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THE SUSPENDED FREE LOOP SPACE OF A SYMMETRIC SPACE

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Abstract

Let M be one of the projective spaces $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ for $n \geq 2$ or the Cayley projective plane $\mathbb{O}\mathrm{P}^2$, and let ΛM denote the free loop space on M. Using Morse theory methods, we prove that the suspension spectrum of $(\Lambda M)_+$ is homotopy equivalent to the suspension spectrum of M_+ wedge a family of Thom spaces of explicit vector bundles over the tangent sphere bundle of M.
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1 Introduction

It has been known at least since Bott and Samelson [1] that it is possible to study the homotopy theory of the free loop space of a symmetric space using tools from differential geometry and Lie group theory. One particularly nice continuation of these ideas is the work by Ziller [16]. He computes the integral homology of ΛM for rank one symmetric spaces, and gives results and techniques that apply to all symmetric spaces.

In [4] we approached the cohomology of a free loop space $H^*(\Lambda M; \mathbb{Z}/2)$ from a totally different angle. We used methods of cosimplicial spaces to set up a spectral sequence which for simply connected M converges to this cohomology. In order to compute the E_2-page of the spectral sequence we needed a version of non-commutative homology, related to André-Quillen homology of a ring. All of this is pure homotopy theory. There is nothing in the methods that needs or uses that M is a smooth manifold. The simplest non trivial case of the spectral sequence is when M is a space such that $H^*(M; \mathbb{Z}/2)$ is a truncated polynomial algebra. In [3] we used this approach to compute $H^*(\Lambda M; \mathbb{Z}/2)$ as a module over the Steenrod algebra for such spaces.

These calculations led to a strange observation. In each case we considered, the cohomology splits as a sum of finite dimensional modules. Moreover, we recognized the pieces as cohomology of spaces closely related to Thom spaces of iterated Whitney sums of the tangent bundle on M. We conjectured that the observed splitting was induced by an actual splitting of the suspension spectrum of ΛM for these spaces.

The class of simply connected spaces such that $H^*(M; \mathbb{Z}/2)$ is a truncated polynomial algebra is quite limited. The complex and quaternionic projective spaces, together with the Cayley projective plane are not the only examples, but clearly stick out as the most interesting ones. And these are exactly the compact, simply connected, globally symmetric spaces of rank one.

The main results of this paper is that we can prove splittings of suspension spectra of ΛM_+ for these rank one symmetric spaces. These theorems are given in section 6. For example:

Theorem 6.1. *Let $p : S(\tau) \to \mathbb{C}\mathrm{P}^n$ be the unit sphere bundle of the tangent bundle τ on $\mathbb{C}\mathrm{P}^n$. Let ξ_m be the vector bundle $(p^*(\tau))^{\oplus(m-1)} \oplus \epsilon$ on $S(\tau)$, where ϵ is a one dimensional trivial bundle. Then, there is a homotopy equivalence of spaces*

$$\Sigma((\Lambda\mathbb{C}\mathrm{P}^n)_+) \simeq \Sigma(\mathbb{C}\mathrm{P}^n_+) \vee \bigvee_{m\geq 1} \Sigma\mathrm{Th}(\xi_m).$$

The proof is based on Ziller's methods. He does not quite formulate it in this way, but he essentially proves that for a general globally symmetric space, there is a splitting of the homology $H_*(\Lambda M; \mathbb{Z}/2)$. The main part of this paper consists of enhancing this argument to the point where we can prove our conjecture on splittings of the suspension spectra. In the tradition of Bott and Samelson, Ziller used a mixture of differential geometry and Morse theory. The extra ingredient we have added to this brew is a pinch of homotopy theory.

The result we obtain is clearly stronger than Ziller's splitting of homology groups. For instance, it follows that there are also splittings for other homology theories. It also follows that the splitting is compatible with the action of cohomology operations.

Previously, various splitting results have been proved, for instance for the free loop space of a suspended space. It is not clear to us that this splitting is related to our splitting.

The proof is not completely formal. At one point, we need a construction, which can only be performed if a certain obstruction vanishes. This obstruction lies in the cokernel of the map of representation rings induced by a certain inclusion of Lie groups. By calculation, this obstruction vanishes in the cases we consider.

We intend to try to extend this result to other symmetric spaces. In this case there is no a priori guarantee that the corresponding obstructions vanish.

In Section 2 we collect some background material on equivariant differential topology and symmetric spaces.

In section 3 we recall how Morse theory is applied to free loop spaces. There is little original here.

In section 4 we reformulate and extend the method Bott and Samelson introduced for using the action of the isometry group to construct interesting subspaces of the free loop space. These spaces can be identified as homogeneous spaces. Much of this must have been known to Ziller.

Section 5 contains the main new toy. We give methods to equivariantly split off the top cell of the homogeneous spaces constructed in section 4. This splitting is often only possible in the stable situation, that is after passing to the suspension spectrum. In all cases, we can only construct the splittings if certain obstructions vanish. Then we show how we can use such splittings to obtain stable splittings of free loop spaces.

In section 6, we apply the theory of the previous sections to the special cases of projective spaces and to the Cayley projective plane. We show that for these spaces the conditions of section 5 are satisfied, and we obtain splittings. We also identify the summands of the splittings. They are essentially Thom spaces of bundles over the total space of the unit sphere tangent bundle over M.

Finally, in section 7 we check that the splitting we obtain here agree with splitting conjectured in [3].

It is a pleasure to thank J. Tornehave for many discussions, which have helped the paper substantially.

2 Technical Recollections

We collect some background material on equivariant differential topology and differential geometry of symmetric spaces, which we will need later.

2.1 Equivariant differential topology

We list a few basic facts about smooth actions of compact Lie groups on manifolds. The tangent bundle of a smooth manifold M is denoted $\tau(M)$.

Theorem 2.1. *Let $\mathbf{H}$ be a compact Lie group with a smooth and free right action on a manifold M. Then the orbit space $M/\mathbf{H}$ has the structure of a smooth manifold, such that the quotient map $M \to M/\mathbf{H}$ is a submersion.*

Proof. By [6], I.3.21 we find that the action is proper since $\mathbf{H}$ is compact. But then it follows from [6], I.5.2 that the orbit space is a smooth manifold, and that the quotient map is a submersion. □

Theorem 2.2. *Let $\mathbf{G}$ be a compact Lie group with a closed subgroup $\mathbf{H}$. Assume that M is a left $\mathbf{H}$-manifold and let $p : \mathbf{G} \times_{\mathbf{H}} M \to \mathbf{G}/\mathbf{H}$ denote the canonical projection $p([g, x]) = g\mathbf{H}$. Then there is an isomorphism of $\mathbf{G}$-vector bundles*

$$\tau(\mathbf{G} \times_{\mathbf{H}} M) \cong p^*(\tau(\mathbf{G}/\mathbf{H})) \oplus (\mathbf{G} \times_{\mathbf{H}} \tau(M)).$$

Proof. We have that $M \xrightarrow{i_g} \mathbf{G} \times_{\mathbf{H}} M \xrightarrow{p} \mathbf{G}/\mathbf{H}$, where $i_g(x) = [g, x]$, is a fiber bundle associated to the principal $\mathbf{H}$-bundle $\mathbf{G} \to \mathbf{G}/\mathbf{H}$. Since a fiber bundle is locally trivial, there exists an open neighborhood U of $g\mathbf{H}$ in $\mathbf{G}/\mathbf{H}$ and a diffeomorphism $p^{-1}(U) \cong U \times M$ which preserves the fiber. So locally our fiber bundle is isomorphic to a product bundle $M \to U \times M \to U$ and therefore we have an exact sequence for all $g \in \mathbf{G}$ and $x \in M$ as follows:

$$0 \longrightarrow T_x(M) \xrightarrow{(i_g)_*} T_{[g,x]}(\mathbf{G} \times_{\mathbf{H}} M) \xrightarrow{p_*} T_{g\mathbf{H}}(\mathbf{G}/\mathbf{H}) \longrightarrow 0.$$

In particular p is a submersion. This implies that the map into the pullback

$$\phi : \tau(\mathbf{G} \times_{\mathbf{H}} M) \to p^*(\tau(\mathbf{G}/\mathbf{H}))$$

is surjective. So ϕ has constant rank and we have a short exact sequence of $\mathbf{G}$-vector bundles

$$0 \longrightarrow \xi \longrightarrow \tau(\mathbf{G} \times_{\mathbf{H}} M) \longrightarrow p^*(\tau(\mathbf{G}/\mathbf{H})) \longrightarrow 0,$$

where $\xi = \ker(\phi)$. Note that the dimension of ξ equals the dimension of M.

For a compact Lie group $\mathbf{G}$ any smooth $\mathbf{G}$-equivariant vector bundle has an invariant inner product ([5] VI.2.1). So the short exact sequences of $\mathbf{G}$-vector bundles splits, and we obtain an isomorphism of $\mathbf{G}$-vector bundles

$$\tau(\mathbf{G} \times_{\mathbf{H}} M) \cong p^*(\tau(\mathbf{G}/\mathbf{H})) \oplus \xi.$$

We only have to identify ξ.

There is a map $\psi : \mathbf{G} \times \tau(M) \to \tau(\mathbf{G} \times_{\mathbf{H}} M)$ given by $\psi(g, v_x) = (i_g)_*(v_x)$. By the identity $i_{gh} \circ (h^{-1}\cdot) = i_g$ for $h \in \mathbf{H}$ we have $(i_{gh})_* \circ (h^{-1}\cdot)_* = (i_g)_*$ so ψ factors through $\mathbf{G} \times_{\mathbf{H}} \tau(M)$ and we obtain a map over $\mathbf{G} \times_{\mathbf{H}} M$:

$$\overline{\psi} : \mathbf{G} \times_{\mathbf{H}} \tau(M) \to \tau(\mathbf{G} \times_{\mathbf{H}} M).$$

The image of $\overline{\psi}$ is contained in ξ and the dimension of its domain vector bundle equals the dimension of ξ. By the exact sequence of tangent spaces above, $\overline{\psi}$ is injective and the result follows. □

Theorem 2.3. *If a compact Lie group $\mathbf{G}$ acts smoothly on a manifold M, then the orbit $\mathbf{G}x \subset M$ is a submanifold of M for each $x \in M$, and the map $\alpha_x : \mathbf{G}/\mathbf{G}_x \to \mathbf{G}x$; $g\mathbf{G}_x \mapsto gx$ is a diffeomorphism, where $\mathbf{G}_x$ denotes the isotropy group.*

Proof. This is proposition I.5.4 of [6]. □

Theorem 2.4. *Let $\mathbf{H}$ be a compact Lie group which acts smoothly on a manifold M with a fixed point $x \in M$.*

1. *The tangent space T_xM is an $\mathbf{H}$-representation and there is an equivariant diffeomorphism $\phi : T_xM \to M$ onto an open neighborhood of x in M such that $\phi(0) = x$.*

2. *If $\mathbf{H}$ is a closed subgroup of another compact Lie group $\mathbf{G}$, then we can consider the $\mathbf{G}$-equivariant section*

$$s : \mathbf{G}/\mathbf{H} \to \mathbf{G} \times_{\mathbf{H}} M \quad ; \quad s(g\mathbf{H}) = [g, x].$$

The image $s(\mathbf{G}/\mathbf{H})$ is a submanifold of $\mathbf{G} \times_{\mathbf{H}} M$ whose normal bundle is $\mathbf{G}$-equivariantly isomorphic to the vector bundle

$$\mathbf{G} \times_{\mathbf{H}} T_xM \to \mathbf{G}/\mathbf{H}.$$

Proof. For the first part, see [6] I.5.6, especially the reference to Bochner.

For the second part, first note that because of theorem 2.1, the three orbit spaces mentioned in the statement are smooth manifolds. Then consider the left action

$$\mathbf{G} \times (\mathbf{G} \times_{\mathbf{H}} M) \to \mathbf{G} \times_{\mathbf{H}} M,$$

and the point $[e, x] \in \mathbf{G} \times_{\mathbf{H}} M$. The isotropy group of this point is $\mathbf{G}_{[e,x]} = \mathbf{H}$. According to [6] I.5.4 we get that s is an embedding and that $s(\mathbf{G}/\mathbf{H})$ is a submanifold which is $\mathbf{G}$ diffeomorphic to $\mathbf{G}/\mathbf{H}$.

We must find the normal bundle of $s(\mathbf{G}/\mathbf{H})$. The map ϕ of part 1. induces a $\mathbf{G}$-equivariant map

$$\mathbf{G} \times_{\mathbf{H}} T_x M \to \mathbf{G} \times_{\mathbf{H}} M$$

This map is a diffeomorphism on its image, and the image is an open neighborhood of $s(\mathbf{G}/\mathbf{H})$. The normal bundle only depends on an open neighborhood of $s(\mathbf{G}/\mathbf{H})$, and the result follows. □

2.2 Differential geometry of symmetric spaces

There are several good expositions of the theory of symmetric spaces, for example in [10] or in [15].

In section 3 we are going to study the Morse theory of closed curves on symmetric spaces. This will lead us to questions on closed geodesic curves on symmetric spaces, and conjugated points on such curves. So as a preliminary and motivating step, we will say a little about this subject.

In section 6 we are going to study symmetric spaces of rank 1. We start this paragraph by discussing these, especially the projective Cayley plane. There does not seem to be a reference to the projective Cayley plane that covers all the properties we need, so we have to collect results from several sources. A part of this section is written in expository style, but we give proofs or precise references for the results we actually need in section 6.

Assume that the Lie group $\mathbf{G}$ is the identity component of the isometry group of a complete Riemannian manifold M. Let $x \in M$ with isotropy group $\mathbf{K} = \mathbf{G}_x$. The group $\mathbf{K}$ acts on $S(T_x M)$ and we let $\mathbf{H}$ denote the isotropy group for this action at a fixed vector $v \in S(T_x M)$.

The set of unit speed geodesics γ with $\gamma(0) = x$ is in one to one correspondence with the unit sphere $S(T_x M)$. The correspondence is given by $\gamma \mapsto \gamma'(0)$. If $\gamma'(0) = v$ we see that $\mathbf{H}$ is the group that fixes γ pointwise.

Lemma 2.5. *Assume that γ is a geodesics on M with $\gamma(0) = x$ and $\gamma'(0) = v \in S(T_x M)$. Let $\mathbf{K}_t = \{g \in \mathbf{K} | g\gamma(t) = \gamma(t)\}$ for $t > 0$.*

1. *If $\dim \mathbf{K}_t > \dim \mathbf{H}$, then $\gamma(t)$ is conjugated to $\gamma(0)$ along γ.*
2. *If $\mathbf{K}$ acts transitively on $S(T_x M)$ and if $\dim(\mathbf{K}_t) = \dim(\mathbf{H})$, then $\gamma(t)$ is not conjugated to $\gamma(0)$ along γ.*

Proof. Let $\exp_x : T_x M \to M$ denote the exponential map. Recall from [12] theorem 18.1 that the point $\gamma(t) = \exp_x(tv)$ is conjugated to $\gamma(0)$ along γ if and only if tv is a critical point for $\exp_x$. Also recall Gauss lemma: The image of the tangential map

$$(\exp_x)_* : T_{tv}(S_t(T_x M)) \to T_{\exp(tv)} M$$

is orthogonal to the tangent vector $\gamma'(t)$. Here $S_t(T_x M)$ denotes the set of tangent vectors at x of length t. We explain how the lemma follows from these two remarks.

To prove the first statement we only use the first remark. Note that the following composite is a constant map, so it induces the trivial map of tangent spaces.

$$\mathbf{K}_t/\mathbf{H} \subset \mathbf{K}/\mathbf{H} \hookrightarrow S_t(T_xM) \xrightarrow{\exp_x} M.$$

Because of theorem 2.3, the differential of the map $\mathbf{K}_t/\mathbf{H} \subset \mathbf{K}/\mathbf{H} \hookrightarrow S_t(T_xM)$ is injective. It follows that if $\dim \mathbf{K}_t > \dim \mathbf{H}$, we obtain a non-trivial element in the kernel of the differential of the exponential map, so $\gamma(tv)$ is conjugated to $\gamma(0)$.

We now consider the second statement of the theorem. If $\gamma(t)$ is conjugated to $\gamma(0)$, the differential at tv of the exponential map $\exp_x : T_xM \to M$ has a non trivial kernel. Write a nontrivial element in this kernel as $w + s(tv)$, where $w \in T_{tv}(S_t(T_xM))$. By Gauss lemma it follows that $s = 0$, so that $w \neq 0$. We conclude that the differential at tv of the composite $S_t(T_xM) \subset T_xM \xrightarrow{\exp_x} M$ has a non-trivial kernel. If $\mathbf{K}$ acts transitively on the unit tangent sphere, by theorem 2.3 the action induces a diffeomorphism $\mathbf{K}/\mathbf{H} \to S(T_xM)$. A non-trivial element of the kernel of the differential defines a 1-parameter subgroup of $\mathbf{K}/\mathbf{H}$ which fixes $\gamma(t)$ (using 2.3 again), so $\dim(\mathbf{K}_t) > \dim(\mathbf{H})$. □

Remark 2.6. In a symmetric space, the relation between the isometry group and Jacobi vector fields is even better, since the symmetry group is big. We will see a version of this improved correspondence in section 4, especially proposition 4.5 and theorem 4.6.

Definition 2.7. A Riemannian manifold M *is isotropic*, if the isometry group of M acts transitively on the total space of the unit sphere bundle of the tangent bundle.

The isotropy condition is quite constraining. It is equivalent to the condition that M is an Euclidean space or a symmetric space of rank 1, and it is also equivalent to the condition that the group $\mathbf{G}$ acts transitively on pairs of points with a fixed distance. These equivalences are proved in [15] corollary 8.12.9.

According to [15] 8.12.2 this class includes spheres, complex and quaternionic projective spaces and the Cayley projective plane, but no other compact, simply connected spaces.

The purpose of the following short general discussion is not to cover all rank 1 spaces simultaneously, but rather an attempt to explain the facts we will need in 6.3 about the most exotic of these spaces, the Cayley projective plane.

Let M be a compact globally symmetric space of rank 1 and dimension m. Since it is compact, it has a finite diameter. By compactness, there exists two points on M with distance equal to the diameter. By the Hopf-Rinow theorem, these two points are joined by a minimal geodesic of length equal to the diameter of M.

Define the *antipodal set* of $x \in M$ by $A_x = \{y \in M | d(x, y) = \operatorname{diam}(M)\}$. according to [10] Ch. IX §5, the isotropy group $\mathbf{K} = \mathbf{G}_x$ acts transitively on A_x so A_x is a compact submanifold of M, diffeomorphic to $\mathbf{K}/\mathbf{K}_1$ where $\mathbf{K}_1$ is the isotropy group of a some point $y \in A_x$ (cf. theorem 2.3).

Lemma 2.8. *A closed geodesic $\gamma(t)$ on M has length $2mL$ for some positive integer m. The point $\gamma(L)$ is a point with maximal distance from $\gamma(0)$, so that diam$(M) = L$. There are at most three conjugacy classes of isotropy groups of pairs of points in M, namely $\mathbf{H} \subset \mathbf{K}_1 \subset \mathbf{K}$.*

1. *If t is not an integer multiple of L, then $\gamma(t)$ is not conjugated to $\gamma(0)$ along γ.*

2. *$\gamma(2kL)$ is conjugated to $\gamma(0)$ along γ for any integer k (and $\gamma(2kL) = \gamma(0)$).*

3. *If $\mathbf{H}$ has positive codimension in $\mathbf{K}_1$, then for every integer k the point $\gamma((2k+1)L)$ is conjugated to $\gamma(0)$ along γ.*

Proof. There exist a simple closed geodesics, and the simply closed geodesics are all of the same length $2L$ by [10] proposition IX.5.3. Because $\mathbf{G}$ acts transitively on geodesics, all closed geodesics will run through a simple closed geodesics an integral number of times. This shows that since any two points are connected by a minimal geodesic, the diameter of M is at most L.

The exponential map is a diffeomorphism from the open ball in the tangent space $B_L(x) = \{v \in T_xM||x| < L\}$ onto the open set $M \setminus A_x$ according to [10] theorem IX.5.4. In particular, this shows that the diameter of M is exactly L.

Since the exponential map is equivariant, it follows that the isotropy subgroup of $\mathbf{K}$ fixing a point in in $B_L(x) \setminus \{x\}$ is exactly $\mathbf{H}$.

Since $\mathbf{K}$ acts transitively on the unit tangent sphere at $x \in M$, lemma 2.5 tells us that a point $\gamma(t)$ cannot be conjugated to $\gamma(0)$ unless either $t = 2kL$ or $t = (2k+1)L$ for some integer k.

The isotropy group of $\gamma(2kL) = \gamma(0)$ is $\mathbf{K}$, and lemma 2.5 tells us that these points are conjugated to $\gamma(0)$.

Finally, the points $\gamma((2k+1)L)$ are conjugated to $\gamma(0)$ if and only if the dimension of the isotropy group of $\gamma(L)$ is bigger than the dimension of $\mathbf{H}$. So, they are all conjugate, or none of them is. And by lemma 2.5, they are conjugated if and only if the dimension of $\mathbf{K}_1$ is strictly greater than the dimension of $\mathbf{H}$. □

Remark 2.9. Actually, we will always have that $\dim(\mathbf{K}_1) > \dim(\mathbf{H})$.

We now specialize this to the Cayley projective plane $\mathbb{O}\mathrm{P}^2$. The exotic Lie group F_4 contains Spin(9) as a subgroup. The homogeneous quotient $\mathbb{O}\mathrm{P}^2 \cong F_4/\mathrm{Spin}(9)$ is an symmetric space of rank 1. This is shown in [15] theorem 8.12.2. The isotropy group Spin(9) acts on the tangent space $T_x\mathbb{O}\mathrm{P}^2$. This is a real 16 dimensional representation.

Remark 2.10. We won't need to determine which representation of Spin(9) this is. But one can show that it is the spinor representation R_9 of Spin(9) (as claimed in [15], proof of theorem 8.12.2).

What we will need are the following facts.

Theorem 2.11. *A minimal closed geodesic γ on $\mathbb{O}\mathrm{P}^2$ has exactly two points $\gamma(L)$ and $\gamma(2L)$ conjugated to $\gamma(0)$. Let $\mathbf{K}_2$ be the isotropy group of $\gamma(0)$, and $\mathbf{K}_1$ be the subgroup of $\mathbf{K}_2$ which fixes the point $\gamma(L)$. There are group isomorphism $\phi_1 : \mathbf{K}_1 \to \mathrm{Spin}(8)$ and $\phi_2 : \mathbf{K}_2 \to \mathrm{Spin}(9)$ so that the composite*

$$\mathrm{Spin}(8) \xrightarrow{\phi_1} \mathbf{K}_1 \subset \mathbf{K}_2 \xrightarrow{\phi_2^{-1}} \mathrm{Spin}(9)$$

is the standard inclusion.

Proof. The calculations of isotropy groups is given in [2] Ch. 1, §5, (3). The model for $\mathbb{O}P^2$ used in this article is Freudenthals model, see also [8]. Points in $\mathbb{O}P^2$ are identified with projections in a Jordan algebra of 3×3 matrices over the octonions. They identify this space with the unique symmetric space of type $F_4/\mathrm{Spin}(9)$ ([2] Ch. 3, §10, at the end of the paragraph).

They pick three points E_1, E_2, E_3 with the property that $E_1+E_2+E_3 = 1$. They compute that the group fixing E_1 is Spin(9). The group fixing E_1 and E_2 actually fixes all three of them since $E_1 + E_2 + E_3 = 1$ is invariant under the action of the isotropy group F_4. In Ch. 1, §5, (3) they compute this group to be Spin(8), and that the inclusion map of this in Spin(9) is the standard inclusion. The fact that this is the standard inclusion is important to us. This was certainly also known to Freudenthal. A more detailed argument for that infinitesimally the inclusion of root systems is the correct one is given in [8] 4.12. But this information determines the inclusion.

It follows from lemma 2.8 that there are at most two points conjugated to $\gamma(0)$. Recall that $\dim(\mathrm{Spin}(n)) = n(n-1)/2$. Since $S(T_x(\mathbb{O}P^2)) \cong \mathrm{Spin}(9)/\mathbf{H}$ we have that $\dim(\mathbb{O}P^2) - 1 = \dim(\mathrm{Spin}(9)) - \dim(\mathbf{H})$ such that $\dim(\mathbf{H}) = 21$. Thus, $\dim(\mathbf{K}_1) = \dim(\mathrm{Spin}(8)) = 28 > \dim(\mathbf{H})$, and by lemma 2.8 we conclude that both $\gamma(L)$ and $\gamma(2L)$ are conjugated to $\gamma(0)$. □

Remark 2.12. We are actually not going to need the following results, so we state them without proof. The group $\mathbf{H}$ is isomorphic to Spin(7). The inclusion $\mathbf{H} \subset \mathbf{K}_1$ is not standard. It is the composite of the usual inclusion with the "triality" automorphism of Spin(8);

$$\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \xrightarrow{\theta} \mathrm{Spin}(8).$$

The inclusion $\mathbf{H} \subset \mathbf{K}_2$ is the composite

$$\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \xrightarrow{\theta} \mathrm{Spin}(8) \subset \mathrm{Spin}(9).$$

The space $\mathbf{K}_1/\mathbf{H}$ is the unit sphere in the standard representation of Spin(7), and the space $\mathbf{K}_2/\mathbf{H}$ is the unit sphere in the tangent representation of Spin(7) at $[e] \in F_4/\mathrm{Spin}(9)$. As we mentioned above, the Spin(9) representation $\mathfrak{f}_4/\mathfrak{spin}(9)$ is the spin representation.

3 Morse theory and geodesics on symmetric spaces

In this section we collect various results. The Morse theory on free loop spaces is a variation on Morse theory of based loop spaces, developed by Bott, Samelson and later Klingenberg. The specialization of this theory to the particularly agreeable case of symmetric spaces is mostly due to Bott, Samelson and Ziller.

3.1 Morse theory of the free loop space

There is a version of Morse theory for the free loop space on a compact Riemannian manifold. In this subsection, we use Klingenberg's book [11] as a standard reference.

In particular, we define the free loop space on a Riemannian manifold M as the space of absolutely continuous maps $\gamma : S^1 \to M$, such the the norm of the derivative $|\gamma'|$ is in $L^2(S^1, \mathbb{R})$. This space is homotopy equivalent to the space of smooth maps (or to the space of continuous maps). For details, see [11] §1.1 and §1.2.

The energy functional E is a C^∞-function on ΛM, and its critical points are the geodesic curves. For a discussion of this in the Hilbert manifold setting, see [11] §1.3. The simplest situation is when the critical points satisfy the Morse non-degeneracy condition. However, we will be looking at cases where the metric on M has a positive dimensional Lie group as symmetries. In this case, the symmetry gives the critical points a strong tendency not to be isolated.

In this situations, sometimes the critical points form critical smooth submanifolds of ΛM. Let N be one of these critical submanifolds. We will assume that we are in the special case where the adjoint $S^1 \times N \to M$ of the inclusion map $i : N \hookrightarrow \Lambda M$ possess a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $[t_{j-1}, t_j] \times N \to M$ is smooth for $j = 0, 1, \dots, k$. That is, each loop is assumed to be piecewise differentiable. By continuity of the derivative, all points in the same component of the critical manifold will have the same energy.

The inclusion of N induces a tangent map $i_* : T_\gamma(N) \to T_\gamma(\Lambda M)$. The tangent vector space $T_\gamma(\Lambda M)$ can be considered as the space of L^2-vector fields along γ. Integration along γ induces an inner product on $T_\gamma(\Lambda M)$, eventually induced from the metric on M.

There is a "normal bundle" ξ defined on each component N. The fiber of ξ at $\gamma \in N$ is the vector space of periodic vector fields along γ.

Assume that the critical manifolds satisfy the Bott non-degeneracy condition. This says that the null space of the Hessian of the energy functional is exactly the tangent directions of N, that is, the image of i_*. The metric on $T_\gamma(\Lambda M)$ induces a splitting (of vector bundles over N) $\xi \cong \xi^- \oplus \xi^+$. The Hessian of the energy function is positive definite on ξ^+, and negative definite on ξ^-. The bundle ξ^+ is infinite dimensional, but ξ^- is a finite dimensional vector bundle.

Let $\Lambda^k M$ be the subspace of loops of energy less than or equal to k. Assume that the critical points of level k form a critical manifold N, and that there are no critical values except k in the interval $(k-\epsilon, k+\epsilon)$ where $\epsilon > 0$. Also assume that all the critical points in $E^{-1}([k-\epsilon, k+\epsilon])$ are situated on a connected, non-degenerate critical manifold N. This manifold will then be isolated.

The main statement of Morse theory on the free loop space is [11] 2.4.10:

Theorem 3.1. *There is a homotopy equivalence*

$$\Lambda^{k+\epsilon} M \simeq \Lambda^{k-\epsilon} M \cup_\sigma D(\xi^-)$$

for some gluing map $\sigma : S(\xi^-) \to \Lambda^{k-\epsilon} M$.

Of course, we can handle the case of several isolated critical manifolds in the same fashion. Assume that k is the only critical value of E in the interval $(k-\epsilon, k+\epsilon)$. Suppose that the critical point with critical value k is a union of non degenerate critical manifolds N_ν, each with a negative bundle ξ_ν^-. Then we have a homotopy equivalence

$$\Lambda^{k+\epsilon} M \simeq \Lambda^{k-\epsilon} M \cup_{\sigma_\nu} D(\xi_\nu^-).$$

We will need more precise information about the gluing map. Let η be a vector bundle over N. We let $D^o(\eta)$ denote the open unit disk bundle and $s : N \to D^o(\eta)$ the 0-section. Assume that $f : D^o(\eta) \to \Lambda M$ is a continuous map, such that $f \circ s = i$ and such that the adjoint $S^1 \times D^o(\eta) \to M$ is smooth in the $D^o(\eta)$ direction and piecewise smooth in the S^1 direction (with respect to some subdivision of the subdivision as we used for the inclusion map i).

The energy functional induces a differentiable map $E \circ f : D^o(\eta) \to \mathbb{R}$ such that each point on N is a critical point.

Lemma 3.2. *Assume that the dimension of η equals the dimension of the negative bundle ξ^-. Assume also that the Hessian of $E \circ f$ is negative definite on each fiber of $D^o(\eta) \to N$. Then, there is a homotopy equivalence*

$$\mathrm{Th}(\eta) \simeq \Lambda^{k+\epsilon} M / \Lambda^{k-\epsilon} M.$$

Proof. By assumption, N is a non-degenerate critical manifold for the function $E \circ f$. For $\gamma \in N$ we consider the restriction $f| : D^o(\eta)_\gamma \to \Lambda M$ to the fiber over γ. The differential of this restriction is an injective map $f|_* : \eta_\gamma \to T_\gamma \Lambda M$. These maps combine to an inclusion of vector bundles $\eta \subset \xi$. The composition with projection onto the negative bundle has trivial kernel, since the Hessian of $E \circ f$ is positive definite on ξ^+, but negative definite on η. Since we are assuming that η and ξ^- have the same dimension, this composite is an isomorphism of vector bundles.

By the proof of 2.4.11. in [11] one has a homotopy equivalence

$$\mathrm{Th}(\xi^-) \simeq \Lambda^{k+\epsilon} M / \Lambda^{k-\epsilon} M$$

and by the isomorphism above we have $\mathrm{Th}(\eta) \cong \mathrm{Th}(\xi^-)$. □

3.2 The free loop space of globally symmetric spaces

From now on, let M be a compact globally symmetric space. Due to the existence of a large isometry group $I(M)$, Morse theory has some very special properties in this situation.

In [16] a space of closed geodesic curves on M is constructed. The construction builds on [1]. We recall the construction, and make some comments on it. Ziller computes the critical submanifolds, and the "negative" bundles over them. He shows that the energy functional satisfies the Bott-Morse non-degeneracy condition on these manifolds.

Let $\gamma \colon [0, a] \to M$ be a geodesic parametrized by arc length which is also a closed curve. It turns out ([16] page 5) that the tangent vectors at the endpoint agree, $\gamma'(0) = \gamma'(a)$, so that γ is actually a periodic geodesic.

The critical submanifolds are determined by geodesic loops $\gamma : [0, 1] \to M$. Let $\mathbf{G} = I_0(M)$ be the identity component of the isometry group and let $\mathbf{H} \subset \mathbf{G}$ be the subgroup that fixes γ pointwise. Then the orbit $\mathbf{G}\gamma$ is diffeomorphic to $\mathbf{G}/\mathbf{H}$ (theorem 2.3).

In [16] the following theorem is proved (Theorem 2):

Theorem 3.3. *Let M be a globally symmetric space. The manifold $\mathbf{G}\gamma$ is a non-degenerate critical manifold in ΛM.*

Obviously every critical point of E is contained in one of these critical manifolds, so we can apply theorem 3.1 to each isolated critical point of the energy functional on ΛM. It follows that ΛM has a filtration, with filtration quotients the Thom spaces of the negative bundles $\mathrm{Th}(\xi^-)$. The filtration is indexed by the value of the energy functional on the components of the critical sets. If there are several components of the same energy level, we chose an order of those, so that we can reindex our filtration by the natural numbers, and the filtration quotients are exactly $\mathrm{Th}(\xi^-)$.

This means that we have the following:

Theorem 3.4. *Let M be a compact globally symmetric space. There is a filtration*

$$\cdots \subseteq F^{\nu} \subseteq F^{\nu'} \subseteq \cdots \subseteq \Lambda M$$

indexed by the $\mathbf{G}$-equivalence classes of geodesic loops. These equivalence classes are ordered in such a way that the energy is weakly increasing. For two consecutive steps $F^{\nu'} \subseteq F^{\nu}$ in the filtration, there is a homotopy equivalence $F^{\nu}/F^{\nu'} \simeq \mathrm{Th}(\xi_{\nu}^-)$ where ξ_{ν}^- is the negative bundle over the critical submanifold indexed by ν.

4 The Bott-Samelson map

This paragraph is a reformulation of [16] §3.1. We intend to analyze closer how F^{ν} is built out of $F^{\nu}/F^{\nu'}$ and $\mathrm{Th}(\xi_{\nu}^-)$. The tool is the construction of a subspace of broken closed geodesics inside F^{ν}. The construction is due to Bott and Samelson [1].

4.1 Bott-Samelson K-cycles

In [1] I 4.4 the so called K-cycles are defined as follows:

Definition 4.1. Let $\mathbf{G}$ be a compact Lie group with a closed subgroup $\mathbf{H}$. Let $\mathbf{K}_\bullet = (\mathbf{K}_1, \ldots, \mathbf{K}_m)$ be an m-tuple of closed subgroups such that $\mathbf{H} \subset \mathbf{K}_i \subset \mathbf{G}$ for $1 \leq i \leq m$. Define a right action of $\mathbf{H}^m = \mathbf{H} \times \cdots \times \mathbf{H}$ on $\mathbf{K}_1 \times \cdots \times \mathbf{K}_m$ as follows:

$$(c_1, \ldots, c_m) * (a_1, \ldots, a_m) = (c_1 a_1, a_1^{-1} c_2 a_2, a_2^{-1} c_3 a_3, \ldots, a_{m-1}^{-1} c_m a_m).$$

The K-cycle is the associated orbit space

$$\mathbf{E}(\mathbf{K}_\bullet; \mathbf{H}) = (\mathbf{K}_1 \times \cdots \times \mathbf{K}_m)/\mathbf{H}^m.$$

Lemma 4.2. *With the above notation one has*

1. *The space $\mathbf{E}(\mathbf{K}_\bullet; \mathbf{H})$ is a smooth manifold.*

2. *The group $\mathbf{K}_1$ acts from the left on this manifold, by the following formula:*

$$k \cdot [c_1, c_2, \ldots, c_m] = [kc_1, c_2, \ldots, c_m].$$

3. There is an inductive formula

$$\mathbf{E}(\mathbf{K}_1, \ldots, \mathbf{K}_m; \mathbf{H}) \cong \mathbf{K}_1 \times_{\mathbf{H}} \mathbf{E}(\mathbf{K}_2, \ldots, \mathbf{K}_m; \mathbf{H})$$

where the right action of $\mathbf{H}$ *on* $\mathbf{K}_1$ *is by right multiplication and the left action of* $\mathbf{H}$ *on* $\mathbf{E}(\mathbf{K}_2, \ldots, \mathbf{K}_m; \mathbf{H})$ *is via 2. In particular, there is a fiber bundle* $\mathbf{E}(\mathbf{K}_1, \ldots, \mathbf{K}_m; \mathbf{H}) \to \mathbf{K}_1/\mathbf{H}$ *with fiber* $\mathbf{E}(\mathbf{K}_2, \ldots, \mathbf{K}_m; \mathbf{H})$.

4. The fiber bundle has a section

$$s : \mathbf{K}_1/\mathbf{H} \to \mathbf{E}(\mathbf{K}_1, \ldots, \mathbf{K}_m; \mathbf{H}) \quad ; \quad k\mathbf{H} \mapsto [k, e, \ldots, e].$$

Proof. An easy computation shows that the action from definition 4.1 is free. The group $\mathbf{H}^m$ is compact. Use theorem 2.1.

One easily checks that the action in 2. is well-defined. In 3. one checks that the identity on $\mathbf{K}_1 \times \cdots \times \mathbf{K}_m$ induces maps in both directions. Finally, $(kh, e, \ldots, e) = (k, e, \ldots, e) * (h, \ldots, h)$ for h in $\mathbf{H}$, which shows that the section in 4. is well-defined. □

Now assume that M is a compact smooth Riemannian manifold. Then the identity component of the isometry group $\mathbf{G} = I_0(M)$ is compact. For a point p in M we write $\mathbf{G}_p = \{g \in \mathbf{G} | gp = p\}$ for the isotropy group. By [6] I.3.5 one has that $\mathbf{G}_p$ is a closed subgroup of $\mathbf{G}$.

Let $\gamma : [0, 1] \to M$ be a geodesic loop. Choose $0 < t_1 < t_2 < \cdots < t_m < 1$ such that $\gamma(t_1), \ldots, \gamma(t_m)$ are the points conjugate to $\gamma(0)$ along γ.

Definition 4.3. Define the following closed subgroups of $\mathbf{G}$:

$$\mathbf{K} = \mathbf{G}_{\gamma(0)}, \quad \mathbf{K}_i = \mathbf{G}_{\gamma(0)} \cap \mathbf{G}_{\gamma(t_i)} \text{ for } 1 \le i \le m, \quad \mathbf{H} = \bigcap_{t \in [0,1]} \mathbf{G}_{\gamma(t)}.$$

Furthermore, let $\Gamma(\mathbf{G}, \gamma)$ denote the smooth manifold

$$\mathbf{E}(\mathbf{G}, \mathbf{K}_\bullet; \mathbf{H}) = \mathbf{E}(\mathbf{G}, \mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m; \mathbf{H}),$$

and $\Gamma(\gamma)$ the fiber $\mathbf{E}(\mathbf{K}_\bullet; \mathbf{H})$.

We will now introduce the Bott-Samelson map. Note that our formula is different from the one in [16] page 14. We have an extra $\mathbf{G}$ factor. Furthermore, Ziller's formula has a minor error. It is not compatible with the action which he defines a few lines before.

Definition 4.4. $\widetilde{BS}_\gamma : \mathbf{G} \times \mathbf{K}_1 \times \cdots \times \mathbf{K}_m \to \Lambda M$ is the map given by

$$\widetilde{BS}_\gamma(g, c_1, \ldots, c_m)(t) = \begin{cases} g\gamma(t) & , 0 \le t \le t_1 \\ gc_1\gamma(t) & , t_1 \le t \le t_2 \\ \quad \vdots & \quad \vdots \\ gc_1 \ldots c_{m-1}\gamma(t) & , t_{m-1} \le t \le t_m \\ gc_1 \ldots c_{m-1}c_m\gamma(t) & , t_m \le t \le 1. \end{cases}$$

Note that the geodesic pieces fit together such that $\widetilde{BS}_\gamma$ takes values in ΛM as stated.

Proposition 4.5. *The map $\widetilde{BS}_\gamma$ is constant on $\mathbf{H}^{m+1}$-orbits so it induces a map*

$$BS_\gamma : \Gamma(\mathbf{G}, \gamma) \to \Lambda M$$

which is called the Bott-Samelson map. This map is $\mathbf{G}$-equivariant where $\mathbf{G}$ acts from the left on $\Gamma(\mathbf{G}, \gamma)$ via Lemma 4.2, 2. and from the left on ΛM by $(g, \eta) \mapsto g\eta$.

Proof. This follows by an easy direct computation. □

By the previous lemma, we have a fiber bundle $\Gamma(\mathbf{G}, \gamma) \to \mathbf{G}/\mathbf{H}$ with fiber $\Gamma(\gamma)$. The fiber is mapped to piece wise geodesic loops with the same initial point $\gamma(0)$ as γ by the Bott-Samelson cycle map. The section $s : \mathbf{G}/\mathbf{H} \to \Gamma(\mathbf{G}, \gamma)$ defines a submanifold of $\Gamma(\mathbf{G}, \gamma)$. The image of this submanifold under the Bott-Samelson map is $\mathbf{G}\gamma$, that is the critical submanifold containing γ.

Any other curve in the image of the map is not a geodesic, but a broken geodesic. In particular, the other critical points for the energy functional is the image of the translates of γ itself. Note that all closed curves in the image of BS_γ will have the same energy.

4.2 The deformation and the normal bundle

We can deform the map BS_γ a little, as done in [16] §3, or [1] §10. We fix a suitably small positive number ϵ. In dependence of this number, we change the curves in the vicinity of a a corner $\gamma(t_i)$. Bott and Samelson replace the curve between $\gamma(t_i - \epsilon)$ and $\gamma(t_i + \epsilon)$ with the unique shortest geodesic between them. This exists, because we did chose ϵ small enough.

We obtain a new map

$$\overline{BS}_\gamma : \Gamma(\mathbf{G}, \gamma) \to \Lambda M.$$

It agrees with BS_γ on the geodesics, that is on the image of the section s.

Bott and Samelson do not discuss the parametrization of the curve we obtain, since they are interested in the length functional L. We use the energy functional E, so the parametrization matters to us. We parametrize all the curves proportional to arc length.

If $x \in \Gamma(\mathbf{G}, \gamma) \setminus s(\mathbf{G}/\mathbf{H})$, then $\overline{BS}_\gamma(x)$ will be a closed curve with strictly lower energy than γ. Moreover, the length functional composed with $\overline{BS}_\gamma$ restricted to $\Gamma(\gamma)$ takes on a non-degenerate maximum at $[e, \dots, e]$.

Because of the relation $E(\gamma) = \frac{1}{2}L^2(\gamma)/\mathrm{Vol}(S^1)$, which is valid for the curves in the image of $\overline{BS}_\gamma$ since they are parametrized proportional to arc length, one also has a non-degenerate maximum for the energy functional.

A formula on page 14 of [16] states that

$$\lambda(c) = \sum \dim \mathbf{K}_i/\mathbf{H}, \tag{1}$$

where $\lambda(c)$ is Ziller's notation for the dimension of the negative bundle. We have used our notation on the right hand side. By this formula we have:

Theorem 4.6. *The dimension of $\Gamma(\gamma)$ agrees with the dimension of the negative bundle.*

We can now prove an addendum to Theorem 3.4.

Theorem 4.7. *Let M be a compact symmetric space. For two consecutive steps $F^{\nu'} \subset F^{\nu}$ in the filtration of ΛM and $\gamma \in \nu$, we have a diagram which commutes up to homotopy*

$$\begin{array}{ccccc} & & \Gamma(\mathbf{G},\gamma) & \xrightarrow{c} & \mathrm{Th}(\xi_\nu^-) \\ & & \downarrow{\scriptstyle \overline{BS}_\gamma} & & \downarrow{\scriptstyle \simeq} \\ F^{\nu'} & \longrightarrow & F^{\nu} & \longrightarrow & F^{\nu}/F^{\nu'} \end{array}$$

The map c is the Thom collapse map that belongs to the embedding of the critical manifold of geodesic loops $\mathbf{G}/\mathbf{H} \hookrightarrow \Gamma(\mathbf{G},\gamma)$.

Proof. It follows from 4.6 and the statement that the energy function has a non-degenerate maximum, that we can apply lemma 3.2 □

To get further, we want to determine the normal bundle of $\mathbf{G}/\mathbf{H}$ in $\Gamma(\mathbf{G},\gamma)$. We now return to the general setting from Definition 4.1. The point $\underline{e} = [e,\dots,e] \in \mathbf{E}(\mathbf{K}_\bullet;\mathbf{H})$ is a fixed point of the left action of $\mathbf{H}$ via Lemma 4.2, 2. In particular, the tangent space of $\mathbf{E}(\mathbf{K}_\bullet;\mathbf{H})$ at this point is an $\mathbf{H}$ representation.

Theorem 4.8. *Let $\mathfrak{h}, \mathfrak{k}_i$ be the Lie algebras of the groups $\mathbf{H}, \mathbf{K}_i$ respectively. Consider these as $\mathbf{H}$-representations with the adjoint action. The $\mathbf{H}$ representation $V := T_{\underline{e}}\mathbf{E}(\mathbf{K}_\bullet;\mathbf{H})$ is equivalent to the $\mathbf{H}$-representation*

$$\mathfrak{k}_1/\mathfrak{h} \oplus \mathfrak{k}_2/\mathfrak{h} \oplus \cdots \oplus \mathfrak{k}_m/\mathfrak{h}.$$

The normal bundle of $s(\mathbf{G}/\mathbf{H}) \subset \mathbf{E}(\mathbf{G},\mathbf{K}_\bullet;\mathbf{H})$ is the bundle $\mathbf{G} \times_{\mathbf{H}} V \to \mathbf{G}/\mathbf{H}$.

Proof. We prove the first statement by induction on m. If $m = 1$, we ask for the tangent space at $[e]$ of $\mathbf{K}/\mathbf{H}$. Let $\tilde{\mathbf{K}}$ be $\mathbf{K}$ considered as a manifold with the conjugation left action of $\mathbf{H}$. The map $\tilde{\mathbf{K}} \to \mathbf{K}/\mathbf{H}$ given by $k \mapsto k\mathbf{H}$ is $\mathbf{H}$ equivariant. The differential of this map at the unit element $e \in \tilde{\mathbf{K}}$ is surjective. Since $e \in \tilde{\mathbf{K}}$ is a fixed point, the source of the differential is a $\mathbf{K}$-representation. By definition, it is exactly $\mathfrak{k}$, with the adjoint representation. The kernel of the differential is $\mathfrak{h}$. This finishes the proof of the induction start $m = 1$.

The induction step is a consequence of the following general argument: Let $\mathbf{K}$ be a compact Lie group with a closed subgroup $\mathbf{H}$. Suppose that $\mathbf{H}$ act smoothly from the left on a manifold M. Assume that $x \in M$ is an $\mathbf{H}$ fixed point. The orbit space $\mathbf{K} \times_{\mathbf{H}} M$ is a left $\mathbf{H}$ space and $[e,x]$ is a fixed point for this action. By 2.2 the tangent $\mathbf{H}$-representation at $[e,x]$ is isomorphic to $\mathfrak{k}/\mathfrak{h} \oplus T_x(M)$.

This finishes the proof of the first statement of the theorem. The second statement follows from theorem 2.4. □

5 Splitting fixed points

5.1 Definition and relation to Thom spaces

We now consider the following general situation. Let $\mathbf{H}$ be a compact Lie group which acts from the left on a based space X (such that the base point is a fixed point). Assume that $x \in X$ is a fixed point different from the basepoint. Also assume that there is an $\mathbf{H}$-invariant open neighborhood U_x of x such that U_x is diffeomorphic to a manifold and the restriction of the action map $H \times U_x \to U_x$ is smooth.

This assumption is satisfied if $X = M_+$ where M is a smooth manifold with a differentiable action of $\mathbf{H}$ and a fixed point $x \in M$ by theorem 2.4.1. This is our main example. But we want to keep the option of changing the topology of M away from the fixed point.

The tangent space T_xX makes sense, and is a representation of the group $\mathbf{H}$. Again by 2.4.1 there is a neighborhood of x which is equivariantly diffeomorphic to T_xX. Collapsing to this neighborhood defines an $\mathbf{H}$-equivariant map from X to the one point compactification of the tangent space:

$$c_x : X \to (T_xX)^+.$$

Definition 5.1. We say that the fixed point x is split up to m-fold suspension if the suspended map $\Sigma^m c_x$ is a surjective retraction in the $\mathbf{H}$-equivariant category. That is, there is a map $s : \Sigma^m(T_xX)^+ \to \Sigma^m X$ such that $(\Sigma^m c_x) \circ s$ is equivariantly homotopic to the identity.

We say that the fixed point x is stably split, if there is a finite dimensional $\mathbf{H}$-representation V and the equivariant map $V^+ \wedge c_x$ is a surjective retraction in the $\mathbf{H}$-equivariant category.

The main example that we will consider is $\mathbf{E}(\mathbf{K}_\bullet; \mathbf{H})$ with the left $\mathbf{H}$ action specified by 2. in lemma 4.2, and where $x = \underline{e} = [e, \ldots, e]$. We are going to give conditions on the groups $\mathbf{K}_i$ and $\mathbf{H}$ that ensures that $\underline{e}$ splits stably or up to suspension.

Now assume that $\mathbf{H}$-acts smoothly from the left on the manifold M, and that $\mathbf{H} \subset \mathbf{G}$ is an inclusion of Lie groups. Then, $\mathbf{G} \times_{\mathbf{H}} M$ is a left $\mathbf{H}$ space with the action $h(g, m) = (hg, m)$. If $x \in M$ is a fixed point for the action of $\mathbf{H}$, it defines an embedding $\mathbf{G}/\mathbf{H} \hookrightarrow \mathbf{G} \times_{\mathbf{H}} M$. Let $\nu(\mathbf{G}/\mathbf{H})$ be the associated normal bundle. By theorem 2.4.2 this normal bundle is the same as the vector bundle $\mathbf{G} \times_{\mathbf{H}} (T_xM) \to \mathbf{G}/\mathbf{H}$.

Lemma 5.2. *Suppose that the fixed point $x \in M$ is m-fold suspension split. Then, the m-fold suspension $\Sigma^m c_x$ of the Thom collapse map*

$$c_x : (\mathbf{G} \times_{\mathbf{H}} M)_+ \to \mathrm{Th}(\nu(\mathbf{G}/\mathbf{H}))$$

is split up to homotopy.

Proof. The Thom collapse map can be identified with the map

$$(\mathbf{G} \times_{\mathbf{H}} M)_+ \to \mathbf{G}_+ \wedge_{\mathbf{H}} (T_xM)^+.$$

Suppose that $s : \Sigma^m(T_xM)^+ \to \Sigma^m M_+$ is a suspension splitting. Then we can split the suspension of the collapse map by

$$\begin{aligned}\Sigma^m(\mathbf{G}_+ \wedge_{\mathbf{H}} (T_xM)^+) &\cong \mathbf{G}_+ \wedge_{\mathbf{H}} \Sigma^m(T_xM)^+ \\ &\xrightarrow{\mathrm{Id}\wedge s} \mathbf{G}_+ \wedge_{\mathbf{H}} (\Sigma^m M_+) \cong \Sigma^m(\mathbf{G}_+ \wedge_{\mathbf{H}} M_+) \cong \Sigma^m(\mathbf{G} \times_{\mathbf{H}} M)_+ \ .\end{aligned}$$

This proves the lemma. □

There is a corresponding statement in the stable case. But to prove it, we need the following neat result about representations of compact Lie groups.

Theorem 5.3. *Let* $\mathbf{G}$ *be a compact Lie group with a closed subgroup* $\mathbf{H}$*. Suppose that* V *is a finite dimensional representation of* $\mathbf{H}$*. There is a finite dimensional representation* W *of* $\mathbf{G}$*, such that* V *is a subrepresentation of the restriction of* W *to* $\mathbf{H}$*.*

For a proof of this, see for instance [7], corollary 4.7.2. The main ingredient of this proof is the Peter-Weyl theorem. Since representations of compact Lie groups are semisimple, we can actually find a finite dimensional $\mathbf{H}$-representation V' such that $V \oplus V'$ is the restriction of a $\mathbf{G}$-representation.

Here is a stable splitting result. Assume that $\mathbf{H}$ acts smoothly on the manifold M.

Lemma 5.4. *Assume that the fixed point* $x \in M$ *is stably split. Then there exist a natural number* m *such that the suspension of the collapse map*

$$\Sigma^m c : \Sigma^m(\mathbf{G} \times_{\mathbf{H}} M)_+ \to \Sigma^m \mathrm{Th}(\nu(\mathbf{G}/\mathbf{H}))$$

is split up to homotopy.

Proof. Let $s : V^+ \wedge (T_xM)^+ \to V^+ \wedge M_+$ be a splitting. Pick a finite dimensional $\mathbf{H}$-representation W (using theorem 5.3), such that $W \oplus V$ is the restriction of a $\mathbf{G}$-representation. Then

$$W^+ \wedge V^+ \wedge s : W^+ \wedge V^+ \wedge (T_xM)^+ \to W^+ \wedge V^+ \wedge M_+$$

is also a splitting which gives a map

$$\mathbf{G}_+ \wedge_{\mathbf{H}} (W \oplus V)^+ \wedge (T_xM)^+ \to \mathbf{G}_+ \wedge_{\mathbf{H}} (W \oplus V)^+ \wedge M_+. \tag{2}$$

On the other hand, if X is any pointed space with an action of $\mathbf{H}$, we have a homeomorphism

$$\begin{gathered}\mathbf{G}_+ \wedge_{\mathbf{H}} ((W \oplus V)^+ \wedge X) \to (W \oplus V)^+ \wedge (\mathbf{G}_+ \wedge_{\mathbf{H}} X) \\ (g, v, x) \mapsto (gv, g, x).\end{gathered} \tag{3}$$

Non-equivariantly we have $S^m \cong (W \oplus V)^+$ for some m. Applying the homeomorphism (2) to the source and to the target of the map (1) gives us a splitting

$$\Sigma^m \mathrm{Th}(\nu(\mathbf{G}/\mathbf{H})) \cong \Sigma^m \mathbf{G}_+ \wedge_{\mathbf{H}} T_x(M)^+ \longrightarrow \Sigma^m \mathbf{G}_+ \wedge_{\mathbf{H}} M_+ \cong \Sigma^m(\mathbf{G} \times_{\mathbf{H}} M)_+.$$

This proves the lemma. □

5.2 Existence of splittings

Here is an elementary example of a fixed point which is 1-fold split.

Example 5.5. Let V be a finite dimensional real inner product space, and let $\mathbf{H}$ be a closed subgroup of the group of orthogonal transformations of V. We have an action $\mathbf{H} \times S(V) \to S(V)$ where $S(V)$ denotes the unit sphere of V. Assume that $x \in S(V)$ is a fixed point. Then, x splits up to suspension by the following argument:

The collapse map takes the form $c_x : S(V)_+ \to (T_x S(V))^+$ and we have a homotopy equivalence $(T_x S(V))^+ \simeq S(V)$. So up to homotopy, the collapse map is part of a cofibration sequence

$$S^0 \xrightarrow{x} S(V)_+ \xrightarrow{\tilde{c}_x} S(V).$$

Here the first map sends the non-basepoint of S^0 to the point $x \in S(V)$ and the last $S(V)$ has x as basepoint. We claim that the map $\Sigma(\tilde{c}_x)$ is an equivariant retraction.

To see this, we note that one can identify it with the quotient map

$$q : S(V \oplus \epsilon)/\{\pm(0,1)\} \to S(V \oplus \epsilon)/\{(ax, b) \mid a^2 + b^2 = 1, a \geq 0\},$$

where ϵ denotes the trivial one dimensional $\mathbf{H}$-representation. Consider the following quotient map:

$$p : S(V \oplus \epsilon) \to S(V \oplus \epsilon)/\{\pm(0,1)\}.$$

The composite $q \circ p$ of the two maps is a quotient map given by dividing out a contractible subspace (actually, an interval). Since the inclusion of the subspace is an equivariant cofibration, this composite is an equivariant homotopy equivalence. Composing the homotopy inverse of this homotopy equivalence with the map p, we get a homotopy left inverse of q and thus also of $\Sigma\tilde{c}_x$ as required.

Here is a more general, stable result.

Lemma 5.6. *Let $\mathbf{G}$ be a compact Lie group with a closed subgroup $\mathbf{H}$ and let W be a finite dimensional representation of $\mathbf{H}$.*

1. *Assume that $[e] \in (\mathbf{G}/\mathbf{H})_+$ is stably split. If the class $[W]$ is contained in the image of the restriction map $\mathrm{RO}(\mathbf{G}) \to \mathrm{RO}(\mathbf{H})$ of real representation rings, then the $\mathbf{H}$ fixed point $[(e,0)] \in \mathbf{G}_+ \wedge_{\mathbf{H}} W^+$ is stably split.*

2. *Assume that $[e] \in (\mathbf{G}/\mathbf{H})_+$ is m-fold suspension split. If the $\mathbf{H}$-representation $T_{[e]}(\mathbf{G}/\mathbf{H}) \oplus W \oplus \epsilon^m$ extends to a $\mathbf{G}$-representation, then $[(e,0)] \in \mathbf{G}_+ \wedge_{\mathbf{H}} W^+$ is m-fold suspension split.*

Proof. 1. The condition on W says that there are $\mathbf{G}$-representations U, U' so that after restricting the action to the subgroup $\mathbf{H}$ we have an isomorphism of $\mathbf{H}$-representations $W \oplus U \cong U'$.

The point $[e] \in (\mathbf{G}/\mathbf{H})_+$ is stably split so there exist an $\mathbf{H}$-representation V and a splitting s of the map $V^+ \wedge c_{[e]} : V^+ \wedge (\mathbf{G}/\mathbf{H})_+ \to V^+ \wedge (T_{[e]}(\mathbf{G}/\mathbf{H}))^+$. For any manifold M and vector space E one has an isomorphism for $x \in M$ as follows:

$$T_{[x,0]}(M_+ \wedge E^+) \cong T_x(M) \oplus E. \tag{4}$$

By this isomorphism and the map $s \wedge W^+ \wedge U^+$ one gets an $\mathbf{H}$-equivariant splitting of the map

$$V^+ \wedge (\mathbf{G}/\mathbf{H})_+ \wedge W^+ \wedge U^+ \xrightarrow{V^+ \wedge c_{[e,0,0]}} V^+ \wedge (T_{[e,0,0]}((\mathbf{G}/\mathbf{H})_+ \wedge W^+ \wedge U^+))^+ \ .$$

This splitting will become useful in a while.

Consider the composite of the following two $\mathbf{G}$-homeomorphisms:

$$\theta : (\mathbf{G}_+ \wedge_{\mathbf{H}} W^+) \wedge U^+ \xrightarrow[\cong]{\phi} \mathbf{G}_+ \wedge_{\mathbf{H}} (W^+ \wedge U^+) \xrightarrow[\cong]{\psi} (\mathbf{G}/\mathbf{H})_+ \wedge W^+ \wedge U^+ \ .$$

Here ϕ is given by $\phi([[g, w], u]) = [g, [w, g^{-1}u]]$ and the map ψ exists since $W \oplus U$ is actually the restriction of a $\mathbf{G}$-representation. Note that we may consider θ a local diffeomorphism near the point $p = [e, 0, 0]$.

We have a homotopy commutative diagram, where horizontal maps are homeomorphisms:

$$\begin{array}{ccc} V^+ \wedge (\mathbf{G}_+ \wedge_{\mathbf{H}} W^+) \wedge U^+ & \xrightarrow{V^+ \wedge \theta} & V^+ \wedge (\mathbf{G}/\mathbf{H})_+ \wedge W^+ \wedge U^+ \\ \downarrow{\scriptstyle V^+ \wedge c_p} & & \downarrow{\scriptstyle V^+ \wedge c_{\theta(p)}} \\ V^+ \wedge T_p((G_+ \wedge_{\mathbf{H}} W^+) \wedge U^+)^+ & \xrightarrow{V^+ \wedge d_p\theta} & V^+ \wedge T_{\theta(p)}((\mathbf{G}/\mathbf{H})_+ \wedge W^+ \wedge U^+)^+ \ . \end{array}$$

The right vertical map has a splitting $s \wedge W^+ \wedge U^+$ as remarked above. So the left vertical map also has a splitting. Using (4) on the lower left corner of the diagram, the result follows.

2. The m-fold suspension case is proved in the same way, using ϵ^m in place of U. □

We can now prove the main result on existence of splittings.

Theorem 5.7. *Let $\mathbf{G}$ be a compact Lie group, and $\mathbf{H} \subset \mathbf{G}$ a closed subgroup. Let $\mathbf{H}$ act on a manifold M with a fixed point x.*

1. *If the $\mathbf{H}$ fixed points $[e] \in \mathbf{G}/\mathbf{H}$ and $x \in M$ are both stably split then the $\mathbf{H}$ fixed point $[e, x] \in \mathbf{G} \times_{\mathbf{H}} M$ is also stably split.*
2. *If $x \in M$ and $[e] \in \mathbf{G}/\mathbf{H}$ are both m-fold suspension split, and if the $\mathbf{H}$-representation $T_x(M) \oplus \epsilon^m$ extends to a $\mathbf{G}$-representation, then $[e, x] \in \mathbf{G} \times_{\mathbf{H}} M$ is also m-fold suspension split.*

Proof. 1. Let $W = T_x M$ be the tangent representation of $\mathbf{H}$. The collapse map $M_+ \to W^+$ is a diffeomorphism in a neighborhood of x, so we have a diagram

$$\begin{array}{ccc} \mathbf{G}_+ \wedge_{\mathbf{H}} M_+ & \xrightarrow{\mathrm{Id} \wedge_{\mathbf{H}} c_x} & \mathbf{G}_+ \wedge_{\mathbf{H}} W^+ \\ \downarrow{\scriptstyle c_{[e,x]}} & & \downarrow{\scriptstyle c_{[e,0]}} \\ (T_{[e,x]}(\mathbf{G} \times_{\mathbf{H}} M))^+ & \xrightarrow{\cong} & (T_{[e,0]}(\mathbf{G}_+ \wedge_{\mathbf{H}} W^+))^+ \end{array}$$

The right vertical map is stably split by lemma 5.6. The upper horizontal map is stably split, since c_x is by assumption. It follows that the left vertical map is stably split.

2. If c_x is m-fold suspension split, we see that the right and upper maps are split in the diagram.

$$\begin{array}{ccc} S^m \wedge \mathbf{G}_+ \wedge_{\mathbf{H}} M_+ & \longrightarrow & S^m \wedge \mathbf{G}^+ \wedge_{\mathbf{H}} W \\ \downarrow & & \downarrow \\ S^m \wedge (T_{[e,x]}(\mathbf{G} \times_{\mathbf{H}} M))^+ & \xrightarrow{\cong} & S^m \wedge (T_{[e,0]}(\mathbf{G}_+ \wedge_{\mathbf{H}} W^+))^+ \end{array}$$

The result follows. □

5.3 Relations to stable equivariant framings

A slightly different approach to the existence of stable splittings is to construct them from stable framings. In the non-equivariant case, it is known that the existence of a stable splitting of the collapse map $M_+ \to (T_xM)^+$ is equivalent to the existence of a stable fiber homotopy trivialization of the tangent sphere bundle. We use an equivariant version of this idea.

Assume that the compact Lie group $\mathbf{H}$ acts smoothly on the compact manifold M. If V is a representation of $\mathbf{H}$ and X is an $\mathbf{H}$-space, let $\epsilon^V(X)$ denote the trivial $\mathbf{H}$-vector bundle $pr_X : X \times V \to X$.

Definition 5.8. We say that M is stably framed if there are $\mathbf{H}$- representations V and W, such that there is an equivalence of $\mathbf{H}$-vector bundles

$$\tau(M) \oplus \epsilon^V(M) \cong \epsilon^W(M).$$

Lemma 5.9. *If M is stably framed, and $x \in M$ is any fixed point, then x is stably split.*

Proof. There is an equivariant embedding $M \hookrightarrow U$ into some $\mathbf{H}$- representation as in [5] theorem VI.4.2. Since M is stably framed, after possibly replacing the embedding by the composite $M \hookrightarrow U \subset U \oplus W$ we can assume that the normal bundle $\nu(M)$ of M in U is a trivial bundle $\nu(M) \cong \epsilon^V(M)$. The Thom collapse map $U^+ \to \mathrm{Th}(\nu(M))$ gives us a map

$$V^+ \wedge (T_xM)^+ \cong (V \oplus T_xM)^+ \cong U^+ \longrightarrow \mathrm{Th}(\nu(M)) \cong \mathrm{Th}(\epsilon^V(M)) \cong V^+ \wedge M_+.$$

This map defines the required splitting for the fixed point x. □

Suppose that $\mathbf{H}$ is a closed subgroup of another compact Lie group $\mathbf{K}$.

Lemma 5.10. *Assume that $\mathbf{K}/\mathbf{H}$ is stably framed as an $\mathbf{H}$-space. Also suppose that M is stably framed as an $\mathbf{H}$-space, such that $\tau(M) \oplus \epsilon^V(M) \cong \epsilon^W(M)$. Assume finally that the element $[W] - [V] \in \mathrm{RO}(\mathbf{H})$ is contained in the image of the natural restriction map $\mathrm{RO}(\mathbf{K}) \to \mathrm{RO}(\mathbf{H})$. Then $\mathbf{K} \times_{\mathbf{H}} M$ is stably framed as an $\mathbf{H}$-space, where the action of $\mathbf{H}$ is given as the restriction of the $\mathbf{K}$-action.*

Proof. For an $\mathbf{H}$-space X we consider the $\mathbf{K}$-space $\mathcal{I}(X) = \mathbf{K} \times_{\mathbf{H}} X$. Similarly, for an $\mathbf{H}$-vector bundle ξ over X, we consider the $\mathbf{K}$-vector bundle $\mathcal{I}(\xi) = \mathbf{K} \times_{\mathbf{H}} \xi$ over $\mathcal{I}(X)$. Let $\pi : M \to *$ be the map to a point. Then $\epsilon^V(M) = \pi^*(\epsilon^V(*))$. The tangent bundle of M is an $\mathbf{H}$-vector bundle, so we can consider the $\mathbf{K}$-vector bundle $\mathcal{I}(\tau(M)) = \mathbf{K} \times_{\mathbf{H}} \tau(M) \to \mathbf{K} \times_{\mathbf{H}} M$. The representations V and W determine $\mathbf{H}$-equivariant vector bundles $\mathcal{I}(\epsilon^V(*)) = \mathbf{K} \times_{\mathbf{H}} V \to \mathbf{K}/\mathbf{H}$ and $\mathcal{I}(\epsilon^W(*)) = \mathbf{K} \times_{\mathbf{H}} W \to \mathbf{K}/\mathbf{H}$.

Our assumption on $\tau(M)$ says that as $\mathbf{H}$-vector bundles,

$$(\mathbf{K} \times_{\mathbf{H}} \tau(M)) \oplus (\mathbf{K} \times_{\mathbf{H}} \epsilon^V(M)) \cong (\mathbf{K} \times_{\mathbf{H}} \epsilon^W(M)).$$

But

$$\mathbf{K} \times_{\mathbf{H}} \epsilon^V(M) = \mathcal{I}(\epsilon^V(M)) = \mathcal{I}(\pi^*(\epsilon^V(*))) = \mathcal{I}(\pi)^*(\mathcal{I}(\epsilon^V(*)))$$

and similarly for W, so we obtain

$$\mathcal{I}(\tau(M)) \oplus \mathcal{I}(\pi)^*(\mathcal{I}(\epsilon^V(*))) \cong \mathcal{I}(\pi)^*(\mathcal{I}(\epsilon^W(*))). \tag{5}$$

We use theorem 5.3 to replace V with a representation which is the restriction of a $\mathbf{K}$-representation. The condition on $[W] - [V] \in \mathrm{RO}(\mathbf{H})$ still holds, and it implies that there are $\mathbf{K}$-representations U_1 and U_2, such that $V \oplus U_1 \cong W \oplus U_2$ as $\mathbf{H}$-representations. Replacing V and W by $V \oplus U_1$ and $W \oplus U_2$, we can assume that both V and W are restrictions of $\mathbf{K}$-representations. But then, $\mathcal{I}(\epsilon^V(*))$ and $\mathcal{I}(\epsilon^W(*))$ are trivial as $\mathbf{H}$-vector bundles, and even as a $\mathbf{K}$-vector bundles. Explicitly, a trivialization for $\mathcal{I}(\epsilon^V(*))$ is given by $\mathbf{K} \times_{\mathbf{H}} V \to \mathbf{K}/\mathbf{H} \times V$ where $[g, v] \mapsto ([g], gv)$.

It now follows from equation (5) that $\mathcal{I}(\tau(M))$ is stably framed.

Finally, we use theorem 2.2 and write

$$\tau(\mathbf{K} \times_{\mathbf{H}} M) \cong \mathcal{I}(\tau(M)) \oplus \mathcal{I}(\pi)^*(\tau(\mathbf{K}/\mathbf{H})).$$

A pullback of a framed bundle is framed. So we see that $\mathcal{I}(\pi)^*(\tau(\mathbf{K}/\mathbf{H}))$ is stably framed. Since both summands are stably framed $\mathbf{H}$-vector bundles, we have equivariantly stably framed the tangent bundle of $\mathbf{K} \times_{\mathbf{H}} M$. □

Remark 5.11. The condition on the representations V, W in the lemma is needed as the following example shows: Let $\mathbf{H} = C_2 = \langle T \rangle$ be the cyclic group of order two inside $\mathbf{K} = S^1$ and let $\mathbf{H}$ act on $\mathbb{R}^2$ by $T \cdot (x, y) = (-x, y)$. Let $M \subset \mathbb{R}^2$ be the unit sphere. Then M is stably $\mathbf{H}$-framed, but $\mathbf{H}$ acts non-orientably on M, so that $\mathbf{K} \times_{\mathbf{H}} M$ is not orientable. This means that it cannot be even non-equivariantly stably framed.

Remark 5.12. Homogeneous spaces $\mathbf{K}/\mathbf{H}$ are not always stably framed. For instance, most projective spaces like

$$\mathbb{R}\mathrm{P}^n = \frac{O(n+1)}{O(n) \times O(1)}, \quad \mathbb{C}\mathrm{P}^n = \frac{U(n+1)}{U(n) \times U(1)}, \quad \mathbb{H}\mathrm{P}^n = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}$$

have non trivial Stiefel Whitney classes, cf. [13], especially corollary 11.15. So they cannot even be non-equivariantly stably framed.

5.4 Application to the Bott-Samelson map

Theorem 5.13. *Let* $\mathbf{H} \subset \mathbf{K}_i \subset \mathbf{G}$ *be compact Lie groups,* $1 \leq i \leq m$*. The normal bundle* $\nu(\mathbf{G}/\mathbf{H})$ *of* $s(\mathbf{G}/\mathbf{H})$ *in* $\mathbf{E}(\mathbf{G}, \mathbf{K}_\bullet; \mathbf{H})$ *is the same as the vector bundle* $\mathbf{G} \times_{\mathbf{H}} V \to \mathbf{G}/\mathbf{H}$*, where* $V = \oplus_{1 \leq i \leq m} \mathfrak{k}_i/\mathfrak{h}$*. We have an associated Thom collapse map* $c : \mathbf{E}(\mathbf{G}, \mathbf{K}_\bullet; \mathbf{H})_+ \to \mathrm{Th}(\nu(\mathbf{G}/\mathbf{H}))$*. Assume that* $\mathbf{K}_i/\mathbf{H}$ *is stably framed as an* $\mathbf{H}$*-manifold for each* i*. Define* $\mathbf{H}$*-representations* V_i *for* $2 \leq i \leq m$ *by* $V_i = \oplus_{i \leq t \leq m}(\mathfrak{k}_t/\mathfrak{h})$*.*

1. *Assume that for each* i*,* $1 \leq i \leq m-1$*, the stable class* $[V_{i+1}] \in \mathrm{RO}(\mathbf{H})$ *is in the image of the restriction map* $\mathrm{RO}(\mathbf{K}_i) \to \mathrm{RO}(\mathbf{H})$*. Then, there is some integer* n *such that the* n*-fold suspended collapse map* $\Sigma^n c$ *splits.*

2. *Assume the further condition that there is a number* k *such that* $[e] \in \mathbf{K}_i/\mathbf{H}$ *is* k *fold suspension split and such that the* $\mathbf{H}$*-representation* $V_{i+1} \oplus \epsilon^k$ *extends to a representation of* $\mathbf{K}_i$ *for* $1 \leq i \leq m-1$*. Then, the* k*-fold suspension of the collapse map* $\Sigma^k c$ *splits.*

Proof. The statement about the normal bundle of $s(\mathbf{G}/\mathbf{H})$ is part of the conclusion of theorem 4.8.

1. Consider $\mathbf{E}[i] = \mathbf{E}(\mathbf{K}_i, \mathbf{K}_{i+1}, \ldots, \mathbf{K}_m; \mathbf{H})$. We claim that $\mathbf{E}[i]$ is stably framed as an $\mathbf{H}$-space for $1 \leq i \leq m$. The argument is by downwards induction. For $i = m$ we have that $\mathbf{E}[m] = \mathbf{K}_m/\mathbf{H}$, which is stably framed by assumption. Assume inductively that $\mathbf{E}[i+1]$ is stably framed. This means that there are representations W and U of $\mathbf{H}$, such that we have an isomorphism of $\mathbf{H}$-vector bundles

$$\tau(\mathbf{E}[i+1]) \oplus \epsilon^W(\mathbf{E}[i+1]) \cong \epsilon^U(\mathbf{E}[i+1])$$

Since $\underline{e} \in \mathbf{E}[i+1]$ is a fixed point, we can restrict this bundle equivalence to $\underline{e}$, and get an isomorphism of $\mathbf{H}$-representations $T_{\underline{e}}\mathbf{E}[i+1] \oplus W \cong U$. But the fixed point representation at $\underline{e}$ is exactly V_{i+1}, according to theorem 4.8, so we obtain the relation

$$[U] - [W] = [V_{i+1}] \in \mathrm{RO}(\mathbf{H}).$$

By our other assumption, $[U] - [W] \in Im(\mathrm{RO}(\mathbf{K}_i) \to \mathrm{RO}(\mathbf{H}))$. Since $\mathbf{K}_i/\mathbf{H}$ is assumed to be stably framed as an $\mathbf{H}$-manifold, We can apply lemma 5.10 to this situation, and get that $\mathbf{E}[i] = \mathbf{K}_i \times_{\mathbf{H}} \mathbf{E}[i+1]$ is indeed stably framed as an $\mathbf{H}$-manifold.

We now know that $\mathbf{E}[1]$ is stably framed, so $\underline{e} \in \mathbf{E}[1]$ is stably split by lemma 5.9. The result follows from lemma 5.4.

2. The unstable case is treated in essentially the same way. We use the second part of theorem 5.7 to show inductively that $\underline{e} \in \mathbf{E}[i]$ is m-fold suspension split. Then the claim follows from lemma 5.2. □

Before we apply this result to the filtration of the free loop space, we show a technical lemma, which we need in the 1-fold suspension split case.

Lemma 5.14. *Let* B *be a connected CW complex with a connected sub complex* A *and let* $i : A \to B$ *denote the inclusion map. Let* q *be the quotient map given by* $A_+ \xrightarrow{i_+} B_+ \xrightarrow{q} B/A$*. Suppose that* $\Sigma q : \Sigma(B_+) \to \Sigma(B/A)$ *is a surjective retraction* *up to homotopy with a right inverse up to homotopy* $\Sigma(B/A) \to \Sigma(B_+)$. *Then* $\Sigma(B_+)$ *is homotopy equivalent to* $\Sigma(B/A) \vee \Sigma(A_+)$.

Proof. Chose a 0-cell $a_0 \in A \subset B$ as base point. Let $f_A : A_+ \to A$ be the based map which is the identity outside the disjoint basepoint $+$. Let $\bar{q} : B \to B/A$ denote the quotient map. There is a commutative diagram

$$\begin{array}{ccccc} \Sigma(A_+) & \xrightarrow{\Sigma i_+} & \Sigma(B_+) & \xrightarrow{\Sigma q} & \Sigma(B/A) \\ \downarrow{\scriptstyle \Sigma f_A} & & \downarrow{\scriptstyle \Sigma f_B} & & \| \\ \Sigma A & \xrightarrow{\Sigma i} & \Sigma B & \xrightarrow{\Sigma \bar{q}} & \Sigma(B/A) \end{array}$$

The bottom line is a cofibration sequence. It splits by the composite map $(\Sigma f_B) \circ s : \Sigma B/A \to \Sigma B$. In particular there is an isomorphism

$$H_*(\Sigma i \vee (\Sigma f_B) \circ s) \colon H_*(\Sigma A \vee \Sigma B/A) \to H_*(\Sigma B).$$

Since $\Sigma i \vee (\Sigma f_B) \circ s$ is a homology equivalence between simply connected spaces, it is a homotopy equivalence. So $\Sigma B \simeq \Sigma A \vee \Sigma B/A$, and obviously $\Sigma B \vee S^1 \simeq \Sigma A \vee S^1 \vee \Sigma B/A$.

We finish the proof of the lemma by noting that for every connected, based CW complex (X, x_0) there are two homotopy equivalences

$$\Sigma(X) \vee S^1 \xleftarrow{\simeq} S(X) \cup_{S^0} D^1 \xrightarrow{\simeq} \Sigma(X_+)$$

which are the quotient maps for the two contractible subspaces $I \times x_0$ and D^1 respectively. (Here $S(X)$ denotes the unreduced suspension of X). □

Now we return to closed geodesics on a symmetric space. For each closed geodesic γ , we consider the isotropy group of the geodesic $\mathbf{H}(\gamma)$. For each point $\gamma(t_i)$ $1 \leq i \leq m$ conjugated to $\gamma(0)$ along γ we consider the group $\mathbf{K}_i(\gamma)$ of isometries fixing both $\gamma(0)$ and $\gamma(t_i)$.

Theorem 5.15. *Let M be a connected, compact globally symmetric space. Assume that all non-trivial critical submanifolds have positive dimensional negative bundles, that is, the manifold of constant paths is the set of all critical points of index 0. Assume also that $\mathbf{K}_i(\gamma)/\mathbf{H}(\gamma)$ is stably framed as an $\mathbf{H}(\gamma)$-manifold for each closed geodesic γ.*

1. *If for each closed geodesic the groups $\mathbf{K}_i(\gamma)$ and $\mathbf{H}(\gamma)$ satisfy the condition in 1. of theorem 5.13, then we have a splitting of suspension spectra up to homotopy*
$$\Sigma^\infty(\Lambda M_+) \simeq \Sigma^\infty M_+ \vee \bigvee_\nu \Sigma^\infty \mathrm{Th}(\xi_\nu^-).$$
2. *If there is a $k \geq 1$ such that for each closed geodesic the groups $\mathbf{K}_i(\gamma)$ and $\mathbf{H}(\gamma)$ satisfy both the condition in 1. and the condition in 2. of theorem 5.13 for this $k \geq 1$, we have a splitting up to homotopy*
$$\Sigma^k(\Lambda M_+) \simeq \Sigma^k M_+ \vee \bigvee_\nu \Sigma^k \mathrm{Th}(\xi_\nu^-).$$

Proof. We prove 2. The proof of 1. is similar but easier. Consider the filtration of theorem 3.4. According to theorem 4.7, if ν' and ν are two consecutive steps in the filtration, we have a diagram

$$\begin{array}{ccccc} & & \Sigma^k(\Gamma(\mathbf{G},\gamma)_+) & \xrightarrow{\Sigma^k c} & \Sigma^k \mathrm{Th}(\xi_\nu^-) \\ & & \downarrow{\scriptstyle \Sigma^k \overline{BS}} & & \downarrow{\scriptstyle f}\,\cong \\ \Sigma^k F^{\nu'} & \longrightarrow & \Sigma^k F^{\nu} & \xrightarrow{\Sigma^k q} & \Sigma^k F^{\nu}/F^{\nu'} . \end{array}$$

By the conclusion of theorem 5.13, $\underline{e} \in \Gamma(\mathbf{G},\gamma)$ is k-fold suspension split. So we have a splitting of $\Sigma^k c$, say $s : \Sigma^k \mathrm{Th}(\xi_\nu^-) \to \Sigma^k(\Gamma(\mathbf{G},\gamma)_+)$. But then the map $\Sigma^k q$ is a split surjection up to homotopy, with right inverse $\Sigma^k \overline{BS} \circ s \circ f^{-1}$.

Each space $\mathrm{Th}(\xi_\nu^-)$ is connected, M is connected, so by induction, using the assumption that there are no critical manifolds of index 0, we see that each space F^ν is connected. Using lemma 5.14 we see inductively that we have a splitting up to suspension

$$\Sigma^k(\Lambda^{e+\epsilon}M)_+ \simeq \Sigma^k M_+ \vee \bigvee_{E(\nu)<e+\epsilon} \Sigma^k \mathrm{Th}(\xi_\nu^-).$$

Passing to the direct limit preserves homotopy equivalences. □

6 Rank one symmetric spaces

We will now assume that M *is isotropic*, so that the isometry group of M acts transitively on the total space of the unit sphere bundle of the tangent bundle.

The isotropy condition is quite constraining. It is equivalent to requiring that the symmetric space has rank one. According to [15] 8.12.2 this class includes spheres, complex and quaternionic projective spaces and the Cayley projective plane, but no other compact, simply connected spaces.

Because of the isotropy condition, the simple geodesics form one critical submanifold N_1, all with the same energy e. The other critical submanifolds will all have energy n^2e for some natural number n, and consist of geodesics N_n obtained by running through a simple geodesic exactly n times.

Let $\mathbf{G}$ be the isometry group of M. Since we are assuming isotropy, it acts transitively on set of pairs (p, v) where $p \in M$, and $v \in T_pM$. It follows that $\mathbf{G}$ acts transitively on the simple geodesics. Such a geodesic has no self intersections (since the tangent vector field along γ is the restriction of a Killing vector field on M). So the space of simple geodesics can be identified as a homogeneous space $\mathbf{G}/\mathbf{H}$ where $\mathbf{H}$ is the subgroup of $\mathbf{G}$ that fixes γ pointwise.

In particular, all geodesics are closed, and all simple closed geodesics have the same length. Let us normalize the metric on M, so that they have length one. We see that we can also think of the space of simple geodesics as the unit sphere bundle of the tangent bundle of M.

We take a close look at the three non-sphere types of compact, simply connected, globally symmetric spaces. We will consider all representations as representations over the real numbers. The dimension of a representation will thus mean its dimension over $\mathbb{R}$.

6.1 The complex projective spaces

If $M = \mathbb{C}\mathrm{P}^n$, we can explicitly compute the groups mentioned above. The unitary group $U(n+1)$ acts on $\mathbb{C}\mathrm{P}^n$ by isometries. A complication is that this action is not effective, the kernel consists of the matrices in $U(n+1)$ of the form zI for $z \in U(1)$. These matrices form the center of the unitary group $\mathbf{D} = \mathbf{D}_{n+1} = Z(U(n+1)) \cong U(1)$. Every isometry of $\mathbb{C}\mathrm{P}^n$ is induced by a unitary matrix, so that the group of isometries is the projective unitary group $\mathbf{G} = PU(n+1) = U(n+1)/\mathbf{D}$.

Let $e_1, \dots, e_{n+1}$ denote the standard basis for $\mathbb{C}^{n+1}$. Consider the point $p = [e_{n+1}] \in \mathbb{C}\mathrm{P}^n$. The isotropy group under the $U(n+1)$ action of this point is $U(n+1)_p = U(n) \times U(1)$. We obtain the isotropy group at the point under the action of the isometry group by factoring out $\mathbf{D}$. This isotropy group is $\mathbf{K}_2 = \mathbf{G}_p = (U(n) \times U(1))/\mathbf{D} \cong U(n)$, where the name of the group is chosen to be compatible with the notation of section 5.

Consider a second point $q = [e_n] \in \mathbb{C}\mathrm{P}^n$. The subgroup of $U(n+1)$ that preserves both p and q is $U(n+1)_p \cap U(n+1)_q = U(n-1) \times U(1) \times U(1)$. The corresponding group of isometries is $\mathbf{K}_1 = U(n-1) \times U(1) \times U(1)/\mathbf{D} \cong U(n-1) \times U(1)$.

Let γ be a geodesic passing through $[e_{n+1}]$ and $[e_n]$. An isometry preserving this geodesic is the same as an isometry preserving both the point $[e_{n+1}]$ and the unit tangent vector of the geodesic at that point. So, if we look at the corresponding unitary transformation $A \in U(n+1)$, we get that there is a complex number λ, so that $Ae_n = \lambda e_n$, and $Ae_{n+1} = \lambda e_{n+1}$. Thus the subgroup of $\mathbf{G}$ that fixes γ pointwise is $\mathbf{H} = U(n-1) \times \mathbf{D}_2/\mathbf{D}_{n+1} \cong U(n-1)$.

After doing the identifications $\mathbf{H} \cong U(n-1)$, $\mathbf{K}_1 \cong U(n-1) \times U(1)$ and $\mathbf{K}_2 \cong U(n)$ the inclusion maps of $\mathbf{H}$ in $\mathbf{K}_1$ respectively $\mathbf{K}_2$ are given by the maps

$$A_{n-1} \mapsto (A_{n-1}, 1) \quad , \quad A_{n-1} \mapsto \begin{pmatrix} A_{n-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

On a simple geodesic γ, the only points conjugated to $\gamma(0)$ is $\gamma(1/2)$ and $\gamma(1)$. We choose our coordinate system such that $\gamma(0) = [e_{n+1}]$ and $\gamma(1/2) = [e_n]$ ([9] example 2.110). If γ_m is a geodesic which runs through a simple geodesic m times, the conjugate points of $\gamma_m(0)$ are $\gamma_m(i/2m)$, where $1 \le i \le 2m$. But $\gamma_m(k/2m) = [e_{n+1}]$ if k is even, and $\gamma_m(k/2m) = [e_n]$ if k is odd. So the groups corresponding to γ_m are given as follows:

$$\mathbf{K}(\gamma_m)_i = \begin{cases} \mathbf{K}_1 & \text{if } i \text{ is odd} \\ \mathbf{K}_2 & \text{if } i \text{ is even} \end{cases} \quad , \quad 1 \le i \le 2m-1.$$

The associated K-cycle is

$$\Gamma(\mathbf{G}, \gamma_m) = \mathbf{E}(\mathbf{G}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_1, \dots, \mathbf{K}_2, \mathbf{K}_1; \mathbf{H}).$$

This manifold contains a copy of $\mathbf{G}/\mathbf{H}$ by the section which maps $g\mathbf{H}$ to $[g, e, \dots, e]$. This section is the inclusion of a submanifold of $\Gamma(\mathbf{G}, \gamma_m)$ with normal bundle ν_m. There is an associated Pontryagin-Thom collapse map $c : \Gamma(\mathbf{G}, \gamma_m)_+ \to \mathrm{Th}(\nu_m)$.

We claim that $[e]$ is a suspension split fixed point for the action of $\mathbf{H}$ on both $\mathbf{K}_1/\mathbf{H}$ and $\mathbf{K}_2/\mathbf{H}$. Notice that $W_1 = T_{[e]}(\mathbf{K}_1/\mathbf{H})$ is a trivial 1-dimensional

$\mathbf{H}$-representation. The $\mathbf{H}$-representation $W_2 = T_{[e]}(\mathbf{K}_2/\mathbf{H})$ is the adjoint action of $U(n-1)$ on the quotient $\mathfrak{u}(n)/\mathfrak{u}(n-1)$ (compare with the proof of 4.8). Since $U(n)/U(n-1) \cong S^{2n-1}$ the representation W_2 is the standard $2n-2$-dimensional $U(n-1)$-representation with a trivial 1-dimensional representation added. In both case we have that $\mathbf{K}_i/\mathbf{H}$ is homeomorphic to $(\mathfrak{k}_i/\mathfrak{h})^+$ as an $\mathbf{H}$-space.

The $\mathbf{H}$-representations W_1 and $W_2 \oplus \epsilon$ both extend to representations of $\mathbf{K}_2$. So, in order to check the conditions of theorem 5.13 for arbitrary $\mathbf{K}_i$ it is enough to see that the $\mathbf{H}$-representations

$$(W_1 \oplus W_2)^{\oplus j} \oplus W_1 \quad , \quad W_2 \oplus (W_1 \oplus W_2)^{\oplus j} \oplus W_1 \quad , j \geq 0$$

extends to $\mathbf{K}_2$ representations. But $W_1 \oplus W_2$ extends to a representation of $\mathbf{K}_2 \cong U(n)$, namely the standard $2n$-dimensional representation of $U(n)$. Since W_1 is just the trivial representation, it can also be extended. We have now checked most of the conditions for theorem 5.15.

Theorem 6.1. *Let $p : S(\tau) \to \mathbb{C}\mathrm{P}^n$ be the unit sphere bundle of the tangent bundle τ on $\mathbb{C}\mathrm{P}^n$. Let ξ_m be the vector bundle $(p^*(\tau))^{\oplus(m-1)} \oplus \epsilon$ on $S(\tau)$, where ϵ is a one dimensional trivial bundle. Then, there is a homotopy equivalence of spaces*

$$\Sigma((\Lambda\mathbb{C}\mathrm{P}^n)_+) \simeq \Sigma(\mathbb{C}\mathrm{P}^n_+) \vee \bigvee_{m\geq 1} \Sigma\mathrm{Th}(\xi_m).$$

Proof. Before we apply theorem 5.15, we have to check that all local minima for the energy function are constant paths. We know all critical points for the energy function by lemma 2.8. The index of one of these points is given by equation (1). By the calculations above, $\dim(\mathbf{K}_1) - \dim(\mathbf{H}) = 1$, so it follows that the index of a critical point which is not a constant path is greater or equal to 1.

We need to identify the negative bundles ξ_ν^- with the bundles ξ_m. Both bundles are bundles over the same space, namely $S(\tau) \cong \mathbf{G}/\mathbf{H}$.

We have identified the negative bundle corresponding to γ_m with the bundle over $\mathbf{G}/\mathbf{H}$ induced by $V = (W_1 \oplus W_2)^{\oplus(m-1)} \oplus W_1$. So it suffices to see that the bundle η on $\mathbf{G}/\mathbf{H}$ induced by $W_1 \oplus W_2$ agrees with $p^*(\tau)$.

Under the identifications given by the transitive action of $\mathbf{G}$ on $S(\tau)$ and $\mathbb{C}\mathrm{P}^n$, the projection p correspond to the standard identification map $\mathbf{G}/\mathbf{H} \to \mathbf{G}/\mathbf{K}_2$, or equivalently to the map

$$U(n+1)/(U(n-1) \times \mathbf{D}) \to U(n+1)/(U(n) \times \mathbf{D}).$$

The bundle η is isomorphic to the pullback under this map of the bundle $\bar{\eta}$ on $\mathbb{C}\mathrm{P}^n \cong U(n+1)/(U(n) \times \mathbf{D})$ induced by the standard representation of $U(n)$. But the bundle induced by the standard representation is isomorphic to the tangent bundle over $\mathbb{C}\mathrm{P}^n$, with fibers given by

$$T_{[w]}\mathbb{C}\mathrm{P}^n = \{v \in \mathbb{C}^{n+1} \mid \langle v, w \rangle = 0\}.$$

An isomorphism is given by the bundle map

$$U(n+1) \times_{(U(n)\times U(1))} \mathbb{C}^n \to \tau(\mathbb{C}\mathrm{P}^n) \quad ; \quad [g, v] \mapsto ([ge_{n+1}], g(v, 0)).$$

This identifies the bundle η as the pullback $p^*(\bar{\eta}) \cong p^*(\tau)$, which finishes the proof of the theorem. □

6.2 The quaternionic projective spaces

The quaternionic projective space is the homogeneous space

$$\mathbb{HP}^n \cong \mathrm{Sp}(n+1)/(\mathrm{Sp}(n) \times \mathrm{Sp}(1)).$$

In this case, the quotient group $\mathrm{Sp}(n+1)/\{\pm I\}$ is the actual group of isometries. The group acts isotropically, and as in the complex case we get that the isotropy group of the geodesic γ starting at $[e_{n+1}]$ in the direction e_n is

$$\mathbf{H} = \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)/\{\pm I\}.$$

The isotropy group of the point $[e_{n+1}]$ is

$$\mathbf{K}_2 = \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\{\pm I\},$$

and the isotropy group of the pair of conjugated points $[e_n], [e_{n+1}]$ is

$$\mathbf{K}_1 = \mathrm{Sp}(n-1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\{\pm I\}.$$

The inclusion maps $\mathbf{H} \subset \mathbf{K}_1$ and $\mathbf{H} \subset \mathbf{K}_2$ are given as follows:

$$(A_1, A_2) \mapsto (A_1, A_2, A_2) \quad , \quad (A_1, A_2) \mapsto (\begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}, A_2)$$

Any closed geodesic will be obtained from a simple closed geodesic by running through it m times. As in the complex case, the isotropy groups associated to the conjugated points will alternate between $\mathbf{K}_1$ and $\mathbf{K}_2$.

The quotient spaces

$$\mathbf{K}_1/\mathbf{H} \cong \mathrm{Sp}(n-1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)/(\mathrm{Sp}(n-1) \times \mathrm{Sp}(1)) \cong \mathrm{Sp}(1),$$
$$\mathbf{K}_2/\mathbf{H} \cong \mathrm{Sp}(n) \times \mathrm{Sp}(1)/(\mathrm{Sp}(n-1) \times \mathrm{Sp}(1)) \cong \mathrm{Sp}(n)/\mathrm{Sp}(n-1) \cong S^{4n-1}$$

are equivariantly unit spheres in $\mathbf{H}$-representations. So by example 5.5 the points $[e] \in \mathbf{K}_1/\mathbf{H}$ and $[e] \in \mathbf{K}_2/\mathbf{H}$ are suspension split. We look closer at these representations.

The $\mathbf{H}$-representation $W_1 = T_{[e]}(\mathbf{K}_1/\mathbf{H}) \cong \mathfrak{sp}(1)$ is induced from the adjoint representation of $\mathrm{Sp}(1)$ under the projection map

$$\mathbf{H} = \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)/\{\pm I\} \xrightarrow{p_2} \mathrm{Sp}(1)/\{\pm I\}.$$

That is, it is the representation by conjugation of the unit quaternions on the 3-dimensional vector space of purely imaginary quaternions $\tilde{\mathbb{H}}$.

The $\mathbf{H}$ representation $W_2 = T_{[e]}(\mathbf{K}_2/\mathbf{H}) \cong \mathfrak{sp}(n)/\mathfrak{sp}(n-1)$ is equivalent to the representation of $\mathbf{H} \cong \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)/\{\pm I\}$ on $\mathbb{H}^{n-1} \oplus \tilde{\mathbb{H}}$, given by

$$(A_{n-1}, B_1)(v_{n-1}, w_1) = (A_{n-1} v_{n-1} B_1^{-1}, B_1 w_1 B_1^{-1}).$$

This representation splits as a sum of two representation, one of them being given by the adjoint representation of the $\mathrm{Sp}(1)$ factor in $\mathbf{H}$.

The representation W_1 extends to a $\mathbf{K}_1 \cong \mathrm{Sp}(n-1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\{\pm I\}$ and to a $\mathbf{K}_2 \cong \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\{\pm I\}$ representation, by projecting the groups onto the last factor $\mathrm{Sp}(1)$, and composing with the adjoint representation.

The representation W_2 does not itself extend, but if you add a trivial 1-dimensional representation to it, it does. Put

$$\mathbb{H}^n = \mathbb{H}^{n-1} \oplus \mathbb{H} = \mathbb{H}^{n-1} \oplus \tilde{\mathbb{H}} \oplus \epsilon \cong W_2 \oplus \epsilon.$$

Theorem 6.2. *Let $p : S(\tau) \to \mathbb{H}\mathrm{P}^n$ be the unit sphere bundle of the tangent bundle τ on $\mathbb{H}\mathrm{P}^n$. Let η be the 3-dimensional bundle on $\mathbb{H}\mathrm{P}^n \cong S^{4n+3}/\mathrm{Sp}(1)$ induced by the adjoint representation of $\mathrm{Sp}(1)$. Let ξ_m be the vector bundle $p^*(\tau^{\oplus(m-1)} \oplus \eta^{\oplus m})$ on $S(\tau)$. Then there is a homotopy equivalence of spectra*

$$\Sigma^\infty(\Lambda\mathbb{H}\mathrm{P}^n_+) \cong \Sigma^\infty(\mathbb{H}\mathrm{P}^n) \vee \bigvee_{m\geq 1} \Sigma^{\infty-m+1}\mathrm{Th}(\xi_m).$$

Proof. We easily check the conditions of theorem 5.15 (compare to theorem 6.1) so we only have to see that there is a bundle equivalence between ξ_m and the pullback of the bundle induced by $(W_2 \oplus \epsilon)^{\oplus(m-1)} \oplus W_1^{\oplus m}$.

Both these bundles are induced from bundles on $\mathbb{H}\mathrm{P}^n$, so it is sufficient to check the isomorphism on the appropriate bundles there.

But the bundle W_1 is by definition η, so we only need to show that the bundle induced by $W_2 \oplus \epsilon$ on $\mathbb{H}\mathrm{P}^n$ is the tangent bundle. We already identified $W_2 \oplus \epsilon$ with the standard representation $\mathbb{H}^n$. And now we can give an isomorphism as a bundle map

$$\mathrm{Sp}(n+1) \times_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)} \mathbb{H}^n \to \tau(\mathbb{H}\mathrm{P}^n) \quad ; \quad [g, v] \mapsto ([ge_{n+1}], g(v, 0))$$

□

6.3 The Cayley projective plane

Now we consider the Cayley projective plane $\mathbb{O}\mathrm{P}^2$. Recall from 2.2 that $\mathbb{O}\mathrm{P}^2$ is a symmetric space of rank 1. The isometry group F_4 acts isotropically. The isotropy group of a point x is $\mathbf{K} = \mathrm{Spin}(9)$, the isotropy group of a geodesic γ is denoted by $\mathbf{H}$. We saw in theorem 2.11 that the isotropy groups of pairs of conjugate points $(\gamma(0), \gamma(t))$ on γ alternate between $\mathbf{K}_1 = \mathrm{Spin}(8)$ and $\mathbf{K}_2 \cong \mathbf{K} = \mathrm{Spin}(9)$. Let W_1 and W_2 denote the $\mathbf{H}$-representations $T_{[e]}(\mathbf{K}_1/\mathbf{H})$ and $T_{[e]}(\mathbf{K}_2/\mathbf{H})$ respectively. Note that $[e] \in \mathbf{K}_i/\mathbf{H}$ is not a fixed point under the $\mathbf{K}_i$-action, so a priory we don't even know that W_i extends to a $\mathbf{K}_i$-representation.

Lemma 6.3. *The action of $\mathbf{H}$ on $W_2 \oplus \epsilon$ can be extended to an action of $\mathbf{K}_2$, and thus to an action of the subgroup $\mathbf{K}_1$. The class $[W_1] \in \mathrm{RO}(\mathbf{H})$ is in the image of the restriction map $\mathrm{RO}(\mathbf{K}) \to \mathrm{RO}(\mathbf{H})$, and thus in the image of $\mathrm{RO}(\mathbf{K}_1) \to \mathrm{RO}(\mathbf{H})$.*

Proof. Even if the action of $\mathbf{H}$ on W_2 does not itself extend to an action of $\mathbf{K}_2$, it does in the stable sense. To see this, note that W_2 is the fixed point representation of the tangent vector in $T_x(\mathbb{O}\mathrm{P}^2)$ corresponding to the geodesic γ. This means that the sum $W_2 \oplus \epsilon$ is the tangent space $\mathbf{H}$-representation $T_x(\mathbb{O}\mathrm{P}^2)$, which by definition is the restriction of a representation of $\mathbf{K}_2$.

The chain of inclusions of Lie groups $\mathbf{H} \subset \mathbf{K}_1 \subset \mathbf{K}_2$ defines a short exact sequence of $\mathbf{H}$-representations.

$$0 \longrightarrow T_{[e]}(\mathbf{K}_1/\mathbf{H}) \longrightarrow T_{[e]}(\mathbf{K}_2/\mathbf{H}) \longrightarrow T_{[e]}(\mathbf{K}_2/\mathbf{K}_1) \longrightarrow 0$$

It follows that $W_2 \cong W_1 \oplus T_{[e]}(\mathbf{K}_2/\mathbf{K}_1)$ as $\mathbf{H}$-representations. So we have to prove that

$$[T_{[e]}(\mathbf{K}_2/\mathbf{K}_1)] \in \mathrm{image}(\mathrm{RO}(\mathbf{K}_2) \to \mathrm{RO}(\mathbf{H}))$$

By definition, $[T_{[e]}(\mathbf{K}_2/\mathbf{K}_1)]$ is the restriction of a $\mathbf{K}_1$-representation, so we have to show that

$$[T_{[e]}(\mathbf{K}_2/\mathbf{K}_1)] \in \mathrm{image}(\mathrm{RO}(\mathbf{K}_2) \to \mathrm{RO}(\mathbf{K}_1))$$

However, according to 2.11, we can identify $\mathbf{K}_1$ with Spin(8) and $\mathbf{K}$ with Spin(9) in such a way that the inclusion $\mathbf{K}_1 \subset \mathbf{K}$ corresponds to the standard inclusion $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$. Under this isomorphism, the representation $T_{[e]}(\mathbf{K}_2/\mathbf{K}_1)$ corresponds to $T_{[e]}(\mathrm{Spin}(9)/\mathrm{Spin}(8))$ which is just the standard 8-dimensional representation ρ_8 of Spin(8). The lemma follows from the fact that the restriction of the standard Spin(9)-representation ρ_9 to Spin(8) is $\rho_8 \oplus \epsilon$. □

Remark 6.4. In the proof of lemma 6.3 we intentionally avoid the discussion of precisely which $\mathbf{H} = \mathrm{Spin}(7)$ representations we are dealing with. This would involve a treatment of triality, which we do not want to bring up here.

Theorem 6.5. *Let $p : S(\tau) \to \mathbb{O}\mathrm{P}^2$ be the unit sphere bundle of the tangent bundle τ on $\mathbb{O}\mathrm{P}^2$. Let η be the 7-dimensional vector bundle induced from W_1 on $\mathbb{O}\mathrm{P}^2 \cong F_4/\mathrm{Spin}(9)$. Let ξ_m be the vector bundle $p^*(\tau^{\oplus(m-1)} \oplus \eta^{\oplus m})$ on $S(\tau(\mathbb{O}\mathrm{P}^2))$. Then there is a homotopy equivalence of spectra*

$$\Sigma^\infty(\Lambda\mathbb{O}\mathrm{P}^2{}_+) \simeq \Sigma^\infty(\mathbb{O}\mathrm{P}^2{}_+) \vee \bigvee_{m\geq 1} \Sigma^{\infty-m+1}\mathrm{Th}(\xi_m).$$

Proof. This is analogous the proof of theorem 6.2. We have to check that the bundle induced by $W_2 \oplus \epsilon$ agrees with the tangent bundle. It is sufficient to check that the $\mathbf{K}_2$ representations on the tangent space $T_{\gamma(0)}(\mathbb{O}\mathrm{P}^2)$ agrees with the $\mathbf{K}_2$-representation $W_2 \oplus \epsilon$. But actually, it is well known that the tangent representation of Spin(9) acting as the isometry group fixing a point on $\mathbb{O}\mathrm{P}^2$ is exactly the sum of a trivial representation and the spinor representation of Spin(9). □

7 Comparison with earlier results

We want to compare the results of this paper with the results of [3], which motivated it. We first recall some notation used in [3].

Let X be either $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ or $\mathbb{O}\mathrm{P}^2$. Let τ be the tangent bundle over X and let $q\tau = \tau \oplus \cdots \oplus \tau$ denote the q-fold Whitney sum. We define $C_q(X)$ to be the cofiber of the cofibration

$$\mathrm{Th}(q\tau) \to \mathrm{Th}((q+1)\tau).$$

The cohomology of X is a truncated polynomial ring. The generator has degree $r(X)$, where $r(X) = 2, 4, 8$ respectively. In [3], we proved the following theorem:

Theorem 7.1. *There is an isomorphism of modules over the mod two Steenrod algebra*

$$H^*(\Lambda X; \mathbb{Z}/2) \cong H^*\Big(X_+ \vee \bigvee_{q\geq 0} \Sigma^{(r(X)-2)(q+1)} C_q(X); \mathbb{Z}/2\Big).$$

To reinterpret this in terms of the bundles considered in this paper, we need the following rewriting: Let X be a space and ξ_1, ξ_2 vector bundles over X. Let $D(\xi_i) \to X$, $S(\xi_i) \to X$, $\mathrm{Th}(\xi_i)$ be the corresponding disk bundle, sphere bundle, and Thom space. Let ϵ denote a trivial line bundle.

Theorem 7.2. *Let $C(\xi_1, \xi_2)$ be the cofiber of the map $\mathrm{Th}(\xi_1) \to \mathrm{Th}(\xi_1 \oplus \xi_2)$ given by inclusion of ξ_1 and the zero section of ξ_2. Let $p : S(\xi_2) \to X$ be the projection map for the sphere bundle of ξ_2. Then there is a homotopy equivalence*

$$C(\xi_1, \xi_2) \simeq \mathrm{Th}(p^*(\xi_1 \oplus \epsilon)).$$

Proof. There is a diagram of cofibrations

$$\begin{array}{ccccc}
S(\xi_1)_+ & \longrightarrow & D(\xi_1)_+ & \longrightarrow & \mathrm{Th}(\xi_1) \\
\downarrow & & \downarrow & & \downarrow \\
S(\xi_1 \oplus \xi_2)_+ & \longrightarrow & D(\xi_1 \oplus \xi_2)_+ & \longrightarrow & \mathrm{Th}(\xi_1 \oplus \xi_2) \\
\downarrow & & \downarrow & & \downarrow \\
S(\xi_1 \oplus \xi_2)/S(\xi_1) & \longrightarrow & D(\xi_1 \oplus \xi_2)/D(\xi_1) & \longrightarrow & C(\xi_1, \xi_2)
\end{array}$$

The inclusion $D(\xi_1) \to D(\xi_1 \oplus \xi_2)$ is a homotopy equivalence, so its cofiber is contractible. Thus by extending the above diagram to the right, we get a homotopy equivalence $C(\xi_1, \xi_2) \simeq \Sigma S(\xi_1 \oplus \xi_2)/S(\xi_1)$. Next, consider the inclusion

$$S(\xi_1) \longhookrightarrow S(\xi_1 \oplus \xi_2) \simeq S(\xi_1) \times_X D(\xi_2) \cup D(\xi_1) \times_X S(\xi_2).$$

Since $S(\xi_1) \hookrightarrow S(\xi_1) \times_X D(\xi_2)$ is a homotopy equivalence we have that the cofiber of the map $f : S(\xi_1) \times_X S(\xi_2) \to D(\xi_1) \times_X S(\xi_2)$ is homotopy equivalent to $S(\xi_1 \oplus \xi_2)/S(\xi_1)$.

But $S(p^*(\xi_1)) \simeq S(\xi_1) \times_X S(\xi_2)$ and $D(p^*(\xi_2)) \simeq D(\xi_1) \times_X S(\xi_2)$ such that $\mathrm{Th}(p^*(\xi_1))$ is the cofiber of f. The result follows. □

We use this result to rewrite theorem 7.1.

Theorem 7.3. *The cohomology $H^*(\Lambda X; \mathbb{Z}/2)$ is as module over the mod two Steenrod algebra isomorphic to the spectrum cohomology*

$$H^*\Big(X_+ \vee \bigvee_{m\geq 1} \Sigma^{-m+1} \mathrm{Th}(p^*((m-1)\tau \oplus m\epsilon^{r(X)-1}); \mathbb{Z}/2\Big).$$

Proof. This follows from applying theorem 7.2 to $\xi_1 = q\tau$ and $\xi_2 = \tau$, rewriting the Thom space a little, and finally substituting $m = q + 1$. □

We now check that this agrees with the results obtained in section 6.

Remark 7.4. The case $X = \mathbb{C}\mathrm{P}^n$, $r(X) = 2$, follows directly from theorem 6.1 after identifying the bundle ξ_m.

Remark 7.5. In the case $X = \mathbb{H}\mathrm{P}^n$, $r(X) = 4$ theorem 6.2 says that if η is the bundle induced from the adjoint representation of S^3 on $\mathbb{H}\mathrm{P}^n$, then $H^*(\Lambda\mathbb{H}\mathrm{P}^n;\mathbb{Z}/2)$ is isomorphic as a module over the Steenrod algebra to

$$H^*(X;\mathbb{Z}/2) \oplus \bigoplus_{m\geq 1} \tilde{H}^*(\Sigma^{-m+1}\mathrm{Th}(p^*((m-1)\tau \oplus m\eta);\mathbb{Z}/2).$$

So it is sufficient to see that there is an isomorphism of modules over the Steenrod algebra

$$H^*(\mathrm{Th}(p^*((m-1)\tau) \oplus mp^*(\eta));\mathbb{Z}/2) \cong H^*(\mathrm{Th}(p^*((m-1)\tau) \oplus m\epsilon^3);\mathbb{Z}/2).$$

According to [13], is is sufficient to see that the Stiefel Whitney classes of $p^*(\eta)$ vanish. By naturality, it is sufficient to see that the Stiefel Whitney classes

$$w_j(\eta) \in H^*(\mathbb{H}\mathrm{P}^n;\mathbb{Z}/2)$$

vanish. But if $j \leq 3$, the group $H^j(\mathbb{H}\mathrm{P}^n)$ vanishes, so $w_j(\eta) = 0$. If $j > 3$, the class $w_j(\eta)$ vanish, since the bundle η is 3 dimensional.

Remark 7.6. As in remark 7.5 we see from theorem 6.5 that in order to check the part of theorem 7.3 involving the Cayley projective plane, it is sufficient to see, that the Stiefel Whitney classes of the bundle η over $S(\tau)$ are trivial. The bundle is 7-dimensional so $w_j(\eta) = 0$ for $j \geq 8$. On the other hand, $w_j \in H^j(S(\tau);\mathbb{Z}/2) = 0$ for $1 \leq j \leq 7$, so all Stiefel Whitney classes are indeed trivial.

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