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A GEOMETRIC THEORY OF HARMONIC AND SEMI-CONFORMAL MAPS

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Abstract. We describe for any Riemannian manifold M a certain scheme M_L, lying in between the first and second neighbourhood of the diagonal of M. Semi-conformal maps between Riemannian manifolds are then analyzed as those maps that preserve M_L; harmonic maps are analyzed as those that preserve the (Levi-Civita-) mirror image formation inside M_L.

Introduction

For any Riemannian manifold M, we describe a subscheme $M_L \subseteq M \times M$, which encodes information about conformal as well as harmonic maps out of M in a succinct geometric way. Thus, a submersion $\phi : M \to N$ between Riemannian manifolds is semi-conformal (=horizontally conformal) iff $\phi \times \phi$ maps M_L into N_L (Theorem 11); and a map $\phi : M \to N$ is a harmonic map if it "commutes with mirror image formation for M_L", where mirror image formation is one of the manifestations of the Levi-Civita parallelism (derived from the Riemannian metric). The mirror image preservation property is best expressed in the set theoretic language for schemes, which we elaborate on in Section 1. Then it just becomes the statement: for $(x, z) \in M_L \subseteq M \times M$, $\phi(z') = (\phi(z))'$, where the primes denote mirror image formation in x (respectively in $\phi(x)$). In particular, when the codomain is $\mathbf{R}$ (the real line with standard metric), this characterization of harmonicity reads

$$\phi(z') = 2\phi(x) - \phi(z),$$

that is, $\phi(x)$ equals the *average value* of $\phi(z)$ and $\phi(z')$, for any z with $(x, z) \in M_L$.

The last section deals with *harmonic morphisms* between Riemannian manifolds, meaning harmonic maps which are at the same time semi-conformal.

This paper has some overlap with [7], but provides a simplification of the construction of M_L, and hence also of the proofs. Theorems 11 and 14 below are new. A novelty in the presentation is a systematic use of the log-exp bijections that relate the infinitesimal neighbourhoods like M_L with their linearized version in the tangent bundle.

The first section is partly expository; it tries to present a (rather primitive) version of the category of (affine) schemes, and the "synthetic" language in which we talk about them.

The paper grew out of a talk presented at the 5th conference "Geometry and Topology of Manifolds", Krynica 2003; I want to thank the organizers for the invitation.

1 The language of schemes

Let M be a smooth manifold. In the ring $C^\infty(M \times M)$, we have the ideal I of functions vanishing on the diagonal $M \subseteq M \times M$. Kähler observed that differential 1-forms on M may be encoded as elements in I/I^2 (the module of Kähler differentials) (here, I^2 is the ideal of functions vanishing to the second order on the diagonal). Similarly, elements of I^2/I^3 encode quadratic differential forms on M. Using the language of schemes will allow us to discuss elements of I/I^2 or of I^2/I^3 in a more geometric way. We summarize here what we need about schemes. First, note that every smooth manifold M gives rise (in a contravariant way) to a commutative $\mathbf{R}$-algebra, the ring $C^\infty(M)$ of (smooth $\mathbf{R}$-valued) functions on it. Grothendieck's bold step was to think of *any* commutative $\mathbf{R}$-algebra as the ring of smooth functions on some "virtual" *geometric* object $\overline{A}$, the *affine scheme* defined by A. So $A = C^\infty(\overline{A})$, by definition, and the category of affine schemes Sch is just the opposite (dual) of the category Alg of (commutative $\mathbf{R}$-)algebras,

$$Sch = (Alg)^{op}.$$

The category of affine schemes contains the category of smooth manifolds as a full subcategory: to the manifold M, associate the scheme $\overline{C^\infty(M)}$ (which we shall not notationally distinguish from M, except for the manifold $\mathbf{R}$, where we write R for $\overline{C^\infty(\mathbf{R})}$).

Some important schemes associated to a manifold M are its infinitesimal "neighbourhoods of the diagonal" $M_{(k)}$, considered classically by Grothendieck [3], Malgrange [12], Kumpera and Spencer [10] and others. For each natural number k, $M_{(k)} \subseteq M \times M$ is the subscheme of $M \times M$ given by the algebra $C^\infty(M \times M)/I^{k+1}$, where I is the ideal of functions vanishing on the "diagonal" $M \subseteq M \times M$; thus I^{k+1} is the ideal of functions vanishing to the $k+1$'st order on the diagonal.

We have $M \subseteq M_{(1)} \subseteq M_{(2)} \subseteq \cdots \subseteq M \times M$, with $M \subseteq M \times M$ identified with the submanifold consisting of "diagonal" points (x, x).

Now, by definition,

$$C^\infty(M \times M)/I^3 = C^\infty(M_{(2)}),$$

so in the language of schemes, we arrive at the following way of speaking: elements of $C^\infty(M \times M)/I^3$ are *functions* on $M_{(2)}$; and elements in $I^2/I^3 \subseteq C^\infty(M \times M)/I^3$ are functions on $M_{(2)}$ which *vanish* on $M_{(1)}$.

(A similar geometric language was presented in [8] for the elements of I/I^2 (=the Kähler differentials): they are functions on $M_{(1)}$ vanishing om $M_{(0)} = M$, i.e. they are *combinatorial differential 1-forms* in the sense of [4].)

Synthetic differential geometry adds one feature to this aspect of scheme theory, namely extended use of set theoretic language for speaking about objects in (sufficiently nice) categories, like Sch. Thus, since $M_{(k)}$ is a subobject of $M \times M$, the synthetic language talks about $M_{(k)}$ *as if* it consisted of pairs of points of M; we shall for instance call such pair "a pair of k'th order neighbours" and write $x \sim_k y$ for $(x, y) \in M_{(2)}$. For instance, the fact that $M_{(k)}$ is stable under the obvious twist map $M \times M \to M \times M$, we express by saying "$x \sim_k y$ implies $y \sim_k x$".

The "set" (scheme) of points $y \in M$ with $x \sim_k y$, we also denote $\mathcal{M}_k(x)$, the *k'th order neighbourhood*, or *k'th monad*, around x. The relation $\sim_k$ is reflexive and symmetric, but not transitive; rather $x \sim_k y$ and $y \sim_l z$ implies $x \sim_{k+l} z$. – Any map f preserves these relations: $x \sim_k y$ implies $f(x) \sim_k f(y)$.

A quadratic differential form on M, i.e. an element of I^2/I^3, can now be expressed: it is a function $g(x, y)$, defined whenever $x \sim_2 y$, and so that $g(x, y) = 0$ if $x \sim_1 y$. If further g is positive definite, then we may *directly* think of $g(x, y) \in R$ as the *square distance* between x and y.

For $M = R^n$, $M_{(k)}$ is canonically isomorphic to $M \times D_k(n)$: $(x, y) \in R^n_{(k)}$ iff $y - x \in D_k(n)$; here, $D_k(n)$ is the "infinitesimal" scheme corresponding to a certain well known Weil-algebra:

Recall that a Weil algebra is a finite dimensional $\mathbf{R}$-algebra, where the nilpotent elements form a (maximal) ideal of codimension one. The most basic Weil algebra is the ring of dual numbers

$$\mathbf{R}[\epsilon] = \mathbf{R}[X]/(X^2) = C^\infty(\mathbf{R})/(x^2);$$

the corresponding affine scheme is often denoted D, and is to be thought of as a "disembodied tangent vector" (cf. Mumford [13], III.4, or Lawvere, [11]). The reason is that maps of schemes $D \to M$ (M a manifold, say) by definition correspond to $\mathbf{R}$-algebra maps $C^\infty(M) \to \mathbf{R}[\epsilon]$, and such in turn correspond, as is known, to tangent vectors of M.

Note that since $\mathbf{R}[\epsilon]$ is a *quotient* algebra of $C^\infty(\mathbf{R})$, D is, by the duality, a *sub*scheme of R; this subscheme may be described synthetically as $\{d \in R \mid d^2 = 0\}$, reflecting the fact that $\mathbf{R}[\epsilon]$ comes about from $C^\infty(\mathbf{R})$ by dividing out x^2.

More generally, for k and n positive integers, $D_k(n)$ is the scheme corresponding to the Weil algebra which one gets from $\mathbf{R}[X_1, \ldots, X_n]$ by dividing out by the ideal generated by monomials of degree $k+1$; or, equivalently, from $C^\infty(\mathbf{R}^n)$ by the ideal of functions that vanish to order $k+1$ at $\underline{0} = (0, \ldots, 0)$ (it is also known as the "algebra of k-jets at $\underline{0}$ in $\mathbf{R}^n$"). – In particular, $D_1(1)$ is the ring of dual numbers described above. Just as D is the subscheme of R described by $D = \{x \in R \mid x^2 = 0\}$, $D_k(n)$ may be described in synthetic language as

$$\{(x_1, \ldots, x_n) \in R^n \mid x_{i_1} \cdot x_{i_2} \cdots x_{i_{k+1}} = 0 \text{ for all } i_1, \ldots, i_{k+1}\}.$$

The specific Weil algebras which form the algebraic backbone of the present paper are the following (first studied for this purpose in [7]). For each natural number $n \geq 2$, we consider the algebra $C^\infty(\mathbf{R}^n)/I_L$, where I_L is the ideal generated by all $x_i^2 - x_j^2$ and all $x_i x_j$ where $i \neq j$. The linear dimension of this algebra is $n + 2$; a basis may be taken to be (the classes mod I_L of) the functions $1, x_1, \ldots, x_n, x_1^2 + \cdots + x_n^2$. The corresponding affine scheme we denote $D_L(n)$ or $D_L(R^n)$; the letter "L" stands for "Laplace", for reasons that will hopefully become clear. Using synthetic language, $D_L(n)$ may be described

$$D_L(n) = \{(x_1, \ldots, x_n) \in R^n \mid x_i^2 = x_j^2;\ x_i x_j = 0 \text{ for } i \neq j\}.$$

Note that $D_1(n) \subseteq D_L(n) \subseteq D_2(n)$. The inclusion $D_1(n) \subseteq D_L(n)$ corresponds to the quotient map

$$C^\infty(\mathbf{R}^n)/I_L \to C^\infty(\mathbf{R}^n)/I_1$$

which in turn comes about because $I_L \subseteq I_1$. The kernel of this quotient map has linear dimension 1; a generator for it is the (class mod I_L of) $x_1^2 + \cdots + x_n^2$.

The following is a *tautological* translation of this fact:

Proposition 1 *Any function $f : D_L(n) \to R$ which vanishes on $D_1(n)$ is of the form $c \cdot (x_1^2 + \cdots + x_n^2)$ for a unique $c \in R$.*

The subscheme $D_k(n) \subseteq R^n$ can be described in coordinate free terms; in fact, it is just the k-monad $\mathcal{M}_k(\underline{0})$ around $\underline{0}$. More generally, for any finite dimensional vector space V, we can give an alternative description of $\mathcal{M}_k(\underline{0})$, which we also denote $D_k(V)$. We only give this description for the case $k = 1$ and $k = 2$, which is all we need:

We have that $\underline{u} \in D_1(V)$ iff for any bilinear $B : V \times V \to R$, $B(\underline{u}, \underline{u}) = 0$; this then also holds for any bilinear $V \times V \to W$, with W a finite dimensional vector space. Similarly $\underline{u} \in D_2(V)$ iff for any trilinear $C : V \times V \times V \to R$, $C(\underline{u}, \underline{u}, \underline{u}) = 0$; this then also holds for any trilinear $V \times V \times V \to W$, with W a finite dimensional vector space.

Any function $f : D_2(V) \to W$ (with W a finite dimensional vector space) can uniquely be written in the form $\underline{u} \mapsto f(\underline{0}) + L(\underline{u}) + B(\underline{u}, \underline{u})$ with $L : V \to W$ linear and $B : V \times V \to W$ bilinear symmetric.

If V is equipped with a positive definite inner product, we shall in the following Section also describe a subscheme $D_L(V)$ with $D_1(V) \subseteq D_L(V) \subseteq D_2(V)$; for $V = R^n$ with standard inner product, it will be the $D_L(n)$ already described.

2 L-neigbours in inner-product spaces

For a 1-dimensional vector space V, we say that $a \in V$ is *L-small* if it is 2-small, i.e. if $a \sim_2 0$.

Given an n-dimensional vector space V ($n \geq 2$) with a positive definite inner product $\langle -, - \rangle$. We call a vector $\underline{a} \in V$ *L-small* if for all $\underline{u}, \underline{v} \in V$

$$\langle \underline{a}, \underline{u} \rangle \langle \underline{a}, \underline{v} \rangle = \tfrac{1}{n} \langle \underline{a}, \underline{a} \rangle \langle \underline{u}, \underline{v} \rangle. \tag{1}$$

The "set" (scheme) of L-small vectors is denoted $D_L(V)$.

It is clear that if $\underline{a} \in D_L(V)$, then $\lambda\underline{a} \in D_L(V)$ for any scalar λ. But $D_L(V)$ will not be stable under addition; it is not hard to prove that if $\underline{a}$ and $\underline{b}$ are L-small vectors, then $\underline{a} + \underline{b}$ is L-small precisely if for all $\underline{u}, \underline{v} \in V$

$$\langle\, \underline{a}, \underline{u}\,\rangle\langle\, \underline{b}, \underline{v}\,\rangle + \langle\, \underline{a}, \underline{v}\,\rangle\langle\, \underline{b}, \underline{u}\,\rangle = \tfrac{2}{n}\langle\, \underline{a}, \underline{b}\,\rangle\langle\, \underline{u}, \underline{v}\,\rangle. \tag{2}$$

Let us analyze these notions for the case of R^n, with its standard inner product. We claim

Proposition 2 *The vector* $\underline{t} = (t_1, \ldots, t_n)$ *belongs to* $D_L(R^n)$ *iff*

$$t_1^2 = \cdots = t_n^2 \text{ ; and } t_i t_j = 0 \text{ for } i \neq j. \tag{3}$$

(So $D_L(R^n)$ equals the $D_L(n)$ described above, or in [7] equation (8).)

Proof. If $\underline{t} \in D_L(R^n)$, we have in particular for each $i = 1, \ldots, n$,

$$t_i^2 = \langle\, \underline{t}, \underline{e}_i\,\rangle\langle\, \underline{t}, \underline{e}_i\,\rangle = \tfrac{1}{n}\langle\, \underline{t}, \underline{t}\,\rangle,$$

where $\underline{e}_1, \ldots, \underline{e}_n$ is the standard (orthonormal) basis for R^n. The right hand side here is independent of i. – Also, if $i \neq j$,

$$t_i t_j = \langle\, \underline{t}, \underline{e}_i\,\rangle\langle\, \underline{t}, \underline{e}_j\,\rangle = \tfrac{1}{n}\langle\, \underline{t}, \underline{t}\,\rangle\langle\, \underline{e}_i, \underline{e}_j\,\rangle = 0,$$

since $\langle\, \underline{e}_i, \underline{e}_j\,\rangle = 0$.

Conversely, assume that (3) holds. Let $\underline{u}$ and $\underline{v}$ be arbitrary vectors, $\underline{u} = (u_1, \ldots, u_n)$, and similarly for $\underline{v}$. Then

$$\langle\, \underline{t}, \underline{u}\,\rangle\langle\, \underline{t}, \underline{v}\,\rangle = \Big(\sum_i t_i u_i\Big)\Big(\sum_j t_j v_j\Big) = \sum_{i,j} t_i t_j u_i v_j = t_1^2 \sum_i u_i v_i,$$

using (3) for the last equality sign. But this is $t_1^2\langle\, \underline{u}, \underline{v}\,\rangle$, and since, again by (3)

$$t_1^2 = \tfrac{1}{n}(t_1^2 + \cdots + t_n^2) = \tfrac{1}{n}\langle\, \underline{t}, \underline{t}\,\rangle,$$

we conclude $\langle\, \underline{t}, \underline{u}\,\rangle\langle\, \underline{t}, \underline{v}\,\rangle = \frac{1}{n}\langle\, \underline{t}, \underline{t}\,\rangle\langle\, \underline{u}, \underline{v}\,\rangle$.

As a Corollary, we get that for $\underline{v} \in V$ (an n-dimensional inner-product space), $\underline{v} \in D_L(V)$ iff for some, or for any, orthonormal coordinate system for V, the coordinates of $\underline{v}$ satisfy the equations (3).

From the coordinate characterization of $D_L(V)$ also immediately follows that $D_L(V) \subseteq D_2(V)$.

Here is an alternative characterization of L-small vectors, for inner product spaces V of dimension ≥ 2 (the word "self-adjoint" may be omitted, but we shall need the Proposition in the form stated).

Proposition 3 *The vector* $\underline{a}$ *belongs to* $D_L(V)$ *if and only if for every self adjoint linear map* $L : V \to V$ *of trace zero,* $\langle\, L(\underline{a}), \underline{a}\,\rangle = 0$.

Proof. We pick orthonormal coordinates, and utilize the "coordinate" description of $D_L(R^n)$. Assume $\underline{a} \in D_L(R^n)$, and assume L is given by the symmetric matrix $[c_{ij}]$ with $\sum c_{ii} = 0$. Then

$$\langle\, L(\underline{a}), \underline{a}\,\rangle = \sum_{ij} c_{ij} a_j a_i;$$

since $a_i a_j = 0$ if $i \neq j$, only the diagonal terms survive, and we get $\langle\, L(\underline{a}), \underline{a}\,\rangle = \sum_i c_{ii} a_i^2 = a_1^2 \sum c_{ii}$, since all the a_i^2 are equal to a_1^2. Since $\sum c_{ii} = 0$, we get 0, as claimed. Conversely, let us pick the L given by the symmetric matrix with $c_{ij} = c_{ji} = 1 (i \neq j)$, and all other entries 0. Then

$$0 = \langle\, L(\underline{a}), \underline{a}\,\rangle = a_i a_j + a_j a_i,$$

whence $a_i a_j = 0$. Next let us pick the L given by the matrix $c_{ii} = 1$, $c_{jj} = -1$ $(i \neq j)$ and all other entries 0. Then

$$0 = \langle\, L(\underline{a}), \underline{a}\,\rangle = a_i a_i - a_j a_j,$$

whence $a_i^2 = a_j^2$. So $\underline{a} \in D_L(R^n)$.

We now consider the question of when a linear map $f : V \to W$ between inner product spaces preserves L-smallness, i.e. when $f(D_L(V)) \subseteq D_L(W)$. Let us call an $m \times n$ matrix *semi-conformal* if the rows are mutually orthogonal, and have same (strictly positive) square norm. (This square norm is then called the *square dilation* of the matrix, and is typically denoted Λ.) The rank of a semi-conformal matrix is m, since its rows, being orthogonal, are linearly independent. It thus represents a surjective linear map $R^n \to R^m$.

We have

Proposition 4 *Let $f : V \to W$ be a surjective linear map between inner product spaces. Then t.f.a.e.*

1) $f(D_L(V)) \subseteq D_L(W)$

2) In some, or any, pair of orthonormal bases for V, W, the matrix expression for f is a semi-conformal matrix

In case these conditions hold, the common square norm Λ of the rows of the matrix is characterized by: for all $\underline{z} \in D_L(V)$

$$\tfrac{1}{m}\langle\, f(\underline{z}), f(\underline{z})\,\rangle = \Lambda \tfrac{1}{n}\langle\, \underline{z}, \underline{z}\,\rangle,$$

(where $n = dim(V)$, $m = dim(W)$).

Proof. Assume 1). Pick orthonormal bases for V and W, thereby identifying V and W with R^n and R^m, with standard inner product. Let the matrix for f be $A = [a_{ij}]$. For all $\underline{z} \in D_L(V)$, we have by assumption that

$$\Big(\sum_j a_{ij} z_j\Big)^2 \text{ is independent of } i.$$

We calculate this expression:

$$\Big(\sum_j a_{ij}z_j\Big)^2 = \Big(\sum_j a_{ij}z_j\Big)\Big(\sum_{j'} a_{ij'}z_{j'}\Big) = \sum_j a_{ij}^2 z_j^2 \tag{4}$$

since the condition $\underline{z} \in D_L(V)$ implies that $z_j z_{j'} = 0$ for $j \neq j'$, so all terms where $j \neq j'$ are killed. Also $z_j^2 = z_1^2$, so bringing this factor outside the sum, we get

$$= z_1^2\Big(\sum_j a_{ij}^2\Big) = \frac{1}{n}\Big(\sum_k z_k^2\Big)\Big(\sum_j a_{ij}^2\Big). \tag{5}$$

Since this is independent of i, then so is $\sum_j a_{ij}^2$, by the uniqueness assertion in Proposition 1. - The proof that the rows of A are mutually orthogonal is similar (or see the proof for Theorem 3.2 in [7]). – Conversely assume 2), and assume that $\underline{z} \in D_L(R^n)$. We prove that $A \cdot \underline{z} \in D_L(R^m)$. The square of the i'th coordinate here is

$$\Big(\sum_j a_{ij}z_j\Big)^2 = z_1^2 \sum_j a_{ij}^2 \tag{6}$$

by the same calculation as before. But now the sum is independent of i, by assumption on the matrix A. – Similarly, if $i \neq i'$, the inner product of the i'th and i''th row of $A \cdot \underline{z}$ is

$$\Big(\sum_j a_{ij}z_j\Big)\Big(\sum_{j'} a_{i'j'}z_{j'}\Big) = z_1^2\Big(\sum_j a_{ij}a_{i'j}\Big),$$

using again the special equations that hold for the z_j's; but now the sum in the parenthesis is 0 by the assumed orthogonality of the rows of A.

Let Λ be the common square norm of the rows of the matrix for f. Then for $\underline{z} \in D_L(V)$,

$$\frac{1}{m}\langle\, f(\underline{z}), f(\underline{z})\,\rangle = \frac{1}{m}\sum_i \Big(\sum_j a_{ij}z_j\Big)\Big(\sum_{j'} a_{ij'}z_{j'}\Big),$$

and multiplying out, only the terms where $j = j'$ survive, since $\underline{z} \in D_L(V)$. Thus we get

$$\frac{1}{m}\sum_i \Big(\sum_j a_{ij}^2 z_j^2\Big) = \frac{1}{m} z_1^2\Big(\sum_i\sum_j a_{ij}^2\Big) = \frac{1}{m} z_1^2\Big(\sum_i \Lambda\Big)$$

but this is $z_1^2\Lambda$, since there are m indices i. On the other hand, $z_1^2 = 1/n(\sum_j z_j^2)$.

We have the following "coordinate free" version of Proposition 1 (derived from it by picking orthonormal coordinates):

Proposition 5 *Let $f_1, f_2 : D_L(V) \to R$ be functions which agree on $D_1(V)$. Then there exists a unique number $c \in R$ so that for all $\underline{z} \in D_L(M)$ we have*

$$f_1(\underline{z}) - f_2(\underline{z}) = c \cdot \langle\, \underline{z}, \underline{z}\,\rangle.$$

Consider a map $f : D_2(V) \to W$ with $f(\underline{0}) = \underline{0}$ and a symmetric bilinear $B : V \times V \to W$. Let $b : V \to W$ denote the "quadratic" map $\underline{u} \mapsto B(\underline{u}, \underline{u})$.

Lemma 6 *The map f takes $D_L(V)$ into $D_L(W)$ if and only if $f + b$ does.*

Proof. This is a simple exercise in degree calculus. Assume f has the property. To prove that $f + b$ does, let $\underline{a} \in D_L(V)$, and let $\underline{u}, \underline{v}$ be arbitrary "test" vectors in W. We consider $\langle\, f(\underline{a}) + b(\underline{a}), \underline{u}\,\rangle\langle\, f(\underline{a}) + b(\underline{a}), \underline{v}\,\rangle$. Using bilinearity of inner product, this comes out as four terms, one of which is $\langle\, f(\underline{a}), \underline{u}\,\rangle\langle\, f(\underline{a}), \underline{v}\,\rangle$, and three of which vanish for degree reasons, thus for instance $\langle\, b(\underline{a}), \underline{u}\,\rangle\langle\, f(\underline{a}), \underline{v}\,\rangle = \langle\, B(\underline{a}, \underline{a}), \underline{u}\,\rangle\langle\, f(\underline{a}), \underline{v}\,\rangle$ which contains $\underline{a}$ in a trilinear way, so vanishes since $\underline{a} \in D_L(V) \subseteq D_2(V)$. So the left hand side in the test equation for L-smallness of $f(\underline{a}) + b(\underline{a})$ equals the left hand side in the test equation for L-smallness of $f(\underline{a})$. The right hand sides of the test equation is dealt with in a similar way.

3 Riemannian metrics

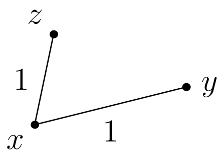
Recall from [6], [7] that a Riemannian metric g on a manifold M may be construed as an R-valued function defined on the second neighbourhood $M_{(2)}$ of the diagonal, and vanishing on $M_{(1)} \subseteq M_{(2)}$; we think of $g(x, y)$ as the *square distance* between x and y. Also g should be *positive definite*, in a sense which is most easily expressed when passing to a coordinatized situation. Since our arguments are all of completely local (in fact infinitesimal) nature, there is no harm in assuming that *one* chart covers all of M, meaning that we have an embedding of M as an open subset of R^n, or of an abstract n-dimensional vector space V. In this case, each T_xM gets canonically identified with V: to $\underline{u} \in V$, associate the tangent vector t at x given by $d \mapsto x + d\cdot\underline{u}$ for $d \in D$. The vector $\underline{u}$ is called the *principal part* of t. In this case g is of the form

$$g(x, z) = G(x; z - x, z - x),$$

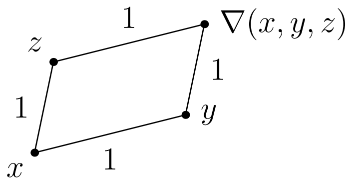
where $G : M \times V \times V \to R$ is bilinear symmetric in the two last arguments. We require each $G(x; -, -)$ to be positive definite, i.e. $G(x; -, -)$ provides V with an inner product (depending on x). Since T_xM is canonically identified with V, each T_xM also acquires an inner product; this inner product can in fact be described in a coordinate free way, in terms of g alone, cf. [7] formula (4).

4 Symmetric affine connections, and the log-exp-bijection

According to [5], an *affine connection* ∇ on a manifold M is a law ∇ which allows one to complete any configuration (with $x \sim_1 y, x \sim_1 z$)



into a configuration



(with $z \sim_1 \nabla(x, y, z) \sim_1 y$), to be thought of as an "infinitesimal parallelogram according to ∇". There is only one axiom assumed:

$$\nabla(x, x, z) = z;\ \nabla(x, y, x) = y.$$

If $\nabla(x, y, z) = \nabla(x, z, y)$ for all $x \sim_1 y$, $x \sim_1 z$, we call the connection *symmetric*.

In a coordinatized situation, i.e. with M identified with an open subset of a finite dimensional vector space V, the data of an affine connection ∇ may be encoded by a map $\Gamma : M \times V \times V \to V$, bilinear in the two last arguments, namely

$$\nabla(x, y, z) = y - x + z + \Gamma(x; y - x, z - x),$$

so that Γ measures the discrepancy between "infinitesimal parallelogram formation according to ∇" and the corresponding parallelograms according to the affine structure of the vector space V. This Γ is the "union of" the Christoffel symbols; and ∇ is symmetric iff $\Gamma(x; -, -)$ is.

A fundamental result in differential geometry is the existence of the Levi-Civita connection associated to a Riemann metric g. This result can be formulated synthetically, without reference to tangent bundles or coordinates, namely: given a Riemann metric g on a manifold, then there exists a unique symmetric connection ∇ on M with the property that for any $x \sim_1 y$, the map $\nabla(x, y, -) : \mathcal{M}_1(x) \to \mathcal{M}_1(y)$ preserves g, i.e. for $z \sim_1 x$, $u \sim_1 x$,

$$g(\nabla(x, y, z), \nabla(x, y, u)) = g(z, u).$$

(This latter condition is equivalent to: the differential of $\nabla(x, y, -)$ at x is an inner-product preserving linear map $T_xM \to T_yM$.)

There is, according to [9] Theorem 4.2, an alternative way of encoding the data of a symmetric affine connection on M, namely as a "partial exponential map", meaning a bijection (for each $x \in M$) $\mathcal{M}_2(x) \cong D_2(T_xM) \subseteq T_x(M)$, with certain properties. We describe how such bijection $\exp_x : D_2(T_xM) \to \mathcal{M}_2(x)$ is related to the connection ∇ (and this equation characterizes $\exp_x$ completely):

$$\exp_x((d_1 + d_2)t) = \nabla(x, t(d_1), t(d_2)),$$

where $t \in T_xM$ and $d_1, d_2 \in D$ (this implies $(d_1 + d_2)t \in D_2(T_xM)$).

Since $\nabla(x, y, x) = y$, it follows by taking $d_2 = 0$ that $\exp(d_1 t) = t(d_1)$, so that the partial exponential map $\mathcal{M}_2(0) \to \mathcal{M}_2(x)$ is an extension of the "first order" partial exponential map $\mathcal{M}_1(0) \to \mathcal{M}_1(x)$, as considered in [8]; the first order exponential map is "absolute" in the sense that its construction does not depend on a metric g on M.

In the coordinatized situation with $M \subseteq V$ an open subset of a vector space V, the second order exponential map corresponding to ∇ is given as follows. Note first that since now M is an open subset of V, $T_x(M)$ may be identified with V canonically, via the usual notion of "principal part" of a tangent vector to V. Let $\underline{u} \in D_2(V)$. Then

$$\exp_x(\underline{u}) = x + \underline{u} + \tfrac{1}{2}\Gamma(x; \underline{u}, \underline{u}).$$

This is an element in $M \subseteq V$, since M is open, in fact, it is an element of $\mathcal{M}_2(x)$.

The inverse of $\exp_x$ we of course have to call $\log_x$; in the coordinatized situation $M \subseteq V$, it is given as follows: let $y \sim_2 x$; then $y = x + \underline{u}$ with $\underline{u} \in D_2(V)$, and

$$\log_x(x + \underline{u}) = \underline{u} - \tfrac{1}{2}\Gamma(x; \underline{u}, \underline{u}).$$

The fact that the map $\log_x$ thus described is inverse for $\exp_x$ is a simple calculation using bilinearity of $\Gamma(x; -, -)$, together with $\Gamma(x; \underline{u}, \Gamma(x; \underline{u}, \underline{u})) = 0$, and $\Gamma(x; \Gamma(x; \underline{u}, \underline{u}), \Gamma(x; \underline{u}, \underline{u})) = 0$, and these follow because they are trilinear (respectively quatrolinear) in the arguments where $\underline{u}$ is substituted.

–The following gives an "isometry" property of the log-exp-bijection. (It does not depend on the relationship between the metric g and the affine connection/partial exponential.)

Proposition 7 *For $z \sim_2 x$, $g(x, z) = \langle \log_x z, \log_x z \rangle$.*

Proof. We work in a coordinatized situation $M \subseteq V$, so that g is encoded by $G : M \times V \times V \to R$, and the connection is encoded by $\Gamma : M \times V \times V \to V$, with both G and Γ bilinear in the two last arguments. Let $z \sim_2 x$, so z is of the form $x + \underline{u}$ with $\underline{u} \in D_2(V)$. Then on the one hand

$$g(x, z) = G(x; \underline{u}, \underline{u}),$$

and on the other hand, $\log_x(z) = \underline{u} - \frac{1}{2}\Gamma(x; \underline{u}, \underline{u})$ so that

$$\langle \log_x z, \log_x z \rangle = G(x; \underline{u} - \tfrac{1}{2}\Gamma(x; \underline{u}, \underline{u}), \underline{u} - \tfrac{1}{2}\Gamma(x; \underline{u}, \underline{u})),$$

and expanding this by bilinearity, we get $G(x; \underline{u}, \underline{u})$ plus some terms which vanish because they are tri- or quatro-linear in $\underline{u}$.

5 Mirror image

Using the (second order) partial exponential map, we can give a simple description of the infinitesimal symmetry ([7]) which any Riemannian manifold has. Let $z \sim_2 x$ in M. Its *mirror image* z' in x is defined by

$$z' := \exp_x(-\log_x(z)).$$

In the coordinatized situation $M \subseteq V$, we can utilize the formulae for log and exp given in terms of Γ to get the following formula for mirror image formation. If $z = x + \underline{u}$ with $\underline{u} \in D_2(V)$, we get

$$z' = x - \underline{u} + \Gamma(x; \underline{u}, \underline{u}).$$

This is a calculation much similar to the one above, namely, cancelling terms of the form $\Gamma(x; \Gamma(x; \underline{u}, \underline{u}), \underline{u})$ or $\Gamma(x; \Gamma(x; \underline{u}, \underline{u}), \Gamma(x; \underline{u}, \underline{u}))$, these being tri- or quatro-linear in $\underline{u}$. A similar calculation will establish that $z'' = z$.

Note also that if $\underline{u} \in D_1(V)$, and $z = x + \underline{u}$, then $z' = x - \underline{u}$.

From this follows

Lemma 8 *Given $x \in M$. Let $f : M \to R$. The function $\tilde{f} : \mathcal{M}_2(x) \to R$ defined by*

$$\tilde{f}(z) = f(z') + f(z) - 2f(x)$$

vanishes on $\mathcal{M}_1(x)$.

For, if df denotes the differential of f at x, and $z = x + \underline{u}$ with $\underline{u} \in D_1(V)$, the right hand side here is

$$(f(x) + df(-\underline{u})) + (f(x) + df(\underline{u})) - 2f(x),$$

and this is 0 since df is linear.

6 L-neighbours in a Riemannian manifold

We consider a Riemannian manifold (M, g), and the various structures on M derived from it, as in the previous sections. In particular, we have the partial exponential map exp, and its inverse log. Using these maps, we shall transport the L-neighbour relation from the inner-product spaces T_xM back to a relation in M. Explicitly,

Definition 1 *Let $x \sim_2 z$ in M. We say that $x \sim_L z$ if $\log_x(z)$ is L-small in the inner product space T_xM (with inner product derived from g).*

Note that this is not apriori a symmetric relation, since $\log(x, z)$ and $\log(z, x)$ are not immediately related – they belong to two different vector spaces T_xM and T_zM; in a coordinatized situation $M \subseteq V$, both these vector spaces may be canonically identified with V, but the notion of exp and log depend on inner products, and V in general gets different inner products from T_xM and T_zM. In [7], the question of symmetry of the relation $\sim_L$ was left open (and the relation $\sim_L$ was defined in a different, more complicated way). We state without proof:

Proposition 9 *The L-neighbour relation is symmetric.*

This fact will not be used in the present paper. It depends on the fact that parallel transport according to ∇ preserves L-smallness, being an isometry.

The following is the fundamental property of L-neighbours, and provides the link to the Laplace operator and harmonic functions, and more generally to harmonic morphisms. It is identical to Theorem 2.4 in [7], but the argument we give presently is more canonical (does not depend on chosing a geodesic coordinate system):

Theorem 10 *For any $f : \mathcal{M}_L(x) \to R$, there exists a unique number c so that for all $z \in \mathcal{M}_L(x)$,*

$$f(z) + f(z') - 2f(x) = c \cdot g(x, z). \tag{7}$$

Proof. Consider the composite of $\exp_x$ with the function $\tilde{f}$ of z described by the left hand side of (7),

$$D_L(T_xM) \xrightarrow{\exp_x} \mathcal{M}_L(x) \xrightarrow{\tilde{f}} R$$

It is a function defined on $D_L(T_xM) \subseteq T_xM$. It follows from Lemma 8 that this function vansihes on $D_1(T_xM)$, and thus is constant multiple of the square-norm function $T_xM \to R$, by Proposition 5,

$$\tilde{f}(\exp_x(\underline{u})) = c \cdot \langle\, \underline{u}, \underline{u} \,\rangle.$$

Apply this to $\underline{u} = \log_x z$ for $z \sim_L x$; we get

$$\tilde{f}(z) = \tilde{f}(\exp_x(\log_x(z))) = c \cdot \langle\, \log_x z, \log_x z \,\rangle,$$

which is $c \cdot g(x, z)$ by Proposition 7.

For any function $f : M \to R$, we can for each $x \in M$ consider the corresponding c, characterized by (7); this gives a function $c : M \to R$, and we define $\Delta(f)$ to be n times this function, in other words, the function $\Delta(f)$ is characterized by: for each pair $x \sim_L z$

$$f(z) + f(z') - 2f(x) = \frac{\Delta(f)(x)}{n} g(x, z), \tag{8}$$

where z' denotes the mirror image of z in x. (This Δ operator can be proved to be the standard Laplace operator, cf. [7].)

We call f a *harmonic* function if $\Delta(f) = 0$. Thus harmonic functions are characterized by the *average value property*: for any $x \sim_L z$, $f(x)$ is the average of $f(z)$ and $f(z')$. This property can also be expressed: for any $z \sim_L x$, $f(z')$ is the mirror image of $f(z)$ in $f(x)$, where mirror image of b in a for $a, b \in R$ means $2a - b$. This is also the mirror-image formation in R w.r.to the standard Riemannian metric given by $g(a, b) = (b - a)^2$.

This observation prompts the following definition:

Definition 2 *Let (M,g) and (N,h) be Riemannian manifolds, and let $\phi : M \to N$ a map. We say that ϕ is a* harmonic map *if it preserves mirror image formation of L-neighbours x, z,*

$$\phi(z') = \phi(z)',$$

where the prime denotes mirror image formation in x w.r.to g and in $\phi(x)$ w.r.to h, respectively.

Note that even if z is an L-neighbour of x, $\phi(z)$ may not be an L-neighbour of $\phi(x)$, but it will be a 2-neighbour of $\phi(x)$, so that the notion of mirror image of it makes sense. – The notion may be localized at x: ϕ is a harmonic map *at x* if for all $z \sim_L x$, $\phi(z') = \phi(z)'$.)

A stronger notion than harmonic *map* is that of harmonic *morphism*; this is a map which is as well a harmonic map, and is also *semi- (or horizontally) conformal* in the sense of the next section. (The terminology is not very fortunate, but classical, cf. [1].)

7 Semi-conformal maps

We consider again two Riemannian manifolds (M,g) and (N,h), and a submersion $\phi : M \to N$. It defines a "vertical" foliation, whose leaves are the (components of) the fibres of ϕ, and hence the transversal distribution consisting of $Ker(df_x)^\perp \subseteq T_xM$. (This "horizontal" distribution can also be described in purely combinatorial terms without reference to the tangent bundle.)

Recall (from [1], say) that ϕ is called *semi-conformal* (or horizontally conformal) at $x \in M$, with square-dilation $\Lambda > 0$, if the linear map $df_x : T_xM \to T_{\phi(x)}N$ is semi-conformal with square-dilation $\Lambda > 0$, in the sense of Section 2. (This property can also be expressed combinatorially.) The following is a generalization of Theorem 3.2 in [7] (which dealt with the case of a diffeomorphism ϕ).

Theorem 11 *Let $\phi : M \to N$ be a submersion, and let $x \in M$. Then t.f.a.e.:*
1) ϕ is semi-conformal at x (for some $\Lambda > 0$)
2) ϕ maps $\mathcal{M}_L(x)$ into $\mathcal{M}_L(\phi(x))$.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{M}_2(x) \arrow[r, "\phi"] \arrow[d, "\log_x"'] & \mathcal{M}_2(\phi(x)) \arrow[d, "\log_{\phi(x)}"] \\
D_2(T_xM) \arrow[r, shift left, "f"] \arrow[r, shift right, "d\phi_x"'] & D_2(T_{\phi(x)}N)
\end{tikzcd}$$

where f is the unique map making the diagram commutative, and where $d\phi_x$ is (the restriction of) the differential of ϕ. It does not make the diagram commutative, but when restricted to $\mathcal{M}_1(x)$, it does, by the very definition of differentials. So f and $d\phi_x$ agree on $\mathcal{M}_1(x)$, and hence differ by a quadratic map b. It then follows from Lemma 6 that f maps $D_L(T_xM)$ into $D_L(T_{\phi(x)})$if and only if $d\phi_x$ does. By definition, $\mathcal{M}_L(x)$ comes about from $D_L(T_xM)$ by transport along the log-exp-bijection, so ϕ preserves $\mathcal{M}_L$ iff f preserves D_L. On the other hand, by the Proposition 4, semi-conformality of $d\phi_x$ is equivalent to $d\phi_x$ preserving D_L.

We may summarize the results of the last two sections by stating the following (which may be taken as definitions of these notions, but couched in purely geometric/combinatorial language): let $\phi : M \to N$ be a submersion between Riemannian manifolds. Then

- ϕ is a *harmonic map* if it preserves mirror image formation of L-neighbours
- ϕ is a *semi-conformal map* if it preserves the notion of L-neighbour
- ϕ is a *harmonic morphism* if it has both these properties.

If the codomain is R, any 2-neigbour is an L-neighbour, so any map to R is automatically semi-conformal, so for codomain R, harmonic map and harmonic morphism means the same thing. Such a map/morphism is in fact exactly a harmonic function $M \to R$.

All three notions make sense "pointwise": ϕ is a harmonic *at* $x \in M$ if it preserves mirror image formation of L-neighbours of x. For this to make sense, we don't need ϕ to be defined on all of M, because the property only depends on the 2-jet of ϕ at x, meaning the restriction of ϕ to $\mathcal{M}_2(x)$.

8 Sufficiency of harmonic 2-jets

By 2-jets, we understand in this Section 2-jets of R-valued functions; so a 2-jet at $x \in M$ is a map $\mathcal{M}_2(x) \to R$. If M is a Riemannian manifold, we say that such a 2-jet f is *harmonic* if it preserves mirror image formation of L-neigbours of x, $f(z') = 2f(x) - f(z)$, for all $z \sim_L x$.

Among such harmonic 2-jets, we have in particular those of the form

$$\mathcal{M}_2(x) \xrightarrow{\log_x} T_xM \xrightarrow{p} R, \tag{9}$$

where the last map p is linear. For, by construction of mirror image in terms of $\log_x$, $\log_x(z') = -\log_x(z)$, and this mirror image formation is preserved by p (here, we don't even need $z \sim_L x$, just $z \sim_2 x$).

Another type of harmonic 2-jet are those of the form

$$\mathcal{M}_2(x) \xrightarrow{\log_x} T_xM \xrightarrow{q} R \tag{10}$$

where q is a "quadratic map of trace 0", meaning $q(\underline{u}) = \langle L(\underline{u}), \underline{u} \rangle$ for some self-adjoint $L : T_xM \to T_xM$ of trace zero. For, $z \sim_L x$ means by definition that $\log_x(z) \in D_L(T_xM)$, and quadratic trace zero maps kill D_L, by Proposition 3.

These two special kinds of harmonic jets are the only ones that we shall use in the proof of the following "recognition Lemma":

Lemma 12 *There are sufficiently many harmonic 2-jets to recognize mirror image formation in x, and to recognize L-neighbours of x.*

Precisely, if z and $\tilde{z}$ are 2-neighbours of x, and $f(\tilde{z}) = 2f(x) - f(z)$ for all harmonic 2-jets f, then $\tilde{z} = z'$; and if z is a 2-neighbour of x such that $f(z) = 0$ for all harmonic 2-jets f which vanish on $\mathcal{M}_1(x)$, then $z \sim_L x$.

Proof. The first assertion follows because $\log_x(z') = -\log_x(z)$, and because there are sufficiently many linear $p : T_xM \to R$ to distinguish any pair of vectors (T_xM being finite-dimensional). The second assertion follows because $\log_x$ maps $\mathcal{M}_L(x)$ bijectively onto $D_L(T_xM)$, and the latter is recognized by quadratic trace zero maps, by Proposition 3.

There is a partial converse:

Proposition 13 *Let $f : \mathcal{M}_2(x) \to R$ be a harmonic 2-jet which vansihes at $\mathcal{M}_1(x)$. Then it vanishes at $\mathcal{M}_L(x)$.*

Proof. Let b denote the composite $f \circ \exp_x : D_2(T_xM) \to R$. The vanishing assumption on f implies that there is a unique quadratic map $T_xM \to R$ extending b. It suffices to prove that $b(\underline{u}) = 0$ for any $\underline{u} \in D_L(T_xM)$. Let z denote $\exp_x(\underline{u})$; then $z \in \mathcal{M}_L(x)$. Harmonicity of f at x implies $f(z)+f(z') = 0$ by Theorem 10, and hence $b(\underline{u}) + b(-\underline{u}) = 0$. But b is a even function, being quadratic, hence $b(\underline{u}) = 0$.

9 Characterization Theorem

The following Theorem is now almost immediate in view of the combinatorial/geometric description of harmonic maps and semi-conformal maps. It is a version of the Characterization Theorem of Fuglede and Ishihara, cf. [1] Theorem 4.2.2.

Theorem 14 *Given a submersion $\phi : M \to N$ between Riemannian manifolds, and let $x \in M$. Then t.f.a.e.*

1) ϕ is a harmonic morphism at x

2) for any harmonic 2-jet f at $\phi(x)$, $f \circ \phi : M \to R$ is a harmonic 2-jet.

(The Theorem in the classical form talks about harmonic *germs* at $\phi(x)$, rather than harmonic 2-*jets*. The "upgrading" of our version to the classical one thus depends on a rather deep existence theorem: any harmonic 2-jet comes about by restriction from a harmonic germ, see Appendix of [1]. Such existence results are beyond the scope of our methods.)

Proof. Assume that ϕ is a harmonic morphism at x, and let f be a harmonic 2-jet. Let $z \sim_L x$. Then $\phi(z') = (\phi(z))'$, since ϕ is a harmonic map; also $\phi(z) \sim_L \phi(x)$ since ϕ is semi-conformal. So f preserves the mirror image of $\phi(z)$. So both ϕ and f

preserve the relevant mirror images, hence so does the composite $f \circ \phi : \mathcal{M}_2(x) \to R$; hence it is a harmonic 2-jet.

Conversely, suppose ϕ has $f \circ \phi$ harmonic for all harmonic 2-jets f at $\phi(x)$. Let $z \sim_L x$. To prove $\phi(z') = (\phi(z))'$, it suffices, by the Recognition Lemma (applied to N) to prove that all harmonic 2-jets f at $\phi(x)$ take $\phi(z')$ to the mirror image of $\phi(z)$. But by assumption $f \circ \phi$ is harmonic at x, so preserves mirror image. – Also, to prove $\phi(z) \sim_L \phi(x)$, it suffices by the Recognition Lemma to prove that any harmonic 2-jet at $\phi(x)$, vanishing on $\mathcal{M}_1(\phi(x))$, kills $\phi(z)$. But by assumption, $f \circ \phi$ is a harmonic 2-jet, and it vanishes at $\mathcal{M}_1(x)$, so by Proposition 13, it kills z. So $f(\phi(z)) = 0$, so $\phi(z) \sim_L \phi(x)$. This proves the Theorem.

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