambda) = 0 \text{ for all } s \geq 0 \text{ unless } \lambda \in W \cdot 0 + l\mathbb{Z}R \tag{1}$$

$$H^s(b_q, \mathbb{C}_0)^{[-l]} \simeq S^{i/2}(u^*),\ i \geq 0 \tag{2}$$

$$H^s(b_q, w \cdot 0 + l\lambda)^{[-l]} \simeq S^{(i-l(w))/2}(u^*) \otimes \lambda,\ i \geq 0. \tag{3}$$

Here u denotes the Lie algebra of the unipotent radical of $\bar{B}$, $S^r u^*$ denotes the r-symmetric power on its dual (interpreted as 0 unless $r \in \mathbb{N}$), and we have written $\mathbb{C}_0$ (instead of just 0) for the trivial 1-dimensional B_q-module determined by $0 \in X$. The isomorphisms in (2) and (3) are $\bar{B}$-isomorphisms.

4.4. Let $\bar{P}$ denote a parabolic subgroup of $\bar{G}$ containing $\bar{B}$. In analogy with Section 2 we get for any $\bar{B}$-module E a vector bundle $\mathcal{L}(E)$ on $\bar{G}/\bar{B}$. The cohomology of the restriction of this bundle to $\bar{P}/\bar{B}$ is denoted $H^i(\bar{P}/\bar{B}, E), i \geq 0$.

If V is a $\bar{P}$-module and E is a $\bar{B}$-module then we have the tensor identities $H^i(\bar{P}/\bar{B}, V \otimes E) \simeq V \otimes H^i(\bar{P}/\bar{B}, E)$ for all i. Moreover, the spectral sequence relating the $\bar{P}$ cohomology of $H^i(\bar{P}/\bar{B}, E)$ to the $\bar{B}$-cohomology of E shows that if there exists an integer i_0 such that $H^i(\bar{P}/\bar{B}, E) = 0$ for $i \neq i_0$ then

$$H^j(\bar{B}, E) \simeq H^{j-i_0}(\bar{P}, H^{i_0}(\bar{P}/\bar{B}, E)),\ j \geq 0. \tag{1}$$

This implies in particular (V still being a $\bar{P}$-module so that the induced vector bundle $\mathcal{L}(V)$ is trivial on $\bar{P}/\bar{B}$)

$$H^j(\bar{B}, V) \simeq H^j(\bar{P}, V),\ j \geq 0. \tag{2}$$

4.5. Suppose α is a simple root. Let $\bar{P}_\alpha$ be the corresponding minimal parabolic subgroup containing $\bar{B}$. In this case we abbreviate and write just H^i_α instead of $H^i(\bar{P}_\alpha/\bar{B}, -)$. Since $\bar{P}_\alpha/\bar{B} = \mathbb{P}^1$ we have $H^i_\alpha = 0$ for $i > 1$. If $\lambda \in X$ then we have $H^1_\alpha(\lambda) = 0$ when $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ and $H^0_\alpha(\lambda) = 0$ when $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$. Moreover, we have $H^0_\alpha(\lambda) \simeq H^1_\alpha(s_\alpha \cdot \lambda)$ for all $\lambda \in X$.

Note now that the line of weight α in u^* is a $\bar{B}$-submodule and that the quotient $V_\alpha = u^*/\alpha$ is a $\bar{P}_\alpha$-module. For each $n > 0$ this leads to an exact sequence of $\bar{B}$-modules

$$0 \to S^{n-1}(u^*) \otimes \alpha \to S^n(u^*) \to S^n(V_\alpha) \to 0. \tag{1}$$

The standard 'Demazure lemma' argument gives

Lemma 4.1. *If $\lambda \in X$ satisfies $\langle \lambda + \rho, \alpha^\vee \rangle < 0$ then $H^j(\bar{B}, S^n V_\alpha \otimes \lambda) \simeq H^{j-1}(\bar{B}, S^n V_\alpha \otimes s_\alpha \cdot \lambda)$ for all j, n.*

Proof: The properties of H^i_α listed above together with 4.4(1–2) give $H^j(\bar{B}, S^n V_\alpha \otimes \lambda) \simeq H^{j-1}(\bar{P}_\alpha, S^n V_\alpha \otimes H^1_\alpha(\lambda)) \simeq H^{j-1}(\bar{P}_\alpha, S^n V_\alpha \otimes H^0_\alpha(s_\alpha \cdot \lambda)) \simeq H^{j-1}(\bar{B}, S^n V_\alpha \otimes s_\alpha \cdot \lambda)$.

4.6. Lemma 4.1 together with the easy fact that $H^j(\bar{B}, \lambda) = 0$ for all j unless $\lambda \leq 0$ imply that in fact $H^j(\bar{B}, \lambda) = 0$ unless $\lambda \in W \cdot 0$ and $H^j(\bar{B}, w \cdot 0) = \mathbb{C}$ for $j = l(w)$ and zero otherwise (This could also be deduced from the Borel-Weil-Bott theorem via 4.4(1), see [6]).

Let $\mathrm{ht} : X \to \mathbb{Z}$ denote the height function on X which takes value 1 on all simple roots. Note that $\mathrm{ht}(w \cdot 0) \leq -l(w)$. So the above tells us in particular that if $H^j(\bar{B}, \lambda) \neq 0$ then $\mathrm{ht}(\lambda) \leq -j$. Clearly any weight of $S^n u^*$ has height at least n and so we deduce

$$H^j(\bar{B}, S^n u^* \otimes \lambda) = 0 \text{ unless } \lambda \leq 0 \text{ with } \mathrm{ht}(\lambda) \leq -n - j. \tag{1}$$

4.7. Let α be a simple root and $\lambda \in X$.

Proposition 4.2. *If $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ and $s_\alpha \cdot \lambda \not\leq 0$ then $H^j(\bar{B}, S^n u^* \otimes \lambda) \simeq H^j(\bar{B}, S^{n-1}u^* \otimes (\lambda + \alpha))$ for all n.*

Proof: By Lemma 4.1 we have $H^j(\bar{B}, S^n V_\alpha \otimes \lambda) \simeq H^{j-1}(\bar{B}, S^n V_\alpha \otimes s_\alpha \cdot \lambda)$. This is clearly 0 for all j when $s_\alpha \cdot \lambda \not\leq 0$. Conclusion via the exact sequence 4.5(2).

4.8. We now combine the above results to obtain

Theorem 4.3. *i) If $\lambda \in X$ then $H^j(B_q, \lambda) = 0$ for all j unless $\lambda \in W \cdot 0 + l\mathbb{Z}R$.*
ii) For each $w \in W$, $\lambda \in \mathbb{Z}R$, $j \geq 0$ we have $H^j(B_q, w \cdot 0 + l\lambda) \simeq H^{j-l(w)}(B_q, l\lambda)$.
iii) If α is a simple root and $\lambda \in \mathbb{Z}R$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ and $s_\alpha \cdot \lambda \not\leq 0$ then $H^j(B_q, l\lambda) \simeq H^{j-2}(B_q, l(\lambda + \alpha))$ for all j.

Proof: We use the spectral sequence 4.2(2) to compute $H^j(B_q, \lambda)$. Then i) is immediate from 4.3(1) and ii) follows from 4.3(3). Finally, iii) is a consequence of Proposition 4.2 and 4.3(2).

4.9. The following result will turn out to be useful in connection with the above

Proposition 4.4. *If α is a simple root and $\lambda \in X$ satisfies $0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq l$ then $H^j(B_q, \lambda) \simeq H^{j+1}(B_q, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

Proof: Let $H^i_{\alpha,q}$ be the quantum analogue of H^i_α. These have properties completely analogous to the ones recalled in 4.5. Then we get – denoting by $P_{\alpha,q}$ the quantum subalgebra analogous to P_α - that $H^j(B_q, \lambda) \simeq H^j(P_{\alpha,q}, H^0_{\alpha,q}(\lambda)) \simeq H^j(P_{\alpha,q}, H^1_{\alpha,q}(s_\alpha \cdot \lambda)) \simeq H^{j+1}(B_q, s_\alpha \cdot \lambda)$.

4.10. The first application of Proposition 4.4 is

Corollary 4.5. *For each $w \in W$ we have*

$$H^j(B_q, w \cdot 0) \simeq \begin{cases} \mathbb{C} & \text{if } j = l(w), \\ 0 & \text{otherwise.} \end{cases}$$

4.11. Let still α be a simple root

Corollary 4.6. *For each $m > 0$ we have*

$$H^j(B_q, w \cdot 0 - ml\alpha) \simeq \begin{cases} \mathbb{C} & \text{if } j = l(w) + 2m, l(w) + 2m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: By Theorem 4.3.ii) it is enough to treat the case $w = 1$. We begin with $m = 1$. By 4.3(2) we have $H^i(b_q, -l\alpha) \simeq (S^{i/2}u^*)^{[l]} \otimes (-l\alpha)$. Now a direct computation based on weight considerations give

$$H^j\big(\bar{B}, S^m u^* \otimes (-\alpha)\big) \simeq \begin{cases} \mathbb{C} & \text{if } m = 0, j = 1; \text{ or } m = 1, j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the spectral sequence 4.2(2) gives

$$H^j(B_q, -l\alpha) \simeq \begin{cases} \mathbb{C} & \text{if } j = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the corollary for $m = 1$. For $m > 1$ we see from Theorem 4.3.iii) (noting that $s_\alpha \cdot (-m\alpha) = (m-1)\alpha \not\leq 0$) $H^i(B_q, -lm\alpha) \simeq H^{i-2}(B_q, -l(m-1)\alpha)$. Hence induction on m finishes the proof.

4.12.

Example 4.7. *When we are in the SL_2-case the above corollary gives for all $m \geq 0$, $j > 0$*

$$H^j(B_q, -lm\alpha) \simeq \begin{cases} \mathbb{C} & \text{if } j = 2m, 2m-1, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$H^j(B_q, -(lm+1)\alpha) \simeq \begin{cases} \mathbb{C} & \text{if } j = 2m, 2m+1, \\ 0 & \text{otherwise.} \end{cases}$$

This accounts for all non-vanishing cohomology in this case.

4.13. When using the spectral sequence 4.2(2) for computing the cohomology

$$H^\bullet(B_q, \lambda),\ \lambda \in X$$

we see via the results in 4.3 that the E_2-terms have the form $H^r(\bar{B}, S^n u^* \otimes \mu)$ for some $r, n \in \mathbb{N},\ \mu \in X$. An alternative way to the computations in 4.5–7 of such terms is to use the spectral sequence 4.4(1) with $\bar{P} = \bar{G}$. Since $\bar{G}$ is reductive this spectral sequence degenerates in this case and gives isomorphisms $H^r(\bar{B}, E) \simeq H^r(\bar{G}/\bar{B}, E)^{\bar{G}}$ for all $r \in \mathbb{N}$ and all $\bar{B}$-modules E. Now for $E = S^n u^* \otimes \mu$ this is related to the cohomology of the line bundle $\mathcal{L}_{\bar{Y}}(\mu)$ on the cotangent bundle $\bar{Y} = T^*(\bar{G}/\bar{B})$. In fact, $\bar{Y} = \bar{G} \times^{\bar{B}} u$ so that for the cohomology of $\mathcal{L}_{\bar{Y}}(\mu)$ we have $H^r(\bar{Y}, \mathcal{L}_{\bar{Y}}(\mu)) = H^r(\bar{G}/\bar{B}, S^\bullet u^* \otimes \mu)$.

Unfortunately, this cohomology is only known when μ is dominant or "almost dominant", see e.g [14], [31], [27] and [32]. In our case the relevant μ's are far from dominant. Our computations in the next section for the SL_3-case show that for such μ there will usually be non-vanishing cohomology for all $r = 0, 1, \cdots, N$.

5. The SL_3-case

In this section we shall compute the cohomology $H^j(B_q, \lambda)$ for all $\lambda \in X$ when B_q is the Borel subalgebra in the quantum group for SL_3 (at a complex root of unity of odd order $l > 3$). We denote the two simple roots α and β.

5.1. As a special case of Corollary 4.6 we have

$$H^j(B_q, s_\beta \cdot 0 - l\alpha) \simeq \begin{cases} \mathbb{C} & \text{if } j = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Now $\langle s_\beta \cdot 0 - l\alpha, \beta^\vee \rangle = l - 2$. Hence by Proposition 4.4 we get $H^j(B_q, s_\beta \cdot 0 - l\alpha) \simeq H^{j-1}(B_q, -l\rho)$ (note that in this case $\rho = \alpha + \beta$). So we find

$$H^j(B_q, -l\rho) \simeq \begin{cases} \mathbb{C} & \text{if } j = 3, 4, \\ 0 & \text{otherwise;} \end{cases}$$

and from this we determine the cohomology of $w \cdot 0 - l\rho$ for all $w \in W$ via Theorem 4.3.ii).

5.2. Our next step will be to treat the weight $-l\alpha - 2l\beta$. We claim

$$H^j(B_q, -l\alpha - 2l\beta) \simeq \begin{cases} \mathbb{C} & \text{if } j = 2,3,5,6, \\ 0 & \text{otherwise.} \end{cases}$$

We shall perform this computation using the spectral sequence 4.2(2). First we observe that since $-\alpha - 2\beta = s_\beta s_\alpha \cdot 0$ we find by the classical analogue of Corollary 4.5

$$H^j(\bar{B}, -\alpha - 2\beta) \simeq \begin{cases} \mathbb{C} & \text{if } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The exact sequence $0 \to \beta \to u^* \to V_\beta \to 0$ together with the observation that $H^j(\bar{B}, -\rho) = 0$ for all j give

$$\begin{aligned} H^j\big(\bar{B}, u^* \otimes (-\alpha - 2\beta)\big) &\simeq H^j(\bar{B}, V_\beta \otimes -\alpha - 2\beta) \\ &\simeq H^{j-1}(\bar{B}, V_\beta \otimes -\alpha) \simeq \begin{cases} \mathbb{C} & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here the second isomorphism comes from Lemma 4.1 (note that $s_\beta \cdot (-\alpha - 2\beta) = -\alpha$ and $\langle -\alpha - l\beta + \rho, \beta^\vee \rangle = -2 < 0$) and the last equality follows by observing that the weights of V_β are ρ and α.

Similarly, we get

$$H^j\big(\bar{B}, S^2(u^*) \otimes (-\alpha - 2\beta)\big) \simeq H^j(\bar{B}, u^* \otimes -\rho) = \begin{cases} \mathbb{C} & \text{if } j = 1, \\ 0 & \text{otherwise;} \end{cases}$$

because $H^i\big(\bar{B}, S^2 V_\beta \otimes (-\alpha - 2\beta)\big) \simeq H^{i-1}(\bar{B}, S^2 V_\beta \otimes -\alpha) = 0$ for all i.

Finally, easy weight considerations give

$$H^j\big(\bar{B}, S^3(u^*) \otimes (-\alpha - 2\beta)\big) \simeq H^j(\bar{B}, S^2 u^* \otimes -\rho) = \begin{cases} \mathbb{C} & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $H^j\big(\bar{B}, S^n(u^*) \otimes (-\alpha - 2\beta)\big) = 0$ for all j when $n > 3$. These calculations now imply the claim.

5.3. The result in 5.2 gives via Theorem 4.3.ii) the cohomology for all the weights $w \cdot 0 - l(\alpha + 2\beta)\, w \in W$. Applying Theorem 4.3.iii) we deduce from this

$$H^j(B_q, w \cdot 0 - l\alpha - lm\beta) \simeq \begin{cases} \mathbb{C} & \text{if } j = l(w) + 2m - 2,\ l(w) + 2m - 1, \\ & \qquad l(w) + 2m + 1,\ l(w) + 2m + 2, \\ 0 & \text{otherwise} \end{cases}$$

for all $m \geq 2$.

5.4. Set $\lambda_1 = s_\alpha \cdot 0 - l\alpha - 3l\beta$. Then $\langle \lambda_1 + \rho, \alpha^\vee \rangle = -2 + l$ and hence by Proposition 4.4 $H^j(B_q, \lambda_1) \simeq H^{j+1}(B_q, s_\alpha \cdot \lambda_1)$. Note that $s_\alpha \cdot \lambda_1 = -2l\alpha - 3l\beta$. Thus the computation (which we did in Section 5.3) of the cohomology for λ_1 leads first to the cohomology for $s_\alpha \cdot \lambda_1$, then via Theorem 4.3.ii) to the cohomology for $w \cdot 0 + s_\alpha \cdot \lambda_1$, and finally (via Theorem 4.3.iii)) to the cohomology for $w \cdot 0 - 2l\alpha - ml\beta$, $m \geq 3$.

The next step is to set $\lambda_2 = s_\alpha \cdot 0 - 2l\alpha - 5l\beta$, pass via the same arguments to $s_\alpha \cdot \lambda_2 = -3l\alpha - 5l\beta$, then to $w \cdot 0 + s_\alpha \cdot \lambda_2$, and finally to $w \cdot 0 - 3l\alpha - ml\beta$, $m \geq 4$. Continuing this we obtain

$$H^j(B_q, w \cdot 0 - nl\alpha - ml\beta) \simeq \begin{cases} \mathbb{C} & \text{if } j = l(w) + 2m + 2n - 4,\ l(w) + 2m + 2n - 3, \\ & \qquad l(w) + 2m + 2n - 1,\ l(w) + 2m + 2n, \\ 0 & \text{otherwise} \end{cases}$$

for all $n \geq 1,\ m \geq n + 1$.

5.5. By symmetry we may of course interchange α and β in the computations above. Our results so far will thus take care of all weights with non-zero cohomology except those of the form $w \cdot 0 - ml\alpha - ml\beta,\ w \in W,\ m \geq 2$. To calculate these we first reduce to the case $w = 1$ (by Theorem 4.3.ii)). Then we resort again to the spectral sequence 4.2(2). Note that $-ml\alpha - ml\beta = -ml\rho$.

First we calculate the $E_2^{0,-}$-terms:

$$H^0\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq \begin{cases} \mathbb{C} & \text{if } n = 2m, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Proof: Clearly, $H^0\big(\bar{B}, S^n V_\alpha \otimes (-m\rho)\big) = 0 = H^0(\bar{B}, S^{n-1} V_\beta \otimes (\alpha - ml\rho)$. Hence via 4.5(1) we get $H^0\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq H^0\big(\bar{B}, S^{n-1} u^* \otimes (\alpha - m\rho)\big) \simeq H^0\big(\bar{B}, S^{n-2} u^* \otimes ((1-m)\rho)\big)$. Repeating this we get

if $n > 2m$ then $H^0\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq H^0(\bar{B}, S^{n-2m} u^*) = 0$,

if $n = 2d \leq 2m$ then $H^0\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq H^0(\bar{B}, (d-m)\rho)$,

and

if $n = 2d + 1 \leq 2m$ then $H^0\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq H^0(\bar{B}, u^* \otimes (d-m)\rho)$.

It is easy to check that $H^0(\bar{B}, u^* \otimes r\rho) = 0$ for all r. So (1) follows.

The next step concerns the $E_2^{1,-}$-terms:

$$H^1\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq \begin{cases} \mathbb{C} & \text{if } n = 2m - 1,\ 2m - 2 > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Proof: Note that $V_\alpha \simeq H^0_\alpha(\rho)$ and $S^n(V_\alpha) \simeq H^0_\alpha(n\rho)$. Therefore $H^0(\bar{B}, S^n(V_\alpha) \otimes \lambda) \simeq \mathbb{C}$ for $\lambda = n\rho - n\alpha$ (respectively 0 for all other λ). Combining this with Lemma 1.6 we get

$$H^1\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq H^1(\bar{B}, S^{n-1} u^* \otimes (\alpha - m\rho)) \text{ for all } n, m \tag{3}$$

and

$$H^1\big(\bar{B}, S^{n-1} u^* \otimes (\alpha - m\rho)\big) \simeq H^1(\bar{B}, S^{n-2} u^* \otimes ((1-m)\rho)) \text{ for all } n \neq m.$$

So if $2 \leq n \neq m$ then $H^1\big(\bar{B}, S^n u^* \otimes (-m\rho)\big) \simeq H^1\big(\bar{B}, S^{n-2} u^* \otimes ((1-m)\rho)\big)$. This gives (2) by induction on n in this case.

For $n = m > 2$ we apply (3) 'twice' and get $H^1\big(\bar{B}, S^n u^* \otimes (-n\rho)\big) \simeq H^1\big(\bar{B}, S^{n-2} u^* \otimes (2\alpha - n\rho)\big)$. This vanish because the weights of $S^{n-2} u^* \otimes (2\alpha - n\rho)$ have heights $\leq 2(n-2) + 2 - 2n = -2$.

Finally, if $n=1$ then we get by (3)

$$H^1\big(\bar B, u^*\otimes(-m\rho)\big)\simeq H^1\big(\bar B,(\alpha-m\rho)\big)=\begin{cases}\mathbb{C} & \text{if } m=1,\\ 0 & \text{otherwise,}\end{cases}$$

and if $n=2$

$$H^1\big(\bar B, S^2u^*\otimes(-m\rho)\big)\simeq H^1\big(\bar B, u^*\otimes(\alpha-m\rho)\big)=0 \text{ for } m\neq 2.$$

For $m=2$ we use the sequence $0\to(\alpha)\oplus(\beta)\to u^*\to\rho\to 0$ to get $H^1\big(\bar B, u^*\otimes(\alpha-2\rho)\big)\simeq\mathbb{C}$ (noticing that $H^i(\bar B,-2\beta)=0=H^i(\bar B,-\rho)$ for all i).

5.6. To compute the remaining E_2-terms we shall take advantage of the fact that for any $\bar B$-module E we have $H^i(\bar B,E)\simeq H^{3-i}(\bar B,E^*\otimes -2\rho)$. So we now replace u^* by u.

Observe that we have an exact $\bar B$-sequence $0\to(-\rho)\to u\to(-\alpha)\oplus(-\beta)\to 0$. We set $E=(-\alpha)\oplus(-\beta)$. Then we have for each n an exact sequence

$$0\to S^{n-1}u\otimes(-\rho)\to S^nu\to S^nE=\bigoplus_{a+b=n}(-a\alpha-b\beta)\to 0. \qquad (1)$$

From this we can then derive

$$H^0(\bar B,S^nu\otimes m\rho)=\begin{cases}\mathbb{C} & \text{if } n=m,\\ 0 & \text{otherwise.}\end{cases} \qquad (2)$$

Proof: Note that $H^0(\bar B,S^nE\otimes m\rho)=0$ unless $n=2m$. Hence (1) gives $H^0(\bar B, S^nu\otimes m\rho)\simeq H^0\big(\bar B,S^{n-1}u\otimes((m-1)\rho)\big)$ for $n\neq 2m$. So (2) follows if we verify $H^0(\bar B,S^{2m}u\otimes m\rho)=0$ for all $m>0$. This we do via the sequence dual to 4.5(1):

$$0\to S^nV_\alpha^*\to S^nu\to S^{n-1}u\otimes(-\alpha)\to 0.$$

Here $S^nV_\alpha^*\simeq H^0_\alpha(-n\beta)$ and therefore $H^0(\bar B,S^nV_\alpha^*\otimes\lambda)=0$ for $\lambda\neq n\rho$. This implies (using the corresponding sequence relative to β)

$$H^0(\bar B,S^{2m}u\otimes m\rho)\subset H^0(\bar B,S^{2m-1}u\otimes(m\rho-\alpha))\subset H^0(\bar B,S^{2m-2}u\otimes((m-1)\rho)).$$

The last module is 0 for $m>1$ by induction. The case $m=1$ is easy.

Next we claim

$$H^1(\bar B,S^nu\otimes m\rho)=\begin{cases}\mathbb{C}^2 & \text{if } m=0,n=1,\\ \mathbb{C} & \text{if } m>0, n=m+1,2m+1,\\ 0 & \text{otherwise.}\end{cases} \qquad (3)$$

Proof: Arguing as above we get for $n\neq 2m$ the exact sequence $0\to H^1(\bar B,S^{n-1}u\otimes(m-1)\rho)\to H^1(\bar B,S^nu\otimes m\rho)\to H^1(\bar B,S^nE\otimes m\rho)$. Note that $H^1(\bar B,S^nE\otimes m\rho)=\mathbb{C}^2$ for $n=2m+1$ and 0 otherwise. So if $n\neq 2m,2m+1$ we have by induction

$$H^1(\bar B,S^nu\otimes m\rho)\simeq H^1(\bar B,S^{n-1}u\otimes(m-1)\rho)=\begin{cases}\mathbb{C}^2 & \text{if } n=m+1,\\ 0 & \text{otherwise.}\end{cases}$$

If $n=2m>0$ we get an exact sequence $0\to H^0(\bar B,S^{2m}E\otimes m\rho)\to H^1(\bar B,S^{2m-1}u\otimes(m-1)\rho)\to H^1(\bar B,S^{2m}u\otimes m\rho)\to 0$, where by (2) and by induction the first two terms are both equal to $\mathbb{C}$. So this gives (3) in this case.

For $n=2m+1$ our induction gives the exact sequence

$$0\to H^1(\bar B,S^{2m+1}u\otimes m\rho)\to\mathbb{C}^2\to H^2(\bar B,S^{2m}u\otimes(m-1)\rho)\to H^2(\bar B,S^{2m+1}u\otimes m\rho).$$

By duality and 5.5(2) we have $H^2(\bar{B}, S^{2m}u \otimes (m-1)\rho) \simeq H^1(\bar{B}, S^{2m}u^* \otimes -(m+1)\rho) = \mathbb{C}$ while $H^2(\bar{B}, S^{2m+1}u \otimes m\rho) \simeq H^1(\bar{B}, S^{2m+1}u^* \otimes (-m-2)\rho) = 0$. So we have the desired result also in this case.

Finally, we have $H^1(\bar{B}, S^n u) = 0$ for $n \neq 1$ (because then no negative simple root is a weight of $S^n u$) while the sequence 5.6(1) with $n = 1$ gives $H^1(\bar{B}, u) \simeq H^1\big(\bar{B}, (-\alpha) \oplus (-\beta)\big) = \mathbb{C}^2$. This verifies (2) for $m = 0$.

5.7. Dualizing 5.6(2)–(3) we get

$$H^3(\bar{B}, S^n u^* \otimes (-m\rho)) = \begin{cases} \mathbb{C} & \text{if } n = m-2, \\ 0 & \text{otherwise;} \end{cases} \tag{1}$$

and

$$H^2(\bar{B}, S^n u^* \otimes (-m\rho)) = \begin{cases} \mathbb{C}^2 & \text{if } m = 2, n = 1, \\ \mathbb{C} & \text{if } m > 2, n = m-1, 2m-3, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It follows from 5.5(1)–(2) and (1)–(2) above that all differentials in the spectral sequence 4.2(2) for the case at hand vanish (for the trivial reason that either their target or source is zero). Hence we obtain

Proposition 5.1. *i)*

$$H^i(B_q, (-2l\rho)) = \begin{cases} \mathbb{C}^2 & \text{if } i = 4, \\ \mathbb{C} & \text{if } i = 3, 5, 7, 8, \\ 0 & \text{otherwise.} \end{cases}$$

ii) For $m > 2$ we have

$$H^i(B_q, (-ml\rho)) = \begin{cases} \mathbb{C} & \text{if } i = 2m-1, 2m, 4m-4, 4m-3, 4m-1, 4m; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.2. By Theorem 4.3.ii) we deduce from this proposition also the cohomology for the weights $w \cdot 0 - ml\rho$ for all $w \in W$, $m \geq 0$. Combined with the computations in 5.1 and 5.4 this describes completely the cohomology $H^i(B_q, \lambda)$ for all i, λ.

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