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Salem Ben Saïd and Bent Ørsted

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*Ny Munkegade, Bldg. 530*
*DK-8000 Aarhus C, Denmark*

*http://www.imf.au.dk*
*institut@imf.au.dk*

# ON BESSEL FUNCTIONS AND DUNKL OPERATORS – THEORY AND APPLICATIONS

SALEM BEN SAÏD AND BENT ØRSTED

Abstract. In this paper we extend previous classes of generalized Bessel functions using the Heckman-Opdam theory of hypergeometric functions, and the theory of Dunkl operators. We furthermore define a Segal-Bargmann transform associated with Coxeter groups, and give an analogue of Howe dual pair theory for Coxeter groups. This paper is a survey of recent results in [B-Ø1], [B-Ø2], and [B-Ø3].

## 1. Introduction

A basic problem in harmonic analysis and applications is to study the Fourier transforms of invariant measures on submanifolds of Euclidean space. For example, let $d\mu$ be the $SO(n+1)$-invariant probability measure on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and consider

$$\psi(y) = \int_{\mathbb{S}^n} e^{i\langle x,y\rangle}\, d\mu(y) \qquad (\mathcal{E}_1)$$

with $\langle\cdot,\cdot\rangle$ the usual inner product. It is well-known that $(\mathcal{E}_1)$ can be found explicitly in terms of the standard Bessel functions

$$I_\nu(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \qquad (\mathcal{E}_2)$$

Indeed, $\psi(y) = \Gamma(n/2)\,(i|y|/2)^{-\frac{n}{2}+1}\, I_{\frac{n}{2}-1}(i|y|)$, and for $n$ odd, these functions are elementary (they may be expressed in terms of polynomials, exponential functions, or their derivatives).

In this paper we shall study integrals of type $(\mathcal{E}_1)$, and special functions as in $(\mathcal{E}_2)$ needed to express them. The main tools are the Dunkl-operators introduced by Dunkl around 1990, and a deformation principle, which we now explain in a simple case. It is well-known that the solutions of Laplace's differential equation on $\mathbb{R}^3$, which are symmetric around the $z$-axis and analytic in a neighborhood of the origin, can be expressed in spherical coordinates $(r,\theta,\phi)$ in the form $r^n P_n(\cos\theta)$. Here $P_n$ is the Legendre polynomial of order $n = 0, 1, 2, \cdots$. Now, consider the nature of the structure of spheres, cones, and planes associated with spherical coordinates in a region of space far from the origin and near the $z$-axis. The spheres approximate to planes and the cones approximate to cylinders, and the structure resembles the one associated with cylindrical coordinates $(\rho,\phi,z)$. The solutions of Laplace's equation referred to such coordinates are of the form $e^{\pm kz}J_0(k\rho)$, where $J_0$ is the Bessel function of order 0, and $k$ is any constant.

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Recall that $J_\nu(r) = i^\nu I_\nu(-ir)$. It is therefore to be expected that, when $r$ and $n$ are large and $\theta$ is small in such a way that $r\sin\theta(=\rho)$ remains bounded, the Legendre polynomial should approximate to a Bessel function. This is equivalent to expect Bessel functions as limits of Legendre polynomials. In 1868, this type of limit was proved by Mehler as follows

$$\lim_{n\to\infty} P_n\Big(\cos\frac{\theta}{n}\Big) = J_0(\theta). \qquad (\mathcal{E}_3)$$

The same formula also appeared in a work of Heine, done independently at about the same time, and it is nowadays known as the Mehler-Heine formula. This result has been extended to generalized Legendre polynomials by Heine and Rayleigh.

In the original proof of $(\mathcal{E}_3)$, the parameter $n$ is assumed to tend to infinity through integral values. One also can prove it when $n$ goes to infinity as a continuous real parameter.

Now, let us give another interpretation of the limit formula $(\mathcal{E}_3)$ in terms of spherical functions associated with non-compact symmetric spaces. In analogue with $(\mathcal{E}_3)$, we get

$$\lim_{\epsilon\to 0} P_{\frac{i\lambda}{\epsilon}}(\operatorname{ch}\epsilon z) = J_0(\lambda z), \quad \lambda\in\mathbb{R}.$$

The functions $z\mapsto P_{i\lambda}(\operatorname{ch} z)$ are the spherical functions of the non-compact Riemannian symmetric space $SO(2,1)/SO(2)$. The functions $x\mapsto J_0(\lambda|x|)$ are the Bessel functions on the tangent space at the origin of $SO(2,1)/SO(2)$, viewed as a flat symmetric space. Thus, we may expect Bessel functions on flat symmetric spaces to be expressed as limits of spherical functions on the corresponding Riemannian symmetric spaces. This was the starting point of our investigation on Bessel functions on flat symmetric spaces by means of Harish-Chandra's spherical functions. In [B-Ø1] (see Section 2 below), we prove the following statement which can be seen as a generalization of the Mehler-Heine formula:

*The Bessel functions on flat symmetric spaces can be obtained as limits of Harish-Chandra's spherical functions on Riemannian symmetric spaces of non-compact type.* $(\mathcal{S})$

In this setting, the integral representation of the Bessel functions is sometimes called Harish Chandra–Itzykson–Zuber (HIZ)-type integral.

The advantage of this approach is that we can derive at least the same amount of explicit information for the Bessel functions, by a limit analysis, as for Harish-Chandra's spherical functions; in some cases even more information is attained. An important motivation to study Bessel functions originates in their relevance for the analysis of quantum many body systems of Calogero-Moser type, and in connection with the study of random matrices.

After this generalization of the Mehler-Heine-type formula in the case of non-compact Riemannian symmetric spaces, we move to other directions were we may expect a statement similar to $(\mathcal{S})$ to hold.

In recent years, Harish-Chandra's theory of spherical functions on Riemannian symmetric spaces has been generalized in three different directions. In the 80s Heckman and Opdam extended the theory of Harish-Chandra to multi-variables hypergeometric functions associated with root systems and depending on additional parameters, namely the multiplicities. Their construction was motivated by the theory of special functions. In one variable, Harish-Chandra's spherical functions on Riemannian symmetric spaces of rank-one are special instances of the Gaussian hypergeometric functions. This was the starting point of Heckman-Opdam's theory of hypergeometric functions. Harish-Chandra's spherical functions can always be regained by specializing the multiplicities. Another line of study consists in extending the theory of Harish-Chandra to a class of non-Riemannian symmetric spaces, called non-compact causal symmetric spaces. This was done in 1994 by Faraut, Hilgert, and Ólafsson. Recently, Pasquale has presented an extension of Heckman-Opdam's theory which also includes the theory of Faraut-Hilgert- Ólafsson, by introducing the so-called Θ-spherical functions.

Using these developments in the theory of spherical functions, in [B-Ø2] (see Section 3, 4, and 5 below), we were able to extend the statement ($\mathcal{S}$) to the above mentioned three directions, namely to Bessel functions related to root systems, to Bessel functions associated with non-compact causal symmetric spaces, and finally to what we shall call the Θ-Bessel functions.

After the important contributions by Heckman and Opdam in the area of special functions related to root systems, the subject has attracted much interest and there has been a rapid development in this area during the last ten years. Around 1990, Dunkl introduced a family of differential-reflection operators associated with Coxeter groups on finite-dimensional Euclidean spaces. These operators are nowadays known as Dunkl operators. They are parameterized deformations of the ordinary derivatives, for which it is still possible to study the spectral problem and develop the theory of the corresponding Fourier transform, which is now called the Dunkl transform. Quite remarkably, it is easier to study the Dunkl operators in the general context than to specialize to invariant objects. The invariant side of Dunkl's theory corresponds to the study of the Bessel functions related to root systems, discussed above. In [B-Ø3] (see Section 6 below), we present various applications of the Dunkl operators, notably with Fock-type spaces, Segal-Bargmann transforms, and an analogue of Howe duality.

The main results of the present paper are: Theorems 2.3, 2.7, 3.3, 3.5, 4.3, 5.4, 6.5, 6.7, and 6.11.

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## 2. Bessel functions on flat symmetric spaces

In this section we shall initiate an investigation of the so-called HIZ-type integral (Harish Chandra-Itzykson-Zuber); these are Fourier transforms of orbits of $K$ in the tangent space at the origin of a semi-simple non-compact symmetric space $G/K$, where $G$ is a connected non-compact semi-simple Lie group with finite center, and $K$ is a maximal compact subgroup. These integrals play a role in the theory of integrable systems in physics, and in connection with the study of random matrices. It is well-known that they correspond to spherical functions on the tangent space, viewed as a flat symmetric space. Our point of view is to see the HIZ-type integrals as limits of spherical functions for $G/K$, and we are able to obtain new and explicit formulas by analyzing the deformation (as the curvature goes to zero) of $G/K$ to its tangent space. We refer to [B-Ø1] for more details and information.

Let $G$ be a connected semisimple Lie group with finite center, and let $K$ be a maximal compact subgroup of $G$. The symmetric space $G/K$ is a Riemannian symmetric space of the non-compact type.

Let $\theta : G \to G$ be the Cartan involution on $G$ corresponding to $K$. Denote by the same letter the derived involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$, and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Fix a Weyl positive chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\Sigma^+$ be the corresponding set of positive roots. For $\alpha \in \Sigma$, let $\mathfrak{g}^{(\alpha)} := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$ be the associated root space, and set $m_\alpha := \dim(\mathfrak{g}^{(\alpha)})$. We denote by $\rho := (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha\alpha$. Let $\mathfrak{n}$ be the sum of the root spaces corresponding to the positive roots. The connected subgroups of $G$ associated with the subalgebras $\mathfrak{a}$ and $\mathfrak{n}$ are denoted by the corresponding capital letters. We have the Iwasawa decomposition $G = KAN$, and the Cartan decomposition $G = KAK$. For $g \in G$, define $H(g) \in \mathfrak{a}$ by $g \in K \exp(H(g))N$.

Denote by $\langle\cdot,\cdot\rangle$ the inner product on $\mathfrak{a}$ induced by the Killing form $B(\cdot,\cdot)$ of $\mathfrak{g}$, and let $\Pi$ be the fundamental system of simple roots associated with $\Sigma^+$. Denote by $W_\Pi$ the Weyl group generated by the reflections $r_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$, with $\alpha \in \Pi$ and $r_\alpha(\lambda) := \lambda - 2\langle\lambda, \alpha\rangle\alpha/\langle\alpha, \alpha\rangle$. The action of $W_\Pi$ on $\mathfrak{a}^*$ extends to $\mathfrak{a}$, and to the complexifications $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}^*_\mathbb{C}$.

Let $\mathscr{D}(G/K)$ be the algebra of $G$-invariant differential operators on $G/K$. Suppose the smooth complex-valued function $\varphi_\lambda$ is an eigenfunction of each $D \in \mathscr{D}(G/K)$

$$D\varphi_\lambda = \gamma_D(\lambda)\varphi_\lambda, \qquad \lambda \in \mathfrak{a}^*. \tag{2.1}$$

Here the eigenfunction is labeled by the parameters $\lambda$, and $\gamma_D(\lambda)$ is the eigenvalue. If in addition $\varphi_\lambda$ satisfies $\varphi_\lambda(\mathbf{e}) = 1$, where $\mathbf{e}$ is the identity element, and $\varphi_\lambda(kgk') = \varphi_\lambda(g)$ for $k, k' \in K$, then the function $\varphi_\lambda$ is called a spherical function. In [HC1], Harish-Chandra proves the following integral representation of the spherical functions.

**Theorem 2.1.** (cf. [HC1]) *As $\lambda$ runs through $\mathfrak{a}^*_\mathbb{C}$, the functions*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)H(gk)}\, dk, \quad g \in G,$$

*exhaust the class of spherical functions on $G$. They are real analytic functions of $g \in G$ and holomorphic functions of $\lambda \in \mathfrak{a}^*_\mathbb{C}$. Moreover, two such functions $\varphi_\lambda$ and $\varphi_\mu$ are identical if and only if $\lambda = \omega\mu$ for some $\omega$ in the Weyl group $W_\Pi$.*

Since $\mathfrak{a}^+$ is the interior of a fundamental domain of $W_\Pi$, the Cartan decomposition implies that $\varphi_\lambda$ is uniquely determined by its restriction to $A^+ := \exp(\mathfrak{a}^+)$. Moreover, a $K$-bi-invariant function $\varphi_\lambda$ is an eigenfunction for $\mathscr{D}(G/K)$ if and only if its restriction to $A^+$ is an eigenfunction for the system of equation on $A^+$ given by the radial components of operators from $\mathscr{D}(G/K)$.

Let $\{H_i\}_{i=1}^N$ be a fixed orthonormal basis of $\mathfrak{a}$. For $H \in \mathfrak{a}$, denote by $\partial(H)$ the corresponding directional derivative in $\mathfrak{a}$. Let $\Delta(m)$ be the radial part of the Laplace-Beltrami operator on $G/K$. Then

$$\Delta(m) = \sum_{i=1}^N \partial(H_i)^2 + \sum_{\alpha\in\Sigma^+} m_\alpha(\coth\alpha)\partial(A_\alpha),$$

where $A_\alpha \in \mathfrak{a}$ is determined by $B(A_\alpha, H) = \alpha(H)$ for $H \in \mathfrak{a}$. In particular

$$\Delta(m)\varphi_\lambda = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)\varphi_\lambda. \tag{2.2}$$

For $\epsilon > 0$, denote by $\mathfrak{g}_\epsilon$ the Lie algebra $\mathfrak{k} \oplus \mathfrak{p}$ with the Lie bracket $[\cdot,\cdot]_\epsilon$ such that

$$\begin{aligned} [X, X']_\epsilon &:= [X, X'] \quad && (X, X' \in \mathfrak{k}), \\ [Y, Y']_\epsilon &:= \epsilon^2 [Y, Y'] \quad && (Y, Y' \in \mathfrak{p}), \\ [X, Y]_\epsilon &:= [X, Y] \quad && (X \in \mathfrak{k},\ Y \in \mathfrak{p}). \end{aligned}$$

Here $[\cdot,\cdot]$ denotes the Lie bracket associated with $\mathfrak{g}$. The following map $\Phi_\epsilon : \mathfrak{g}_\epsilon \to \mathfrak{g}$ defined by

$$\Phi_\epsilon(X) := X, \ \text{ if } X \in \mathfrak{k} \qquad \text{and} \qquad \Phi_\epsilon(Y) := \epsilon^{-1} Y, \ \text{ if } Y \in \mathfrak{p},$$

is an isomorphism from $\mathfrak{g}_\epsilon$ to $\mathfrak{g}$. Further, if $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ then $\epsilon\alpha \in \Sigma(\mathfrak{g}_\epsilon, \mathfrak{a})$, and the corresponding root space $\mathfrak{g}_\epsilon^{(\epsilon\alpha)}$ is given by

$$\mathfrak{g}_\epsilon^{(\epsilon\alpha)} = \left\{ \epsilon X_k + X_p \ \mid \ X_k + X_p \in \mathfrak{g}^{(\alpha)} \text{ where } X_k \in \mathfrak{k},\ X_p \in \mathfrak{p} \right\}.$$

Let $G_\epsilon$ be the analytic Lie group with Lie algebra $\mathfrak{g}_\epsilon$ via the Baker-Campbell-Hausdorff formula. Denote by $\Delta^{(\epsilon)}(m)$ the radial part of the Laplace-Beltrami operator on $G_\epsilon/K$ given by

$$\Delta^{(\epsilon)}(m) = \sum_{i=1}^{N} \partial(H_{i,\epsilon})^2 + \sum_{\alpha \in \Sigma^+} m_\alpha \coth(\epsilon\alpha)\partial(A_{\epsilon\alpha}),$$

where $\{H_{i,\epsilon}\}$ is a fixed orthonormal basis of $\mathfrak{a}$ in $\mathfrak{g}_\epsilon$, and $A_{\epsilon\alpha} \in \mathfrak{a}$ is determined by $B_\epsilon(A_{\epsilon\alpha}, H) = \epsilon\alpha(H)$ for $H \in \mathfrak{a}$. Here $B_\epsilon(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}_\epsilon$. The above relationship between $\mathfrak{g}^{(\alpha)}$ and $\mathfrak{g}_\epsilon^{(\epsilon\alpha)}$ yields to the fact that $B_\epsilon(A_{\epsilon\alpha}, H) = \epsilon^2 B(A_{\epsilon\alpha}, H)$, which implies that

$$A_{\epsilon\alpha} = \epsilon^{-1} A_\alpha, \quad \text{and} \quad H_{i,\epsilon} = \epsilon^{-1} H_i. \tag{2.3}$$

Now, the following theorem holds.

**Theorem 2.2.** (cf. [B-Ø1]) *Let*

$$\Delta^\circ(m) := \sum_{i=1}^{N} \partial(H_i)^2 + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{\alpha} \partial(A_\alpha).$$

*The following limit holds*

$$\lim_{\epsilon \to 0} \epsilon^2 \Delta^{(\epsilon)}(m) = \Delta^\circ(m).$$

By (2.3), one may check the relation $\Delta^{(\epsilon)}(m) = (\epsilon^*)^{-1} \circ \Delta(m) \circ (\epsilon^*)$, where $\epsilon^* f(X) := f(\epsilon X)$, whilst $\varphi_\lambda$ is an eigenfunction for $\Delta(m)$. This observation suggests to obtain the eigenfunctions of $\Delta^\circ(m)$ as an appropriate limit of the Harish-Chandra's spherical functions; which turns out to be true.

In terms of symmetric spaces, consider the limit $\epsilon \to 0$ as equivalent to letting the curvature of $G/K$ tend to zero. This scaling removes the curvature so that in the limit we recover the tangent space at the origin of $G/K$, viewed as a flat symmetric space in the following sense: The symmetric spaces fall into three different categories: the compact-type, the non-compact-type, and the flat symmetric space. The three cases can be distinguished by means of their curvature. In the class of compact-type, the symmetric space has sectional curvature everywhere positive. In the class of non-compact-type, the symmetric space has sectional curvature everywhere negative, and in the class of flat symmetric spaces, the sectional curvature is zero.

Actually, if $\mathfrak{g}$ is the Lie algebra of a Lie group $G$ with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then, if $G_0 := K \ltimes \mathfrak{p}$, the flat symmetric space $G_0/K$ can be identified with $\mathfrak{p}$. The elements $g_0 = (k, p) \in G_0$ act on $G_0/K$ in the following way

$$g_0(p') = \mathrm{Ad}(k)p' + p, \qquad k \in K, \quad p, p' \in G_0/K.$$

An example of a zero curvature symmetric space is the flat Euclidean space in four dimensions. It is known that this space can be realized as the coset of the Euclidean Poincaré group $\widetilde{P}$ with respect to $SO(4)$, $\mathfrak{p} \simeq \widetilde{P}/SO(4)$. The translations of the Poincaré group play the role of $\mathfrak{p}$, and they are isomorphic to Euclidean space and have all the characteristics of a zero curvature symmetric space. The fact that the zero curvature spaces can be obtained as limits of positive curvature spaces can be exemplified as follows. We can realize the Euclidean Poincaré group as a suitable limit of the $SO(5)$ group. In this limit the coset $SO(5)/SO(4)$, which is the four dimensional unit sphere, becomes the Euclidean four-dimensional space.

For the spherical functions on the three categories of symmetric spaces, it is well-known that the spherical functions on symmetric spaces of the non-compact-type can be obtained from the spherical functions on symmetric spaces of the compact-type, and vice versa, via some analytic continuation. Next we prove that the spherical functions associated with flat symmetric spaces can also be obtained from the spherical functions on symmetric spaces of the non-compact-type, or the compact-type, by letting the curvature tend to zero from the left, or from the right, respectively.

For $\epsilon > 0$, write $g_\epsilon = k \exp(\epsilon X)$ where $k \in K$ and $X \in \mathfrak{p}$. Using the fact that $\Delta^{(\epsilon)}(m) = (\epsilon^*)^{-1} \circ \Delta(m) \circ (\epsilon^*)$, and $\varphi_\lambda$ is an eigenfunction of $\Delta(m)$ with the eigenvalue $-(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)$, we obtain

$$\Delta^{(\epsilon)}(m)\varphi_{\frac{\lambda}{\epsilon}}(g_\epsilon) = -\Big(\Big\langle \frac{\lambda}{\epsilon}, \frac{\lambda}{\epsilon} \Big\rangle + \Big\langle \rho, \rho \Big\rangle\Big)\varphi_{\frac{\lambda}{\epsilon}}(g_\epsilon).$$

Denote by

$$\psi(\lambda, X) := \lim_{\epsilon \to 0} \varphi_{\frac{\lambda}{\epsilon}}(g_\epsilon).$$

Now, we summarize the consequence of all the above discussions in the light of Theorem 2.1 and Theorem 2.2.

**Theorem 2.3.** (cf. [B-Ø1]) *The limit $\psi(\lambda, X)$, and its derivatives exist. Its integral representation is given by*

$$\psi(\lambda, X) = \int_K e^{iB(\mathrm{Ad}(k)X, A_\lambda)}\, dk, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ X \in \mathfrak{p}.$$

*Moreover, $\psi(\lambda, X)$ satisfies*

$$\Delta^\circ(m)\psi(\lambda, X) = -\langle \lambda, \lambda \rangle \psi(\lambda, X).$$

*The limit $\psi(\lambda, X)$ is the so-called Harish Chandra-Itzykson-Zuber (HIZ)-type integral, and it is well known that it corresponds to spherical functions, or the Bessel functions, on the flat symmetric space $\mathfrak{p}$.*

Briefly we give the argument used to obtain the integral representation of $\psi(\lambda, X)$. Denote by $\mathbb{P} : \mathfrak{p} \rightarrow \mathfrak{a}$ the orthogonal projection on $\mathfrak{a}$ for the scalar product associated with the Killing form. Notice that $H(\exp(\epsilon X)k) = H(\exp(\epsilon k^{-1} \cdot X))$ where $k^{-1} \cdot X = \mathrm{Ad}(k)X$. Write $k^{-1} \cdot X = \mathbb{P}(k^{-1} \cdot X) + Y \in \mathfrak{a} \oplus \mathfrak{a}^{\perp}$, where $\mathfrak{a}^{\perp}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$, and use the fact that $Y \in \mathfrak{a}^{\perp}$ can be written as $Y = Y_k + Y_n \in \mathfrak{k} \oplus \mathfrak{n}$. Then $k^{-1} \cdot X = \mathbb{P}(k^{-1} \cdot X) + Y_k + Y_n$. Now one may check that the functions $\epsilon \mapsto H(\exp(\epsilon Y_k) \exp(\epsilon \mathbb{P}(k^{-1} \cdot X)) \exp(\epsilon Y_n))$ and $\epsilon \mapsto H(\exp(\epsilon k^{-1} \cdot X))$ have the same derivative at $\epsilon = 0$, and therefore

$$\lim_{\epsilon \to 0} \frac{\lambda}{\epsilon} H\big(\exp(\epsilon k^{-1} \cdot X)\big) = \lambda \mathbb{P}\big(k^{-1} \cdot X\big).$$

***Remark*** 2.4. (i) Since $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = \omega\mu$ for $\omega \in W_\Pi$, the same assertion holds for $\psi(\lambda, X)$.

(ii) The spherical functions $\psi(\lambda, X)$ are symmetric with respect to $\lambda$ and $X$, while $\Delta^\circ(m)$ is not symmetric under interchange of $\lambda$ and $X$.

(iii) The contraction principle was used earlier in [Do-Ri] for understanding the relationship between the representation theories of $K \ltimes \mathfrak{p}$ and $G$. The limit approach was also used in [Ø-Z] to define the Weyl transform on flat symmetric spaces where $G/K$ is a Hermitian symmetric space. Another application of the limit approach can be found in [Cl].

It is remarkable that in spite of many results about the analysis of spherical functions on $G/K$, their Fourier analysis and asymptotic properties, it is only for very few cases that explicit formulas exist for these functions. From Theorem 2.3, one can see that for flat symmetric spaces we may derive at least the same amount of explicit information by a limit analysis, as the curvature goes to zero, of spherical functions; and in some cases even more information is attained. In some interesting cases, for instance $SU^*(2n)/Sp(n)$ and $SU(p,q)/S(U(p) \times U(q))$, we are able to give in [B-Ø1] explicit formulas for the spherical functions $\psi(\lambda, X)$. In particular, these formulas give concrete solutions for problems of many body systems, which are related to quantum mechanics. We refer to [B-Ø1] for further details. Other interesting cases are also investigated. After [B-Ø1] was completed, we were able to give in [B-Ø2] a unified formula for the Bessel functions on flat symmetric spaces when $m_\alpha \in 2\mathbb{N}$ for all $\alpha \in \Sigma$. See Theorem 5.4, Table I, and Table II below for more information.

***Example* 2.5.** (The real rank-one case) This case corresponds to Riemannian symmetric spaces $G/K$ of non-compact type for which $\mathfrak{a}$ is one dimensional. There are only four type of groups $G$ with real rank-one, namely $SO_0(n,1)$, $SU(n,1)$, $Sp(n,1)$, and $F_{4(-20)}$.
Fix $\alpha = 1$. The set $\Sigma^+$ consists at most of two elements $\alpha$ and, possibly, $2\alpha$. We may identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}$, and their complexifications $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}^*_\mathbb{C}$ with $\mathbb{C}$.
For symmetric spaces of real rank-one, the algebra $\mathscr{D}(G/K)$ of $G$-invariant differential operators on $G/K$ is generated by the Laplace-Beltrami operator, where its radial part defines the differential equation

$$\left\{\frac{d^2}{dt^2} + (m_\alpha \coth t + 2m_{2\alpha} \coth 2t)\frac{d}{dt}\right\} y = -(\lambda^2 + \rho^2) y \qquad (t \in \mathbb{R}), \tag{2.4}$$

with $\lambda \in \mathbb{C}$ and $\rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha})$. The spherical function

$$\varphi_\lambda(t) = {}_2F_1\left(\frac{i\lambda + \rho}{2}, \frac{-i\lambda + \rho}{2}; \frac{m_\alpha + m_{2\alpha} + 1}{2}; -\operatorname{sh}^2 t\right)$$

is the unique solution of (2.4) that satisfies $\varphi_\lambda(\mathbf{e}) = 1$ in the unit $\mathbf{e}$ of $G$. Here $m_\alpha = n-1$ and $m_{2\alpha} = 0$ for $G = SO_0(n,1)$; $m_\alpha = 2(n-1)$ and $m_{2\alpha} = 1$ for $G = SU(n,1)$; $m_\alpha = 4(n-1)$ and $m_{2\alpha} = 3$ for $G = Sp(n,1)$; and $m_\alpha = 8$ and $m_{2\alpha} = 7$ for $G = F_{4(-20)}$. Using the following well-known fact

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1 + \frac{1}{2z}(a-b)(a+b-1) + \mathcal{O}(z^{-2})\right\}, \quad \text{if } z \to \infty,$$

one can prove that

$$\psi(\lambda, t) = \Gamma\left(\frac{m_\alpha + m_{2\alpha} + 1}{2}\right)\left(\frac{\lambda t}{2}\right)^{-\frac{m_\alpha + m_{2\alpha} - 1}{2}} J_{\frac{m_\alpha + m_{2\alpha} - 1}{2}}(\lambda t),$$

where $J_\nu$ is the Bessel function of the first kind.

***Example* 2.6.** (The complex case) Let $G$ be a Lie group with complex structure. Complex symmetric spaces are mainly characterized by the fact that $\Sigma$ is reduced and $m_\alpha = 2$ for all $\alpha \in \Sigma$. For the complex case, an explicit formula for the spherical functions $\varphi_\lambda$ on $G/K$ was given by Harish-Chandra, namely

$$\varphi_\lambda(\exp(X)) = \frac{\prod_{\alpha\in\Sigma^+}\langle\alpha,\rho\rangle \sum_{\omega\in W_\Pi}(\det\omega)e^{\langle i\omega\lambda, X\rangle}}{\prod_{\alpha\in\Sigma^+}\langle\alpha,i\lambda\rangle \sum_{\omega\in W_\Pi}(\det\omega)e^{\langle\omega\rho, X\rangle}}, \quad X \in \mathfrak{a},$$

(cf. [He]). Notice that

$$\sum_{\omega\in W_\Pi}(\det\omega)e^{\langle\omega\rho, X\rangle} = e^{\langle\rho, X\rangle}\prod_{\alpha\in\Sigma^+}(1 - e^{-2\langle\alpha, X\rangle}).$$

Using Theorem 2.3, we obtain

$$\int_K e^{iB(A_\lambda, \mathrm{Ad}(k)X)}\,dk = \frac{\prod_{\alpha\in\Sigma^+}\langle\alpha,\rho\rangle}{\prod_{\alpha\in\Sigma^+}\langle\alpha,i\lambda\rangle\prod_{\alpha\in\Sigma^+}2\langle\alpha,X\rangle}\sum_{\omega\in W_\Pi}(\det\omega)e^{\langle i\omega\lambda, X\rangle}. \tag{2.5}$$

In this case, i.e. when $G$ is complex, the integral representation of $\psi$ is the so-called Harish-Chandra integral, and its explicit expression (2.5) was proved earlier by Harish-Chandra in [HC2] using other techniques. Another proof is given in Berline-Getzler-Vergne's book [Be-G-V] by using the orbit method. Our approach gives a new and simple proof of the Harish-Chandra integral.

We close this section by giving the Taylor expansion of $\psi(\lambda, X)$ in a series of the Jack polynomials, when $G/K$ admits a root system of type $A_{N-1}$ $(N = 2, 3, \ldots)$. This follows from Theorem 2.3 by using the generalized binomial formula of the spherical functions on $G/K$ proved in [Ok-Ol]

Consider the following list of non-compact symmetric spaces with root system of type $A_{N-1}$ ($N = 2, 3, \dots$)

$$GL(N,\mathbb{R})/O(N), \quad GL(N,\mathbb{C})/U(N), \quad GL(N,\mathbb{H})/Sp(N),$$
$$E_{6(-26)}/F_4, \quad O(1,N)/O(N).$$

Let $\wp$ be a strictly positive parameter. Let $P_N = \mathbb{C}[x_1, \dots, x_N]$ be the polynomial algebra in $N$ independent variables and $\Lambda_N \subset P_N$ be the algebra of symmetric polynomials. A partition is any sequence $\lambda = (\lambda_1, \dots, \lambda_N, \dots)$ of nonnegative integers in decreasing order $\lambda_1 \geq \cdots \geq \lambda_N \geq \cdots$ containing only finitely many nonzero terms. The number of nonzero terms in $\lambda$ is the length of $\lambda$ denoted by $l(\lambda)$. The sum $|\lambda| = \lambda_1 + \cdots + \lambda_N + \cdots$ is called the weight of $\lambda$. The set of partitions of weight $N$ is denoted by $\mathcal{P}_N$. On this set there is a natural involution: in the standard diagrammatic representation it corresponds to the transposition (reflection in the main diagonal). The image of a partition $\lambda$ under this involution is called the conjugate of $\lambda$ and is denoted by $\lambda'$.

An important example of symmetric functions are Jack polynomials. We give here their definition. Recall that on the set of partitions $\mathcal{P}_N$ there is the following dominance partial ordering: we write $\mu \leq \lambda$ if for all $i \geq 1$

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

Consider the following Calogero-Moser-Sutherland operator

$$\Delta_\wp^N = \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i}\right)^2 + 2\wp \sum_{i \neq j} \frac{x_i x_j}{x_i - x_j} \frac{\partial}{\partial x_i}.$$

If $\wp$ is not a negative rational number or zero, then for any partition $\lambda$ such that $l(\lambda) \geq N$, there is a unique polynomial $P_\lambda(x, \wp) \in \Lambda_N$, called the Jack polynomial, such that

(i) $P_\lambda(x, \wp)$ is an eigenfunction of the $\Delta_\wp^N$ operator.

(ii) $P_\lambda(x, \wp) = e_\lambda + \sum_{\mu < \lambda} v_{\lambda,\mu} e_\mu$, where $v_{\lambda,\mu} \in \mathbb{C}$ and $e_\mu$ is the elementary symmetric polynomial.

Now we discuss the so-called shifted Jack polynomials investigated recently by Knop, Sahi, Okounkov, and Olshanski (cf. [Kn-Sa] [Ok-Ol]). Let us denote by $\Lambda_{\wp,N}$ the algebra of polynomials $f(x_1, \dots, x_N)$ which are symmetric in the shifted variables $x_i \wp(1-i)$. Let us introduce the following function on the set of partition

$$H(\lambda, \wp) = \prod_{\square \in \lambda} (c_\wp(\square) + 1),$$

where

$$c_\wp(\square) = \lambda_i - j + \wp(\lambda'_j - i).$$

Here we identify a partition $\lambda$ with its diagram

$$\lambda = \{\, \square = (i,j) : 1 \leq i \leq l(\lambda),\ 1 \leq j \leq \lambda_i \,\}.$$

Let $\lambda$ be a partition with $\lambda_{N+1} = 0$. There exists a unique shifted symmetric polynomial $P^*_\lambda(x, \wp) \in \Lambda_{N,\wp}$, called shifted Jack polynomial, such that $\deg(P^*_\lambda) \leq |\lambda|$, and

$$P^*_\lambda(\mu, \wp) = \begin{cases} H(\lambda, \wp), & \mu = \lambda \\ 0, & |\mu| \leq |\lambda|, \quad \mu \neq \lambda, \quad \mu_{N+1} = 0. \end{cases}$$

Knop and Sahi proved that the shifted Jack polynomial $P^*_\lambda(x,\wp)$ satisfies the extra vanishing property $P^*_\lambda(\mu,\wp)=0$ unless the diagram of $\mu$ is a subset of the diagram of $\lambda$, i.e. $\mu_i\le\lambda_i$ for all $i\ge 1$, and that $P^*_\lambda(x,\wp)$ is the usual Jack polynomial $P_\lambda(x,\wp)$ plus lower order terms. We shall write $\mu\subset\lambda$ to mean that the diagram of $\lambda$ contains the diagram of $\mu$.

By [M] and [St], we have the following branching rule for the Jack polynomials

$$P_\lambda(x_1,x_2,\dots,x_N,\wp)=\sum_{\mu\prec\lambda}\varphi_{\lambda/\mu}(\wp)x_1^{|\lambda/\mu|}P_\mu(x_2,\dots,x_N;\wp), \tag{2.6}$$

where $\mu\prec\lambda$ stands for the inequalities of interlacing

$$\lambda_1\ge\mu_1\ge\lambda_2\ge\mu_2\ge\cdots\ge\mu_{N-1}\ge\lambda_N,$$

the weight of the skew diagram $\lambda/\mu$ equals $|\lambda|-|\mu|$, and $\varphi_{\lambda/\mu}(\wp)$ is the following coefficient

$$\varphi_{\lambda/\mu}(\wp)=\prod_{1\le i\le j\le N-1}\frac{(\mu_i-\mu_j+\wp(j-i)+\wp)_{\mu_j-\lambda_{j+1}}}{(\mu_i-\mu_j+\wp(j-i)+1)_{\mu_j-\lambda_{j+1}}}\times\frac{(\lambda_i-\mu_j+\wp(j-i)+1)_{\mu_j-\lambda_{j+1}}}{(\lambda_i-\mu_j+\wp(j-i)+\wp)_{\mu_j-\lambda_{j+1}}}.$$

For the shifted Jack polynomial $P^*_\lambda$, Okounkov proved in [Ok] the following formula

$$P^*_\lambda(x_1,\dots,x_N;\wp)=\sum_{\mu\prec\lambda}\varphi_{\lambda/\mu}(\wp)\prod_{\square\in\lambda/\mu}(x_1-c'_\wp(\square))P^*_\mu(x_2,\dots,x_N;\wp), \tag{2.7}$$

where $\varphi_{\lambda/\mu}(\wp)$ is the same as for $P_\lambda$, and

$$c'_\wp(\square)=(j-1)-\wp(i-1),\qquad \square=(i,j).$$

Next, we review the spherical functions on symmetric cones. For details, we refer to Faraut-Korányi's book [F-K].

Let $\Omega$ be an open and convex cone associated with an Euclidean Jordan algebra $\mathbb{V}$. The Riemannian symmetric space $\Omega$ can be identified with one of the symmetric spaces $G/K$ listed above.

Assume that $\mathbb{V}$ is a simple Euclidean Jordan algebra, i.e. $\mathbb{V}$ does not contain non-trivial ideals. Let $N$ be the rank of $\mathbb{V}$, and let $\{c_1,\dots,c_N\}$ be a complete system of orthogonal idempotents elements. Each element $x$ in $\mathbb{V}$ can be written as $x=k\sum_{i=1}^N x_ic_i$, with $k\in K$ and $x_i\in\mathbb{R}$.

For $\mathbf{m}=(m_1,\dots,m_N)$ and $x=\sum_{j=1}^N x_jc_j$, the spherical functions on $\Omega$ are given by

$$\varphi_{\mathbf{m}}(x)=\int_K\Delta_1^{m_1-m_2}(kx)\cdots\Delta_{N-1}^{m_{N-1}-m_N}(kx)\Delta_N^{m_N}(kx)\,dk,$$

where $\Delta_j(y)$ is the principal minor of order $j$ of $y$, and $dk$ denotes the normalized Haar measure on $K$. This formula corresponds to the classical Harish-Chandra's formula for the spherical functions on $G/K$ with $\mathbf{m}=\frac{\lambda+\rho}{2}$ where $\rho=(\rho_1,\dots,\rho_N)$, $\rho_j=\frac12(2j-N-1)$, and $\lambda\in\mathbb{C}^N$.

If $\mathbf{m}=(m_1,\dots,m_N)\in\mathbb{N}^N$ such that $m_1\ge\cdots\ge m_N\ge 0$, the spherical function $\varphi_{\mathbf{m}}$ is a polynomial, and can be written in terms of the Jack polynomials.

If $x = k\sum_{j=1}^N x_j c_j$

$$\varphi_{\mathbf{m}}(x) = \frac{P_{\mathbf{m}}(x_1,\ldots,x_N;\wp)}{P_{\mathbf{m}}(1,\ldots,1;\wp)}, \quad \wp = \frac{2}{d},$$

where

$$\begin{cases} d=1 & \text{for} \quad GL(N,\mathbb{R})/O(N),\\ d=2 & \text{for} \quad GL(N,\mathbb{C})/U(N),\\ d=4 & \text{for} \quad GL(N,\mathbb{H})/Sp(N),\\ d=8 & \text{for} \quad E_{6(-26)}/F_4, \qquad \text{and}\\ d=N & \text{for} \quad O(1,N)/O(N).\end{cases}$$

By [Ok-Ol, (2.6)], we have the following binomial formula

$$\frac{P_{\mathbf{m}}(1+x_1,\ldots,1+x_N;\wp)}{P_{\mathbf{m}}(1,\ldots,1;\wp)} = \sum_{\mu\subset\mathbf{m}} \frac{P^*_\mu(\mathbf{m},\wp)P_\mu(x,\wp)}{P_\mu(1,\ldots,1;\wp)H(\mu,\wp)} \tag{2.8}$$

For $\wp = 1$, formula (2.8) reduces to the usual binomial formula

$$\frac{S_{\mathbf{m}}(1+x_1,\ldots,1+x_N)}{S_{\mathbf{m}}(1,\ldots,1)} = \sum_{\mu\subset\mathbf{m}} \frac{S^*_\mu(\mathbf{m})S_\mu(x)}{\prod_{\square=(i,j)\in\mu}(N+j-i)}$$

where $S_\mu$ is the Schur function

$$S_\mu(x) = \frac{\det\left(x_i^{\mu_j+N-j}\right)_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(x_i-x_j)}$$

and

$$S^*_\mu(\mathbf{m}) = \frac{\det((m_i+N-i)\cdots(m_i-i+j-\mu_j+1))_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(m_i-i-m_j+j)}.$$

Henceforth, $\varsigma$ denotes a very large integer. After using formula (2.6) $N$-times, we obtain

$$P_\mu(1-e^{s_1/\varsigma},\ldots,1-e^{s_N/\varsigma};\wp) \sim \varsigma^{-|\mu|}P_\mu(s_1,\ldots,s_N;\wp) \quad \text{as} \ \sigma\to\infty.$$

The same argument for the shifted Jack polynomials gives

$$P^*_\mu(m_1\varsigma,\ldots,m_N\varsigma;\wp) \sim \varsigma^{|\mu|}P_\mu(m_1,\ldots,m_N;\wp), \quad \text{as} \ \varsigma\to\infty.$$

Thus, by (2.8) the following theorem holds

**Theorem 2.7.** (cf. [B-Ø1]) *For* $X = \sum_{i=1}^N x_i c_i$, $\mathbf{m} = (m_1,\ldots,m_N) \in \mathbb{N}^N$ *such that* $m_1 \ge \cdots \ge m_N$, *and* $\varsigma \in \mathbb{N}$, *the following Taylor series holds*

$$\lim_{\varsigma\to\infty} \varphi_{\mathbf{m}\varsigma}(\exp(\varsigma^{-1}X)) = \sum_{\mu\in\mathcal{P}_N} \frac{P_\mu(m_1,\ldots,m_N;\frac{2}{d})P_\mu(x_1,\ldots,x_N;\frac{2}{d})}{P_\mu(1,\ldots,1;\frac{2}{d})H(\mu;\frac{2}{d})}.$$

***Remark*** 2.8. (i) For a real vector $\nu = (\nu_1,\ldots,\nu_N)$ such that $\nu_1 \ge \cdots \ge \nu_N$, write

$$\psi_c(\nu,X) := \int_K e^{B(A_\nu,\mathrm{Ad}(k)X)}\,dk.$$

As in Theorem 2.3, one can prove that

$$\begin{aligned}\psi_c(\nu, X) &= \lim_{\varsigma\to\infty} \varphi_{[\varsigma\nu]}(\exp(\varsigma^{-1}X)) \\ &= \sum_{\mu\in\mathcal{P}_N} \frac{P_\mu(\nu_1,\dots,\nu_N;\frac{2}{d})P_\mu(x_1,\dots,x_N;\frac{2}{d})}{P_\mu(1,\dots,1;\frac{2}{d})H(\mu;\frac{2}{d})} \qquad \text{(by Theorem 2.7)}\end{aligned}$$

where $[\varsigma\nu] = ([\varsigma\nu_1],\dots,[\varsigma\nu_N])$ is the $N$-vector of integral parts.

(ii) Using [Sw], one can prove that for $\mathbf{m} = (m_1, m_2)$

$$\varphi_{\mathbf{m}}(x_1, x_2) = e^{m_1x_1}e^{m_2x_2}{}_2F_1\left(m_2 - m_1, \frac{d}{2}; d; 1 - e^{x_2-x_1}\right).$$

Hence, if $N = 2$, we may rewrite the Bessel function $\psi_c(\nu_1, \nu_2; x_1, x_2)$ explicitly as

$$\psi_c(\nu_1, \nu_2; x_1, x_2) = e^{\nu_1x_1}e^{\nu_2x_2}{}_1F_1\left(\frac{d}{2}, d; (\nu_1 - \nu_2)(x_1 - x_2)\right),$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

## 3. Bessel functions related to root systems

As stated in Example 2.5, the spherical functions $\varphi_\lambda$ on Riemannian symmetric spaces $G/K$ of rank-one are a special type of hypergeometric functions. The specialization occurs with the choice of the multiplicities $m_\alpha$ and $m_{2\alpha}$ as the dimension of the root spaces $\mathfrak{g}^{(\alpha)}$ and $\mathfrak{g}^{(2\alpha)}$, respectively. This specialization is still effective even for symmetric spaces with higher-rank by constraining the root multiplicities $m_\alpha$ to assume certain positive integer values. The spherical functions are determined as well by the geometry, since $\varphi_\lambda$ is an eigenfunction of each $G$-invariant differential operator that belongs to $\mathscr{D}(G/K)$. Notice though, the differential system (2.2) makes perfect sense without the geometric restriction on the multiplicities $m_\alpha$. This was the starting point of Heckman-Opdam's theory on hypergeometric functions. Their objective was to generalize Harish-Chandra's theory of spherical functions for arbitrary complex values of multiplicities associated with root systems (cf. [Hec-O, Hec1, O1, O2]).

In Heckman-Opdam's theory, the Riemannian symmetric spaces $G/K$ are replaced by the following ingredients: a $N$-dimensional real Euclidean vector space $\mathfrak{a}$ with fixed inner product $\langle\cdot,\cdot\rangle$, a root system $\mathcal{R}$ in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ – which is assumed to satisfy the crystallographic condition, i.e. $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathcal{R}$, and finally a multiplicity function $k : \mathcal{R} \to \mathbb{C}$ invariant under the action of the Weyl group associated with $\mathcal{R}$.

The symbols $\mathfrak{a}_\mathbb{C}, \mathfrak{a}^*_\mathbb{C}, \mathfrak{a}^+, A, A_\mathbb{C}, A^+$ shall have the same meaning as in the previous section.

Let $\mathcal{R}^+$ be a choice of positive roots in $\mathcal{R}$, and denote by $\Pi$ the corresponding fundamental system of simple roots. The Weyl group $W_\Pi$ is generated by the reflections $r_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$, with $\alpha \in \Pi$ and $r_\alpha(\lambda) = \lambda - \lambda(\check{\alpha})\alpha \in \mathfrak{a}^*$. Here $\lambda(\check{\alpha}) := 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$. The action of $W_\Pi$ extends to $\mathfrak{a}$ by duality, to $\mathfrak{a}^*_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}$ by $\mathbb{C}$-linearity, and to $A_\mathbb{C}$ and $A$ by the exponential map.

Let $k : \mathcal{R} \to \mathbb{C}$ be a multiplicity function. Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathcal{R}$, we have $k_{w\alpha} = k_\alpha$ for all $w \in W_\Pi$. Denote by $\mathscr{K}$ the set of all multiplicity functions on $\mathcal{R}$. If $m = \sharp\{W_\Pi\text{−orbits in } \mathcal{R}\}$, then $\mathscr{K} \cong \mathbb{C}^m$.

The set $A_{\mathbb{C}}^{\mathrm{reg}} := \{a \in A_{\mathbb{C}} \mid e^{\alpha(\log a)} \neq 1 \;\forall\, \alpha \in \mathcal{R}\}$ consists of the regular elements of $A_{\mathbb{C}}$ for the $W_\Pi$-action. Notice that $A^+$ is a subset of $A_{\mathbb{C}}^{\mathrm{reg}}$. Denote by $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$ the algebra of regular functions on $A_{\mathbb{C}}^{\mathrm{reg}}$.

Let $S(\mathfrak{a}_{\mathbb{C}})$ be the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomials functions on $\mathfrak{a}_{\mathbb{C}}$, and let $S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}$ be the subalgebra of $W_\Pi$-invariant elements. Each $p \in S(\mathfrak{a}_{\mathbb{C}})$ defines a constant-coefficient differential operator $\partial(p)$ on $A_{\mathbb{C}}$ and on $\mathfrak{a}_{\mathbb{C}}$ such that $\partial(H)$ is the directional derivative in the direction of $H$ for all $H \in \mathfrak{a}$. The algebra of the differential operators $\partial(p)$, with $p \in S(\mathfrak{a}_{\mathbb{C}})$, will also be denoted by $S(\mathfrak{a}_{\mathbb{C}})$. Let $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) := \mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$ be the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\mathrm{reg}}]$. The Weyl group action on $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ is given by

$$w(f \otimes \partial(p)) = wf \otimes \partial(wp), \qquad w \in W_\Pi.$$

Set $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{W_\Pi}$ to be the subspace of $W_\Pi$-invariant elements.

For $k \in \mathscr{K}$ and for a fixed orthonormal basis $\{\xi_i\}_{i=1}^N$ of $\mathfrak{a}$, write

$$\Delta(k) := \sum_{j=1}^N \partial_{\xi_j}^2 + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \coth\left(\frac{\alpha}{2}\right) \partial_\alpha. \tag{3.1}$$

Here we write $\partial_{\xi_i}$ for $\partial(\xi_i)$. Denote by $\widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ the commutator of $\Delta(k)$ in $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{W_\Pi}$. In [Hec-O], Heckman and Opdam proved that $\widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ is a commutative algebra and parameterized by the elements of $S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}$. In [C], Cherednik was able to give an algebraic algorithm for constructing the operators $D(k,p) \in \widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ corresponding to $p \in S(A_{\mathbb{C}})^{W_\Pi}$ by using the so-called Dunkl-Cherednik operators. We recall briefly this algorithm following [Hec2]. For $k \in \mathscr{K}$ and $\xi \in \mathfrak{a}_{\mathbb{C}}$, the Dunkl-Cherednik operator $T(\xi,k) \in \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W_\Pi]$ is defined by

$$T(\xi,k) := \partial_\xi - \rho(k)(\xi) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha(\xi)(1-e^{-\alpha})^{-1} \otimes (1 - r_\alpha), \tag{3.2}$$

where $\rho(k) := (1/2)\sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha \in \mathfrak{a}_{\mathbb{C}}^*$. In particular, for all $\xi, \nu \in \mathfrak{a}_{\mathbb{C}}$ and $k \in \mathscr{K}$, $[T(\xi,k), T(\nu,k)] = 0$. Due to the commutativity of the Dunkl-Cherednik operators, the map $\mathfrak{a}_{\mathbb{C}} \to \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W_\Pi]$, $\xi \mapsto T(\xi,k)$, can be extended in a unique way to an algebra homomorphism $S(\mathfrak{a}_{\mathbb{C}}) \to \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W_\Pi]$. The image of $p \in S(\mathfrak{a}_{\mathbb{C}})$ will be denoted by $T(p,k)$. Suppose $p \in S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}$, then by [Hec2]

$$T(p,k) = \sum_{w \in W_\Pi} D(w,p,k) \otimes w \in \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W_\Pi].$$

Moreover, if we denote by "Proj" the map from $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[W_\Pi]$ to $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ given by $\mathrm{Proj}(\sum_i D_i \otimes w) = \sum_i D_i$, then

$$D(p,k) := \mathrm{Proj}(T(p,k)) = \sum_{w \in W_\Pi} D(w,p,k) \in \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{W_\Pi}.$$

The operator $D(p,k)$ is the unique element in $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{W_\Pi}$ which has the same restriction to $\mathbb{C}[A_{\mathbb{C}}]^{W_\Pi}$ as $T(p,k)$. By [Hec2] the element $D(p,k)$ preserves $\mathbb{C}[A_{\mathbb{C}}]^{W_\Pi}$, and
$D(p,k)D(q,k) = D(pq,k)$ for $p,q \in S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}$. Thus the set $\{D(p,k) \mid p \in S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}\}$ is a commutative algebra of differential operators. For instance, if $p_0 = \sum_{j=1}^N \xi_j^2$ where $\{\xi_i\}_{i=1}^N$ is the fixed orthonormal basis of $\mathfrak{a}$, then $D(p_0,k) = \Delta(k) + \langle \rho(k), \rho(k) \rangle$, where $\Delta(k)$ is the Laplacian operator (3.1).

***Example* 3.1.** Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace in $\mathfrak{p}$, and $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system associated with $\mathfrak{a}$. If we put $\mathcal{R} = 2\Sigma(\mathfrak{g}, \mathfrak{a})$ and $k_\alpha = \frac{1}{2} m_\alpha$, where $m_\alpha$ is the multiplicity of the root $\alpha$, then $\Delta(k)$ coincides with the radial part of the Laplace-Beltrami operator on the symmetric space $G/K$. The set $\widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ represents the commutative algebra of radial parts on $A^+$ of the differential operators in $\mathscr{D}(G/K)$. This setting corresponds to the situation in the previous section which we shall refer to it by the geometric case.

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the following system of differential equations

$$D(p,k)F = p(\lambda)F, \qquad p \in S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}, \tag{3.3}$$

is the so-called hypergeometric system of differential equations associated with the root system $\mathcal{R}$. In particular, if $p_0 = \sum_{j=1}^N \xi_j^2$, the differential equations (3.3) becomes

$$\Delta(k)F = (\langle \lambda, \lambda \rangle - \langle \rho(k), \rho(k) \rangle)F. \tag{3.4}$$

In the geometric case, the hypergeometric system (3.3) coincides with the system of differential equations (2.1) defining Harish-Chandra's spherical functions $\varphi_{i\lambda}$.

By the explicit expression of the differential equation (3.4), Heckman and Opdam searched for solutions for the hypergeometric system on $A^+ = \exp(\mathfrak{a}^+)$ of the form

$$\Phi(\lambda, k, a) = \sum_{\ell > 0} \Gamma_\ell(\lambda, k) e^{\lambda - \rho(k) - \ell(\log a)}, \qquad a \in A^+$$

where $\Gamma_0(\lambda, k) = 1$ and $\Gamma_\ell(\lambda, k) \in \mathbb{C}$ satisfying some recurrence relations [Hec-O]. Using $\Phi(\lambda, k, \cdot)$, Heckman and Opdam were able to build a basis for the solution space of the entire hypergeometric system with spectral parameter $\lambda$. This is possible if $\lambda$ is generic, i.e. $\lambda(\breve{\alpha}) \notin \mathbb{Z}$ for all $\alpha \in \mathcal{R}$. To write the main result of Heckman and Opdam, set

$$\widetilde{c}(\lambda, k) := \prod_{\alpha \in \mathcal{R}^+} \frac{\Gamma(\lambda(\breve{\alpha}) + \frac{1}{2} k_{\frac{\alpha}{2}})}{\Gamma(\lambda(\breve{\alpha}) + \frac{1}{2} k_{\frac{\alpha}{2}} + k_\alpha)}, \qquad (\lambda, k) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathscr{K}, \tag{3.5}$$

and define the following meromorphic $c$-function

$$c(\lambda, k) := \frac{\widetilde{c}(\lambda, k)}{\widetilde{c}(\rho(k), k)}.$$

**Theorem 3.2.** (cf. [Hec-O, Hec-S]) *Let $S := \{$ zeros of the entire function $\tilde{c}(\rho(k), k) \}$. There exists a $W_\Pi$-invariant tubular neighborhood $U$ of $A$ in $A_{\mathbb{C}}$ such that the hypergeometric function*

$$F(\lambda, k, a) := \sum_{w \in W_\Pi} c(w\lambda, k) \Phi(w\lambda, k, a),$$

*is a holomorphic function on $\mathfrak{a}_{\mathbb{C}}^* \times (\mathscr{K} \setminus S) \times U$. Moreover*

$$F(w\lambda, k, a) = F(\lambda, k, wa) = F(\lambda, k, a),$$

*for all $w \in W_\Pi$ and $(\lambda, k, a) \in \mathfrak{a}_{\mathbb{C}}^* \times (\mathscr{K} \setminus S) \times U$.*

The functions $F(\lambda, k, a)$ are nowadays known as Heckman-Opdam hypergeometric functions.

To obtain the Bessel functions related to the root system $\mathcal{R}$ as an appropriate limit of Heckman-Opdam hypergeometric functions, we will proceed as in Section 2.

For strictly positive small real $\epsilon$, we introduce the following principle

$$(\mathcal{P}) \qquad \begin{array}{l} \text{– substitute } \alpha \text{ by } \epsilon\alpha \\ \text{– substitute } \langle\cdot,\cdot\rangle \text{ by } \epsilon^2\langle\cdot,\cdot\rangle \text{ on } \mathfrak{a}. \end{array}$$

Applying the principle $(\mathcal{P})$ to the definition of the operator $T(\xi,k)$ in (3.2), we obtain

$$T^{(\epsilon)}(\xi,k) \;=\; \frac{1}{\epsilon}\partial_\xi - \rho(k)(\xi) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha(\xi)}{\epsilon\alpha} \sum_{m=0}^{\infty} \frac{B_m(1)\epsilon^m\alpha^m}{m!}(1-r_\alpha),$$

where $B_m(1)$ is the Bernoulli number. Clearly the following limit exists

$$\lim_{\epsilon\to 0} \epsilon T^{(\epsilon)}(\xi,k) = T^\circ(\xi,k),$$

with

$$T^\circ(\xi,k) = \partial_\xi + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha(\xi)}{\alpha}(1-r_\alpha). \tag{3.6}$$

The differential operator $T^\circ(\xi,k)$ is the so-called Dunkl operator [Du1]. We are not aware of any previous work mentioning a connection of this type between the Dunkl-Cherednik operators $T(\xi,k)$ and the Dunkl operators $T^\circ(\xi,k)$. A similar relation between $T(\xi,k)$ and $T^\circ(\xi,k)$ is however given in [T].

Further, by the principle $(\mathcal{P})$, the operator $D(p_0,k) = \mathrm{Proj}(T(p_0,k)) = \Delta(k) + \langle\rho(k),\rho(k)\rangle$ becomes $\Delta^{(\epsilon)}(k) + \langle\rho(k),\rho(k)\rangle$ where

$$\Delta^{(\epsilon)}(k) = \frac{1}{\epsilon^2}\sum_{j=1}^{N} \partial^2_{\xi_j} + \frac{1}{\epsilon^2}\sum_{\alpha\in\mathcal{R}^+} \frac{2k_\alpha}{\alpha}\partial_\alpha + \sum_{\alpha\in\mathcal{R}^+}\sum_{m=1}^{\infty} \frac{k_\alpha B_{2m}(0)}{(2m!)} \left(\frac{\alpha}{2}\right)^{2m-1} \epsilon^{2(m-1)}\partial_\alpha,$$

and $B_{2m}(0)$ is the Bernoulli number. Thus

$$\lim_{\epsilon\to 0} \epsilon^2\big[\Delta^{(\epsilon)}(k) + \langle\rho(k),\rho(k)\rangle\big] = \sum_{j=1}^{N} \partial^2_{\xi_j} + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \left(\frac{2}{\alpha}\right)\partial_\alpha. \tag{3.7}$$

In the geometric case, this is equivalent to Theorem 2.2. We will denote by $\Delta^\circ(k)$ the right-hand side of (3.7).

To derive the eigenfunctions of $\Delta^\circ(k)$ by a limit analysis of Heckman-Opdam hypergeometric functions, we will use the same argument to that in the previous section for Harish-Chandra's spherical functions. We should notice that $F(\lambda,k,a)$ does not have an integral representation, since there is no longer a group theory behind. To get over this missing fact, we will proceed by induction on the multiplicity functions $k$.

From the definition of the $\tilde{c}$-function, one can see that $\tilde{c}(\lambda,0) = 1$. By [Hec-S, (3.5.14)] $\lim_{k\to 0}\tilde{c}(\rho(k),k) = |W_\Pi|$ and therefore

$$F(\lambda,0,a) = \frac{1}{|W_\Pi|}\sum_{w\in W_\Pi} e^{w\lambda(\log a)}, \qquad a\in A.$$

In particular, the following limit formula holds

$$\lim_{\epsilon\to 0} F\Big(\frac{\lambda}{\epsilon}, 0, \exp(\epsilon X)\Big) = \frac{1}{|W_\Pi|}\sum_{w\in W_\Pi} e^{w\lambda(X)}, \quad X\in\mathfrak{a}. \tag{3.8}$$

Next, for $X\in\mathfrak{a}$ and $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, we will denote by $F^\circ(\lambda,k,X)$ the limit of $F(\frac{\lambda}{\epsilon},k,\exp(\epsilon X))$ as $\epsilon\to 0$, if it exists.

Recall that $\mathscr{K}\cong\mathbb{C}^m$ where $m=\sharp\{W_\Pi-\text{orbits in }\mathcal{R}\}$. In the remaining part of this section we will assume that $k\in\mathscr{Z}:=\mathscr{K}\cap\mathbb{Z}^m$, i.e. $k_\alpha\in\mathbb{Z}$ for all $\alpha\in\mathcal{R}$. Denote by $\mathscr{Z}^+$ the set of positive-integer valued multiplicity functions.

The main tool in the induction process is to use the so-called Opdam's shift operators $G_\pm(\pm\ell,k)$ of shifts $\pm\ell$, where $\ell\in\mathscr{Z}^+$ and $k\in\mathscr{Z}$. These operators satisfy

$$\begin{aligned} G_-(-\ell,k)\Phi(\lambda,k,a) &= \frac{\tilde{c}(\lambda,k-\ell)}{\tilde{c}(\lambda,k)}\Phi(\lambda,k-\ell,a)\\ G_+(\ell,k)\Phi(\lambda,k,a) &= \frac{\tilde{c}(-\lambda,k)}{\tilde{c}(-\lambda,k+\ell)}\Phi(\lambda,k+\ell,a). \end{aligned}$$

We refer to [O1] and [O2] for more details on $G_\pm(\pm\ell,k)$.

Using the explicit forms of $G_\pm(\pm\ell,k)$ and the principle $(\mathcal{P})$, we obtain two deformed operators $G_\pm^{(\epsilon)}(\pm\ell,k)$, and we prove that the following limits exist

$$G_+^\circ(\ell,k) := \lim_{\epsilon\to 0}\epsilon^{2\sum_{\alpha>0}\ell_\alpha} G_+^{(\epsilon)}(\ell,k), \tag{3.9}$$

$$G_-^\circ(-\ell,k) := \lim_{\epsilon\to 0} G_-^{(\epsilon)}(-\ell,k). \tag{3.10}$$

We refer to [B-Ø2] for the explicit expressions of $G_\pm^\circ(\pm\ell,k)$. Further results on $G_\pm^\circ(\pm\ell,k)$ are also obtained. The shift operators $G_\pm^\circ(\pm\ell,k)$ can also be constructed by composing fundamental shift operators of shifts $\pm 1$.

Recall that $F^\circ(\lambda,k,X)$ exists for $k\equiv 0$. Using the shift operators $G_\pm^\circ(\pm\ell,k)$, we prove the following theorem by induction on $k$.

**Theorem 3.3.** (cf. [B-Ø2]) *For all $k\in\mathscr{Z}\setminus S$, the following limit and its derivatives exist*

$$\lim_{\epsilon\to 0} F\Big(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)\Big) = F^\circ(\lambda,k,X),$$

*and it satisfies the following Bessel system of differential equations on $W_\Pi\setminus\mathfrak{a}_{\mathbb{C}}$*

$$T^\circ(p,\xi)_{\big|\mathbb{C}[\mathfrak{a}_{\mathbb{C}}]^{W_\Pi}}\Psi = p(\lambda)\Psi, \qquad \forall\, p\in S(\mathfrak{a}_{\mathbb{C}})^{W_\Pi}. \tag{3.11}$$

*Moreover, for $\ell\in\mathscr{Z}^+$*

$$\begin{aligned} G_+^\circ(\ell,k)F^\circ(\lambda,k,X) &= \lambda^{2\sum_{\alpha\in\mathcal{R}^+}\ell_\alpha}\frac{\tilde{c}(\rho(k+\ell),k+\ell)}{\tilde{c}(\rho(k),k)}F^\circ(\lambda,k+\ell,X),\\ G_-^\circ(-\ell,k)F^\circ(\lambda,k,X) &= \frac{\tilde{c}(\rho(k-\ell),k-\ell)}{\tilde{c}(\rho(k),k)}F^\circ(\lambda,k-\ell,X). \end{aligned}$$

**Corollary 3.4.** (cf. [B-Ø2]) *The Bessel function $F^\circ(\lambda,k,X)$ satisfies*

$$\begin{aligned} &F^\circ(w\lambda,k,X) = F^\circ(\lambda,k,wX) = F^\circ(\lambda,k,X) \quad \textit{for all } w\in W_\Pi,\\ &F^\circ(\lambda,k,0)=1. \end{aligned}$$

The above corollary follows immediately from the fact that a similar statement for Heckman-Opdam hypergeometric functions holds, by taking the limit. As one can notice, we may derive at least the same amount of explicit information for the Bessel functions $F^\circ$ by a limit analysis of the hypergeometric functions $F$. Indeed, in [Ó-P], the authors were able to give an explicit formula for Heckman-Opdam hypergeometric functions when the root system $\mathcal{R}$ is reduced and $k \in \mathscr{Z}^+ \setminus S$, which we use to prove the following theorem.

**Theorem 3.5.** (cf. [B-Ø2]) *Assume that $k \in \mathscr{Z}^+ \setminus S$, and $\mathcal{R}$ is reduced. There exists a differential operator $\mathbb{D}(k) \in \mathbb{C}[\mathfrak{a}_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$, and a $W_\Pi$-invariant tubular neighborhood $\mathfrak{u}$ of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$ such that*

$$F^\circ(\lambda, k, X) = \frac{(-1)^{\sum_{\alpha>0} 1+k_\alpha} 2^{\sum_{\alpha>0} 1-2k_\alpha}}{\widetilde{c}(\rho(k), k)} \frac{\mathbb{D}(k)\Big(\sum_{w\in W_\Pi} \epsilon(w) e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathcal{R}^+} \langle \alpha, X\rangle^{2k_\alpha} \prod_{\alpha\in\mathcal{R}^+} \langle \alpha, \lambda\rangle^{2k_\alpha - 1}},$$

*for all $(\lambda, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathfrak{u}$.*

Explicit expression for $\mathbb{D}(k)$, which is given in terms of the Dunkl operators, can be found in [B-Ø2].

In [O4, Theorem 3.15] and for $k \in \mathscr{K}^+$, Opdam proved that the hypergeometric functions can be written as $F(\lambda, k, a) = \frac{1}{|W_\Pi|} \sum_{w\in W_\Pi} G(w\lambda, k, a)$ where $G(\lambda, k, a)$ is an eigenfunction for the Dunkl-Cherednik operator $T(\xi, k)$ with eigenvalue $\lambda(\xi)$. Using Theorem 3.3, and the fact that $\lim_{\epsilon\to 0} \epsilon T^{(\epsilon)}(\xi, k) = T^\circ(\xi, k)$, we obtain the following connection between the Bessel functions $F^\circ(\lambda, k, X)$ and the eigenfunctions of $T^\circ(\xi, k)$ with spectral parameter $\lambda$. The proof of the following theorem can be found in [B-Ø2], Theorem 4.6 and Proposition 4.7.

**Theorem 3.6.** (cf. [B-Ø2]) *Assume that $k \in \mathscr{Z}^+$.*

(i) *There exists a unique holomorphic function $G^\circ(\lambda, k, \cdot)$ in a tubular neighborhood $\mathfrak{u}$ of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$ such that*

$$T^\circ(\xi, k) G^\circ(\lambda, k, X) = \lambda(\xi) G^\circ(\lambda, k, X), \quad \xi \in \mathfrak{a}_\mathbb{C},$$
$$G^\circ(\lambda, k, 0) = 1.$$

(ii) *The Bessel functions can be written as*

$$F^\circ(\lambda, k, X) = \frac{1}{|W_\Pi|} \sum_{w\in W_\Pi} G^\circ(w\lambda, k, X).$$

(iii) *For all $w \in W_\Pi$, $G^\circ(w\lambda, k, wX) = G^\circ(\lambda, k, X)$ and $G^\circ(\lambda, k, 0) = 1$.*

We should note that the above theorem (other than (ii)) is also proved in [O3], where the author uses a different approach. In [O3], the statement (ii) appears as the definition of the Bessel functions.

The eigenfunctions $G^\circ(\lambda, k, X)$ are known as the Dunkl kernels. This kernel plays a major role in the theory of special functions related to Coxeter groups, which has been a rapid development in this area in the last few years. In the last section of the present paper, we will give new applications for the Dunkl kernels in connection with Hilbert spaces of holomorphic functions and Segal-Bargmann transforms.

***Example* 3.7.** (Generalization of Example 2.5) Assume that $\mathcal{R}$ is a rank-one root system of type $BC_1$, i.e. $\mathcal{R} = \{\pm\alpha, \pm 2\alpha\}$. In this example we have $A_\mathbb{C} \simeq \mathbb{C}^*$

and $\mathbb{C}[A_\mathbb{C}] = \mathbb{C}[x^{-1}, x]$ where $x = e^\alpha$. The nontrivial Weyl group element acts by $x \mapsto x^{-1}$ on $A_\mathbb{C}$.
If $\xi = (2\alpha)\check{}$, then $\partial_\xi = x\partial_x$. We will normalize the inner product on $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}^*_\mathbb{C}$ by $\langle \alpha, \alpha \rangle = 1$. In the $x$ coordinate, the differential operators (3.1) and (3.2) become

$$\Delta(k) = (x\partial_x)^2 - \left( k_\alpha \frac{1+x}{1-x} + 2k_{2\alpha} \frac{1+x^2}{1-x^2} \right) x\partial_x,$$

$$T(\xi, k) = x\partial_x + \left( \frac{k_\alpha}{1-x^{-1}} + \frac{2k_{2\alpha}}{1-x^{-2}} \right)(1-r) + (\frac{1}{2}k_\alpha + k_{2\alpha}),$$

where $r(x^m) = x^{-m}$. Opdam's shift operators $G_\pm(\pm 1, k)$ are given by [Hec-S]

$$G_+(+1, k) = \frac{x\partial_x}{x - x^{-1}},$$
$$G_-(-1, k) = (x^2-1)\partial_x + (k_\alpha + 2k_{2\alpha} - 1)(x + x^{-1}) + 2k_\alpha.$$

Let $z = -\frac{1}{4}x^{-1}(1-x)^2$ be a coordinate on $W_\Pi \setminus A_\mathbb{C}$. Put $\gamma_1 = \lambda + \frac{1}{2}k_\alpha + k_{2\alpha}$, $\gamma_2 = -\lambda + \frac{1}{2}k_\alpha + k_{2\alpha}$, and $\gamma_3 = \frac{1}{2} + k_\alpha + k_{2\alpha}$. The functions $F(\lambda, k, a)$ and $G(\lambda, k, a)$ are given by

$$F(\lambda, k, a) = {}_2F_1(\gamma_1, \gamma_2; \gamma_3; z),$$
$$G(\lambda, k, a) = {}_2F_1(\gamma_1, \gamma_2; \gamma_3; z) + \frac{\gamma_1}{4\gamma_3}(x - x^{-1})\,{}_2F_1(\gamma_1 + 1, \gamma_2 + 1; \gamma_3 + 1; z),$$

where ${}_2F_1(\gamma_1, \gamma_2; \gamma_3; z)$ is the Gauss hypergeometric function.
On $\mathfrak{a}_\mathbb{C}$, the infinitesimal operator (3.6) associated with $T(\xi, k)$ is given by

$$T^\circ(\xi, k) = \partial_X + \frac{k_\alpha + k_{2\alpha}}{X}(1-r),$$

and the shift operators $G^\circ_\pm(\pm 1, k)$ are given by

$$G^\circ_+(+1, k) = \frac{1}{2X}\partial_X, \qquad G^\circ_-(-1, k) = (2X)\partial_X + 2(2k_\alpha + 2k_{2\alpha} - 1).$$

Using the fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1 + \mathcal{O}(z^{-1})\right\}, \qquad \text{if } z \to \infty,$$

we obtain

$$\begin{aligned} F^\circ(\lambda, k, X) &= \lim_{\epsilon \to 0} F\left(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)\right) \\ &= \Gamma\left(\frac{1}{2} + k_\alpha + k_{2\alpha}\right) \left(\frac{\lambda X}{2}\right)^{\frac{1}{2} - k_\alpha - k_{2\alpha}} I_{k_\alpha + k_{2\alpha} - \frac{1}{2}}(\lambda X), \end{aligned}$$

where $I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz)$, with $J_\nu(z)$ is the Bessel function of the first kind. To regain Example 2.5 one needs to assume $k_\alpha = m_\alpha/2$ and $k_{2\alpha} = m_{2\alpha}/2$. As we can see, the Bessel function $F^\circ$ generalizes $\psi(\lambda, t)$ from Example 2.5, since we do not constrain the root multiplicities $k_\alpha$ and $k_{2\alpha}$ to assume certain specific values. From classical analysis on special functions, it is a well-known fact that for fixed

$\lambda \in \mathbb{C}$, the function $\zeta(X) := \Gamma(\frac{1}{2} + k_\alpha + k_{2\alpha}) \left(\frac{\lambda X}{2}\right)^{\frac{1}{2} - k_\alpha - k_{2\alpha}} I_{k_\alpha + k_{2\alpha} - \frac{1}{2}}(\lambda X)$ is the unique analytic solution of the differential equation

$$\zeta'' + \frac{2(k_\alpha + k_{2\alpha})}{X}\zeta' = \lambda^2 \zeta,$$

which is even and normalized by $\zeta(0) = 1$. The eigenfunction $G^\circ(\lambda, k, X)$ for $T^\circ(\xi, k)$ is given by

$$\begin{aligned} G^\circ(\lambda, k, X) &= \lim_{\epsilon \to \infty} G\left(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)\right) \\ &= \Gamma\left(\frac{1}{2} + k_\alpha + k_{2\alpha}\right) \left(\frac{\lambda X}{2}\right)^{\frac{1}{2} - k_\alpha - k_{2\alpha}} \left\{ I_{k_\alpha + k_{2\alpha} - \frac{1}{2}}(\lambda X) + I_{k_\alpha + k_{2\alpha} + \frac{1}{2}}(\lambda X) \right\}. \end{aligned}$$

## 4. Bessel functions associated with non-compact causal symmetric spaces

The theory of Harish-Chandra's spherical functions depends mainly on a compact subgroup $K$ of a Lie group $G$, i.e. a Cartan involution $\theta$; on the fact that the algebra $\mathscr{D}(G/K)$ contains an elliptic differential operator, and therefore all the joint eigenfunctions are real analytic; and finally on the Iwasawa decomposition $KAN$ and the Cartan decomposition $KAK$ of the Lie group $G$. Now, if we substitute $\theta$ by an arbitrary involution $\tau : G \to G$, and the subgroup $K$ by $H := \{h \in G \mid \tau(h) = h\}$. Then $H$ is no longer compact and, in general, there are no elliptic invariant differential operators on $G/H$. Further, $G \neq HAN$ and $G \neq HAH$. However, in [F-H-Ó] Faraut, Hilgert, and Ólafsson were able to prove that there exists a class of symmetric spaces, which is the so-called non-compact causal symmetric spaces, where an analogue theory of spherical functions defined on open $H$-invariant conal subset of $G/H$ can be developed. We refer to [H-Ó] for information on causal symmetric spaces.

Let $(G, H)$ be a symmetric pair, i.e. $G$ is a connected semisimple Lie group with finite center, $H$ is a closed subgroup, and there exists an involutive automorphism $\tau$ of $G$ such that

$$(G^\tau)_0 \subset H \subset G^\tau,$$

where $G^\tau := \{ g \in G \mid \tau(g) = g \}$, and $(G^\tau)_0$ is the identity component in $G^\tau$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$ and denote the differential of $\tau$ also by the same letter. Therefore $\mathfrak{h} = \mathfrak{g}^\tau$. Set $\mathfrak{q} = \mathfrak{g}^{-\tau}$. Let $x_0 = \mathbf{e}H$ where $\mathbf{e}$ is the unit element in $G$. The tangent space at $x_0$ can be identified with $\mathfrak{q}$. Let $\theta$ be a Cartan involution of $G$ commuting with $\tau$, and $K$ be the corresponding maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k} = \mathfrak{g}^\theta$. Put $\mathfrak{p} = \mathfrak{g}^{-\theta}$.

Assume that there exists in $\mathfrak{p} \cap \mathfrak{q}$ a non-zero vector $X_0$ which is invariant under $\mathrm{Ad}(H \cap K)$ and such that the projection on every irreducible component is non zero. Thus if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, then $X_0 \in \mathfrak{a}$ and $\mathfrak{a}$ is a maximal abelian in $\mathfrak{p}$.

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system and choose a positive system $\Sigma^+$ in $\Sigma$. Notice that $\Sigma$ is always reduced. Therefore one obtains in $\mathfrak{q}$ a closed convex $H$-invariant cone $C_{\max}$ such that $C_{\max} \cap \mathfrak{a} \neq \emptyset$. Put as usual $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}^{(\alpha)}$, and $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ where $m_\alpha = \dim(\mathfrak{g}^{(\alpha)})$. On the Lie algebra level, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{h}$ and the map $N \times A \times H \ni (n, a, h) \mapsto nah \in G$ is a diffeomorphism onto an open subset of $G$. From this it follows that the map

$N \times \mathfrak{a} \to G/H, \qquad (n, X) \mapsto n \exp(X) \cdot x_0$ is a diffeomorphism of $N \times \mathfrak{a}$ onto the open set $NA \cdot x_0$. For $x$ in this set, $x = n \exp(X) \cdot x_0$, we set $A(x) := X$. Note that the map $A$ is right $H$-invariant.

Denote by $\mathcal{E} = \{ \lambda \in \mathfrak{a}_{\mathbb{C}} \mid \mathrm{Rel}\langle \lambda + \rho, \alpha \rangle < 0 \; \forall \alpha \in \Sigma^+ \}$, and write $S$ the semi-group given by $S = \exp(C_{\max})H$. For $\lambda \in \mathcal{E}$, the spherical function $\varphi_\lambda$ is defined on the interior $S^0$ of $S$ by

$$\varphi_\lambda(x) = \int_H e^{\langle \rho - \lambda, A(hx) \rangle} \, dh, \tag{4.1}$$

(cf. [F-H-Ó]). (The measures are normalized via the Killing form.)

For $\epsilon > 0$, write $\gamma_\epsilon = \exp(\epsilon X) h$ with $h \in H$ and $X \in C^0_{\max}$. For arbitrary fixed $\lambda \in \mathcal{E}$, denote by

$$\psi(\lambda, X) := \lim_{\epsilon \to 0} \varphi_{\frac{\lambda}{\epsilon}}(\gamma_\epsilon).$$

Using (4.1), a similar argument to that of Theorem 2.3 gives the following integral representation of the Bessel functions associated with non-compact causal symmetric spaces.

**Theorem 4.1.** (cf. [B-Ø2]) *Let $G/H$ be a non-compact causal symmetric space, $\lambda \in \mathcal{E}$, and $X \in C^0_{\max}$. The limit $\psi(\lambda, X)$ and its derivatives exist. Its integral representation is given by*

$$\psi(\lambda, X) = \int_H e^{-B(A_\lambda, \mathrm{Ad}(h)X)} \, dh.$$

***Example* 4.2.** Let $G = SO_0(1, n)$ and let $H = SO_0(1, n-1)$, $n \geq 2$. Let $\mathfrak{a} = \mathbb{R}X_0$ where $X_0 = E_{1,n+1} + E_{n+1,1}$. Here we use the standard notations for the matrix element $E_{i,j}$. We choose the positive roots such that $\alpha(X_0) = 1$ and identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ via $z \mapsto -z\alpha$. Then $\rho = -(n-1)/2$. For $t > 0$ and $\mathrm{Rel}(\lambda) < -(n-3)/2$, we have

$$\begin{aligned} \varphi_\lambda(\exp(tX_0)) = {} & \pi^{-1/2} 2^{n/2-1} e^{-i\pi(n/2-1)} \Gamma\Big(\frac{n}{2} - \frac{1}{2}\Big) (\mathrm{sh}\, t)^{-(n/2-1)} \\ & \times \frac{\Gamma(\lambda - \frac{n}{2} + \frac{3}{2})}{\Gamma(\lambda + \frac{n}{2} - \frac{1}{2})} Q^{n/2-1}_{\lambda - 1/2}(\mathrm{ch}\, t), \end{aligned}$$

where $Q^\mu_\nu$ is the Legendre function of the second kind (cf. [F-H-Ó]). Using [E, 3.2 (10)], we can rewrite the spherical function $\varphi_\lambda$ as $\varphi_\lambda(\exp(tX_0)) = \varphi^{(1)}_\lambda(\exp(tX_0)) + \varphi^{(2)}_\lambda(\exp(tX_0))$, where

$$\begin{aligned} \varphi^{(1)}_\lambda(\exp(tX_0)) = {} & (i)^{\frac{n}{2}-1} \pi^{\frac{-1}{2}} 2^{n-3} \Gamma\Big(\frac{n}{2} - \frac{1}{2}\Big) \Gamma\Big(\frac{n}{2} - 1\Big) (\mathrm{sh}\, t)^{-(n-2)} \frac{\Gamma(\lambda - \frac{n}{2} + \frac{3}{2})}{\Gamma(\lambda + \frac{n}{2} - \frac{1}{2})} \\ & \times {}_2F_1\Big(\frac{\lambda}{2} - \frac{n}{4} + \frac{3}{4}, -\frac{\lambda}{2} - \frac{n}{4} + \frac{3}{4}, -\frac{n}{2} + 2; -\mathrm{sh}^2\, t\Big), \end{aligned}$$

and

$$\begin{aligned} \varphi^{(2)}_\lambda(\exp(tX_0)) = {} & (i)^{\frac{n}{2}-1} \pi^{\frac{-1}{2}} 2^{-1} \Gamma\Big(\frac{n}{2} - \frac{1}{2}\Big) \Gamma\Big(-\frac{n}{2} + 1\Big) \\ & \times {}_2F_1\Big(\frac{\lambda}{2} + \frac{n}{4} - \frac{1}{4}, -\frac{\lambda}{2} + \frac{n}{4} - \frac{1}{4}, \frac{n}{2}; -\mathrm{sh}^2\, t\Big). \end{aligned}$$

Using the fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1+\mathcal{O}(z^{-1})\right\}, \quad \text{if } \ z\to\infty,$$

we obtain

$$\lim_{\epsilon\to 0}\varphi^{(1)}_{\frac{\lambda}{\epsilon}}(\exp(\epsilon t X_0)) = -(i)^{\frac{n}{2}-1}\pi^{\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)(\lambda t)^{-\frac{n}{2}+1}\frac{I_{1-\frac{n}{2}}(\lambda t)}{\sin(\pi(1-\frac{n}{2}))},$$

and

$$\lim_{\epsilon\to 0}\varphi^{(2)}_{\frac{\lambda}{\epsilon}}(\exp(\epsilon t X_0)) = (i)^{\frac{n}{2}-1}\pi^{\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)(\lambda t)^{-\frac{n}{2}+1}\frac{I_{\frac{n}{2}-1}(\lambda t)}{\sin(\pi(1-\frac{n}{2}))}.$$

Here $I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz)$, where $J_\nu$ is the Bessel function of the first type. In conclusion

$$\psi(\lambda,t) = \lim_{\epsilon\to 0}\varphi_{\frac{\lambda}{\epsilon}}(\exp(\epsilon t X_0)) = (i)^{\frac{n}{2}-1}\pi^{-\frac{1}{2}}2^{\frac{n}{2}-1}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)(\lambda t)^{-\frac{n}{2}+1}\mathscr{K}_{1-\frac{n}{2}}(\lambda t),$$

where the function

$$\mathscr{K}_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}$$

is known as Macdonald's function, or the Bessel function of the third kind.

Put $\Sigma_0 := \Sigma(\mathfrak{k}\cap\mathfrak{h}\oplus\mathfrak{p}\cap\mathfrak{q},\mathfrak{a})$, and let $\Pi_0$ be the corresponding fundamental system of simple roots. Denote by $W_{\Pi_0}$ the Weyl group generated by $\{r_\alpha \mid \alpha\in\Pi_0\}$, where $r_\alpha : \mathfrak{a}^*\to\mathfrak{a}^*$ is the reflection $r_\alpha(\lambda) = \lambda - \lambda(\check{\alpha})\alpha$ (recall the definition of $\lambda(\check{\alpha})$).

In the next section we will investigate a more general class of Bessel functions related to arbitrary root subsystem, which will be called the $\Theta$-Bessel functions. This new class encloses both, the Bessel functions $F^\circ$ discussed in Section 3 and the present theory of Bessel functions associated with non-compact causal symmetric spaces. Indeed, in [B-Ø2] we were able to give explicit formulas for the $\Theta$-Bessel functions under certain conditions on the multiplicity functions. See the next section for more details. In particular, the general expression of the $\Theta$-Bessel functions for non-compact causal symmetric spaces reduces to:

**Theorem 4.3.** (cf. [B-Ø2]) *Let $G/H$ be a non-compact causal symmetric space such that $m_\alpha\in 2\mathbb{Z}$ for all $\alpha\in\Sigma$. For $(\lambda,X)\in\mathcal{E}\times(\mathfrak{a}\cap C^0_{\max})$, there exists a differential operator $\mathbb{D}(m)\in\mathbb{C}[\mathfrak{a}_\mathbb{C}]\otimes S(\mathfrak{a}_\mathbb{C})$ such that*

$$\int_H e^{-B(A_\lambda,\mathrm{Ad}(h)X)}\,dh = c_0(m)\frac{\mathbb{D}(m)\Big(\sum_{w\in W_{\Pi_0}}\epsilon(w)e^{-w\lambda(X)}\Big)}{\prod_{\alpha\in\Sigma^+}\langle\alpha,X\rangle^{m_\alpha}\prod_{\alpha\in\Sigma^+}\langle\alpha,\lambda\rangle^{m_\alpha-1}},$$

*where $c_0(m)$ is a constant that depends only on $m=(m_\alpha)_{\alpha\in\Sigma}$, and one may find it explicitly in Theorem 5.4 below (recall that for all non-compact causal symmetric spaces, $\Sigma$ is always reduced).*

See Table III and Table IV below for the list of all possible causal symmetric spaces for which the assumption of the above theorem holds. Explicit expression for $\mathbb{D}(m)$, which is given in terms of the Dunkl operators, can be found in [B-Ø2].

In the case when $G$ is a connected semisimple Lie group such that $G_\mathbb{C}/G$ is ordered, the above theorem gives a similar formula to the one for the character of discrete series representations of $G$. This case is mainly characterized by the fact that $m_\alpha = 2$ for all $\alpha\in\Sigma$.

**Theorem 4.4.** (cf. [B-Ø2]) *Let $G$ be a connected semisimple Lie group such that $G_{\mathbb{C}}/G$ is ordered. For $\lambda \in \mathcal{E}$ and $X \in C^0_{\max}$*

$$\int_G e^{-B(A_\lambda, \mathrm{Ad}(g)X)}\, dg = 2^{-2|\Sigma^+|} \prod_{\alpha \in \Sigma^+} \rho(\check{\alpha}) \frac{\sum_{w \in W_{\Pi_0}} \epsilon(w) e^{-w\lambda(X)}}{\prod_{\alpha \in \Sigma^+} \langle \alpha, \lambda \rangle \prod_{\alpha \in \Sigma^+} \langle \alpha, X \rangle}.$$

## 5. The Θ-Bessel functions

By introducing the so-called $\Theta$-spherical functions, in [P], Pasquale presents an extension of the theory of Heckman-Opdam hypergeometric functions associated with root systems, so that it encloses Harish-Chandra's theory of spherical functions and the theory of spherical functions on non-compact causal symmetric spaces. Using Pasquale's results, we introduce a new class of Bessel functions related to root systems, which we shall call the $\Theta$-Bessel functions. The theory of $\Theta$-Bessel functions extends naturally the theory of Bessel functions discussed in Section 3, and therefore it encloses the geometric case investigated in Section 2. It also covers the theory of Bessel functions associated with non-compact causal symmetric spaces.

The symbols $\mathcal{R}, \mathcal{R}^+, \mathfrak{a}, \mathfrak{a}_{\mathbb{C}}, A, A^+, W_\Pi, \mathscr{K}, r_\alpha, \lambda(\check{\alpha})$ shall have the same meaning as in Section 3. Recall that $\mathcal{R}$ is supposed to satisfy the crystallographic condition.

Let $\Pi = \{\alpha_1, \dots, \alpha_N\}$ be the system of simple roots associated with $\mathcal{R}^+$. Let $\Theta \subset \Pi$ be an arbitrary subset of $\Pi$. The set $\langle \Theta \rangle$ of elements in $\mathcal{R}$, which can be written as linear combinations of elements from $\Theta$, is a subsystem of $\mathcal{R}$. Its Weyl group $W_\Theta$ is generated by the reflections $r_{\alpha_j}$ with $\alpha_j \in \Theta$.

For a multiplicity function $k \in \mathscr{K}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we set

$$c^+_\Theta(\lambda, k) = \prod_{\alpha \in \langle \Theta \rangle^+} \frac{\Gamma(\lambda(\check{\alpha}) + \frac{1}{2}k_{\frac{\alpha}{2}})}{\Gamma(\lambda(\check{\alpha}) + \frac{1}{2}k_{\frac{\alpha}{2}} + k_\alpha)}$$

$$c^-_\Theta(\lambda, k) = \prod_{\alpha \in \mathcal{R}^+ \setminus \langle \Theta \rangle^+} \frac{\Gamma(-\lambda(\check{\alpha}) - \frac{1}{2}k_{\frac{\alpha}{2}} - k_\alpha + 1)}{\Gamma(-\lambda(\check{\alpha}) - \frac{1}{2}k_{\frac{\alpha}{2}} + 1)}$$

$$c^{+,c}_\Theta(\lambda, k) = \prod_{\alpha \in \mathcal{R}^+ \setminus \langle \Theta \rangle^+} \frac{\Gamma(\lambda(\check{\alpha}) + \frac{1}{2}k_{\frac{\alpha}{2}})}{\Gamma(\lambda(\check{\alpha}) + \frac{1}{2}k_{\frac{\alpha}{2}} + k_\alpha)}$$

with the conventions

$$c^+_\emptyset = c^{+,c}_\Pi = 1, \qquad \text{and} \qquad c^-_\Pi = 1.$$

If $\Theta = \Pi$, the function $c^+_\Pi(\lambda, k)$ coincides with the $\widetilde{c}$-function (3.5).

Let $U$ be a connected and simply connected open subset of $\exp(i\mathfrak{a})$ containing the identity element. The function on $A^+U$ defined for generic $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ by

$$\varphi_\Theta(\lambda, k, a) = c^-_\Theta(\lambda, k) \sum_{w \in W_\Theta} c^+_\Theta(w\lambda, k) \Phi(w\lambda, k, a), \quad a \in A^+U,$$

is called the $\Theta$-spherical function of spectral parameter $\lambda$ (see page 12 for the definition of $\Phi(\lambda, k, a)$). We refer to [P] for more details on $\Theta$-spherical functions. As a linear combination of the Harish-Chandra series $\Phi(w\lambda, k, a)$, the $\Theta$-spherical function is by construction a solution of the hypergeometric system (3.3).

***Example* 5.1.** When $\Theta = \Pi$, the ratio $\varphi_\Pi(\lambda, k, a)/c^+_\Pi(\rho(k), k)$ coincides with the Heckman-Opdam hypergeometric function $F(\lambda, k, a)$. In the geometric case, the ratio coincides with Harish-Chandra's spherical function.

***Example* 5.2.** For non-compact causal symmetric spaces, recall that $\Pi_0$ stands for the fundamental system for the positive compact roots in $\Sigma_0^+$. If $\Theta = \Pi_0$, then the ratio $\varphi_{\Pi_0}(\lambda, k, a)/c^+_{\Pi_0}(\rho(k), k)c^-_{\Pi_0}(\rho(k), k)$ coincides with the spherical function $\varphi_\lambda(a)$ on $G/H$ investigated in the previous section.

Recall that $\mathscr{Z}^+$ denotes the set of positive integer-valued multiplicity functions, and $S$ is the set of zeros of $\widetilde{c}(\rho(k), k) = c^+_\Pi(\rho(k), k)$. Further, put $d(\Theta, k) = \sum_{\alpha\in\mathcal{R}^+\setminus\langle\Theta\rangle^+} k_\alpha$, and define

$$\mathfrak{a}_{\Theta,+} := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \mathcal{R}^+ \setminus \langle\Theta\rangle^+\}.$$

In [Ó-P, Theorem 5.1], Ólafsson and Pasquale give an explicit global formulas for the $\Theta$-spherical functions for $k \in \mathscr{Z}^+$ by means of Opdam's shift operators $G_\pm(\pm\ell, k)$. Using our results on the shift operators $G^\circ_\pm(\pm\ell, k)$ (see (3.9) and (3.10)) together with the explicit expressions of the $\Theta$-spherical functions, we prove that for $X \in \mathfrak{a}_{\Theta,+}$ and $k \in \mathscr{Z}^+$

$$\varphi_\Theta\Big(\frac{\lambda}{\epsilon}, k, \exp(\epsilon X)\Big) \text{ conveges as } \epsilon \to 0.$$

Denote by $\widetilde{F}^\circ_\Theta(\lambda, k, X)$ its limit. We shall call $\widetilde{F}^\circ_\Theta$ the $\Theta$-Bessel functions, which are solutions for the Bessel system of differential equations (3.11). With the notations of Section 3, we have $F^\circ(\lambda, k, X) = \widetilde{F}^\circ_\Pi(\lambda, k, X)/c^+_\Pi(\rho(k), k)$ for $X \in \mathfrak{a}_{\Pi,+}$ and $k \in \mathscr{Z}^+ \setminus S$. The above transition relation linking $F^\circ$ to $\widetilde{F}^\circ_\Pi$ can be generalized by linking the $\Theta$-Bessel functions for arbitrary $\Theta$ to the Bessel functions $F^\circ$.

**Lemma 5.3.** (cf. [B-Ø2]) *There exists a $W_\Theta$-invariant tubular neighborhood $\mathfrak{u}_\Theta$ of $\mathfrak{a}_{\Theta,+}$ in $\mathfrak{a}_\mathbb{C}$ such that for $(\lambda, k, X) \in \mathfrak{a}^*_\mathbb{C} \times (\mathscr{Z}^+ \setminus S) \times \mathfrak{u}_\Theta$*

$$F^\circ(\lambda, k, X) = \frac{(-1)^{d(\Theta,k)}}{c^+_\Pi(\rho(k), k)} \sum_{w\in W_\Theta\setminus W_\Pi} \widetilde{F}^\circ_\Theta(w\lambda, k, X).$$

Put $d'(\Theta, k) := \sum_{\alpha\in\langle\Theta\rangle^+} k_\alpha$. The main result of this section is the following explicit expression for the $\Theta$-Bessel functions when $k \in \mathscr{Z}^+$ and $\mathcal{R}$ is reduced.

**Theorem 5.4.** (cf. [B-Ø2]) *Let $k \in \mathscr{Z}^+$ and let $\mathcal{R}$ be a reduced root system. There exists a differential operator $\mathbb{D}(k) \in \mathbb{C}[\mathfrak{a}_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$ such that*

$$\widetilde{F}^\circ_\Theta(\lambda, k, X) = (-1)^{d'(\Theta,k)+|\mathcal{R}^+|}\, 2^{\sum_{\alpha>0} 1-2k_\alpha} \frac{\mathbb{D}(k)\Big(\sum_{w\in W_\Theta} \epsilon(w) e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathcal{R}^+}\langle\alpha, X\rangle^{2k_\alpha} \prod_{\alpha\in\mathcal{R}^+}\langle\alpha, \lambda\rangle^{2k_\alpha-1}},$$

*for all $(\lambda, X) \in \mathfrak{a}^*_\mathbb{C} \times \mathfrak{u}_\Theta$.*

Explicit expression for $\mathbb{D}(k)$, which is given in terms of the Dunkl operators, and more information on the proof can be found [B-Ø2].

As we mentioned above, this new class of $\Theta$-Bessel functions encloses the Bessel functions $\psi(\lambda, X)$ associated with both, Riemannian symmetric spaces $G/K$ and non-compact causal symmetric spaces $G/H$. In order to obtain the expressions of $\psi(\lambda, X)$ from the above theorem, one needs to assume that $k_\alpha = m_\alpha/2 \in \mathbb{N}$ for all $\alpha \in \mathcal{R}(= 2\Sigma(\mathfrak{g}, \mathfrak{a}))$. Here $m_\alpha$ and $\Sigma(\mathfrak{g}, \mathfrak{a})$ have the same meaning as in Section 2 and in Section 4. We close this section by giving the list of all possible symmetric spaces $G/K$ and $G/H$ where $m_\alpha \in 2\mathbb{N}$ for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. This list has been extracted from the classification due to Oshima and Sekiguchi in [Os-Se] and to Hilgert and Ólafsson in [H-Ó].

***Riemannian symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{k}$ | $\Sigma$ | $m_\alpha$ | Comments |
|---|---|---|---|---|
| $\mathfrak{sl}(n, \mathbb{C})$ | $\mathfrak{su}(n)$ | $A_{n-1}$ | 2 | $n \geq 2$ |
| $\mathfrak{so}(2n+1, \mathbb{C})$ | $\mathfrak{so}(2n+1)$ | $B_n$ | 2 | $n \geq 2$ |
| $\mathfrak{sp}(n, \mathbb{C})$ | $\mathfrak{sp}(n)$ | $C_n$ | 2 | $n \geq 3$ |
| $\mathfrak{so}(2n, \mathbb{C})$ | $\mathfrak{so}(2n)$ | $D_n$ | 2 | $n \geq 4$ |
| $\mathfrak{so}(2n+1, 1)$ | $\mathfrak{so}(2n+1)$ | $A_1$ | $2n$ | $n \geq 3$ |
| $\mathfrak{su}^*(2n)$ | $\mathfrak{sp}(n)$ | $A_{n-1}$ | 4 | $n \geq 2$ |
| $(\mathfrak{e}_6)_\mathbb{C}$ | $\mathfrak{e}_6$ | $E_6$ | 2 | |
| $(\mathfrak{e}_7)_\mathbb{C}$ | $\mathfrak{e}_7$ | $E_7$ | 2 | |
| $(\mathfrak{e}_8)_\mathbb{C}$ | $\mathfrak{e}_8$ | $E_8$ | 2 | |
| $(\mathfrak{f}_4)_\mathbb{C}$ | $\mathfrak{f}_4$ | $F_4$ | 2 | |
| $(\mathfrak{g}_2)_\mathbb{C}$ | $\mathfrak{g}_2$ | $G_2$ | 2 | |
| $\mathfrak{e}_{6(-26)}$ | $\mathfrak{f}_{4(-20)}$ | $A_2$ | 8 | |

***Table I***

***Special isomorphisms of Riemannian symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{k}$ |
|---|---|
| $\mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{sp}(1) \approx \mathfrak{su}(2)$ |
| $\mathfrak{so}(3, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{so}(3) \approx \mathfrak{su}(2)$ |
| $\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C})$ | $\mathfrak{sp}(2) \approx \mathfrak{so}(5)$ |
| $\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C})$ | $\mathfrak{so}(6) \approx \mathfrak{su}(4)$ |
| $\mathfrak{so}(3, 1) \approx \mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{so}(3) \approx \mathfrak{su}(2)$ |
| $\mathfrak{so}(5, 1) \approx \mathfrak{su}^*(4)$ | $\mathfrak{so}(5) \approx \mathfrak{sp}(2)$ |

***Table II***

***Non-compact causal symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{h}$ | $\Sigma$ | $m_\alpha$ | Comments |
|---|---|---|---|---|
| $\mathfrak{sl}(n,\mathbb{C})$ | $\mathfrak{su}(n-j,j)$ | $A_{n-1}$ | 2 | $n\geq 2,\ 1\leq j\leq [n/2]$ |
| $\mathfrak{so}(2n+1,\mathbb{C})$ | $\mathfrak{so}(2n-1,2)$ | $B_n$ | 2 | $n\geq 2$ |
| $\mathfrak{sp}(n,\mathbb{C})$ | $\mathfrak{sp}(n,\mathbb{R})$ | $C_n$ | 2 | $n\geq 3$ |
| $\mathfrak{so}(2n,\mathbb{C})$ | $\mathfrak{so}(2n-2,2)$ | $D_n$ | 2 | $n\geq 4$ |
| $\mathfrak{so}(2n,\mathbb{C})$ | $\mathfrak{so}^*(2n)$ | $D_n$ | 2 | $n\geq 5$ |
| $\mathfrak{so}(2n+1,1)$ | $\mathfrak{so}(2n,1)$ | $A_1$ | $2n$ | $n\geq 3$ |
| $\mathfrak{su}^*(2n)$ | $\mathfrak{sp}(n-j,j)$ | $A_{n-1}$ | 4 | $n\geq 2,\ 1\leq j\leq [n/2]$ |
| $(\mathfrak{e}_6)_\mathbb{C}$ | $\mathfrak{e}_{6(-14)}$ | $E_6$ | 2 | |
| $(\mathfrak{e}_7)_\mathbb{C}$ | $\mathfrak{e}_{7(-25)}$ | $E_7$ | 2 | |
| $\mathfrak{e}_{6(-26)}$ | $\mathfrak{f}_{4(-20)}$ | $A_2$ | 8 | |

***Table III***

***Special isomorphisms of non-compact causal symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{h}$ |
|---|---|
| $\mathfrak{sp}(1,\mathbb{C})\approx\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{sp}(1,\mathbb{R})\approx\mathfrak{su}(1,1)$ |
| $\mathfrak{so}(3,\mathbb{C})\approx\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{so}(1,2)\approx\mathfrak{su}(1,1)$ |
| $\mathfrak{sp}(2,\mathbb{C})\approx\mathfrak{so}(5,\mathbb{C})$ | $\mathfrak{sp}(2,\mathbb{R})\approx\mathfrak{so}(3,2)$ |
| $\mathfrak{so}(6,\mathbb{C})\approx\mathfrak{sl}(4,\mathbb{C})$ | $\mathfrak{so}(4,2)\approx\mathfrak{su}(2,2)$ |
| $\mathfrak{so}(6,\mathbb{C})\approx\mathfrak{sl}(4,\mathbb{C})$ | $\mathfrak{so}^*(6)\approx\mathfrak{su}(3,1)$ |
| $\mathfrak{so}(8,\mathbb{C})=\mathfrak{so}(8,\mathbb{C})$ | $\mathfrak{so}^*(8)\approx\mathfrak{so}(2,6)$ |
| $\mathfrak{so}(3,1)\approx\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{so}(2,1)\approx\mathfrak{su}(1,1)$ |
| $\mathfrak{so}(5,1)\approx\mathfrak{su}^*(4)$ | $\mathfrak{so}(4,1)\approx\mathfrak{sp}(1,1)$ |

***Table IV***

## 6. Fock spaces and Segal-Bargmann transforms associated with Coxeter groups

In Section 3, Theorem 3.6, we proved that the Bessel functions can be written as an avreage over the Weyl group of the Dunkl kernels. In this section we shall give some applications for the Dunkl kernels in the theory of Hilbert spaces of holomorphic functions and Segal-Bargmann transforms. This section can be read independently from the previous sections.

Around 1928, in [Fo], Fock has introduced a Hilbert space of holomorphic functions on $\mathbb{C}^N$ which are square integrable with respect to the Gaussian measure $e^{-\|z\|^2}dz$, where $\|z\|^2=\sum_{i=1}^N z_i\overline{z_i}$. These spaces are nowadays known as the Fock spaces. After Bargmann's elegant paper [Ba], the Fock spaces have attracted much interest and have played an important role in a number of developments, namely in physics and mathematical physics. The remarkable invention of Bargmann is the construction of a unitary map from the Schrödinger model to the Fock model intertwining the action of the Heisenberg group. This idea also appeared in the work of Segal [Seg], done independently at about the same time. This intertwining operator is the so-called Segal-Bargmann transform. For more details, we refer to Bargmann's paper [Ba] which is still the best introduction to this matter.

In this section, we will investigate a generalization of both, the Fock spaces and the Segal-Bargmann transform, in the setting of Coxeter groups and Dunkl operators. Furthermore, a branching decomposition of the generalized Fock spaces will be given.

To simplify the presentation of our results, we will identify the $N$-dimensional Euclidean spaces $\mathfrak{a}$ and $\mathfrak{a}^*$ from the previous sections with $\mathbb{R}^N$. The setting of this section is slightly more general than the one of Section 3 as following: First we assume that $\mathcal{R}$ is a root system without the extra crystallographic condition, and therefore the Weyl group $W_\Pi$ will be replaced by an arbitrary Coxeter group. Secondly, we will assume that the multiplicity functions $k : \mathcal{R} \to \mathbb{C}$ are positive-real valued.

For $\alpha \in \mathcal{R}$, recall that the reflection $r_\alpha$ is given by

$$r_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha \qquad \alpha \in \mathbb{R}^N.$$

Henceforth, we will normalize $\mathcal{R}$ in the sense that $\langle \alpha, \alpha\rangle = 2$. This simplifies formulas, with no loss of generality for our purposes. We will use the same notation $\langle \cdot, \cdot\rangle$ for the bilinear extension of the Euclidean scalar product to $\mathbb{C}^N \times \mathbb{C}^N$.

A Coxeter group $G$ is a finite subgroup of the orthogonal group $O(N)$ generated by the reflections $\{r_\alpha \,|\, \alpha \in \mathcal{R}\}$. As we mentioned above, Coxeter groups generalize Weyl groups since there is not the additional crystallographic condition for $\mathcal{R}$.

As usual, $\mathcal{R}^+$ denotes a choice of positive roots in $\mathcal{R}$, and $\mathscr{K}$ is the set of multiplicity functions $k : \mathcal{R} \to \mathbb{C}$.

According to the standard notations in Dunkl's theory, we denote the Dunkl operator by

$$T_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \langle \alpha, \xi\rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle}, \qquad f \in \mathscr{C}^1(\mathbb{R}^N),$$

instead of $T^\circ(\xi, k)$ in (3.6). For any orthonormal basis $\{\xi_i\}_{i=1}^N$ of $\mathbb{R}^N$, set

$$\Delta_k f(x) := \sum_{i=1}^N T_{\xi_i}(k)^2 = \Delta f(x) + 2\sum_{\alpha\in\mathcal{R}^+} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle^2} \right\},$$

where $\Delta$ and $\nabla$ denote the usual Laplacian and gradient, respectively. The restriction of $\Delta_k$ to $G$-invariant functions coincides with $\Delta^\circ(k)$ in (3.7). For all $i$-th basis vector $\xi_i$, we will use the abbreviation $T_{\xi_i}(k) = T_i(k)$.

When $k$ is a positive-integer valued multiplicity function, and $G$ is a Weyl group, we proved in Theorem 3.6 that the Dunkl system of differential equations admits a solution, which is known as the Dunkl kernel. This result was also proved by Opdam for positive-real valued multiplicity functions and for all Coxeter groups $G$. The following theorem summarizes Opdam's result where we write $E_k(z, w)$ instead of $G^\circ(\lambda, k, X)$ in Theorem 3.6, based on the notations in Dunkl's theory (recall that $G^\circ(\lambda, k, X)$ is symmetric under the interchange of $\lambda$ and $X$).

**Theorem 6.1.** (cf. [O3]) *For $k \geq 0$, there exists a unique meromorphic function $E_k$ on $\mathbb{C}^N \times \mathbb{C}^N$ characterized by:*
(i) $T_\xi(k)E_k(z,w) = \langle \xi, w\rangle E_k(z,w)$; *and*
(ii) $E_k(z,0) = 1$.
*Moreover, this function satisfies*
(iii) $E_k$ *is holomorphic on* $\mathbb{C}^N \times \mathbb{C}^N$; *and*
(iv) $E_k(g_0 \cdot z, g_0 \cdot w) = E_k(z,w)$ *for all* $g_0 \in G$.

***Remark*** 6.2. If $k \equiv 0$, we have $E_0(z,w) = e^{\langle z,w\rangle}$ for $z, w \in \mathbb{C}^N$.

For $k \geq 0$, let $w_k$ be the weight function on $\mathbb{R}^N$ defined by

$$w_k(x) := \prod_{\alpha\in\mathcal{R}^+} |\langle \alpha, x\rangle|^{2k_\alpha}.$$

Further, let

$$c_k := \int_{\mathbb{R}^N} e^{-\langle x,x\rangle/2} w_k(x)\, dx,$$

which is called the Macdonald-Metha-Selberg integral. In [O3] Opdam gives a closed form for $c_k$ for finite Coxeter groups. The following proposition is crucial in Dunkl's theory and its applications.

**Proposition 6.3.** (cf. [Du2]) *Let $z, w \in \mathbb{C}^N$. For non-negative multiplicity function $k$*

$$\int_{\mathbb{R}^N} E_k(x,z)E_k(x,w)e^{-\langle x,x\rangle/2} w_k(x)\, dx = c_k e^{(\langle z,z\rangle + \langle w,w\rangle)/2} E_k(z,w). \tag{6.1}$$

For $z, w \in \mathbb{C}^N$, define

$$\mathbb{K}_{k,w}(z) = \mathbb{K}_k(z,w) := E_k(z,\bar{w}).$$

As $k$ will be fixed, we will write $\mathbb{K}$ for $\mathbb{K}_k$. By Theorem 6.1, one may check that $\mathbb{K}$ is continuous and $\mathbb{K}_w$ is holomorphic for all $w \in \mathbb{C}^N$. Further $\mathbb{K}(z,w) = \overline{\mathbb{K}(w,z)}$. Another crucial property is that $\mathbb{K}(z,w)$ is a positive definite kernel, i.e. for all $z^{(1)}, \dots, z^{(\ell)} \in \mathbb{C}^N$ and $a_1, \dots a_\ell \in \mathbb{C}$

$$\sum_{i,j=1}^{\ell} a_i \overline{a_j} \mathbb{K}(z^{(i)}, z^{(j)}) \geq 0.$$

These properties of $\mathbb{K}$ lead to the following result.

**Theorem 6.4.** (cf. [B-Ø3]) (i) *There exists a Hilbert space $\mathscr{F}_k(\mathbb{C}^N)$ of holomorphic functions, such that $\mathbb{K}$ is its reproducing kernel.*
(ii) *The Hilbert space $\mathscr{F}_k(\mathbb{C}^N)$ contains the $\mathbb{C}$-algebra $\mathscr{P}(\mathbb{C}^N)$ of polynomial functions on $\mathbb{C}^N$ as a dense subspace.*

In particular, if we denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_k$ the inner product in $\mathscr{F}_k(\mathbb{C}^N)$, then

$$\langle\!\langle p, q \rangle\!\rangle_k = p(T(k))\overline{q(\bar{z})}_{\big| z=0}, \qquad \forall\, p, q \in \mathscr{P}(\mathbb{C}^N),$$

where $p(T(k))$ is the operator formed by replacing $z_i$ by $T_i(z)$ for $1 \leq i \leq N$.

If $k \equiv 0$, $\mathscr{F}_0(\mathbb{C}^N)$ coincides with the classical Fock space. We shall call $\mathscr{F}_k(\mathbb{C}^N)$ the Fock space associated with the Coxeter $G$.

The study of several generalizations of the classical Segal-Bargmann transform has a long and rich history in many different settings [Ba, Seg, Ó-Ø, Hal, D-Ó-Z1, D-Ó-Z2, Z]. There are many ways of computing the integral kernel appearing in the Segal-Bargmann transform and showing the unitarity of this transform. One unifying tool is the restriction principle, i.e. polarization of a suitable restriction map [Ø-Z, Ó-Ø]. We will use this idea to construct the Segal-Bargmann transform associated with $G$. The main tool is the heat-kernel analysis for Coxeter groups [R2].

For $t > 0$ and $z, w \in \mathbb{C}^N$, set

$$\Gamma_k(t, z, w) = \frac{1}{(2t)^{\gamma_k + N/2} c_k} e^{-(\langle z,z\rangle + \langle w,w\rangle)/4t} E_k\Big(\frac{z}{\sqrt{2t}}, \frac{w}{\sqrt{2t}}\Big).$$

The kernel $\Gamma_k(t, z, w)$ was introduced in [R2] as a generalized heat kernel.

Let $\mathscr{L}^2(\mathbb{R}^N, w_k)$ be the space of $\mathscr{L}^2$-functions on $\mathbb{R}^N$ with respect to the weight function $w_k$. The notation $\|\cdot\|$ will be set for the norm in $\mathscr{L}^2(\mathbb{R}^N, w_k)$.

Let $\mathscr{R}_k$ be the restriction map $\mathscr{R}_k : \mathscr{F}_k(\mathbb{C}^N) \to \mathscr{L}^2(\mathbb{R}^N, w_k)$, given by

$$\mathscr{R}_k f(x) := e^{-\langle x,x\rangle/2} f(x), \quad x \in \mathbb{R}^N.$$

The map $\mathscr{R}_k$ is a closed, densely defined operator from $\mathscr{F}_k(\mathbb{C}^N)$ into $\mathscr{L}^2(\mathbb{R}^N, w_k)$ with dense image (see for instance [R1, Theorem 3.2]). Consider the adjoint $\mathscr{R}_k^* : \mathscr{L}^2(\mathbb{R}^N, w_k) \to \mathscr{F}_k(\mathbb{C}^N)$ as a densely defined operator. Since $\mathbb{K}$ is the reproducing kernel of $\mathscr{F}_k(\mathbb{C}^N)$, one can prove that for $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$, the integral

$$\mathscr{R}_k \mathscr{R}_k^* f(y) = c_k \int_{\mathbb{R}^N} f(x) \Gamma_k\Big(\frac{1}{2}, x, y\Big) w_k(x)\, dx$$

converges absolutely for a.e. $y \in \mathbb{R}^N$. The function $\mathscr{R}_k \mathscr{R}_k^* f$ thus defined is in $\mathscr{L}^2(\mathbb{R}^N, w_k)$ and $\|\mathscr{R}_k \mathscr{R}_k^*\| \le c_k$. We can therefore define $\sqrt{\mathscr{R}_k \mathscr{R}_k^*}$. Thus there exists an isometry $\mathscr{B}_k$ so that $\mathscr{R}_k^* = \mathscr{B}_k \sqrt{\mathscr{R}_k \mathscr{R}_k^*}$. Since $\mathscr{R}_k = \sqrt{\mathscr{R}_k \mathscr{R}_k^*} \mathscr{B}_k^*$ and Image($\mathscr{R}_k$) is dense, it follows that $\mathscr{B}_k$ is a unitary isomorphism. We shall call $\mathscr{B}_k$ the Segal-Bargmann transform associated with $G$. Using the positivity of the heat kernel $\Gamma(t, x, y)$ as an operator [R2], we obtain the following integral representation of the Segal-Bargmann transform $\mathscr{B}_k$.

**Theorem 6.5.** (cf. [B-Ø3]) *The unitary isomorphism* $\mathscr{B}_k : \mathscr{L}^2(\mathbb{R}^N, w_k) \to \mathscr{F}_k(\mathbb{C}^N)$ *is given by*

$$\mathscr{B}_k f(z) = 2^{\gamma_k + N/2} c_k^{-1/2} e^{-\langle z,z\rangle/2} \int_{\mathbb{R}^N} f(x) E_k(\sqrt{2}x, \sqrt{2}z) e^{-\langle x,x\rangle} w_k(x)\, dx,$$

*where* $\gamma_k := \sum_{\alpha \in \mathcal{R}^+} k_\alpha$.

***Remark*** 6.6. (i) For the special case $k \equiv 0$

$$\mathscr{B}_0 f(z) = (2/\pi)^{N/4} \int_{\mathbb{R}^N} e^{-\langle x,x\rangle + 2\langle x,z\rangle - \langle z,z\rangle/2} f(x)\, dx.$$

This compares well with the classical Segal-Bargmann transform (cf. [Fol, p. 40]).

(ii) When $N = 1$ and $G = \mathbb{Z}/2\mathbb{Z}$, Cholewinski [Ch] has investigated the Segal-Bargmann transform only in the Hilbert space of even functions in $\mathscr{F}_k(\mathbb{C})$, by employing another approach. Recently, in [Si-So] and always for $N = 1$ and $G = \mathbb{Z}/2\mathbb{Z}$, Sifi and Soltani use Cholewinski's method to obtain the Segal-Bargmann transform for $\mathscr{F}_k(\mathbb{C})$. Recently, we learned about Soltani's preprint [So] which contains some results on Segal-Bargmann transform for Coxeter groups, using Cholewinski's approach.

The Dunkl transform, which shares many properties with the classical Fourier transform, was introduced in [Du2] and further studied in [J]. For our convenience, we will write the Dunkl transform as

$$\mathscr{D}_k f(\xi) = c_k^{-1} 2^{-\gamma_k - N/2} \int_{\mathbb{R}^N} f(x/2) E_k(-i\xi, x) w_k(x)\, dx, \quad \xi \in \mathbb{R}^N.$$

**Theorem 6.7.** (cf. [B-Ø3]) *The following diagram commutes*

$$\begin{array}{ccc} \mathscr{L}^2(\mathbb{R}^N, w_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \\ {\scriptstyle \mathscr{D}_k}\downarrow & & \downarrow{\scriptstyle (-i)^*} \\ \mathscr{L}^2(\mathbb{R}^N, w_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \end{array}$$

*where* $(-i)^* f(z) := f(-iz)$ *for* $f \in \mathscr{F}_k(\mathbb{C}^N)$.

The above theorem gives a simple alternative proof for the unitarity of the transform $\mathscr{D}_k$, which was proved earlier by Dunkl [Du2] using a different approach. See also [J]. Our proof uses only the integral formula (6.1).

For $\xi \in \mathbb{C}^N$, denote by $M_\xi$ the operator $M_\xi(f)(z) := \langle z, \xi\rangle f(z)$. Define the lowering and the raising operators on $\mathscr{L}^2(\mathbb{R}^N, w_k)$ by

$$A_\xi^- := \frac{1}{\sqrt{2}}(M_{2\xi} + T_\xi(k)), \qquad A_\xi^+ := \frac{1}{\sqrt{2}}(M_{2\xi} - T_\xi(k)).$$

These two operators were introduced by Rösler [R3] in connection with Rodrigues-type formulas for the eigenfunctions of the Calogero-Moser systems. Next we will see that these two operators, in the Fock model, are also the lowering and the raising operators on $\mathscr{F}_k(\mathbb{C}^N)$ in a more natural way.

Below, we will exhibit some relationships between operators on $\mathscr{L}^2(\mathbb{R}^N, w_k)$ and on $\mathscr{F}_k(\mathbb{C}^N)$. For an operator $\mathcal{O}$ on $\mathscr{L}^2(\mathbb{R}^N, w_k)$, we define the operator $\breve{\mathcal{O}}$ on $\mathscr{F}_k(\mathbb{C}^N)$ by

$$\breve{\mathcal{O}} = \mathscr{B}_k \circ \mathcal{O} \circ \mathscr{B}_k^{-1}.$$

Further, as usual, $[A, B] = AB - BA$ for $A, B \in \mathrm{End}(\mathscr{P}(\mathbb{C}^N))$.

**Theorem 6.8.** (cf. [B-Ø3]) *The following properties hold:*

(i) $\breve{T}_\xi(k) = T_\xi(k) - M_\xi$ *for* $\xi \in \mathbb{C}^N$;
(ii) $[\breve{T}_\xi(k), \breve{T}_\eta(k)] = 0$ *for* $\xi, \eta \in \mathbb{C}^N$;
(iii) $\breve{M}_{2\xi} = T_\xi(k) + M_\xi$ *for* $\xi \in \mathbb{C}^N$;
(iv) $[\breve{M}_{2\xi}, \breve{M}_{2\eta}] = 0$ *for* $\xi, \eta \in \mathbb{C}^N$;
(v) $[\breve{T}_\xi(k), \breve{M}_{2\eta}] = 2\langle \xi, \eta\rangle + 2\sum_{\alpha\in\mathcal{R}^+} k_\alpha \langle \alpha, \xi\rangle\langle \alpha, \eta\rangle r_\alpha$;
(vi) $\breve{A}_\xi^- = \sqrt{2} T_\xi(k)$, *and* $\breve{A}_\xi^+ = \sqrt{2} M_\xi$.

*Notice that, as the Dunkl operators are homogeneous of degree $-1$ on polynomials, and since $M_\xi$ are the multiplication operators, now obviously $\breve{A}_\xi^-$ and $\breve{A}_\xi^+$ are the lowering and the raising operators on $\mathscr{P}(\mathbb{C}^N)$.*

The above theorem, which is of independent interest, is mainly useful to obtain the quantum Calogero-Moser (CM) rational system in the Fock model. We refer to [B-Ø3] for more details on this matter.

Let

$$\mathscr{L}_k := \Delta - 2\sum_{\alpha\in\mathcal{R}^+} \frac{1}{\langle \alpha, x\rangle^2} k_\alpha (k_\alpha - r_\alpha),$$

and consider the following gauge equivalent version

$$\mathscr{H}_k := \tfrac{1}{4} w_k^{-1/2}(-\mathscr{L}_k + 4\langle x, x\rangle) w_k^{1/2} = \tfrac{1}{4}(-\Delta_k + 4\langle x, x\rangle)$$

of the CM Hamiltonian with harmonic confinement and reflection terms.

**Theorem 6.9.** (cf. [B-Ø3]) *Let $\{\xi_1, \ldots, \xi_N\}$ be any orthonormal basis of $\mathbb{C}^N$. On $\mathscr{F}_k(\mathbb{C}^N)$, the corresponding operator to the Hamiltonian $\mathscr{H}_k$ is given by*

$$\breve{\mathscr{H}}_k = (\gamma_k + N/2) + \sum_{i=1}^{N} \xi_i \partial_{\xi_i},$$

*where $\gamma_k = \sum_{\alpha \in \mathcal{R}^+} k_\alpha$. Clearly, the study of the Hamiltonian $\mathscr{H}_k$ in the Fock model is rather easy.*

We close this section by describing the structure of a representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on $\mathscr{P}(\mathbb{C}^N)$. This representation, together with the left regular action of the Coxeter group $G$, allows to obtain the branching decomposition of the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ under the action of $G \times \widetilde{SL(2,\mathbb{R})}$. Those readers who are familiar with the theory of Howe reductive dual pairs [Ho1, Ho2] will find that our formulation can be thought of as an analogue of this theory. Hecke's formula for the Dunkl transform holds immediately from our $\widetilde{SL(2,\mathbb{R})}$-representation.

Choose $z_1, z_2, \ldots, z_N$ as the usual system of coordinates on $\mathbb{C}^N$. Let

$$E = \frac{1}{2}\sum_{i=1}^{N} z_i^2, \quad F = -\frac{1}{2}\Delta_k, \quad H = N/2 + \gamma_k + \sum_{i=1}^{N} z_i \partial_{z_i}.$$

In the notation of Theorem 6.9, the operator $H = \breve{\mathscr{H}}_k$. Then $E$ (resp. $F$) acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a creation (resp. annihilation) operator, and $H$ acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a number operator. If $\mathscr{P}(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{P}_m(\mathbb{C}^N)$ is the natural grading on $\mathscr{P}(\mathbb{C}^N)$, it is clear that $E$ raises $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m+2}(\mathbb{C}^N)$, $F$ lowers $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m-2}(\mathbb{C}^N)$, and $H$ multiplies (elementwise) $\mathscr{P}_m(\mathbb{C}^N)$ by the number $(N/2+\gamma_k+m)$. In [Hec1], Heckman showed the following commutation relations

$$[E, F] = H, \quad [E, H] = -2E, \quad [F, H] = 2F. \tag{6.2}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (6.2) gives raise to a unitary representation $\omega$ of $\mathfrak{sl}(2,\mathbb{R})$. The unitarty of $\omega$ follows from the fact that $E^* = -F$ and $H = H^*$ (cf. [B-Ø3, Theorem 3.7]). Notice also that $H$ has discrete spectrum bounded below.

On $\mathscr{P}(\mathbb{C}^N)$, the representation $\omega$ can be described as

$$\omega(\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}) = \mathfrak{sl}_2^{(2,0)} \oplus \mathfrak{sl}_2^{(1,1)} \oplus \mathfrak{sl}_2^{(0,2)}, \tag{6.3}$$

where

$$\mathfrak{sl}_2^{(2,0)} = \mathrm{Span}\{E\}, \quad \mathfrak{sl}_2^{(1,1)} = \mathrm{Span}\{H\}, \quad \mathfrak{sl}_2^{(0,2)} = \mathrm{Span}\{F\}.$$

The decomposition (6.3) is an instance of the Cartan decomposition

$$\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$$

where $\mathfrak{sl}_2^{(2,0)} \simeq \omega(\mathfrak{p}^+)$, $\mathfrak{sl}_2^{(1,1)} \simeq \omega(\mathfrak{k}_{\mathbb{C}})$, and $\mathfrak{sl}_2^{(0,2)} \simeq \omega(\mathfrak{p}^-)$. Here $\mathfrak{k} = \mathfrak{u}(1)$, the Lie algebra of the compact group $U(1)$. The integrated form of the Lie algebra representation $\omega$ is an analogue of the metaplectic representation, or the oscillator representation, of the universal covering $\widetilde{SL(2,\mathbb{R})}$ of the group $SL(2,\mathbb{R})$. Notice that if $N/2+\gamma_k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ we obtain a unitary representation of the double covering $Mp(2,\mathbb{R})$ of $SL(2,\mathbb{R})$, and if $N/2+\gamma_k \in \mathbb{Z}$ we obtain a representation of $SL(2,\mathbb{R})$. By applying the Segal-Bargmann transform, one obtains the Schrödinger picture of this representation of $\widetilde{SL(2,\mathbb{R})}$. However, for our purpose, its infinitesimal action (6.3) is enough.

Since $\omega$ is a unitary representation, and the operator $H$, which is the generator of $\mathfrak{k}$, has a positive spectrum, then the representation contains vectors $v_0$ such that $\omega(\mathfrak{p}^-)v_0 = 0$ and $\omega(\mathfrak{k})v_0 = (m + N/2 + \gamma_k)v_0$ for some positive integer $m$. The vector $v_0$ is the so-called lowest weight vector for a representation, and the number $(m + N/2 + \gamma_k)$ is the lowest weight. The space of representation then has an orthonormal basis consisting of vectors $v_\ell \in \omega(\mathfrak{p}^+)^\ell v_0$. It is easy to check that each vector $v_\ell$ is an eigenvector for $\omega(\mathfrak{k})$ with eigenvalue $(m+2\ell+N/2+\gamma_k)$. Denote by $\mathscr{W}_{m+N/2+\gamma_k}$ the unitary representation of $\widetilde{SL(2,\mathbb{R})}$ with lowest weight $m + N/2 + \gamma_k$.

For $m \in \mathbb{N}$, set $\mathscr{H}_m(\subset \mathscr{P}(\mathbb{C}^N))$ to be the space of harmonic homogeneous polynomials of degree $m$, i.e. all functions $p \in \mathscr{P}_m(\mathbb{C}^N)$ such that $\Delta_k p = 0$. It is clear that $p \in \mathscr{H}_m$ if and only if $\omega(\mathfrak{k})p = (m + N/2 + \gamma_k)p$ and $\omega(\mathfrak{p}^-)p = 0$.

Now one of the key features in this formalism is the following branching decomposition. We refer to [B-Ø3] for its proof, which was inspired by Sobolev's argument in the classical case [Sob]. The notation $[m/2]$ stands for the integer part of $m/2$.

**Theorem 6.10.** (cf. [B-Ø3]) *The space $\mathscr{P}_m(\mathbb{C}^N)$ of homogeneous polynomials of degree $m$ has a unique decomposition of the form*

$$\mathscr{P}_m(\mathbb{C}^N) = \sum_{\mu=0}^{[m/2]}{}^{\oplus} \langle z, z\rangle^\mu \mathscr{H}_{m-2\mu},$$

*where $\mathscr{H}_{m-2\mu}$ denotes the space of harmonic homogeneous polynomials of degree $m-2\mu$. Moreover, each homogeneous polynomial $p \in \mathscr{P}_m(\mathbb{C}^N)$ can be written in a unique way as*

$$p(z) = \sum_{\mu=0}^{[m/2]} \frac{\Gamma(N/2+m-\mu+\gamma_k-1)}{4^\mu\Gamma(\mu+1)\Gamma(N/2+m+\gamma_k-1)}\langle z, z\rangle^\mu h_{m-2\mu}(z),$$

*where $h_{m-2\mu} \in \mathscr{H}_{m-2\mu}$ and is given explicitly by*

$$h_{m-2\mu}(z) = \sum_{\nu=0}^{[m/2]-\mu} \frac{(-1)^\nu\Gamma(N/2+m-2\mu-\nu-1+\gamma_k)}{4^\nu\Gamma(\nu+1)\Gamma(N/2+m-2\mu+\gamma_k-1)}\langle z, z\rangle^\nu \Delta_k^{\mu+\nu}p(z).$$

For $g \in G$, denote by $\pi(g)$ the left regular action of $G$ on $\mathscr{F}_k(\mathbb{C}^N)$

$$\pi(g)f(z) = f(g^{-1}z).$$

The actions of $G$ and $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{F}_k(\mathbb{C}^N)$ commute.

We now summarize the consequences of all the above computations and discussions in the light of Theorem 6.10.

**Theorem 6.11.** (cf. [B-Ø3]) *As a $G \times \widetilde{SL(2,\mathbb{R})}$-module, the Fock space admits the following multiplicity-free decomposition*

$$\mathscr{F}_k(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{H}_m \otimes \mathscr{W}_{m+N/2+\gamma_k}, \tag{6.4}$$

*where $\mathscr{W}_{m+N/2+\gamma_k}$ is the $\widetilde{SL(2,\mathbb{R})}$-representation with lowest weight $m+N/2+\gamma_k$. We also have the following separation of variables decomposition*

$$\mathscr{P}(\mathbb{C}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \sum_{\mu=0}^{[m/2]}{}^{\oplus} \langle z, z\rangle^{\mu} \mathscr{H}_{m-2\mu}.$$

***Remark*** 6.12. (i) Recall that the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ was defined for non-negative multiplicity functions $k$. Now, notice that the right hand side of (6.4) exists for all $\gamma_k > -N/2$, where $\gamma_k = \sum_{\alpha\in\mathcal{R}^+} k_\alpha$, which implies that $k = (k_\alpha)_{\alpha\in\mathcal{R}}$ could have negative-values up to a certain point. By analytic continuation, it follows that the left hand side of (6.4), i.e. the Fock space $\mathscr{F}_k(\mathbb{C}^N)$, exists also for these negative-valued multiplicity functions $k$.

(ii) The space $\mathscr{H}_m$ is a unitary representation of $G$, in general not irreducible. It would be interesting to decompose it further.

As an application of the above demonstrated $\mathfrak{sl}(2,\mathbb{R})$-representation theory, we obtain the Hecke's formula for the Dunkl transform as following: Recall that $H = \breve{\mathscr{H}}_k = \mathscr{B}_k \circ \left\{\frac{1}{4}(-\Delta_k + 4\langle x, x\rangle)\right\} \circ \mathscr{B}_k^{-1}$ (see Theorem 6.8). Therefore

$$\left\{\tfrac{1}{4}(-\Delta_k + 4\langle x, x\rangle)\right\} \mathscr{B}_k^{-1}(p) = (m + N/2 + \gamma_k)\mathscr{B}_k^{-1}(p), \qquad \forall\, p \in \mathscr{P}_m(\mathbb{C}^N).$$

Further, in [B-Ø3, Corollary 4.6], we proved that the restriction of the inverse transform $\mathscr{B}_k^{-1}$ to $\mathscr{H}_m$ coincides with $2^{\gamma_k+N/2}c_k^{-1/2}e^{-\langle x,x\rangle}$, which implies that $e^{-\langle x,x\rangle}p$, for $p \in \mathscr{H}_m$, is an eigenvector for $\left\{\frac{1}{4}(-\Delta_k + 4\langle x, x\rangle)\right\}$ with eigenvalue $(m + N/2 + \gamma_k)$. On the other hand, by [B-Ø3, Corollary 4.14], the Dunkl transform $\mathscr{D}_k$ can be written as

$$\mathscr{D}_k = e^{i\frac{\pi}{2}(\gamma_k+N/2)} e^{-\frac{\pi}{8}(-\Delta_k+4\langle x,x\rangle)},$$

whilst $\left\{\frac{1}{4}(-\Delta_k + 4\langle x, x\rangle)\right\}$ is the generator of the Lie algebra $\mathfrak{k} \cong \mathfrak{so}(2)$. Thus, for $p \in \mathscr{H}_m$

$$\begin{aligned}\mathscr{D}_k(e^{-\langle x,x\rangle}p) &= e^{i\frac{\pi}{2}(\gamma_k+N/2)} e^{-\frac{\pi}{2}\mathfrak{k}}(e^{-\langle x,x\rangle}p)\\ &= e^{i\frac{\pi}{2}(\gamma_k+N/2)} e^{-i\frac{\pi}{2}(m+N/2+\gamma_k)} e^{-\langle x,x\rangle}p\\ &= e^{-i\frac{\pi}{2}m} e^{-\langle x,x\rangle}p,\end{aligned}$$

and the following theorem has been established.

**Theorem 6.13.** (cf. [B-Ø3]) *The following Hecke-type formula holds for the Dunkl transform*

$$\mathscr{D}_k(e^{-\langle x,x\rangle}p) = e^{-i\frac{\pi}{2}m} e^{-\langle x,x\rangle}p, \qquad \forall\, p \in \mathscr{H}_m.$$

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S. Ben Saïd: University of Aarhus, Department of Mathematical Sciences, Building 530, Ny Munkegade, DK-8000, Aarhus C, Denmark

*E-mail address*: `ssaid@imf.au.dk`

B. Ørsted: University of Aarhus, Department of Mathematical Sciences, Building 530, Ny Munkegade, DK-8000, Aarhus C, Denmark

*E-mail address*: `orsted@imf.au.dk`