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CATEGORICAL DISTRIBUTION THEORY; HEAT EQUATION

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Introduction

The simplest notion allowing a theory of function spaces to be formulated is that of cartesian closed categories.

In a cartesian closed category, containing in a suitable sense the ring R of real numbers, a notion of "distribution of compact support" on any object M can be defined, because the object of R-linear functionals on the ring R^M can be formulated, cf. e.g. [24], [21]. Thus, a "synthetic" theory of "distributions-of-compact support", and models for it, do exist (we exploited this fact in [15]).

The content of the present note is to provide a similar theory, as well as models, for distributions which are not necessarily of compact support. This amounts to describing synthetically the notion of "test function" of compact support. The R-linear dual of the vector space of test functions then is then a synthetic version of the space of distributions.

When we say "model", we mean more precisely a cartesian closed category, containing as full subcategories both the category of smooth manifolds, and also some suitable category of topological vector spaces, in such a way that the synthetic contructs alluded to agree with the classical functional analytic ones.

The category of "suitable" topological vector spaces will be taken to be the category of Convenient Vector Spaces, in the sense of [5], [16]. With the smooth (not necessarily linear) maps, this category $\underline{\text{Con}}^\infty$ is already cartesian closed, cf. loc.cit. We exhibited in 1986-1987 ([11], [13]) a full embedding of this category into a certain topos (the "Cahiers Topos" of Dubuc [3]). It is this embedding that we here shall prove is a model for a synthetic theory of distributions. The point about the Cahiers Topos is that it is also a well-adapted model for Synthetic Differential Geometry (meaning in essence that $\mathbf{R}$ acquires sufficiently many nilpotent elements).

The functional-analytic spadework that we provide also gives, – with much less effort than what is needed for the Cahiers Topos –, a simpler model, namely Grothendieck's "Smooth Topos". However, a main purpose of distribution theory is to account for partial differential equations, and therefore a synthetic theory of *differentiation* should preferably be available in the model, as well, which it is in the Cahiers Topos, but not in the Smooth Topos (at least such theory has not yet been developed, and is anyway bound to be less simple).

As a pilot project for our theory, we shall finish by showing that the Cahiers topos does admit a fundamental (distributional) solution of the heat equation on the unlimited line. (Here clearly distributions of compact support will not suffice.)

Solutions of the heat equation model evolution through time of a *heat distribution*. A heat distribution is an *extensive* quantity and does not necessarily have a density *function*, which is an *intensive* quantity (cf. [18]). The most important of all distributions, the point- or Dirac- distributions, do not. For the heat equation, it is well known that the evolution through time of any distribution leads 'instantaneously' (i.e., after any *positive* lapse of time $t > 0$) to distributions that do have smooth density functions. Indeed, the evolution through time of the Dirac distribution $\delta(0)$ is given by the map ("heat kernel", "fundamental solution")

$$K : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R}) \tag{1}$$

defined by cases by the classical formula

$$K(t) = \begin{cases} e^{-x^2/4t}/\sqrt{4\pi t} & \text{if } t > 0 \\ \delta(0) & \text{if } t = 0 \end{cases} ; \tag{2}$$

here $\mathcal{D}'(\mathbf{R})$ denotes a suitable space of distributions (in the sense of [25], [26]); notice that in the first clause we are identifying distributions with their density functions (when such density functions exist).

The fundamental mathematical object given in (2) presents a challenge to the synthetic kind of reasoning in differential geometry, where a basic tenet is "everything is smooth"; therefore, definition by cases, as in (2), has a dubious status. This challenge was one of the motivations for the present study.

One may see another lack of smoothness in (2), namely "$\delta(0)$ is not smooth"; but this "lack of smoothness" is completely spurious, when one firmly stays in the space of distributions and their intrinsic "diffeology", in particular avoids viewing distributions as generalized functions. We describe in Section 2 the distribution theory that is adequate for the purpose. In fact, as will be seen in Section 5 and 6, this theory is forced on us by synthetic considerations in the Smooth Topos, respectively in the Cahiers topos.

We want to thank Henrik Stetkær for useful conversations on the topic of distributions.

1 Diffeological spaces and convenient vector spaces

A *diffeological space* is a set X equipped with a collection of *smooth plots*, a plot p being a map from (the underlying set of) an open set U of some $\mathbf{R}^n$ into X, $p : U \to X$; the collection should satisfy certain stability properties: a smooth plot precomposed with an ordinary smooth map $U' \to U$ is again a smooth plot; and the property of being a smooth plot is a *local* property (local on the domain). These properties are conceptualized by considering the following site $\underline{\text{mf}}$: its objects are open subsets of $\mathbf{R}^n$, the maps are smooth maps between such sets; a covering is a jointly surjective family of local diffeomorphisms. (This site is a site of definition of the "Smooth Topos" of Grothendieck et al., [1] p. 318; and is one of the first examples of what they call a "Gros Topos".) Any set X gives rise to a presheaf $c(X)$ on this site, namely $c(X)(U) := \mathrm{Hom}_{\mathrm{sets}}(U, X)$. A diffeological structure on the set X is a subsheaf P of the presheaf $c(X)$, the elements of $P(U)$ are called the *smooth* U-plots on X. A set theoretic map $f : X \to X'$ between diffeological spaces is called *(plot-) smooth* if $f \circ p$ is a smooth plot on X' whenever p is a smooth plot on X.

Any smooth manifold M carries a canonical diffeology, namely with $P(U)$ being the set of smooth maps $U \to M$. We have full inclusions of categories: smooth manifolds into diffeological spaces into the smooth topos, (= the topos of sheaves on the site $\underline{\text{mf}}$),

$$\underline{\text{Mf}} \subseteq \underline{\text{Diff}} \subseteq \underline{\text{sh}}(\underline{\text{mf}}).$$

The category of diffeological spaces $\underline{\text{Diff}}$ is cartesian closed (in fact, it is a concrete quasi-topos). Thus, if X and Y are diffeological spaces, Y^X has for its underlying set the set of smooth maps $X \to Y$; and a map $U \to Y^X$ is declared to be a smooth plot if its transpose $U \times X \to Y$ is smooth. The inclusion of $\underline{\text{Diff}}$ into the smooth topos preserves the cartesian closed structure.

For any smooth manifold M, we have in particular a diffeology on $C^\infty(M) = \mathbf{R}^M$, namely a map $g : U \to C^\infty(M)$ is declared to be a smooth plot iff its transpose $U \times M \to \mathbf{R}$ is smooth.

Topological vector spaces X carry a canonical diffeology: a plot $f : U \to X$ is declared to be smooth if for every continuous linear functional $\phi : X \to \mathbf{R}$, $\phi \circ f : U \to \mathbf{R}$ is smooth in the standard sense of multivariable calculus. (Note that the diffeology on a topological vector space X only depends on the dual space X'.) – Continuous linear functionals $X \to \mathbf{R}$ are, almost tautologically, (plot-)smooth; on the other hand, (plot-)smooth linear functionals $X \to \mathbf{R}$ need not be continuous.

A *convenient vector space* (cf. [5]) is a (Hausdorff) locally convex topological vector space X such that plot smooth linear functionals are continuous, and which have a completeness property. The completeness property may be stated in several ways, cf. [5], [16]; for the purposes here, the most natural formulation is: for any smooth curve $f : U \to X$ (where U is an open interval), there exists a smooth curve $f' : U \to X$ which is *derivative* of f in the scalarwise sense that for any continuous linear $\phi : X \to \mathbf{R}$, $\phi \circ f' = (\phi \circ f)'$.

– More generally, if $U \subseteq \mathbf{R}^n$ is open, and $f : U \to X$ is a smooth plot, then partial derivatives f^α of f exist, in the scalarwise sense; and they are smooth. Here α is a multi-index; and to say that f^α is an iterated partial derivative of f, in the *scalarwise* sense, is to say: for each $\phi \in X'$, $\phi \circ f$ has an α'th iterated derivative, and $(\phi \circ f)^\alpha = \phi \circ f^\alpha$.

The category of convenient vector spaces which we deal with here is $\underline{\text{Con}}^\infty$ (cf. [5]), whose objects are the convenient vector spaces and whose morphisms are *all* smooth maps in between them, not just the smooth *linear* ones. The category $\underline{\text{Con}}^\infty$ is a full subcategory of the category $\underline{\text{Diff}}$ of diffeological spaces, and is cartesian closed; the inclusion functor preserves the cartesian closed structure.

(In [16], the notion of Convenient Vector Space is taken in a slightly wider sense: it is not required that (plot-) smooth linear functionals are continuous. The resulting category of "convenient" vector spaces and smooth maps in [16] is therefore larger, but *equivalent* to the one of [5]. Every convenient vector space in the "wide" sende is smoothly (but not necessarily topologically) isomorphic to one in the "narrow" sense of [5].)

Let $i : X \to Y$ be a (plot-) smooth linear map between convenient vector spaces. Then i preserves differentiation of smooth plots $U \to X$, in an obvious sense. For instance, if $f : U \to X$ is a smooth curve, i.e. $U \subseteq \mathbf{R}$ an open interval, then for any $t_0 \in U$,

$$(i \circ f)'(t_0) = i(f'(t_0)).$$

For, it suffices to test this with the elements $\psi \in Y'$. If $\psi \in Y'$, then $\psi \circ i \in X'$ since i is smooth and linear, and the result then follows by definition of being a scalarwise derivative in X.

(Another aspect of the completeness of convenient vector spaces is: if U is an open interval, and $u_0 \in U$, there is a unique smooth primitive G ($G' = g$) of g, with $G(u_0) = 0$. This is the basis for constructing "Hadamard remainders" with values in a convenient vector space, and hence for the comparisons of the present Section 6.)

2 The basic vector spaces of distribution theory; test plots

Let M be a smooth (paracompact) manifold M. Distribution theory starts out with the vector space $C^\infty(M)$ of smooth real valued functions on M, and the linear subspace $\mathcal{D}(M) \subseteq C^\infty(M)$ consisting of functions with compact support ($\mathcal{D}(M)$ is the "space of *test functions*"). The topology relevant for distribution theory is described (in terms of convergence of sequences) in [26], p. 79 and 108, respectively. Note that topology on $\mathcal{D}(M)$ is finer than the one induced from the topology on $C^\infty(M)$. The sheaf semantics which we shall consider in Section 5 and 6 will justify the choice of this topology.

We shall describe the diffeological structure, arising from the topology on $\mathcal{D}(M)$, and utilize the fact ([5], Remark 3.5) that it is a convenient vector space.

We cover M by an increasing sequence K_b of compact subsets, each contained in the interior of the next, and with $M = \cup K_b$; the notions that we now describe are independent of the choice of these K_b. For $M = \mathbf{R}^n$, we would typically take $K_b = \{x \in \mathbf{R}^n \mid |x| \leq b\}$, $b \in \mathbf{N}$.

Consider a smooth map $f : U \times M \to \mathbf{R}$, where U is an open subset of some $\mathbf{R}^n$. We say that it is of *uniformly bounded support* if there exists b so that

$$f(u, x) = 0 \text{ for all } u \in U \text{ and all } x \text{ with } x \notin K_b$$

We say that f is *locally* of uniformly bounded support ("l.u.b.s.") if U can be covered by open subsets U_i such that for each i, the restriction of f to $U_i \times M$ is of uniformly bounded support. (We may use the phrase "f is l.u.b.s., *locally in the variable* $u \in U$") – Alternatively, we say that $f : U \times M \to \mathbf{R}$ is of uniformly bounded support *at* $u \in U$ if there is an open neighbourhood U' around u such that the restriction of f to $U' \times M$ is of uniformly bounded support; and f is l.u.b.s. if it is of uniformly bounded support at u, for each u. (For yet another description of the notion, see Lemma 5.2 below.)

We let $\hat{f}$ denote the transpose of f, so $\hat{f} : U \to C^\infty(M)$.

Theorem 2.1 *Let $f : U \times M \to \mathbf{R}$ be smooth, and pointwise of bounded support (so that $\hat{f}$ factors through $\mathcal{D}(M)$). Then t.f.a.e.:*
1) f is locally of uniformly bounded support
2) $\hat{f} : U \to \mathcal{D}(M)$ is continuous.

We may use the term *test plot* for functions f satisfying the conditions of the Proposition. In particular, they are pointwise test *functions* in the sense of distribution theory.

Proof of the Theorem. We first prove that 1) implies 2). Since the question is local in U, we may assume that f is of uniformly bounded support, i.e. there exists a compact $K \subseteq M$ so that $f(t,x) = 0$ for $x \notin K$ and all t. The same K applies then to all the iterated partial derivatives f_α of f in the M-directions (α denoting some multi-index). So f and all the f_α factor through $\mathcal{D}_K$, the subset of $C^\infty(M)$ of functions vanishing outside K. Now to say that $\hat{f} : U \to \mathcal{D}_K$ is continuous is by definition of the topology on $\mathcal{D}_K$ equivalent to saying that for each α, $(f_\alpha)\hat{}$ is continuous as a map into $\mathbf{R}^K$, the space of continuous maps $K \to \mathbf{R}$, with the topology of uniform convergence. This topology is the categorical exponent (= compact open topology) (cf. [6] Ch. 7 Thm. 11), which implies that $(f_\alpha)\hat{} : U \to \mathbf{R}^K$ is continuous iff $f_\alpha : U \times K \to \mathbf{R}$ is continuous, iff $f_\alpha : U \times M \to \mathbf{R}$ is continuous. But f_α is indeed continuous, by the smoothness assumption on f. So $\hat{f} : U \to \mathcal{D}(M)$ is continuous.

For proving that 2) implies 1), we prove that if not 1), then not 2), i.e. we consider a function $f : U \times M \to \mathbf{R}$ which is smooth and of pointwise bounded support, but not l.u.b.s. Then there is a $t_0 \in U$ and a sequence $t_k \to t_0$, as well as a sequence $x_k \in M \setminus K_k$ with $f(t_k, x_k) \neq 0$, denote this number c_k. Let N be a number so that the support of $f(t_0, -)$ is contained in K_N. We consider the (non-linear) functional $T : \mathcal{D}(M) \to \mathbf{R}$ given by

$$g \mapsto \sum_{n=N}^{\infty} c_n^{-2} g(x_n)^2.$$

Note that for g of compact support, this sum is finite, since the x_n's "tend to infinity". Also, the functional $\mathcal{D}(M) \to \mathbf{R}$ is continuous; for the topology on $\mathcal{D}(M)$ is the inductive limit of the topology $\mathcal{D}(K_k)$, and the restriction of T to this subspace equals a *finite* algebraic combination of the Dirac distributions. Now it is easy to see that T takes $f(t_0, -)$ to 0, by the choice of N, whereas T applied to $f(t_k, -)$ for $k > N$ yields a sum of non-negative terms, one of which has value 1, namely the one with index k, which is $c_k^{-2} f(t_k, x_k)^2 = 1$. So $T \circ \hat{f}$ is not continuous, hence $\hat{f}$ is not continuous.

This proves the Theorem.

It has the following Corollary:

Theorem 2.2 *Let $f : U \times M \to \mathbf{R}$ be smooth and of pointwise bounded support (U an open subset of some $\mathbf{R}^n$). Then t.f.a.e.:*
1) f is locally of uniformly bounded support
2) $\hat{f} : U \to \mathcal{D}(M)$ is smooth.

Recall that assertion 2) means "in the scalarwise sense", i.e. $\phi \circ \hat{f}$ is smooth for any continuous linear functional, i.e. for any distribution ϕ.

Proof. The implication 2) $\Rightarrow$ 1) is a consequence of Theorem 2.1, since smoothness implies continuity. (This is not completely evident. "Smooth" means "scalarwise smooth", and this of course implies scalarwise continuity; now, scalarwise continuity means continuity w.r.to the *weak* topology, but Theorem 2.1 deals with the classical (inductive limit) topology. We don't know at present whether these two topologies agree. However, since $\mathcal{D}(M)$ is a Montel space ([7] p. 197), these topologies agree on *bounded* subsets, ([7] p. 196), which suffices here since U is locally compact.)

Conversely, assume 1), i.e. assume f is smooth and l.u.b.s. Then we also have that $\partial^\alpha f/\partial t^\alpha$ is smooth (iterated partial derivative in the U-directions, α a multi-index) and l.u.b.s., and so its transpose is a continuous maps $U \to \mathcal{D}(M)$, by Theorem 2.1; it serves as scalarwise iterated partial derivative (cf. the argument in [16] p. 20-21).

The vector space of distributions $\mathcal{D}'(M)$ is, in diffeological terms, the linear subspace of the diffeological space $\mathbf{R}^{\mathcal{D}(M)}$ consisting of the smooth *linear* maps $\mathcal{D}(M) \to \mathbf{R}$. They are the same as the continuous linear maps, since $\mathcal{D}(M)$ is a convenient vector space. (So the vector space of distributions $\mathcal{D}'(M)$ (as an abstract vector space) is the same in the diffeological and the topological context.) A map $U \to \mathcal{D}'(M)$ is smooth iff it is smooth as a map into $\mathbf{R}^{\mathcal{D}(M)}$; this defines a diffeology on $\mathcal{D}'(M)$. With this diffeology, $\mathcal{D}'(M)$, too, is convenient.

3 Functions as distributions

Any sufficiently nice function $f : \mathbf{R}^n \to \mathbf{R}$ gives rise to a distribution $i(f) \in \mathcal{D}'(\mathbf{R}^n)$ in the standard way "by integration over $\mathbf{R}^n$"

$$\langle\, i(f), \phi \,\rangle := \int_{\mathbf{R}^n} f(s) \cdot \phi(s)\, ds.$$

This also applies if $\mathbf{R}^n$ is replaced by another smooth manifold M equipped with a suitable measure. For simplicity of notation, we write M for $\mathbf{R}^n$ in the following. – All smooth functions $f : M \to \mathbf{R}$ are "sufficiently nice"; so we get a map (obviously linear)

$$i : C^\infty(M) \to \mathcal{D}'(M). \tag{3}$$

It is also easy to see that this map is injective.

Theorem 3.1 *The map i is smooth.*

Proof. Let $g : V \to C^\infty(M)$ be smooth, (V an open subset of some $\mathbf{R}^n$), we have to see that $i \circ g : V \to \mathcal{D}'(M)$ is smooth, which in turn means that its transpose

$$(i \circ g)\hat{}\, : V \times \mathcal{D}(M) \to \mathbf{R}$$

is smooth. So consider a smooth plot $U \to V \times \mathcal{D}(M)$, given by a pair of smooth maps $h : U \to V$ and $\hat{\Phi} : U \to \mathcal{D}(M)$. Here U is again an open subset of some $\mathbf{R}^k$.

Let us write $\hat{F}$ for $g \circ h : U \to C^\infty(M)$. It is transpose of a map $F : U \times M \to \mathbf{R}$. Also, let us write Φ for the transpose of $\hat{\Phi}$; thus Φ is a map

$$\Phi : U \times M \to \mathbf{R}$$

which is locally (in U) of uniformly bounded support, by Theorem 2.2. We have to see that $(i \circ g)\hat{} \circ < h, \Phi >$ is smooth (in the usual sense). By unravelling the transpositions, one can easily check that

$$((i \circ g)\hat{} \circ \langle h, \Phi \rangle)(t) = \langle i(F(t, -)), \Phi(t, -) \rangle$$

The conclusion of the Theorem is thus the assertion that the composite map $U \to \mathbf{R}$ given by

$$t \mapsto \int_M F(t, s) \cdot \Phi(t, s)\, ds \tag{4}$$

is smooth (in the standard sense of finite dimensional calculus). To prove smoothness at $t_0 \in U$, we may find a neigbourhood U' of t_0 and a b such that

$$\Phi(t, s) = 0 \text{ if } t \in U' \text{ and } s \notin K_b,$$

because Φ is l.u.b.s. We thus have, for any $t \in U'$, that the expression in (4) is $\int_{K_b} F(t, s) \cdot \Phi(t, s)\, ds$, but since K_b is compact, differentiation and other limits in the variable t may be taken inside the integration sign.

Since $i : C^\infty(M) \to \mathcal{D}'(M)$ is smooth and linear, it preserves differentiation. In particular, if $f : U \to C^\infty(M)$ is a smooth curve, and $t_0 \in U$, we have that $(i \circ f)'(t_0) = i(f'(t_0))$. However, f' is explicitly calculated in terms of the partial derivative of the transpose $\hat{f} : U \times M \to \mathbf{R}$, namely as the function $s \mapsto \partial f(t, s)/\partial t\,|_{(t_0,s)}$. This is the reason that ordinary (evolution-) differential equations for curves $f : U \to \mathcal{D}'(M)$ manifest themselves as *partial* differential equations, as soon as the values of f are distributions represented by smooth functions.

4 Smoothness of the heat kernel

We consider the heat equation on the line,

$$\partial f/\partial t = \partial^2 f/\partial x^2.$$

Recall that the classical distribution solution of this equation, having $\delta(0)$ as initial distribution, is the map

$$K : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$$

whose value at $t \geq 0$ takes a test function ϕ to

$$\langle K(t), \phi \rangle = \begin{cases} \int_{-\infty}^{\infty} e^{-s^2/4t}/\sqrt{4\pi t}\, \phi(s)\, ds & \text{if } t > 0 \\ \phi(0) & \text{if } t = 0 \end{cases} \tag{5}$$

We need the smoothness of K in the diffeological sense. The diffeology on $\mathbf{R}_{\geq 0}$ is induced by the inclusion of it into $\mathbf{R}$.

The following is a special case of [16] Theorem 24.5 and Proposition 24.10 (which in turn is a generalization of Seeley's Theorem, [27]).

Theorem 4.1 *Let X be a convenient vector space, and let $K : \mathbf{R}_{\geq 0} \to X$ be a map. Then K is smooth in the diffeological sense iff its restriction to $\mathbf{R}_{>0}$ is smooth, and for all n, $\lim_{t\to 0+} K^{(n)}(t)$ exists (w.r.to the weak topology on X). In this case, K extends to a smooth map on all of $\mathbf{R}$, (whose n'th derviative at 0 then equals $\lim_{t\to 0+} K^{(n)}(t)$).*

We shall apply this Theorem to the heat kernel K described in (5), so X is $\mathcal{D}'(\mathbf{R})$. For $t > 0$, the smooth two-variable function $K(t,x)$ satisfies the heat equation as a partial differential equation $\partial/\partial t\ K(t,x) = \partial^2/\partial x^2\ K(t,x)$. Since, by Section 3, the inclusion $i : C^\infty(\mathbf{R}) \to \mathcal{D}'(\mathbf{R})$ preserves differentiation ("in the t-direction"), we get, by iteration, that for any test function ϕ, and any $t > 0$,

$$d^n/dt^n \langle\, K(t), \phi \,\rangle = \langle\, K(t), \phi^{(2n)} \,\rangle. \tag{6}$$

Also, it is well known that for any smooth ψ,

$$\lim_{t\to 0+} \langle\, K(t), \psi \,\rangle = \psi(0)(= \langle\, \delta, \psi \,\rangle), \tag{7}$$

where δ is the Dirac distribution at 0.

To prove that the conditions for smoothness in the above Theorem are satisfied, we shall prove that

$$\lim_{t\to 0+} K^{(n)}(t) = \delta^{(2n)}.$$

Since the topology on $\mathcal{D}'(\mathbf{R})$ is the weak one and $\mathcal{D}(\mathbf{R})$ is reflexive, it suffices to prove that for each $\phi \in \mathcal{D}(\mathbf{R})$,

$$\lim_{t\to 0+} d^n/dt^n \langle\, K(t), \phi \,\rangle = \phi^{2n}(0).$$

But this is immediate from (6) and (7).

5 Distributions in the Smooth Topos

Recall from section 1 that the Smooth Topos is the topos $\mathcal{S} = \underline{\mathrm{sh}}(\underline{\mathrm{mf}})$ of sheaves on the site $\underline{\mathrm{mf}}$ of open subsets of coordinate vector spaces $\mathbf{R}^n$. It contains the category of diffeological spaces (and hence also $\underline{\mathrm{Con}}^\infty$) as a full subcategory, and the inclusion preserves exponentials. Let us denote the embedding $\underline{\mathrm{Diff}} \subseteq \mathcal{S}$ by h. – We write R instead of $h(\mathbf{R})$.

We want to give a synthetic status to $h(\mathcal{D}(M))$ and to $h(\mathcal{D}(M)')$. Here M is any paracompact smooth manifold, and for the synthetic description, one needs to cover M by an increasing sequence of compacts K_b, as in Section 2. The predicate of "belonging to $K_b \subseteq M$" will have to be part of the language. In order not to load the exposition too heavily, we shall consider the case of $M = \mathbf{R}$ only, with K_b the closed interval from $-b$ to b ($b \in \mathbf{N}$).

Because h preserves exponentials, and $R = h(\mathbf{R})$, R^R is $h(C^\infty(\mathbf{R}))$. (For, the latter with its standard Frechet topology, is the exponential in $\underline{\mathrm{Con}}^\infty$, by [16], Theorem 3.2.)

The following is a formula with a free variable f that ranges over R^R:

$$\exists b > 0 \; [\; \forall x, (x < -b \vee x > b) \Rightarrow f(x) = 0 \;]. \tag{8}$$

Let us write $|x| > b$ as shorthand for the formula $x < -b \vee x > b$ (so, in spite of the notation, we don't assume an "absolute value" function). Then the formula (8) gets the more readable appearance:

$$\exists b > 0 \; [\; \forall x, |x| > b \Rightarrow f(x) = 0 \;]. \tag{9}$$

(verbally: "f is a function $R \to R$ of bounded support" (namely support contained in the interval $[-b, b]$). Its extension is a subobject $\mathcal{D}(R) \subseteq R^R$.

Theorem 5.1 (Test functions in the Smooth Topos) *The inclusion $\mathcal{D}(\mathbf{R}) \subseteq C^\infty(\mathbf{R})$ goes by $h : \underline{\mathrm{Con}}^\infty \to \mathcal{S}$ to the inclusion $\mathcal{D}(R) \subseteq R^R$.*

Proof. We shall freely use sheaf semantics, cf. e.g. [9], [21], and thus consider "generalized elements" or "elements defined at different stages", the stages being the objects of the site $\underline{\mathrm{mf}}$.

Consider an element $f \in_U R^R$ (a generalized element at stage U). This means a map $h(U) \to R^R$ in $\mathcal{C}$, and this in turn corresponds, by transposition, and by fullness of the embedding h, to a smooth map

$$\check{f} : U \times \mathbf{R} \to \mathbf{R}.$$

Now we have that

$$\vdash_U \exists b > 0 \; [\, \forall x, |x| > b \Rightarrow f(x) = 0 \,]$$

if and only if there is a covering U_i of U $(i \in I)$ and witnesses $b_i \in_{U_i} R_{>0}$, so that for each i

$$\vdash_{U_i} \forall x, \; |x| > b_i \Rightarrow f(x) = 0$$

Externally, this implies that $b_i : U_i \to \mathbf{R}$ is a smooth function with positive values, with the property that for all $t \in U_i$, if x has $x > b_i(t)$, then $\check{f}(t, x) = 0$. The following Lemma then implies that $\check{f}$ is of l.u.b.s. on U_i, and since the U_i's cover K, $\check{f}$ is of l.u.b.s. on K.

Lemma 5.2 *Let $g : U \times \mathbf{R} \to \mathbf{R}$ have the property that there exists a smooth (or just continuous) $b : U \to R_{>0}$ so that for all $t \in U$ $|x| > b(t)$ implies $g(t, x) = 0$. Then g is l.u.b.s.*

Proof. For each $t \in U$, let c_t denote $b(t) + 1$. There is a neighbourhood V_t around t such that $b(y) < c_t$ for all $y \in V_t$. The family of V_t's, together with the constants c_t now witness that g is l.u.b.s. For, for all $y \in V_t$ and any x with $|x| > c_t$, we have $|x| > c_t > b(y)$, so $g(y, x) = 0$.

Conversely, if $\check{f}$ is l.u.b.s., it is easy to see that the element $f \in_K U^R$ satisfies the formula (reduce to the uniformly bounded case, and write the condition as existence of a commutative square).

So we conclude that for $f \in_U R^R$, $f \in_U \mathcal{D}(R)$ iff the external function $\check{f} : U \times \mathbf{R} \to \mathbf{R}$ is l.u.b.s., i.e., by Theorem 2.2, iff $\hat{f} : U \to C^\infty(\mathbf{R})$ factors by a (diffeologically!) smooth map through the inclusion $\mathcal{D}(\mathbf{R}) \subseteq C^\infty(\mathbf{R})$, i.e. belongs to $C^\infty(U, \mathcal{D}(\mathbf{R})) = h(\mathcal{D}(\mathbf{R}))(C^\infty(U)$. This proves that $h(\mathcal{D}(\mathbf{R})) = \mathcal{D}(R)$.

We next can consider the synthetic status in $\mathcal{S}$ of the space of distributions $\mathcal{D}(\mathbf{R})'$.

If R is a commutative ring object in a topos, and it is equipped with a compatible preorder $\leq$, we have already described the R-module $\mathcal{D}(R)$, (space of test functions). For any R-module object Y, we may then form its R-linear dual object $Y' = Lin_R(Y, R)$ as a subobject of R^Y; in particular, we may form $(\mathcal{D}(R))'$ which is then the internal object of distributions on R, as alluded to in the Introduction.

Theorem 5.3 (Distributions in the Smooth Topos) *The convenient vector space $(\mathcal{D}(\mathbf{R}))'$ goes by $h : \underline{\mathrm{Con}}^\infty \to \mathcal{S}$ to the internal object of distributions $(\mathcal{D}(R))'$.*

We first make an analysis of $h(Y')$ for a general convenient vector space Y. (Here, Y' denotes the diffeological dual consisting of smooth linear functionals.) Recall that the diffeology on Y' is inherited from that of $C^\infty(Y, \mathbf{R})$, so that (for an open $U \subseteq \mathbf{R}^k$), the smooth plots $U \to Y'$ are in bijective correspondence with smooth maps $U \times Y \to \mathbf{R}$, which are $\mathbf{R}$-linear in the second variable $y \in Y$. It follows that the elements at stage U are in bijective correspondence with smooth maps $U \times Y \to \mathbf{R}$, $\mathbf{R}$-linear in the second variable, or equivalently, with smooth $\mathbf{R}$-linear maps $Y \to C^\infty(U, \mathbf{R})$.

On the other hand, an element of $R^{h(Y)}$ defined at stage U is a morphism $h(U) \to R^{h(Y)}$, hence by double transposition it corresponds to a map $h(Y) \to R^{h(U)}$; and it belongs to the subobject $Lin_R(h(Y), R)$ iff its double transpose is R-linear. Since h is full and faithful, and preserves the cartesian closed structure (hence the transpositions), this double transpose corresponds bijectively to a smooth map $Y \to C^\infty(U, \mathbf{R}) = C^\infty(U)$, and R-linearity is equivalent to $\mathbf{R}$-linearity, by the following general

Lemma 5.4 *Let X and Y be convenient vector spaces. Then a smooth map $f : Y \to X$ is $\mathbf{R}$-linear iff $h(f) : h(Y) \to h(X)$ is R-linear.*

Proof. The implication $\Rightarrow$ is a consequence of the fact that h preserves binary cartesian products (and of $h(\mathbf{R}) = R$). For the implication $\Leftarrow$, we just apply the global sections functor Γ; note that $\Gamma(Y)$ is the underlying set of the vector space Y, and similar for X; and $\Gamma(R) = \mathbf{R}$.

The Theorem now follows from Theorem 5.1.

We have in particular:

Proposition 5.5 *There is a natural one-to one correspondence between distributions on $\mathbf{R}$, and R-linear maps $\mathcal{D}(R) \to R$*

Proof. This follows from fullness of the embedding h.

This result should be compared to the Theorem of [24], or Proposition II.3.6 in [21], where a related assertion is made for distributions-with-compact-support, i.e. where $\mathcal{D}(R)$ is replaced by the whole of R^R, (– or even with R^M, with M an arbitrary smooth manifold; the generalization of our theory is straightforward). Distributions with compact support are generally easier to deal with synthetically (as we did in [15]).

6 Cahiers Topos

This topos was constructed by Dubuc [3] in order to get what he called a well-adapted model for Synthetic Differential Geometry (SDG). The site of definition should contain not only a suitable representative category of smooth manifolds, but also encode the infinitesimal objects (of "nilpotent elements"), like D, which are crucial in SDG. The category of infinitesimal objects is taken to be the dual of the category of Weil-algebras (i.e. finite dimensional commutative real algebras, where the nilpotent elements form an ideal of codimension 1). This prompts us to replace also the representative category of smooth manifolds with a the dual of a category of (C^∞-) algebras, capitalizing on the fact that smooth maps $U \to V$ correspond bijectively to $C^\infty-$algebra maps $C^\infty(V) \to C^\infty(U)$.

To conform with our exposition in [13], we take the representative smooth manifolds just to be the coordinate vector spaces $\mathbf{R}^k$, rather than all open subsets U of such. (We could, by suitable comparison theorem of site theory, have used the category of just these $\mathbf{R}^k$ for the Smooth Topos also.)

We recall the site of definition $\underline{D}$ for the Cahiers Topos $\mathcal{C}$. The underlying category is the dual of a certain category of C^∞-rings, namely those that are of of the form $C^\infty(\mathbf{R}^{l+k})/J$ where J is a semi-Weil ideal; we explain this notion: a Weil ideal $I \subseteq C^\infty(\mathbf{R}^l)$ is an ideal such that the residue ring is a Weil algebra (in particular, I is of finite codimension). A semi-Weil ideal $J \subseteq C^\infty(\mathbf{R}^{l+k})$ is an ideal which comes about from a Weil ideal I in $C^\infty(\mathbf{R}^l$ as I^*, where I^* is the ideal of functions of the form $\sum f_i(x,y) \cdot g_i(x)$ with $g_i \in I$.

To describe and analyze the embedding h of $\underline{\mathrm{Con}}^\infty$ into $\mathcal{C}$, we need a more elaborate account of the relationship between semi-Weil ideals and convenient vector spaces:

6.1 Ideals and differential operators

Let $x \in \mathbf{R}^n$. By a *differential operator supported at* x, we understand a map $d : C^\infty(\mathbf{R}^n) \to \mathbf{R}$ which is a linear combination of operators $f \mapsto \partial^{|\alpha|} f / \partial t^\alpha(x)$, where α is a multi-index and $t = (t_1, \ldots, t_n)$. (The notion can be defined in a coordinate free way; it is actually the same as a distribution with point-support.) In particular, d is linear.

Any such d defines, because of its explicit form, for each convenient vector space Y a linear $d_Y : C^\infty(\mathbf{R}^n, Y) \to Y$ with the property that for $f : \mathbf{R}^n \to Y$

$$d(\phi \circ f) = \phi(d_Y(f))$$

for all $\phi \in Y'$. The maps d_Y are natural in Y w.r.to smooth *linear* maps:

Proposition 6.1 *If $F : Y \to X$ is a smooth linear map, then for any differential operator d, and any $f \in C^\infty(\mathbf{R}^n, Y)$, $d_X(F \circ f) = F(d_Y(f))$*

Proof. It suffices to test with an arbitrary $\phi \in X'$; by replacing F by $\phi \circ F$, this reduces the problem to the case where the dodomain X is $\mathbf{R}$, and here, the result follows from the very characterization of Y-valued derivatives in "scalarwise" terms.

Let us also note that "partial derivatives are transposable" (cf. [16] Section 3). For simplicity, we state it for functions in two variables s, t only:

Proposition 6.2 *Let $f(s,t) : \mathbf{R}^2 \to Y$ be a smooth function with values in a convenient vector space Y. Then the $\partial f(s,t)/\partial s$ is smooth in s, t, and its transpose is the derivative $(\hat{f})'(s)$ of the transposed function $\hat{f} : \mathbf{R} \to C^\infty(\mathbf{R}, Y)$.*

Proof. The function $(\hat{f})'(s)$ exists and is smooth, and characterized in terms of the smooth linear functionals on $C^\infty(\mathbf{R}, Y)$. But among these are those of the form (for $t \in \mathbf{R}$)

$$C^\infty(\mathbf{R}, Y) \xrightarrow{ev_t} Y \xrightarrow{\psi} \mathbf{R},$$

and these are enough to recognize the transpose of $\hat{f}'(s)$ as $\partial f(s,t)$, for each t.

Let N be a manifold, and let $I \subseteq C^\infty(N)$ be an ideal. For each convenient vector space Y, (in fact for any dualized vector space (Y, Y')) we define two linear subspaces of $C^\infty(N, Y)$, the "weak" and the "strong" $I(Y)$, denoted $I_w(Y)$ and $I_s(Y)$, respectively. To say that $f : N \to Y$ is in $I_w(Y)$ is to say that for every $\phi \in Y'$, $\phi \circ f \in I$; and to say that $f : N \to Y$ is in $I_s(Y)$ is to say that f may be written

$$f(s) = \sum h_i(s) k_i(s),$$

with the h_i's scalar valued functions belonging to I, and the k_i's smooth Y-valued functions. It is clear that $I_s(Y) \subseteq I_w(Y)$. We are interested in when the converse implication holds.

A main result in [10] (Theorem 2.11) says that this is the case for the ideal $\mathcal{M}^r \subseteq C^\infty(\mathbf{R}^l)$ of functions vanishing to order r at 0. In [13] (Proposition 1), we generalized this to any proper ideal $I \subseteq C^\infty(\mathbf{R}^l)$ which contains an ideal $\mathcal{M}^r$. We call such ideals *Weil ideals*; they are of finite codimension, and the algebra $C^\infty(\mathbf{R}^l)/I$ is a Weil algebra (in the sense of [9] or [21], say); any Weil algebra arises this way. (Note that a Weil ideal is contained in $\mathcal{M}$, since the only maximal ideal containing $\mathcal{M}^r$ is $\mathcal{M}$. So if $f \in I$, $f(0) = 0$.)

We shall generalize this result further to semi-Weil ideals J (whose definition we shall recall), and at the same time provide a simpler proof of the result quoted from [13].

If $I \subseteq C^\infty(N)$ is an ideal and if $p : P \to N$ is a smooth map (P and N manifolds), we get an ideal $p^*(I) \subseteq C^\infty(P)$ consisting of functions $f : P \to \mathbf{R}$ which can be written $\sum (h_i \circ p) \cdot k_i$ with the h_i's in I (and the k_i's in $C^\infty(P)$). This is clearly a "transitive" construction, in an evident sense, $q^*(p^*(I)) = (p \circ q)^*(I)$. On the other hand, since $C^\infty(M)$ is a convenient vector space, we may consider $I_s(C^\infty(M)) \subseteq C^\infty(N, C^\infty(M))$. Under the isomorphism $C^\infty(N, C^\infty(M)) \cong C^\infty(N \times M)$ it is clear that $I_s(C^\infty(M))$ corresponds to $p^*(I)$, where $p : N \times M \to N$ denotes the projection.

If I is a Weil ideal $\subseteq C^\infty(\mathbf{R}^l)$, and $p : \mathbf{R}^{l+k} \to \mathbf{R}^l$ the projection, we get by the above procedure an ideal $J = p^*(I)$ in $C^\infty(\mathbf{R}^{l+k})$, and ideals J of this form, we call *semi-Weil ideals.* (If p is understood from the context, we may write I^* instead of $p^*(I)$.)

The basis of monomials s^α (where α is a multi-index of order $< r$) for $C^\infty(\mathbf{R}^l)/(\mathcal{M}^r)$ gives rise to a dual basis for the linear dual $(C^\infty(\mathbf{R}^l)/\mathcal{M}^r))^*$, and this dual basis consists of differential operators supported at 0,

$$f \mapsto \frac{\partial^\alpha f(0)}{|\alpha|!\partial s^\alpha}.$$

So $f \in \mathcal{M}^r$ iff $\frac{\partial^\alpha f(0)}{\partial s^\alpha} = 0$ for such multi-indices α.

We consider functions $f(s,t) : \mathbf{R}^{l+k} \to Y$, where Y is a convenient vector space; s denotes a variable ranging over $\mathbf{R}^l$ and t a variable ranging over $\mathbf{R}^k$. We then have

Proposition 6.3 *Let $f : \mathbf{R}^{l+k} \to Y$ be a smooth function. Then*

$$f \in (p^*(\mathcal{M}^r))_w(Y)$$

if and only if

$$\frac{\partial^\alpha f(0,t)}{\partial s^\alpha} = 0 \text{ for all } \alpha \text{ with } |\alpha| < r \text{ and all } t.$$

Proof. The Y-valued partial derivatives here are determined scalarwise, i.e. determined by testing with the $\phi \in Y'$, and since these ϕ are linear, the problem immediately reduces to the case of $Y = \mathbf{R}$, i.e. to the assertion $f(s,t) \in p^*(\mathcal{M}^r)$ iff $\frac{\partial^\alpha f(0,t)}{\partial s^\alpha} = 0$ for all α with $|\alpha| < r$ and all t. This is well known (or can be deduced from Theorem 2.11 in [10], by passing to the transpose function $\hat{f} : \mathbf{R}^l \to C^\infty(\mathbf{R}^k)$).

The following is now a Corollary of Theorem 2.11 in [10]:

Proposition 6.4 *For any convenient vector space Y, we have $(p^*(\mathcal{M}^r))_w(Y) = (p^*(\mathcal{M}^r))_s(Y)$.*

Proof. It suffices to prove the inclusion $\subseteq$. If f is in the left hand side, it satisfies the equational conditions of Proposition 6.3, but then its transpose $\hat{f} : \mathbf{R}^l \to C^\infty(\mathbf{R}^k, Y)$ has $\frac{\partial^\alpha \hat{f}(0)}{\partial s^\alpha} = 0$ for all α with $|\alpha| < r$. Now we apply Proposition 6.3 again, this time for the convenient vector space $C^\infty(\mathbf{R}^k, Y)$, and with no p^* involved, and conclude $\hat{f} \in (\mathcal{M}^r)_w(C^\infty(\mathbf{R}^k, Y))$. Then, by the Theorem quoted, $\hat{f} \in (\mathcal{M}^r)_s(C^\infty(\mathbf{R}^k, Y))$ (strong instead of weak), and this in turn implies that $f \in (p^*(\mathcal{M}^r))_s(Y)$, proving the Proposition.

Consider a Weil ideal I i.e. an ideal $I \subseteq C^\infty(\mathbf{R}^l)$ containing some $\mathcal{M}^r$. There is a (finite) basis A for the dual vector space $(C^\infty(\mathbf{R}^l)/\mathcal{M}^r)^*$ consisting of differential operators D^α at 0 (with $\mathcal{M}^r$ the common nullspace of these). (Here, α is just an abstract index, and D^α does not necessarily mean $\frac{\partial^\alpha}{\partial s^\alpha}$.) Since $(C^\infty(\mathbf{R}^l)/I)^* \subseteq (C^\infty(\mathbf{R}^l)/\mathcal{M}^r)^*$, we may, by suitable change of basis, organize ourselves so that the basis A for $(C^\infty(\mathbf{R}^l)/\mathcal{M}^r)^*$ contains a subset B which is a basis for $(C^\infty(\mathbf{R}^l)/I)^*$. It follows that I is the common null space of the collection B of differential operators.

The dual basis $\hat{A}$ for $C^\infty(\mathbf{R}^l)/\mathcal{M}^r$ consists (modulo $\mathcal{M}^r$) of polynomials h_α of degree $< r$, $(\alpha \in A)$. The fact that the bases A and A' are dual implies that for any $f \in C^\infty(\mathbf{R}^l)$,

$$f(s) \equiv \sum_\alpha D^\alpha f \cdot h_\alpha(s),$$

mod $\mathcal{M}^r$ (as functions of $s \in \mathbf{R}^l$). If now $f \in I$, the terms $D^\alpha f$ vanish for $\alpha \in B$. With $A - B$ as index set for the index γ, we therefore have

Proposition 6.5 *Given a Weil-ideal $I \in C^\infty(\mathbf{R}^l)$ containing $\mathcal{M}^r$. There is a finite family of differential operators D^γ and a family of polynomials $h_\gamma(s)$ in $s \in \mathbf{R}^l$ so that for any $f \in I$,*

$$f(s) - \sum_\gamma D^\gamma f \cdot h_\gamma(s) \in \mathcal{M}^r.$$

(If for instance $I = \mathcal{M}^{r-1} \supseteq \mathcal{M}^r$, the h_γ's may be taken to be the monomials s^α, where α ranges over multi-indices with $|\alpha| = r - 1$.)

Because differentiation of functions $\mathbf{R}^l \to Y$ (with Y a convenient vector space) makes sense, and because of the explicit way (in terms of D^γ's) in which functions in I get transformed into functions in $\mathcal{M}^r$, this Proposition immediately extends to functions $\mathbf{R}^l \to Y$; let I and h_γ be as above, and let the D_Y^γ denote the Y-valued differential operators corresponding to the $\mathbf{R}$-valued D^γ's considered.

Proposition 6.6 *For any $f \in I_w(Y)$, the difference*

$$f(s) - \sum_\gamma D^\gamma f \cdot h_\gamma(s)$$

belongs to $\mathcal{M}_w^r(Y)$ (which equals $\mathcal{M}_s^r(Y)$) by the Theorem [10] 2.11 quoted).

Proof. We test with arbitrary $\phi \in Y'$; since ϕ is linear, and since ϕ commutes with differentiation, the result follows by applying the result of the previous Proposition to the smooth function $\phi \circ f$, which is in I by assumption.

Now let J denote the semi-Weil ideal $p^* I \subseteq C^\infty(\mathbf{R}^{l+k})$ given by the Weil ideal $I \subseteq C^\infty(\mathbf{R}^l)$. Then

Proposition 6.7 *Let $f : \mathbf{R}^{l+k} \to \mathbf{R}$ be a function in J. Then*

$$f(s,t) - \sum (D^\gamma f)(t) \cdot h_\gamma(s)$$

is in $p^*(\mathcal{M}^r)$ (where $p : \mathbf{R}^{l+k} \to \mathbf{R}^l$ is the projection).

(Here, s and t denote variables ranging over $\mathbf{R}^l$ and $\mathbf{R}^k$, respectively. The differential operators D^γ operate in the s-variable and then $s = 0$ is substituted, so a function $D^\gamma f$ of t remains, as indicated.)

Proof. We pass to the transpose function $\hat{f} : \mathbf{R}^l \to Y$, where Y is the convenient vector space $C^\infty(\mathbf{R}^k)$. To say $f \in J$ is equivalent to saying $\hat{f} \in I_s(C^\infty(\mathbf{R}^k))$, in particular $\hat{f} \in I_w(C^\infty(\mathbf{R}^k))$, and so Proposition 6.6 may be applied, reducing $\hat{f}$ to $\mathcal{M}_s^r(C^\infty(\mathbf{R}^k)$, which by transposition corresponds to $p^*(\mathcal{M}^r)$. This proves the Proposition.

We generalize this further to the case of functions with values in a convenient vector space Y.

Proposition 6.8 *Let $J \subseteq C^\infty(\mathbf{R}^{l+k})$ be the semi-Weil ideal given by the Weil ideal $I \in C^\infty(\mathbf{R}^l)$. Let $g(s,t) \in J_w(Y)$. Then*

$$g(s,t) - \sum (D^\gamma g)(t) \cdot h_\gamma(s) \tag{10}$$

is in $(p^*(\mathcal{M}^r))_w(Y)$ (hence, by Proposition 6.4, in $(p^*(\mathcal{M}^r))_s(Y)$).

Proof. Testing with $\phi \in Y'$ reduces the problem to showing that

$$\phi(g(s,t)) - \sum D^\gamma(\phi \circ g)(t) \cdot h_\gamma(s) \tag{11}$$

is in $p^*(\mathcal{M}^r)$, but this follows from Proposition 6.7, applied to $f = \phi \circ g$.

Theorem 6.9 *If J is a semi-Weil ideal, and Y a convenient vector space, $J_s(Y) = J_w(Y)$ (as linear subspaces of $C^\infty(\mathbf{R}^{l+k}, Y)$).*

Proof. Let $g = g(s,t)$, $g : \mathbf{R}^l \times \mathbf{R}^k \to Y$, be a map in $J_w(Y)$. Since the $h_\gamma(s)$ are in I, the sum $\sum (D^\gamma g)(t) \cdot h_\gamma(s)$ in (10) is in $J_s(Y)$. The whole expression in (10) is in $(p^*(\mathcal{M}^r))_w(Y)$, by Proposition 6.8, and hence, by Proposition 6.4, in $(p^*(\mathcal{M}^r))_s(Y)$ which in turn is contained in $J_s(Y)$. This proves the Theorem.

From now on, we write $J(Y)$ instead of $J_w(Y)$ or $J_s(Y)$, in case J is a semi-Weil ideal and Y a convenient vector space; for, they agree, by the Theorem.

We now discuss the description of semi-Weil ideals in terms of differential operators.

If $I \subseteq C^\infty(\mathbf{R}^n)$ is an ideal which is the null space of a family of differential operators $\{d^\beta \mid \beta \in B\}$ (not necessarily supported at the same $x \in \mathbf{R}^n$), then it follows from Proposition 6.1 that $I_w(Y) \subseteq C^\infty(\mathbf{R}^n, Y)$ is the null space of the family of the d^β_Y.

If I is a Weil ideal in $C^\infty(\mathbf{R}^l)$, null space of a finite family $\{d^\beta \mid \beta \in B\}$ of differential operators supported at $0 \in \mathbf{R}^l$, then $J \subseteq C^\infty(\mathbf{R}^{l+k})$ is the null space of the (infinite) family of differential operators $d^{\beta,x}$, $\beta \in B$, $x \in \mathbf{R}^k$, where for a function $f(s,t) \in C^\infty(\mathbf{R}^{l+k})$, $d^{\beta,x}(f)$ takes the relevant partial derivatives in the s-directions, and then substitutes 0 for s and x for t.

It follows that $J(Y)$, for Y a convenient vector space, may be described as the null space of the $B \times \mathbf{R}^k$-indexed family of differential operators $d^{\beta,x}_Y : C^\infty(\mathbf{R}^{l+k}, Y) \to Y$.

Also, it follows from Theorem 6.9 that under the transposition isomorphism $C^\infty(\mathbf{R}^{l+k}, Y) \cong C^\infty(\mathbf{R}^l, C^\infty(\mathbf{R}^k, Y))$, the line ar subspace $J(Y)$ on the left corresponds to the linear subspace $I(C^\infty(\mathbf{R}^k, Y))$ on the right.

Let $I \subseteq \mathbf{R}^l$ be a Weil ideal, $I \supseteq \mathcal{M}^r$. Let $\{D^\beta \mid \beta \in B\}$ be a family of differential operators at 0, of degree $< r$, forming a basis for $(C^\infty(\mathbf{R}^l)/I)^*$. Note that B is a finite set. Let the dual basis for $(C^\infty(\mathbf{R}^l)/I$ be represented by polynomials of degree $< r$, $\{p_\beta(s) \mid \beta \in B\}$. Then we can construct a linear isomorphism

$$C^\infty(\mathbf{R}^l, Y)/I(Y) \to \prod_B Y,$$

by sending the class of $f : \mathbf{R}^l \to Y$ into the B-tuple $D^\beta_Y(f)$. Its inverse is given by sending a B-tuple $y_\beta \in Y$ to $\sum_B p_\beta(s) \cdot y_\beta$.

It follows that for a semi-Weil ideal $J = p^*(I) \subseteq \mathbf{R}^{l+k}$, as above,

$$C^\infty(\mathbf{R}^{k+l}, Y)/J(Y) \cong \prod_B C^\infty(\mathbf{R}^k, Y). \tag{12}$$

(The isomorphism is not canonical but depends on the choice of a linear basis $p_\beta(s)$ for the Weil algebra $C^\infty(\mathbf{R}^l)/I$.)

6.2 The embedding into $\mathcal{C}$

The full embedding h, described in [13], of $\underline{\mathrm{Con}}^\infty$ into $\mathcal{C}$ is, on objects, given by sending a convenient vector space X into the presheaf on $\underline{D}$ given by

$$C^\infty(\mathbf{R}^{l+k})/J \mapsto C^\infty(\mathbf{R}^{l+k}, X)/J(X).$$

The fundamental observation in [13] is that for smooth maps $f : X \to Y$, composing with f preserves the property of "being congruent mod J", (provided that J is a semi-Weil ideal), cf. Coroll. 2 in [13]; and this describes the functorality. (This fundamental observation, in turn, is a generalization of the theory from [11] that the category of Weil algebras acts "by Weil prolongation" on the category $\underline{\mathrm{Con}}^\infty$; this prolongation construction is expounded also in [16] Section 31.) – For finite dimensional vector spaces X, $h(X) = i(X)$.

The embedding h is full. It preserves the exponentials in $\underline{\mathrm{Con}}^\infty$, and furthermore, if X is a convenient vector space, the R-module $h(X)$ in $\mathcal{C}$ "satisfies the vector form of Axiom 1" (generalized Kock-Lawvere Axiom), so that in particular synthetic calculus for curves $R \to h(X)$ is available; cf. the final remark in [11]. From this, one may deduce that the embedding h preserves differentiation, i.e. for $f : \mathbf{R} \to X$ a smooth curve, its derivative $f' : \mathbf{R} \to X$ goes by h to the synthetically defined derivative of the curve $h(f) : R = h(\mathbf{R}) \to h(X)$. This follows by repeating the argument for Theorem 1 in [8] (the Theorem there deals with the case where the codomain of f is $\mathbf{R}$, but it is valid for X as well because $h(X)$ satisfies the vector form of Axiom 1).

We note the following aspect of the embedding h. Let X be a convenient vector space. Each $\phi \in X'$ is smooth linear $X \to \mathbf{R}$ and hence defines a map $h(\phi) : h(X) \to h(\mathbf{R}) = R$ in $\mathcal{C}$. This map is R-linear.

Proposition 6.10 *The maps $h(\phi) : h(X) \to R$, as ϕ ranges over X', form a jointly monic family.*

Proof. The assertion can also be formulated: the natural map

$$e : h(X) \to \prod_{\phi \in X'} R$$

is monic (where $proj_\phi \circ e := h(\phi)$). To prove that this (linear) map is monic, consider an element a of the domain, defined at stage $C^\infty(\mathbf{R}^{l+k})/J$, where J is a semi-Weil ideal. So $a \in C^\infty(\mathbf{R}^{l+k}, X)/J(X)$. Let $\alpha \in C^\infty(\mathbf{R}^{l+k}, X)$ be a smooth map representing the class a, $a = \alpha + J(X)$. The element $e(a)$ is the X' tuple $a_\phi + J(X)$, where $a_\phi \in C^\infty(\mathbf{R}^{l+k})/J(X)$ is represented by the smooth map $\phi \circ \alpha : \mathbf{R}^{l+k} \to \mathbf{R}$. To say that a maps to 0 by e is thus to say that for each $\phi \in X'$, $\phi \circ \alpha \in J$. But this is precisely the defining property for α itself to be in $J_w(X) = J(X)$, i.e. for a to be the zero as an element of $h(X)$ (at the given stage $C^\infty(\mathbf{R}^{l+k})/J$).

6.3 Distributions in the Cahiers Topos

We begin by analyzing the object of test functions. We shall prove the analogue of Theorem 5.1, now for the embedding $h : \underline{\text{Con}}^\infty \to \mathcal{C}$. The object $\mathcal{D}(R)$ is defined synthetically by the same formula (8) as in Section 5. Part of the proof of the Theorem 5.1 there can be "recycled". In fact, letting U be $\mathbf{R}^k$, the proof recycles to give information about the elements of $\mathcal{D}(R)$ defined at stage $C^\infty(\mathbf{R}^k)$; they are the same as the elements of $h(\mathcal{D})(\mathbf{R})$, more precisely,

$$h(\mathcal{D}(\mathbf{R}))(C^\infty(\mathbf{R}^k)) = \mathcal{D}(R)(C^\infty(\mathbf{R}^k)). \tag{13}$$

To get a similar conclusion for elements of $\mathcal{D}(R)$ (as synthetically defined by (9)), defined at stage $C^\infty(\mathbf{R}^{l+k})/J$, we shall prove that such can be represented by B-tuples of elements defined at stage $C^\infty(\mathbf{R}^k)$; we shall prove that such a B-tuple defines an element of $\mathcal{D}(R)$ precisely if each of these B elements is an element in $\mathcal{D}(R)$. This proof is a piece of purely synthetic reasoning:

We consider an $\mathbf{R}$-algebra object R in a topos $\mathcal{C}$, and assume that R satisfies the general "Kock-Lawvere" (K-L) axiom (recalled below), and is equipped with a strict order relation $<$. Because the reasoning is purely synthetic, we don't have to think in terms of sheaf semantics, so for instance we don't have to be specific at what "stages", the "elements" in question are defined; we reason *as if* all elements are global elements. For $b > 0$, we write $|x| > b$ as shorthand for $x < -b \vee x > b$ as before; and we stress again that we don't assume any absolute-value function (it does not exist in the Cahiers topos). We argue in $\mathcal{C}$ as if it were the category of sets, making sure to use only intuitionistically valid reasoning.

A Weil algebra $C^\infty(\mathbf{R}^l)/I$, as above, gives rise to an "infinitesimal" subobject $W \subseteq R^l$: pick a (finite) set of differential operators D_β ($\beta \in B$) forming a basis for $(C^\infty(\mathbf{R}^l)/I)^*$, and take the dual basis for $C^\infty(\mathbf{R}^l)/I$, whose elements are represented mod I by polynomials $p_\beta(s)$ in l variables. Then $W \subseteq R^l$ is the extension of the formulas $p_\beta(s) = 0$, s being a variable ranging over R^l (note that real polynomials in l variables define functions $R^l \to R$ in $\mathcal{C}$).

We assume that such W's are internal atoms, in a sense we partially recall below; this is so for all interesting models $\mathcal{C}$ of SDG, including the Cahiers Topos.

To say that an R-module object Y in $\mathcal{C}$ satisfies the general K-L axiom is to say that for each such Weil algebra, the map

$$\prod_B Y \to Y^W$$

given by

$$(y_\beta)_{\beta\in B} \mapsto \left[s \mapsto \sum_B p_\beta(s) \cdot y_\beta \right]$$

is an isomorphism.

We assume that R itself satisfies K-L. This immediately implies that R^M does for any $M \in \mathcal{C}$. We shall consider R^R.

Now recall that $\mathcal{D}(R) \subseteq R^R$ was the subobject which is the extension of the formula (9) (with free variable f ranging over R^R) $\exists b > 0 : |x| > b \Rightarrow f(x) = 0$.

Proposition 6.11 *Let a B-tuple of elements f_β in R^R represent an element in $(R^R)^W$. Then it defines an element in the sub "set" $(\mathcal{D}(R))^W$ if and only if each f_β is in $\mathcal{D}(R)$.*

Proof. Assume first that all f_β are in $\mathcal{D}(R)$. For each β there exists a witnessing $b_\beta > 0$ witnessing that the formula (9) holds for f_β, but since there are only finitely many β's, we may assume one common witness $b > 0$. So for all β, and for all x with $|x| > b$, $f_\beta(x) = 0$. But then for each such x, the function of $s \in W$ given by

$$s \mapsto \sum_\beta p_\beta(s) \cdot f_\beta(x)$$

is the zero function. The sum here, as a function of s and x, is the element of $(R^R)^W$ corresponding to the B-tuple f_β, and for $|x| > b$, it is the zero. So for each s, the given fixed b witnesses that the sum, as a function of x, is in $\mathcal{D}(R)$.

Conversely, assume that the f_β's are such that the corresponding function $W \to R^R$ factors through $\mathcal{D}(R)$. So for each $s \in W$, the function

$$x \mapsto \sum_\beta p_\beta(s) \cdot f_\beta(x)$$

belongs to $\mathcal{D}(R)$. So

$$\forall s \in W \; \exists b > 0 : |x| > b \Rightarrow \sum_\beta p_\beta(s) \cdot f_\beta(x) = 0. \tag{14}$$

We would like to *pick* for each $s \in W$ a $\tilde{b}(s)$ such that

$$\forall s \in W : |x| > \tilde{b}(s) \Rightarrow \sum_\beta p_\beta(s) \cdot f_\beta(x) = 0;$$

the existence of such a *function* $\tilde{b}$ follows from (14) by a use of the Axiom of Choice, so in general is not possible in a topos. But since W is an internal atom, and s ranges over W, such a function $\tilde{b}$ exists after all. (See the Appendix for a general formulation and proof of this principle.)

But now $|x| > \tilde{b}(0) \Rightarrow |x| > \tilde{b}(s)$ for all $s \in W$, because $\tilde{b}$, as does any function, preserves infinitesimals, and because strict inequality is unaffected by infinitesimals. So we have a b, namely $\tilde{b}(0)$, so that

$$\forall s \in W : |x| > b \Rightarrow \sum p_\beta(s) \cdot f_\beta(x) = 0.$$

So for $|x| > b$,

$$\forall s \in W, \sum p_\beta(s) \cdot f_\beta(x) = 0.$$

Thus, for fixed x with $|x| > b$, the function of s here is constantly 0. But functions $W \to R$ can uniquely be described as linear combinations of the $p_\beta(s)$'s (this is a verbal rendering of the K-L axiom for R). So for such x each $f_\beta(x)$ is 0. So b witnesses, for each β, that $f_\beta \in \mathcal{D}(R)$. This proves the Proposition.

Combining (12) (with $\mathcal{D}(\mathbf{R})$ for Y) with (13) and Proposition 6.11, we get

Theorem 6.12 (Test functions in the Cahiers Topos) *The inclusion $\mathcal{D}(\mathbf{R})$ goes by $h : \underline{\mathrm{Con}}^{\infty} \to \mathcal{C}$ to the inclusion $\mathcal{D}(R) \subseteq R^R$*

We proceed to use this result as the tool analyze the object of distributions $\mathcal{D}(R)'$. We could proceed along the lines of the proof of Theorem 5.1, but a more elegant argument is available. For any convenient vector space Y, the dual Y' is a not only a subspace of $\mathbf{R}^Y$, it is even a retract, namely the fixpoint set of the smooth linear endomap d_0 on $\mathbf{R}^Y$ given by $f \mapsto d_0 f$, the differential of f at $0 \in Y$. In the Cahiers topos $\mathcal{C}$, synthetic differential calculus is available, and there is a similar retraction operator d_0 on R^Z, for any vector space (R-module) Z, and in fact, the object $Lin_R(Z, R) \subseteq R^Z$ is the fixpoint object for this operator (this follows from elementary synthetic differential calculus, cf. [17] 1.2.3 and 1.2.4). But the embedding h takes the "external" d_0 to the internal one, and any functor preserves fixpoint objects for idempotent endomaps. Thus h takes the subobject $Y' \subseteq \mathbf{R}^Y$ to the subobject $Lin_R(h(Y), R)$. If we apply this observation to the case of $Y = \mathcal{D}(\mathbf{R})$, and use the Theorem 6.12 above, we get

Theorem 6.13 (Distributions in the Cahiers Topos) *The embedding*

$$h : \underline{\mathrm{Con}}^{\infty} \to \mathcal{C}$$

takes the convenient vector space $\mathcal{D}(\mathbf{R})'$ of distributions on $\mathbf{R}$ into the internal object of distributions $\mathcal{D}(R)'$.

We get in particular

Proposition 6.14 *There is a bijective correspondence between distributions on $\mathbf{R}$, and R-linear maps in $\mathcal{C}$, $\mathcal{D}(R) \to R$.*

6.4 Half line in $\mathcal{C}$

By Theorem 4.1, the two C^∞-rings $C^\infty(\mathbf{R})/\mathcal{M}^\infty_{\geq 0}$ and $C^\infty(\mathbf{R}_{\geq 0})$ are isomorphic, where $\mathcal{M}^\infty_{\geq 0}$ is the ideal of smooth functions vanishing on the non-negative half line, and $C^\infty(\mathbf{R}_{\geq 0})$ is the ring of smooth functions $\mathbf{R}_{\geq 0} \to \mathbf{R}$. Being a quotient of the ring $C^\infty(\mathbf{R})$ which represents $R \in \mathcal{C}$, it defines a subobject of R, which we denote $R_{\geq 0}$ (also considered in [12][1]). – Thus, $R_{\geq 0}$ is "represented from the outside" by the C^∞-ring $C^\infty(\mathbf{R})/\mathcal{M}^\infty_{\geq 0} \cong C^\infty(\mathbf{R}_{\geq 0})$.

Proposition 6.15 *Let $I \subseteq C^\infty(\mathbf{R}^l)$ be a Weil ideal and let $f : \mathbf{R}^l \times \mathbf{R}^k \to \mathbf{R}$ be a smooth function. Then the following are equivalent:*

1. *$f(\underline{0}, x) \geq 0$ for all $x \in \mathbf{R}^k$*
2. *$\rho(f(\underline{w}, x)) \in I^*$ for all $\rho \in \mathcal{M}^\infty_{\mathbf{R}\geq 0}$.*

[1] The ring representing $R_{\geq 0}$, was in loc.cit. defined using the ideal $\mathcal{M}^g_{\geq 0}$ of functions vanishing on an open neighbourhood of $\mathbf{R}_{\geq 0}$, rather than the ideal $\mathcal{M}^\infty_{\geq 0}$ considered here. But it can be proved that they represent (from the outside) the same object in the Cahiers topos.

Proof. "not 1" implies "not 2"; for, if $f(0,x) < 0$, we may find a function ρ vanishing on $\mathbf{R}_{\geq 0}$ and with value 1 at $f(0,x)$. Then $f \notin I^*$ (recall that any Weil ideal I consists of functions vanishing at 0). – On the other hand, "1" implies "2": For, by Taylor expansion,

$$(\rho \circ f)(\underline{w}, x) = (\rho \circ f)(\underline{0}, x) + \sum_i w_i (\rho \circ f)'_i(\underline{0}.x) + \sum_{i,j} w_i w_j (\rho \circ f)'_{i,j}(\underline{0}, x) + \cdots$$

where $(-)_i = \partial/\partial x_i,\ (-)_{i,j} = \partial^2/\partial x_i x_j$ etc.

This series finishes after finitely many terms modulo I^*, since a product of powers of w_i's belong to the ideal I. But each of its terms is 0: Indeed, so is the term without derivatives, by hypothesis. But so are the others. For instance. $(\rho \circ f)'_i(\underline{0}, x) = \rho'(f(\underline{0}, x)\partial f/\partial x_i(\underline{0}, x)$ is 0, since the derivative of ρ is zero on non-negative reals (by definition of $m^{\infty}_{\mathbf{R}\geq 0}$).

Let J denote I^*. Then an element F of $R_{\geq 0}$ defined at stage $C^\infty(\mathbf{R}^{l+k})/J$ is represented by a function f satisfying the conditions of the Proposition.

Proposition 6.16 *There is a bijection between the set of smooth maps $K : \mathbf{R}_{\geq 0} \to X$ and the set of maps $\overline{K} : R_{\geq 0} \to h(X)$ in $\mathcal{C}$.*

Proof/Construction. Passing from $\overline{K}$ to K is just by taking global sections. – Conversely, given K, we extend it (using Theorem 4.1) to a smooth map $K_1 : \mathbf{R} \to X$, and apply the embedding h to get a map $h(K_1) : R \to h(X)$ in $\mathcal{C}$; its restriction to $R_{\geq 0}$ is the desired $\overline{K}$. We have to see that this $\overline{K}$ does not depend on the choice of the extension K_1. Given some other extension K_2. We should prove that for any generalized element F of $R_{\geq 0}$, $h(K_1)(F) = h(K_2)(F)$. Suppose F is an element of stage $C^\infty(\mathbf{R}^{l+k})/J$, where J is the semi-Weil ideal I^* considered in the Proposition above. Thus, as a generalized element of R, it is identified with $f + J$, where $f \in C^\infty(\mathbf{R}^{l+k})$, and it satisfies condition 2. of the Proposition, being an element of $R_{\geq 0}$.

We should prove that $K_1 \circ f \cong K_2 \circ f$ modulo $J(X)$. Since J is a semi-Weil ideal, it suffices by Theorem 6.9 to prove, for any $\phi \in Y'$ that

$$\phi \circ K_1 \circ f \cong \phi \circ K_2 \circ f$$

modulo J. But subtracting the two entries to be compared yields, by linearity of ϕ the map

$$\phi \circ (K_1 - K_2) \circ f,$$

and since $K_1 - K_2$ vanishes on $\mathbf{R}_{\geq 0}$, then so does $\phi \circ (K_1 - K_2)$. We may thus take $\rho = \phi \circ (K_1 - K_2)$ in the condition 2. in the Proposition, and conclude that $\phi \circ (K_1 - K_2) \circ f$ is in $I^* = J$, as desired.

– Uniqueness is easy, using Proposition 6.10, together with the fullness result from [23] on manifolds with boundary.

The Proposition is a "mixed fullness" result; we have that Con^∞ and Mf (= smooth manifolds), (even the category of smooth manifolds with boundary), embed fully in The Cahiers Topos; but at present we do not at present have a general result about what can be said about $C^\infty(M, X)$, for M a manifold with boundary and X a convenient vector space.

7 Heat Equation in the Cahiers Topos

For any topos $\mathcal{C}$ with a ring object R with a preorder $\leq$, we may form the R-module $\mathcal{D}'(R^n)$ of distributions on R^n, as explained in Section 5 and 6. If $\mathcal{C}, R$ is a model of SDG, then $\mathcal{D}'(R^n)$ automatically satisfies the "vector form" of the general Kock-Lawvere axiom, so that (synthetic) differentiation of functions $K : R \to \mathcal{D}'(R^n)$ is possible – it is even enough that K be defined on suitable ("formally étale") subobjects of R, like $R_{\geq 0}$. We think of the domain R or $R_{\geq 0}$ as "time", and denote the differentiation of curves K w.r. to time by the Newton dot, $\dot{K}$. On the other hand, we think of R^n as a space, and the various partial derivatives $\partial/\partial x_i$ ($i = 1, \ldots, n$), as well as their iterates, we call spatial derivatives; in case $n = 1$, they are just denoted $(-)'$, $(-)''$, etc. They live on $\mathcal{D}'(R^n)$ as well, by the standard way of differentiating distributions (which immediately translates into the synthetic context, cf. e.g. [15]). The heat equation for (Euclidean) space in n dimensions says $\dot{K} = \Delta \circ K$, where Δ is the Laplace operator; in one dimension it is thus the equation

$$\dot{K} = K''.$$

We can summarize the constructions into an general existence theorem about models for SDG:

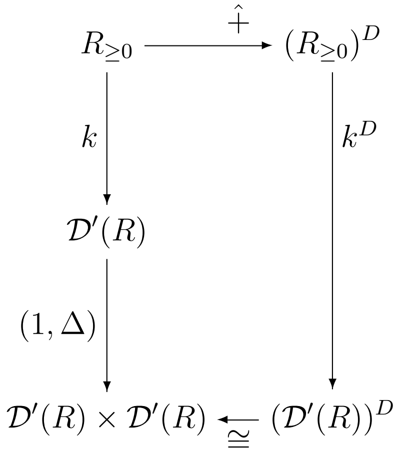
Theorem 7.1 *There exists a well-adapted model for SDG (with a preorder $\leq$ on R), in which the heat equation on the (unlimited) line R has a unique solution $k : R_{\geq 0} \to \mathcal{D}'(R)$ with initial value $k(0) = \delta(0)$ (the Dirac distribution).*

Proof. The well adapted model witnessing the validity of the Theorem is the Cahiers Topos $\mathcal{C}$. Consider the classical heat kernel, viewed, as we did in Section 4, as a map $\mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$. By Section 4, this map is smooth, hence by Proposition 6.16, it defines a morphism in $\mathcal{C}$, $\bar{K} : R_{\geq 0} \to h(\mathcal{D}'(\mathbf{R}))$. This $\bar{K}$ is going to be our k. By Theorem ??, its codomain is the desired $\mathcal{D}'(R)$. We prove that this k satisfies the heat equation $\dot{k} = \Delta \circ k$. This is a purely formal argument from the fact that K does, and the fact that h takes "analytic" differentiation into the "synthetic" differentiation in $\mathcal{C}$. We give this argument. Synthetically, we want to prove that for all $x \in \mathbf{R}_{\geq 0}$ and $d \in D$

$$k(x + d) = k(x) + d \cdot \Delta(k(x)).$$

Universal validity of this equation means that a certain diagram, with domain $R_{\geq 0} \times D$ and codomain $\mathcal{D}'(R)$, commutes. Taking the transpose of this diagram, we get a diagram with domain $R_{\geq 0}$ and codomain $(\mathcal{D}'(R))^D \cong \mathcal{D}'(R) \times \mathcal{D}'(R)$ (by K-L for

$\mathcal{D}'(R)$):



When the global sections functor Γ is applied to this diagram, the left hand column yields $(K, \Delta \circ K)$, because $\Gamma(k) = K$; the composite of the other maps is $(K, \dot{K})$ because Γ takes synthetic differentiation into usual differentiation. Since K satisfies $\dot{K} = \Delta \circ K$, we conclude that Γ applied to the exhibited diagram commutes. Now Γ is not faithful, but because of the special form of the domain and codomain of the two maps to be compared, we may still get the conclusion, by virtue of the following

Proposition 7.2 *Given a map $a : R_{\geq 0} \to h(X)$, where X is a convenient vector space. If $\Gamma(a) = 0$, then $a = 0$.*

Proof. Since the $h(\phi) : h(X) \to R$ are jointly monic as ϕ ranges over X', by Proposition 6.10, it suffices to see that each $h(\phi) \circ a$ is 0. Since $\Gamma(h(\phi) \circ a) = \phi \circ \Gamma(a)$, this reduces the question to the case where $X = \mathbf{R}$. A map $a : R_{\geq 0} \to R$ is tantamount to an element in $\bar{a} : C^\infty(\mathbf{R}_{\geq 0})$, and the assumption $\Gamma(a) = 0$ is tantamount to $\bar{a}(t) = 0$ for all $t \in \mathbf{R}_{\geq 0}$. But this clearly implies that $\bar{a}$, and hence a, is 0.

The uniqueness assertion in the Theorem is likewise an easy consequence of this Proposition.

Appendix

Recall that an *atom* A in a cartesian closed category $\mathcal{C}$ is an object so that the exponential functor $(-)A$ has a right adjoint; in particular, it takes epimorphisms to epimorphisms. The following says that "axiom of choice" holds for "A"-tuples sets:

Proposition 7.3 *Assume that A is an atom, B arbitrary and $R \subseteq A \times B$. Then*

$$(\forall a \in A)(\exists b \in B)\ R(a, b) \Longrightarrow (\exists \tilde{b} \in B^A)(\forall a \in A)\ R(a, \tilde{b}(a))$$

Proof: The hypothesis means that the composite $R \to A \times B \xrightarrow{\pi_1} A$ is surjective. By exponentiation, and the assumption that A is an atom, the composite $R^A \to A^A \times B^A \xrightarrow{\pi_1} A^A$ is surjective. In particular, $1_A \in A^A$ must have a pre-image $(1_A, \tilde{b})$. This $\tilde{b}$ obviously does the job.

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