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On the quadratic functional equation on groups

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Abstract

We study the solutions $f : G \to H$ of the quadratic functional equation on G, where G and H are groups, H abelian. We show that any solution f is a function on the quotient group $[G, [G, G]]$. By help of this we find sufficient conditions on G for all solutions to satisfy Kannappan's condition. We use this to derive explicit formulas for the solutions on various groups like, e.g., the $(ax + b)$-group and $GL(n, \mathbb{R})$.

Keywords: Quadratic functional equation, Cauchy difference, Kannappan condition, semidirect product.
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1 Introduction

A norm $\|\cdot\|$ on a vector space V stems from an inner product $(\cdot, \cdot)$ on V, i.e. $\|x\|^2 = (x, x)$ for all $x \in V$, if and only if it satisfies the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V. \tag{1.1}$$

This is classical knowledge, going back to the theorem of Apollonius in Euclidean geometry: The sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides. A special case of Apollonius' theorem is Pythagoras' theorem that asserts the same for rectangles. Jordan and von Neumann [12] proved the result for vector spaces.

Generalizing the parallelogram identity from a vector space V to a group G we are led to *the quadratic functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \quad x, y \in G, \tag{1.2}$$

where $f : G \to H$ is to be determined. In the present paper we allow G to be any group and the range space H of f to be any abelian group. The functional equation (1.2) also turns up in discussions of other functional equations (See for example [2, Lemma 2] and [20, Corollary III.8]).

A generalization of (1.1), similar to (1.2), is

$$f(xy) + f(y^{-1}x) = 2f(x) + 2f(y), \quad x, y \in G. \tag{1.3}$$

However the two functional equations (1.2) and (1.3) have the same solutions because any solution f of one of them satisfies that $f(xy) = f(yx)$ for all $x, y \in G$. This is because the right hand side is symmetric in x and y (See Section 9 for a general result). So (1.3) is not really a new generalization.

In [13] Kannappan listed a number of functional equations that are equivalent to the quadratic functional equation if G is abelian or just if f satisfies Kannapan's condition $f(xyz) = f(xzy)$, $x, y, z \in G$. Kannappan's condition on f is equivalent to f being a function on the abelian group $G/[G, G]$.

To describe the existing results about the quadratic functional equation on groups in a short way we introduce the following terminology:

Definition 1.1. Let G be a group and H be an abelian group. We will say that a map $f : G \to H$ is *a quadratic function* if there exists a symmetric bimorphism $Q : G \times G \to H$ such that $f(x) = Q(x, x)$ for all $x \in G$.

We choose the word function to avoid confusion with [15], [25] and [13] in which a *quadratic form* or a *quadratic functional* by definition is any solution of the quadratic functional equation.

There may be other solutions than the quadratic functions even for $H = \mathbb{C}$. See the remarks prior to Corollary 6.7 for an example.

The basic result for abelian groups is Theorem 1.2 below. It is due to Aczél [1]. In [13, Result 1] it is mentioned that the assumption of G being abelian can be replaced by Kannappan's condition.

Theorem 1.2 (Aczél 1965). *Let G be an abelian group and let H be an abelian group in which every equation of the form $2x = h \in H$ has one and only one solution $x \in H$. Then any solution $f : G \to H$ of the quadratic functional equation on G is a quadratic function.*

Motivated by the original situation (1.1) Kannappan studied in [13] the equation (1.2) when G is the additive group of a linear space, to find out when solutions arise from bilinear functionals. So there is not only the group structure to take into account, but also the multiplication by scalars. It might here be mentioned that Kurepa [15, Theorem 5], also inspired by the original equation (1.1), studied when the square $f(x) = d(x, e)^2$ of a right-invariant pseudometric $d(\cdot, \cdot)$ on a group G (instead of the square of a norm on a vector space) solves (1.2). He found in particular that such an f is a quadratic function. For literature about the vector space situation we refer to [3], [13] and the recent monograph [7] and their references, because we shall here discuss the pure group case, where the group may even be non-abelian.

The quadratic functional equation was generalized by Chung, Ebanks, Ng and Sahoo [5]. Their paper [5] derives formulas for the complex-valued solutions of the generalization, assuming Kannappan's condition. Our paper does not impose Kannappan's condition in the general set-up, and we do not restrict ourselves to complex-valued solutions, but on the other hand we study exclusively the quadratic functional equation (1.2) and not any generalizations of it.

For any function $f : G \to H$, where G is a group and H is an abelian group, we introduce the Cauchy-difference Cf of f by

$$Cf(x, y) = f(xy) - f(x) - f(y), \quad x, y \in G. \tag{1.4}$$

The key to the studies on a non-abelian group G is the following observation: If $f : G \to H$ is a solution of the quadratic equation (1.2), then its Cauchy-difference Cf satisfies Jensen's functional equation in each of its variables when the other variable is fixed. That is why Hosszú [11] and Kurepa [14], [15] found relations between solutions of the quadratic functional equation and their Cauchy-differences. In [15, Theorem 1] Kurepa shows that any solution of the quadratic functional equation is a quadratic function if G is generated by 2 elements and H has no elements of order 2.

Corovei considered in [6, Lemma 4] the quadratic functional equation when G was a P_3-group and H a quadratically closed field of characteristic different from 2 and 3. He found that the quadratic functions are the only solutions.

Using Ng's works [16], [17], [18] on Jensen's functional equation Di-Lian Yang [25] derived a number of basic formulas for the solutions of the quadratic functional equation and applied them to solve the equation on free groups and on the general linear group $GL_n(\mathbb{Z})$ over the integers. Like Ng, but in contrast to the other works mentioned above, she imposed no assumptions on the range group H but that it should be an abelian group. Her formulas can be used to derive results by Hosszú [11] and Kurepa [14], [15].

The purpose of the present paper is to continue the investigations of the quadratic functional equation on groups that are not necessarily abelian. We let the range group H be any abelian group, even though many of our statements would simplify, if we assumed, e.g., that H had no elements of order 2.

To formulate our results we introduce the following notation and terminology: If $x, y \in G$ we let $[x, y] = xyx^{-1}y^{-1}$. And if A and B are subsets of G we let $[A, B]$ denote the subgroup of G generated by the elements $[a, b]$, where $a \in A$ and $b \in B$. Of particular interest are $[G, G]$ and $[G, [G, G]]$ which are normal subgroups of G.

Let G_0 be a subgroup of G. A function $f : G \to H$ is said to be a function on G/G_0 if $f(xx_0) = f(x)$ for all $x \in G$ and $x_0 \in G_0$. In that case we will not distinguish between $f : G \to H$ and the function $F : G/G_0 \to H$ defined on the coset space G/G_0 by $F(xG_0) = f(x)$, $x \in G$.

Building upon certain of the formulas derived in [25] we get the following results:

(a) We refine Aczél's basic result for abelian groups (Theorem 1.2) by allowing any abelian group as range group (Proposition 5.1 and Corollary 5.2).

(b) We generalize Corovei's result [6, Lemma 4] on P_3-groups, i.e. groups in which each commutator has order ≤ 2: It suffices that each commutator has finite order (see the remark after Proposition 5.3 for the precise statement).

(c) Kurepa's result for groups with 2 generators is extended: It suffices that the quotient group $G/(Z(G)[G,G])$ is generated by 2 elements, and Kurepa's assumption about H having no elements of order 2 is removed (Corollary 6.7).

(d) We prove that any solution of the quadratic functional equation on a group G is a function on the quotient group $G/[G,[G,G]]$ (Theorem 3.2). This fact is mentioned without proof in [25, Remark 4.4].

(e) We show that a solution of the quadratic equation on a product of groups satisfies Kannappan's condition if and only if its restriction to each of the subgroups satisfies Kannappan's condition (Corollary 6.3).

(f) We give sufficient conditions on the group G to ensure that all solutions of the quadratic equation on it are quadratic functions. One such condition is that $[G,[G,G]] = [G,G]$, which is satisfied for certain non-abelian groups like $GL(n,\mathbb{R})$, $n \geq 2$. (See Theorem 5.2 for details).

(g) We solve the quadratic functional equation on selected groups (mainly semidirect products), that are of interest in other connections. Among the examples are the $(ax+b)$-group, the Heisenberg group and $GL(n,\mathbb{R})$, $n \geq 2$, where we find simple explicit formulas for the continuous solutions of the quadratic functional equation (Section 8).

Throughout the present paper (except for Section 9) we let G denote a group with neutral element e, and we let H denote an abelian group.

2 Formulas and preliminary results

This section contains results that will be needed later. We mention in particular the indispensable Theorem 2.6.

Definition 2.1. We say that a map $\Phi : G \times G \times \cdots \times G \to H$ (n factors) is

1. a *multimorphism*, if it is a homomorphism in any of its variables when the remaining $n-1$ variables are fixed. If $n = 2$ a multimorphism is called a *bimorphism.*

2. *alternating*, if $\Phi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \operatorname{sgn}(\sigma)\Phi(x_1, \dots, x_n)$ whenever σ is a permutation of n elements and $x_1, \dots, x_n \in G$.

We recall ([25, formula (3)]) that the map $B_f : G \times G \times G \to H$ corresponding to any $f : G \to H$ is for $x, y, z \in G$ defined by

$$\begin{aligned} B_f(x,y,z) &= f(xyz) - f(xy) - f(xz) - f(yz) + f(x) + f(y) + f(z) \\ &= Cf(xy,z) - Cf(x,z) - Cf(y,z). \end{aligned} \tag{2.1}$$

The second line shows that B_f is the 2^{nd} Cauchy-difference of f.

We will often use the following easy result from [25] without explicit mentioning. It allows us in many contexts to assume that $f(e) = 0$, which is the standard normalization of f in the formulation of the results in [25].

Lemma 2.2. *Let $f : G \to H$ be a solution of the quadratic functional equation (1.2). Then $2f(e) = 0$, and the decomposition $f = (f - f(e)) + f(e)$ splits f into two solutions, such that the first one vanishes at the neutral element e, while the second one is constant. Furthermore $2B_{f-f(e)} = 2B_f$.*

The following relations between solutions of (1.2) and Cauchy-differences are easy to derive from [25].

Lemma 2.3. *If $f : G \to H$ is a solution of the quadratic functional equation, then*

(a) $[f - f(e)](x^n) = n^2[f - f(e)](x)$ for all $n \in \mathbb{Z}$ and $x \in G$.

(b) $2f(x) = C(f - f(e))(x, x)$ for all $x \in G$.

(c) f is invariant under inner automorphisms, i.e. $f(xy) = f(yx)$ for all $x, y \in G$.

(d) Cf is symmetric, i.e. $Cf(x, y) = Cf(y, x)$ for all $x, y \in G$.

(e) $Cf(\cdot, y)$ is a solution of Jensen's functional equation for each fixed $y \in G$, and $Cf(e, y) = f(e)$.

(f) If f is a quadratic function, say $f(x) = Q(x, x)$, where $Q : G \times G \to H$ is a symmetric bimorphism, then $2Q = Cf$.

Proof. (a) [25, formula (13)].
(b) If $f(e) = 0$, then (b) can be found as [25, formula (15)]. Using that, we get for a general solution f that $2f(x) = 2(f(x) - f(e)) = C(f - f(e))(x, x)$ for all $x \in G$.
(c) follows from Proposition 9.1.
(d) follows from (c).
(e) is a simple computation (Details can be found in the proof of [25, formula (8)]).
(f) The identity $f(xy) = Q(xy, xy) = Q(x, x) + Q(y, y) + 2Q(x, y) = f(x) + f(y) + 2Q(x, y)$ implies that $2Q(x, y) = f(xy) - f(x) - f(y) = Cf(x, y)$. □

A frequently encountered condition on G is that it shall be 2-divisible, i.e. that $G = \{x^2 \mid x \in G\}$. A weaker condition is that G is generated by its squares, i.e. that $G = \langle G^2 \rangle$, where $\langle G^2 \rangle$ denotes the subgroup of G generated by the set of squares $\{x^2 \mid x \in G\}$. Parnami and Vasudeva have in [19] a still weaker condition, viz. that $[G/\langle G^2 \rangle] \leq 2$. To put their condition into perspective we observe the following: Let G be a Lie group, and let G^o denote the connected component of G containing the identity element. As is well known $\langle G^2 \rangle \supseteq G^o$ [Let $\exp : \mathfrak{g} \to G$ be the exponential map. Then $U = \exp(\mathfrak{g})$ is a neighborhood of $\{e\}$ consisting of squares. Hence $\langle U \rangle \subseteq \langle G^2 \rangle$. But, U being a neighborhood of $\{e\}$, we have $\langle U \rangle \supseteq G^o$ by [10, Theorem 7.4]]. If G is connected then $G = \langle G^2 \rangle$, and if G has two components then $[G/\langle G^2 \rangle] \leq [G/G^o] = 2$. So the condition is automatically satisfied for any Lie group with at most two connected components like $GL(n, \mathbb{R})$. In this context we note the following technical lemma that holds for any group G:

Lemma 2.4. *Assume that $[G/\langle G^2\rangle] \leq 2$. If $f : G \to H$ is a solution of the quadratic functional equation on G, then $Cf(x,y) \in f(e) + 2H$ for all $x, y \in G$.*

Proof. Possibly replacing f by $f - f(e)$ we may assume that $f(e) = 0$ (note that $C(f - c) = Cf + c$).

We shall prove that $Cf(u, \xi) \in 2H$ and $Cf(\xi, u) \in 2H$ for any $\xi \in \langle G^2\rangle$ and $u \in G$. By the symmetry of Cf (Lemma 2.3(d)) we need only prove the second claim. Now $Cf(\cdot, u)$ is by Lemma 2.3(e) a solution of Jensen's functional equation on G such that $Cf(e, u) = 0$. By [16, formula (2.2)] we have that

$$Cf(xy^n z, u) = nCf(xyz, u) - (n-1)Cf(xz, u) \quad \text{for all } x, y, z \in G.$$

Using this formula with $n = 2$ we get by induction on k that $Cf(x_1^2 x_2^2 \cdots x_k^2, u) \in 2H$.

We use below that any element $\xi \in \langle G^2\rangle$ may be written in the form $\xi = x_1^2 x_2^2 \cdots x_n^2$ where $x_1, x_2, \ldots, x_n \in G$.

If $[G/\langle G^2\rangle] = 1$ then $G = \langle G^2\rangle$, so $Cf(\xi, u) \in 2H$ for all $\xi \in \langle G^2\rangle$ and $u \in G$.

We next write down the proof in the case of $[G/\langle G^2\rangle] = 2$, so we assume that there exists an $x_0 \in G \setminus \langle G^2\rangle$ such that any element in G may be written in the form $x_0^k \xi$, where $k \in \{0, 1\}$ and where $\xi \in \langle G^2\rangle$.

It is left to prove that $Cf(x_0\xi, x_0\eta) \in 2H$ for any $\xi, \eta \in \langle G^2\rangle$. Now,

$$\begin{aligned}
&Cf(x_0\xi, x_0\eta) = f(x_0\xi, x_0\eta) - f(x_0\xi) - f(x_0\eta) \\
&= f(x_0^2(x_0^{-1}\xi x_0)\eta) \\
&\quad - [Cf(x_0, \xi) + f(x_0) + f(\xi)] - [Cf(x_0, \eta) + f(x_0) + f(\eta)] \\
&= f(x_0^2(x_0^{-1}\xi x_0)\eta) \\
&\quad - Cf(x_0, \xi) - Cf(x_0, \eta) - 2f(x_0) - f(\xi) - f(\eta),
\end{aligned}$$

so modulo $2H$ we have that

$$Cf(x_0\xi, x_0\eta) = f(x_0^2(x_0^{-1}\xi x_0)\eta) - f(\xi) - f(\eta).$$

Noting that $x_0^{-1}\xi x_0 \in \langle G^2\rangle$, because $\langle G^2\rangle$ is normal, we see that it suffices to prove that $f(\langle G^2\rangle) \subseteq 2H$. Any element in $\langle G^2\rangle$ may be written in the form $x_1^2 x_2^2 \cdots x_n^2$ where $x_1, x_2, \ldots, x_n \in G$, so we will establish $f(\langle G^2\rangle) \subseteq 2H$ by induction on $n = 1, 2, \ldots$. Since $f(x^2) = 4f(x)$ by Lemma 2.3(a) (this is $n = 1$) we find with $a = x_1^2 x_2^2 \cdots x_n^2$, computing modulo $2H$, that

$$f(ax_{n+1}^2) = f(a) + f(x_{n+1}^2) + Cf(a, x_{n+1}^2) = f(a) + 4f(x_{n+1}) + 0 = f(a),$$

from which the induction step follows. □

Lemma 2.5. *If $f : G \to H$ is a solution of the quadratic functional equation, then*

(a) Let Γ be a subgroup of G. If f is a function on G/Γ, then Cf is a function on the coset space $G/\Gamma \times G/\Gamma$.

(b) *Assume that* $[G/\langle G^2\rangle] \leq 2$*, and that* H *has the property that* $2h = 0$ *implies* $h = 0$*. Then there exists exactly one map* $A : G \times G \to H$ *such that* $Cf = 2A$*. The map* A *is symmetric, and* $A(\cdot, u)$ *is a solution of Jensen's functional equation on* G *such that* $A(e, u) = 0$ *for each* $u \in G$*. Furthermore* $f(x) = A(x, x)$ *for all* $x \in G$*.*

(c) *Assume that* f *is a function on* G/Γ*, where* Γ *is a subgroup of* G*. Assume also that* f *is a quadratic function, say* $f(x) = Q(x, x)$*, where* $Q : G \times G \to H$ *is a symmetric bimorphism. Assume finally that* H *has the property that* $2h = 0$ *implies* $h = 0$*. Then* Q *is a symmetric function on* $G/\Gamma \times G/\Gamma$*.*

Proof. (a) Due to the symmetry of Cf (Lemma 2.3(d)) we only need to verify that $Cf(x, y\gamma) = Cf(x, y)$ for all $x, y \in G$ and $\gamma \in \Gamma$. And $Cf(x, y\gamma) = f(xy\gamma) - f(x) - f(y\gamma) = f(xy) - f(x) - f(y) = Cf(x, y)$.
(b) The only problem is to show that $Cf(x, y) \in 2H = \{2h \mid h \in H\}$ for all $x, y \in G$, the unique divisibility by 2 taking care of the rest. But this is done in Lemma 2.4.
(c) Immediate from (a) and Lemma 2.3(f). □

In Theorem 2.6 we list pertinent basic properties of any solution $f : G \to H$ of the quadratic functional equation and the corresponding map $B_f : G \times G \times G \to H$. These properties are consequences of [25, Theorem 2.1]. Of central interest for us is the formula [25, formula (17)]:

$$f(xuvy) = f(xvuy) + 2B_f(u, v, yx), \quad x, y, u, v \in G, \tag{2.2}$$

because it deals with interchange of elements and hence with commutators. It is derived in [25] under the hypothesis that $f(e) = 0$, but it remains true without this assumption.

Theorem 2.6. *If* $f : G \to H$ *is a solution of the quadratic functional equation, then*

(a) $f(x[y, z]) = f(x) + 2B_f(x, y, z)$ *for all* $x, y, z \in G$*.*

(b) $2B_f : G \times G \times G \to H$ *is a multimorphism.*

(c) $B_f : G \times G \times G \to H$ *is alternating.*

(d) $2B_f(x, y, z) = 0$ *if any two of the elements* $x, y, z \in G$ *commute.*

(e) $B_f(x, y, z) = f(e)$ *if any two of the elements* $x, y, z \in G$ *are equal.*

Proof. We may in (a) and (b) assume that $f(e) = 0$, since $B_{f-f(e)} = B_f + f(e)$, so that $2B_{f-f(e)} = 2B_f$, because $2f(e) = 0$ Lemma 2.2.
(b) The multimorphism property is proved in [25, formula (16)].
(e) is immediate from [25, formula (20)].
(a) Put $y = u^{-1}v^{-1}$ in (2.2) and use (b) and (e).
(c) By [25, formula (22)] we have that $B_{f-f(e)}(x, y, z) + B_{f-f(e)}(x, z, y) = 0$ for all $x, y, z \in G$. Substituting $B_{f-f(e)}(x, y, z) = B_f(x, y, z) + f(e)$ we get that $B_f(x, y, z)+$

$B_f(x,z,y)=0$. Similarly arguments show that $B_f(x,y,z)+B_f(z,y,x)=0$ and that $B_f(x,y,z)+B_f(y,x,z)=0$.
(d) Assume first that y and z commute. Using the definition (2.1) of B_f and (c) we find that that $B_f(x,y,z)=B_f(x,z,y)=-B_f(x,y,z)$, proving (d) in this case. Similar arguments work if other elements commute, because B_f is alternating.

$\square$

3 Solutions are functions on the quotient group $G/[G,[G,G]]$

Given any function f on G we introduced in [21] the normal subgroup

$$Z(f)=\{u\in G \mid f(xuy)=f(xyu) \text{ for all } x,y\in G\}$$

of G. The notation $Z(f)$ reflects the fact that if an element $u\in Z(f)$ occurs in an argument for f then it may be moved around as though residing in the center $Z(G)$ of G, i.e. $f(xuyz)=f(xyuz)$ for all $x,y,z\in G$.

Lemma 3.1. *If $f: G\to H$ is a solution of the quadratic functional equation, then $[G,G]\subseteq Z(f)$. Furthermore $2B_f(x,y,z)=0$, if one of the elements $x,y,z\in G$ belongs to $Z(f)$.*

Proof. Consider the formula (2.2) with $y=e$, i.e.

$$f(xuv)=f(xvu)+2B_f(u,v,x) \text{ for all } x,u,v\in G. \tag{3.1}$$

Any $u\in[G,G]$ may be written in the form $u=\prod_{i=1}^n[a_i,b_i]$ where $a_i,b_i\in G$. We see that the last term of (3.1) vanishes, because $2B_f$ is a homomorphism in its first variable (Theorem 2.6(b)): $2B_f(u,v,x)=\sum_{i=1}^n 2B_f([a_i,b_i],v,x)=0$. It follows that $u\in Z(f)$.

If $u\in Z(f)$, i.e. $f(xuv)=f(xvu)$, then we get from (3.1) that $2B_f(u,v,x)=0$, so the statement is true in the first variable. It follows for the two other variables, because B_f is alternating (Theorem 2.6(c)). $\square$

Theorem 3.2. *Any solution $f: G\to H$ of the quadratic functional equation* (1.2) *is a function on the quotient group $G/[G,[G,G]]$, i.e. $f(xv)=f(x)$ for all $x\in G$ and $v\in[G,[G,G]]$.*

Proof. Any element $v\in[G,[G,G]]$ is a product of factors of the form $[y,u]$ and $[u,y]$, where $y\in G$ and $u\in[G,G]$. Elements in $[G,G]$ behave according to Lemma 3.1 as in the center of G when occurring in an argument of f, so moving $u\in[G,G]$ one position to the right, a factor of the form $[y,u]=yuy^{-1}u^{-1}$ may be replaced by $yy^{-1}uu^{-1}=e$ without affecting the value of f. Similarly for the factors of the form $[u,y]$, where $y\in G$ and $u\in[G,G]$. So each of the factors of v may be replaced by e. $\square$

The result of Theorem 3.2 is mentioned without proof in [25, Remark 4.4], and is also true for solutions of Jensen's functional equation [22, Theorem 2.2(c)] (It is derived in [22] for $H=\mathbb{C}$, but the proof holds for any abelian range group H).

4 Solutions are constant on the commutator subgroup

The following Proposition 4.1 can be found as [25, Corollary 2.3] under the assumption that $f(e) = 0$. We believe that our proof is simpler.

Proposition 4.1. *If $f : G \to H$ is a solution of the quadratic functional equation on a group G, then $f = f(e)$ on $[G, G]$.*

Proof. From Theorem 2.6(a) and the fact that $2B_f$ is a multimorphism we get by induction on n the formula

$$f(x[y_n, z_n][y_{n-1}, z_{n-1}] \cdots [y_1, z_1]) = f(x) + \sum_{i=1}^{n} 2B_f(x, y_i, z_i)$$
$$\text{for } x, y_i, z_i \in G,\ i = 1, 2, \ldots, n.$$

The proposition is the special case of $x = e$, because $B_f(e, y_i, z_i) = f(e)$ by the defining identity (2.1), so $2B_f(e, y_i, z_i) = 2f(e) = 0$. □

If G is a connected semisimple Lie group like $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $Sp(n)$ etc, then $[G, G] = G$ by [24, Corollary 3.18.10], so on such groups each solution f of the quadratic functional equation is constant by Proposition 4.1. However, certain groups are by their nature not connected, but have several components. To take an example the pseudo-orthogonal group $O(p, q)$ has 4 connected components when $p, q \geq 1$. The following Corollary 4.2 addresses that situation.

Corollary 4.2. *If $f : G \to \mathbb{C}$ is a solution of the quadratic functional equation on a semisimple Lie group G with only finitely many connected components, then $f = 0$.*

Proof. Let G^o denote identity component of G. It is a normal subgroup of G by [10, Theorem 7.1]. Then G/G^o is a finite group, the order, say n, of which is the number of connected components of G. Now $[G^o, G^o] = G^o$ by [24, Corollary 3.18.10], so

$$G/[G, G] \cong (G/[G^o, G^o])/([G, G]/[G^o, G^o]) = (G/G^o)/([G, G]/G^o),$$

so $G/[G, G]$ is a finite group the order of which divides n. Hence $x^n \in [G, G]$ for any $x \in G$. By Proposition 4.1 and Lemma 2.3(a) we get $0 = f(x^n) = n^2 f(x)$, so that $f(x) = 0$. □

5 Conditions for all solutions to be quadratic functions

As is easy to check any quadratic function is a solution of (1.2). We shall in this section give sufficient conditions on G to ensure the converse, i.e. that all solutions of the quadratic functional equation are quadratic functions, at least for range groups in which no element has order 2.

Our first result refines Aczél's classical result (Theorem 1.2) by allowing any abelian group H as range group.

Proposition 5.1. *Let $f : G \to H$ be a solution of the quadratic functional equation satisfying Kannappan's condition. Then*

(a) $4f$ is a quadratic function. Indeed, $2C(f - f(e)) : G \times G \to H$ is a symmetric biadditive map, and $2f(x) = C(f - f(e))(x, x)$ for all $x \in G$.

(b) If H has the property that $2h = 0$ implies $h = 0$, then $2f$ is a quadratic function.

(c) If the map $h \mapsto 2h$ is a bijection of H onto H, then f is a quadratic function.

(d) Let $\langle G^2 \rangle$ denote the subgroup of G generated by the squares $\{x^2 \mid x \in G\}$. If $[G/\langle G^2 \rangle] \le 2$, then $2f$ is a quadratic function.

(e) If H has the property that $2h = 0$ implies $h = 0$ and if furthermore $[G/\langle G^2 \rangle] \le 2$, then f is a quadratic function.

Proof. (a) From Lemma 2.3(b) we recall the formula

$$2f(x) = C(f - f(e))(x, x) \quad \text{for all } x \in G. \tag{5.1}$$

$C(f - f(e))(\cdot, y)$ is a solution of Jensen's functional equation (Lemma 2.3(e)) vanishing at e and satisfying Kannappan's condition, so $2C(f - f(e))(\cdot, y)$ is a homomorphism [2, Proof of Lemma 1]. The Cauchy difference $C(f - f(e))$ of $f - f(e)$ is symmetric by Lemma 2.3(d), so $2C(f - f(e)) : G \times G \to H$ is a symmetric biadditive map. Multiplying (5.1) by 2 we get that $4f$ is a quadratic function as desired.
(b) is immediate from (a).
(c) follows immediately from (b).
(d) Since f satisfies Kannappan's condition we may, possibly replacing G by $G/[G, G]$, assume that G is abelian. Actually in this replacement we use the fact that $[G, G] \subseteq \langle G^2 \rangle$, which follows from the formula $[x, y] = (xy)^2 y^{-2} (yx^{-1}y^{-1})^2$. Indeed,

$$G/\langle G^2 \rangle = (G/[G, G])/(\langle G^2 \rangle/[G, G])$$

By [19, Theorem 6] $Cf(\cdot, \cdot)$ is a homomorphism in each variable, when G is abelian.
(e) According to Lemma 2.5(b) there exists a symmetric map $A : G \times G \to H$ such that $f(x) = A(x, x)$ for all $x \in G$ and $Cf = 2A$. By assumption f satisfies Kannappan's condition, so hence does Cf. Being a solution of Jensen's equation Cf is a homomorphism in each variable. Hence so is $2A$. The unique divisibility by 2 implies that A is a homomorphism in each variable. □

The assumption $[G, [G, G]] = [G, G]$ of the following Corollary 5.2 is clearly satisfied for abelian groups, so the corollary contains the classical result stated above in Theorem 1.2. In Section 8 we give examples of important non-abelian groups, for which the assumption holds. Let us here note that the condition holds on semisimple Lie groups (that a Lie group is semisimple means by definition that its Lie algebra is semisimple) with at most two connected components like $SO(p, q)$ when $0 < p < p+q$ (see [9, Lemma X.2.4]): Let G^o be the connected component of the identity for such a group G. It is a normal subgroup of G by [10, Theorem 7.1]. The possible other component must be of the form aG^o for some $a \notin G^o$. It follows that $a^2 \in G^o$. Since G has at most two connected components we see by small calculations that $[G, G] \subseteq G^o$. And since G^o is a connected semisimple Lie group, we have $[G^o, G^o] = G^o$ (see [24, Corollary 3.18.10]). Now

$$[G, [G, G]] \supseteq [G^o, [G^o, G^o]] = [G^o, G^o] = G^o \supseteq [G, G]$$

implies that $[G, [G, G]] = [G, G]$. See also Corollary 4.2.

Corollary 5.2. *Let $f : G \to H$ be a solution of the quadratic functional equation on G. Let G satisfy that $[G, [G, G]] = [G, G]$. Then f satisfies Kannappan's condition. Furthermore*

(a) $4f$ is a quadratic function. Indeed, $2C(f - f(e)) : G \times G \to H$ is a symmetric biadditive map, and $2f(x) = C(f - f(e))(x, x)$ for all $x \in G$.

(b) If H has the property that $2h = 0$ implies $h = 0$, then $2f$ is a quadratic function.

(c) If the map $h \mapsto 2h$ is a bijection of H onto H, then f is a quadratic function.

(d) If $[G/\langle G^2\rangle] \leq 2$, then $2f$ is a quadratic function.

(e) If H has the property that $2h = 0$ implies $h = 0$ and if furthermore $[G/\langle G^2\rangle] \leq 2$, then f is a quadratic function.

Proof. The condition $[G, [G, G]] = [G, G]$ is equivalent to the quotient group $G/[G, [G, G]]$ being abelian. f is a function on $G/[G, [G, G]]$ according to Theorem 3.2, so by our assumption f is a function on the abelian group $G/[G, [G, G]]$. In particular f satisfies Kannappan's condition, so that we may apply Proposition 5.1. □

Proposition 5.3. *If*

(a) each commutator in G has finite order, or

(b) $[G, G]/[G, [G, G]]$ is a torsion group, or

(c) G is a semi-direct product of a topological abelian group (the normal part) and a semi-simple Lie group with at most finitely many connected components,

then any solution $f : G \to \mathbb{C}$ of the quadratic equation (1.2) on G is a quadratic function.

Proof. (a): Using Lemma 2.2 we get that $f(e) = 0$, because $H = \mathbb{C}$. From Lemma 2.3 we get that $f(x) = \frac{1}{2}Cf(x, x)$. The Cauchy-difference Cf of f is symmetric by Lemma 2.3, so it suffices to prove that $Cf(\cdot, y) = 0$ is additive for each $y \in G$. Now $Cf(\cdot, y)$ is a solution of Jensen's functional equation (Lemma 2.3) vanishing at $e \in G$. By [22, Proposition 6.3(c)] we get the additivity.
(b) resp. (c): The same as for (a) except that the last reference shall be to [22, Proposition 3.3(c)], resp. [22, Proposition 6.5(b)]. □

Proposition 5.3(a) covers for complex-valued solutions the case of a P_3-group that was studied by Corovei [6, Lemma 4].

6 Kannappan's condition and the 2^{nd} Cauchy difference

There are close relations between Kannappan's condition and the 2^{nd} Cauchy difference B_f for a solution f of the quadratic functional equation. The present section describes and exploits some of these relations. The key observation is the following lemma.

Lemma 6.1. *Let $f : G \to H$ be a solution of the quadratic functional equation on a group G. Then f satisfies Kannapan's condition if and only if $2B_f = 0$.*

Proof. Immediate from the formula (2.2) with $y = e$. □

Earlier papers (see [25, Remark 3.3] for references) stated Lemma 6.1 in an equivalent way, namely that $2Cf$ should be a bimorphism. The formulation of Lemma 6.1 has certain advantages as for example demonstrated by our proof of Corollary 6.2.

Corollary 6.2. *Let $f : G \to H$ be a solution of the quadratic functional equation on G. Let G_i, $i \in I$ be subgroups of G. Assume that*

(a) G is generated by the G_i, $i \in I$

(b) G_i and G_j commute when $i \neq j$

(c) For each $i \in I$ the restriction of f to G_i satisfies Kannappan's condition

Then f satisfies Kannappan's condition on G.

Proof. Any 3 elements $x, y, z \in G$ can be written in the form $x = x_{i_1}x_{i_2}\cdots x_{i_n}$, $y = y_{i_1}y_{i_2}\cdots y_{i_n}$, $z = z_{i_1}z_{i_2}\cdots z_{i_n}$, where $x_{i_l}, y_{i_l}, z_{i_l} \in G_{i_l}$ for $l = 1, 2, \ldots, n$. Now

$$2B_f(x, y, z) = 2B_f(x_{i_1}x_{i_2}\cdots x_{i_n}, y_{i_1}y_{i_2}\cdots y_{i_n}, z_{i_1}z_{i_2}\cdots z_{i_n}).$$

Due to the multi-additivity of $2B_f$ (Theorem 2.6(b)) we get that $2B_f(x, y, z)$ is a sum of terms of the form $2B_f(x_{i_k}, y_{i_l}, z_{i_m})$. Factors from subgroups with different index commute by assumption. So if two of the indices in $2B_f(x_{i_k}, y_{i_l}, z_{i_m})$ are different then this term is 0 by Theorem 2.6(d). Left are the terms of the form $2B_f(x_{i_k}, y_{i_k}, z_{i_k})$. But they are 0 by Lemma 6.1. Hence $2B_f(x, y, z) = 0$ for all $x, y, z \in G$, and the result is a consequence of Lemma 6.1. □

The Examples 8.4 and 8.5 show that Corollary 6.2 can be applied to the Heisenberg groups $H_{2n+1}(\mathbb{R})$ and $H_{2n+1}(\mathbb{Z})$. The G_i will be copies of the non-abelian groups $H_3(\mathbb{R})$ and $H_3(\mathbb{Z})$ respectively.

If $\{G_i \mid i \in I\}$ is a non-void family of groups we let $\prod_{i\in I} G_i$ denote the direct product of the groups G_i. We let $\prod_{i\in I}^* G_i$ denote the weak direct product of the groups G_i, i.e. the subgroup of $\prod_{i\in I} G_i$ of all $(x_i)_{i\in I}$ such that $x_i = e$ for all but a finite set of indices (this set varying with $(x_i)_{i\in I}$). If the index set I is finite, say $I = \{1, 2, \ldots, n\}$, then $\prod_{i\in I} G_i = \prod_{i\in I}^* G_i = G_1 \times G_2 \times \cdots \times G_n$.

Corollary 6.3. *Let $G = \prod_{i\in I}^{*} G_i$ be the weak direct product of the subgroups G_i. Let $f : G \to H$ be a solution of the quadratic functional equation on G. Then f satisfies Kannappan's condition on G if and only if the restriction of f to each of the subgroups G_i, $i \in I$, satisfies Kannapan's condition on that subgroup.*

Proof. Immediate from Corollary 6.2. □

Corollary 6.4. *Let $G = \prod_{i\in I} G_i$ be the direct product of the subgroups G_i. We assume that each G_i is a topological group and that G has the product topology. We let H be a Hausdorff topological group.*

Let finally $f : G \to H$ be a continuous solution of the quadratic functional equation on G.

Then f satisfies Kannappan's condition on G if and only if the restriction of f to each of the subgroups G_i, $i \in I$, satisfies Kannapan's condition on that subgroup.

Proof. If f satisfies Kannappan's condition on G, then it clearly does so on each subgroup of G. Let us conversely assume that $f|G_i$ satisfies Kannappan's condition for each $i \in I$. According to Corollary 6.3 f satisfies Kannappan's condition on the subgroup $\prod_{i\in I}^{*} G_i$ of G. Combining that this subgroup is dense in G (by [10, Theorem 6.2]) with the continuity of f we get that f satisfies Kannappan's condition on all of G. □

Lemma 6.5. *If $f : G \to H$ is a solution of the quadratic functional equation on a group G, then $2B_f : G \times G \times G \to H$ is actually a function on $G/Z(f) \times G/Z(f) \times G/Z(f)$, i.e. $2B_f(xz_1, yz_2, zz_3) = 2B_f(x, y, z)$ for all $x, y, z \in G$ and all $z_1, z_2, z_3 \in Z(f)$.*

Proof. By Lemma 3.1 we see that $2B_f(x, y, z) = 0$ for all $x, y, z \in G$ such that $z \in Z(f)$. $2B_f$ being a homomorphism in its third variable we find that

$$2B_f(x, y, zz_3) = 2B_f(x, y, z) + 2B_f(x, y, z_3) = 2B_f(x, y, z) + 0 = 2B_f(x, y, z).$$

The result for the two other variables follows from the fact that B_f is alternating (Theorem 2.6(c)). □

By Lemma 3.1 we have that $[G, G] \subseteq Z(f)$ for any solution $f : G \to H$ of the quadratic functional equation on the group G. Since also $Z(G) \subseteq Z(f)$ we get

Corollary 6.6. *If $f : G \to H$ is a solution of the quadratic functional equation on a group G, then $2B_f$ is an alternating multimorphism of $G/(Z(G)[G, G]) \times G/(Z(G)[G, G]) \times G/(Z(G)[G, G])$ into H.*

Corollary 6.7 below gives a transparent proof that Kannappan's condition automatically holds if the group G is generated by 2 elements. It generalizes [15, Theorem 1] and [25, Remark 3.3], that impose the stronger condition that G, and not just $G/(Z(G)[G, G])$, is generated by 2 elements.

The corresponding result is not valid in general for groups with 3 or more generators. Indeed, let G be the free group generated by 3 elements $\{a_1, a_2, a_3\}$ and define $f : G \to \mathbb{C}$ by

$$f(a_{s_1}^{m_1} a_{s_2}^{m_2} \cdots a_{s_l}^{m_l}) = \sum_{1 \le i < j < k \le l} m_i m_j m_k B(a_{s_i}, a_{s_j}, a_{s_k}),$$

where $B(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = \operatorname{sgn}(\sigma)$ for any permutation σ of three objects and 0 otherwise (see [25, Remark 3.3]). Then f is a solution of the quadratic functional equation (1.2), which does not satisfy Kannappan's condition. f is by the way an example of a solution of (1.2) that is not a quadratic function.

Corollary 6.7. *If the quotient group $G/(Z(G)[G,G])$ is generated by 2 elements, in particular if so is G, then any solution of the quadratic functional equation on G satisfies Kannappan's condition.*

Proof. Let a and b be generators of $G/(Z(G)[G,G])$. Then any element may be written as a product of factors of the form $a^m b^n$. $2B_f$ is an alternating multimorphism of $G/(Z(G)[G,G]) \times G/(Z(G)[G,G]) \times G/(Z(G)[G,G])$ into H, so $2B_f(x,y,z)$ becomes for any $x, y, z \in G$ a linear combination of the terms $2B_f(a,a,a)$, $2B_f(a,a,b)$, $2B_f(a,b,b)$ and $2B_f(b,b,b)$. But these terms vanish according to Theorem 2.6(e), each containing at least 2 identical elements, and hence so does $2B_f(x,y,z)$. We now refer to Lemma 6.1. $\square$

Corollary 6.7 is used in Example 8.5 (The Heisenberg group with integer entries).

7 On certain semi-direct products

Many important groups are semi-direct products. In the present short section we study a special class of semi-direct products. The class contains examples like the $(ax+b)-$group and $GL(n,\mathbb{R})$ that are treated in detail in Section 8 below.

Let $G = NK$ be the semi-direct product of a normal subgroup N and a subgroup K. Thus $N \cap K = \{e\}$, and each element $x \in G$ may in exactly one way be written as $x = nk$, where $n \in N$ and $k \in K$.

We have $[G,N] \subseteq N$, because N is a normal subgroup of G. Assuming more, namely equality $[G,N] = N$, we get that $N = [G,N] = [G,[G,N]] \subseteq [G,[G,G]]$. Let $f : G \to H$ be a solution of the quadratic functional equation on G. From Theorem 3.2 we read that f is a function on $G/[G,[G,G]]$, so that $f(xk) = f(kx) = f(k)$ for all $x \in [G,[G,G]]$ and $k \in K$. In particular $f(nk) = f(k)$ for all $n \in N$ and $k \in K$. The restriction F of f to the subgroup K of G is of course a solution of the quadratic functional equation on K. Conversely, if F is a solution of the quadratic functional equation on K, then the function f defined by $f(nk) = F(k)$ is a solution of the quadratic functional equation on G, so we get all the solutions of the quadratic functional equation on G from the solutions on K. Let $\pi_K : G \to K$ denote the homomorphism given by $\pi(nk) = k$ for $n \in N$, $k \in K$. Then the solutions of the quadratic functional equation on G are the functions of the form $F \circ \pi_K$, where F ranges over the solutions of the quadratic functional equation on K.

Let us furthermore take G to be a topological group, and let N and K be closed subgroups of G. Equipping N and K with the topology from G we also assume that the topology on G is the product topology from N and K (by the open mapping theorem for groups [9, Corollary II.3.3] the last statement is automatically true if G is a locally compact, second countable Hausdorff group), so that the map $\pi_K : G \to K$ is continuous. Still enforcing $[G, N] = N$ we find that the continuous solutions of the quadratic functional equation on G are the functions of the form $F \circ \pi_K$, where F ranges over the continuous solutions of the quadratic functional equation on K.

The group $\mathbb{R}^n \times_s O(n)$ of rigid motions of $\mathbb{R}^n$ for $n \geq 2$ is an example of a semi-direct product such that $[G, N] = N$. The Heisenberg group is an example that does not satisfy this condition (see Example 8.4 for details).

Summing up we have

Proposition 7.1. *Let $G = NK$ be the semi-direct product of a normal subgroup N and a subgroup K, and assume that $[G, N] = N$. Let $\pi_K : G \to K$ denote the projection on K. Then*

(a) The solutions of the quadratic functional equation on G are the functions of the form $F \circ \pi_K$, where F ranges over the solutions of the quadratic functional equation on K.
F satisfies Kannappan's condition on K if and only if $F \circ \pi_K$ satisfies Kannappan's condition on G.

(b) Assume furthermore that G is a topological group, N and K are closed subgroups of G, and that the topology on G is the product topology from N and K. Then the continuous solutions of the quadratic functional equation on G are the functions of the form $F \circ \pi_K$, where F ranges over the continuous solutions of the quadratic functional equation on K.

Proposition 7.1 will be useful several times in the discussions of the specific examples in Section 8.

8 Examples

We have already above encountered general examples of various types: The remarks right after Proposition 5.1, Proposition 5.3 and Corollary 4.2. In this section we discuss some specific examples.

Example 8.1. The $(ax + b)-$group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R},\ a > 0,\ b \in \mathbb{R} \right\}$$

is the semi-direct product of the subgroups

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\} \text{ and } K = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R},\ a > 0 \right\} \cong \mathbb{R}^+,$$

N being the normal part. Since $[G, N] = N$, we can apply Proposition 7.1.

In the notation of Proposition 7.1 we have that

$$\pi_K \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \pi_K \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Any continuous solution $f : G \to \mathbb{C}$ of the quadratic functional equation on G has the form $f = F \circ \pi_K$, where $F : K \cong \mathbb{R}^+ \to \mathbb{C}$ is a continuous solution of the quadratic functional equation on $\mathbb{R}^+$, i.e.

$$F(ac) + F(ac^{-1}) = 2F(a) + 2F(c), \quad a, c > 0.$$

The continuous function $\Phi(x) := F(\exp x)$, $x \in \mathbb{R}$, satisfies

$$\Phi(x + y) + \Phi(x - y) = 2\Phi(x) + 2\Phi(y), \quad x, y \in \mathbb{R}.$$

It follows that Φ has the form $\Phi(x) = cx^2$ for some constant $c \in \mathbb{C}$ [7, Corollary 10.1], so that $F(a) = c(\log a)^2$ for $a > 0$.

We conclude that the continuous solutions of the quadratic functional equation on the $(ax + b)-$group are the functions f_c of the form

$$f_c \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = c(\log a)^2, \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G,$$

where c ranges over $\mathbb{C}$.

Example 8.2. Let $n \geq 3$. For any ring with unit 1 the subgroup $E_n(R)$ of $GL(n, R)$ generated by the elementary matrices satisfies $[E_n(R), E_n(R)] = E_n(R)$ (see [4]). From now on we assume that R is a field or a commutative Euclidean ring. In those cases $E_n(R) = SL(n, R)$, so $[SL(n, R), SL(n, R)] = SL(n, R)$.

According to [25, Corollary 2.3] the only solution of the quadratic functional equation on $SL(n, R)$ is $f = 0$.

We will consider the group $G = GL(n, R)$. Decomposing the matrix $A = \{a_{ij}\} \in GL(n, R)$ as follows

$$\begin{Bmatrix} a_{11}(\det A)^{-1} & a_{12} & \cdots & a_{1n} \\ a_{21}(\det A)^{-1} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\det A)^{-1} & a_{n2} & \cdots & a_{nn} \end{Bmatrix} \begin{Bmatrix} \det A & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{Bmatrix}$$

we see that $GL(n, R)$ is the semi-direct product of $SL(n, R)$ (the normal part) and the group U of units of the ring R.

It follows from Proposition 7.1 that the solutions $f : G \to H$ of the quadratic functional equation on $GL(n, R)$ are the functions of the form $f = \phi \circ \det$, where $\phi : U \to H$ ranges over the solutions of the quadratic functional equation on the abelian group U.

For $R = \mathbb{Z}$ we have $U = \{\pm 1\}$ and so we get [25, Theorem 4.2] in the case of $n \geq 3$.

Example 8.3. We will find the continuous solutions $f : G \to \mathbb{C}$ of the quadratic functional equation on $G = GL(n, \mathbb{R})$ for $n \geq 2$. For $n \geq 3$ we can apply the results of the previous Example 8.2. Actually the crucial point in Example 8.2 is that $[SL(n, R), SL(n, R)] = SL(n, R)$, which is true for $R = \mathbb{R}$, because $SL(n, \mathbb{R})$ is a connected semi-simple Lie group (see [24, Corollary 3.18.10]). So we may also take $n = 2$. According to Example 8.2 the solutions of the quadratic functional equation on G are the functions of the form $f = \phi \circ \det$, where $\phi : U \to \mathbb{C}$ ranges over the continuous solutions of the quadratic functional equation on the group U of units of $\mathbb{R}$, i.e. the multiplicative group $\mathbb{R}^*$ of all non-zero real numbers.

The continuous quadratic forms on $\mathbb{R}^*$ are the functions of the form $\phi(t) = c(\log|t|)^2$, $t \in \mathbb{R}^*$, where $c \in \mathbb{C}$ is an arbitrary constant. This formula was on the subgroup $t > 0$ derived in Example 8.1. To get it on the negative half-axis as well, we first note that

$$0 = 2\phi(1) = \phi((-1)(-1)) + \phi((-1)(-1)^{-1}) = 2\phi(-1) + 2\phi(-1) = 4\phi(-1),$$

so $\phi(-1) = 0$. And then we get for any $t > 0$ that

$$2\phi(-t) = \phi(t(-1)) + \phi(t(-1)^{-1}) = 2\phi(t) + 2\phi(-1) = 2\phi(t) + 0 = 2\phi(t),$$

so $\phi(-t) = \phi(t)$.

Combining the above we get that the continuous solutions of the quadratic functional equation on $GL(n, \mathbb{R})$ are the functions

$$f(x) = c(\log|\det x|)^2, \quad x \in GL(n, \mathbb{R}),$$

where $c \in \mathbb{C}$ is an arbitrary constant.

This may be compared with the case of $GL(n, \mathbb{Z})$, where $f = 0$ is the only complex-valued quadratic form [25, Theorem 4.2].

Example 8.4. We will find all continuous solutions $f : G \to \mathbb{C}$ of the quadratic functional equation on the Heisenberg group $G = H_3(\mathbb{R})$, defined by

$$H_3(\mathbb{R}) = \left\{ (x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\},$$

but until further notice we consider solutions $f : G \to H$, where H is any abelian group.

The Heisenberg group is the semi-direct product of the abelian subgroups $N = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ and $K = \{(x, 0, 0) \mid x \in \mathbb{R}\}$, but it does not satisfy the condition $[G,N]=N$ from Proposition 7.1. Indeed, $[G,N]=\{(0,0,z)\mid z\in\mathbb{R}\}\neq N$.

It is easy to check that $e=(0,0,0)$, and that $[G,G]=\{(0,0,z)\in G\,|\,z\in\mathbb{R}\}$. Furthermore $[G,[G,G]]=\{e\}$, so $[G,[G,G]]\neq[G,G]$. Thus the condition of Corollary 5.2 is not satisfied either.

Nevertheless, as we shall see, all solutions $f:G\to H$ of the quadratic functional equation on G satisfy Kannappan's condition. We find this surprising, because the Heisenberg group provides a counter-example to the conjecture that all solutions of Jensen's functional equation are affine functions. This was observed in [8, Proposition 4.3] and [22, Example 5.1].

G is 2-divisible, and so in particular generated by its squares, because

$$\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}-\frac{xy}{8}\right)^2=(x,y,z)\quad\text{for all } x,y,z\in G.$$

Furthermore we get by an easy calculation for any $x',y',z',x,y,z\in\mathbb{R}$ that

$$(x',y',z')(x,y,z)(x',y',z')^{-1}=(x,y,z-xy'+yx'). \tag{8.1}$$

Let $f:G\to H$ be a solution of the quadratic functional equation. Since f is invariant under inner automorphisms we have from (8.1) that

$$\begin{aligned} f(x,y,z)&=f((x',y',z')(x,y,z)(x',y',z')^{-1})\\ &=f(x,y,z-xy'+yx'),\ \text{ for all } x',y',z',x,y,z\in\mathbb{R},\end{aligned}$$

so that $f(x,y,z)=f(x,y,0)$ if $(x,y)\neq(0,0)$. The same conclusion holds if $(x,y)=(0,0)$, because f by Theorem 4.1 is equal to $f(e)=f(0,0,0)$ on $[G,G]=\{(0,0,z)\in G\,|\,z\in\mathbb{R}\}$. Thus f is a function on $G/[G,G]$ which is an abelian group. In particular f satisfies Kannappan's condition. Via Proposition 5.1(e) we see that f is a quadratic function, if H has the property that $2h=0$ implies $h=0$. Assuming this we let $f(x)=Q(x,x)$, where $Q:G\times G\to H$ is a symmetric bimorphism. f is a function on $G/[G,G]$, so Q is by Lemma 2.5 a map $G/[G,G]\times G/[G,G]\to H$. Thus we shall find all symmetric bimorphisms of $G/[G,G]\simeq(\mathbb{R}^2,+)$ into H.

We do this for $H=\mathbb{C}$ and $f:G\to\mathbb{C}$ continuous. Since f is continuous, so is $Q:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{C}$. Now Q is bilinear, being continuous and biadditive. This gives $f(x,y,z)=f(x,y,0)=Q((x,y),(x,y))=ax^2+bxy+cy^2$ for some constants $a,b,c\in\mathbb{C}$.

Combining Corollary 6.2 with the result just obtained about $H_3(\mathbb{R})$ we get more generally that any solution of the quadratic functional equation on the $(2n+1)$-dimensional Heisenberg group $H_{2n+1}(\mathbb{R})$ satisfies Kannappan's condition.

Example 8.5. Let us consider the Heisenberg group $G=H_3(\mathbb{Z})$ with integer entries, defined by

$$H_3(\mathbb{Z})=\left\{(x,y,z)=\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}\,\middle|\,x,y,z\in\mathbb{Z}\right\}.$$

Here $[G,G] = \{(0,0,k) \mid k \in \mathbb{Z}\}$, so $G/[G,G] \cong \mathbb{Z}^2$. This group is generated by two elements, so all solutions of the quadratic functional equation on G satisfy Kannappan's condition according to Corollary 6.7 (or [25, Remark 3.3]). Thus any solution f is a function on $G/[G,G]$. We get that

$$\begin{aligned}
f(m,n,k) &= f((0,n,0)(m,0,0)(0,0,k)) = f((0,n,0)(m,0,0)) \\
&= [f - f(e)]((0,n,0)(m,0,0)) + f(e) \\
&= C(f - f(e))((0,n,0),(m,0,0)) \\
&\quad + [f - f(e)](0,n,0) + [f - f(e)](m,0,0) + f(e) \\
&= C(f - f(e))((0,1,0)^n,(1,0,0)^m) \\
&\quad + [f - f(e)]((0,1,0)^n) + [f - f(e)]((1,0,0)^m) + f(e).
\end{aligned}$$

Here we use that the first term $C(f - f(e))(\cdot,\cdot)$ is a solution of Jensen's functional equation in each variable, vanishing at the neutral element, and that each such solution g of Jensen's equation has the property $g(x^l) = lg(x)$ for any $l \in \mathbb{Z}$ and $x \in G$ ([16, Formula (2.2)]). On the second and the third term we apply Lemma 2.3(e). We get

$$\begin{aligned}
f(m,n,k) &= nmC(f - f(e))((0,1,0),(1,0,0)) \\
&\quad + n^2[f - f(e)]((0,1,0)) + m^2[f - f(e)]((1,0,0)) + f(e).
\end{aligned}$$

Thus there exist $h_0, h_1, h_2, h_3 \in H$ with $2h_0 = 0$ such that

$$f(m,n,k) = m^2 h_1 + n^2 h_2 + mnh_3 + h_0, \quad m,n,k \in \mathbb{Z}. \tag{8.2}$$

Conversely, any map of $H_3(\mathbb{Z})$ into H of this form is a solution.

This result could alternatively have been derived from [25, Corollary 2.2].

We saw above that any solution of the quadratic functional equation on the Heisenberg group $H_3(\mathbb{Z})$ with integer entries satisfies Kannappan's condition. Combining this with Corollary 6.2 we get more generally that any solution of the quadratic functional equation on the Heisenberg group $H_{2n+1}(\mathbb{Z}) = \{(x,y,z) \in H_{2n+1}(\mathbb{R}) \mid x,y,z \in \mathbb{Z}\}$ with integer entries satisfies Kannappan's condition. This can be used to derive a formula similar to (8.2) for the general solution of the quadratic functional equation on $H_{2n+1}(\mathbb{Z})$.

9 The case of a symmetric right hand side

The results of this section have been noted in various special instances. For example in [23, Proposition B.1] and [25, Remark 4.3].

Throughout this section we let G and H be two sets with binary operations $(x,y) \mapsto xy$ and $(a,b) \mapsto a * b$ respectively. We assume that the operation in G has a unit $e \in G$, and that $* : H \times H \to H$ has the cancellation properties that for all $a,b,c \in H$ we have $a * b = a * c$ implies $b = c$ and $b * a = c * a$ implies $b = c$. These assumptions are of course satisfied if G and H are groups. Finally $\sigma : G \to G$ is an involution, i.e. $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$ and $\sigma(e) = e$. If G is a group then σ could be $\sigma(x) = x^{-1}$, $x \in G$, which is the involution met above in this paper.

Proposition 9.1. *Let $f : G \to H$ and $F : G \times G \to H$ satisfy*

$$f(xy) * f(x\sigma(y)) = F(x, y) \text{ for all } x, y \in G.$$

Then the two statements

(a) $f(xy) = f(yx)$ for all $x, y \in G$, and $f \circ \sigma = f$.

(b) $F(x, y) = F(y, x)$ for all $x, y \in G$.

are equivalent.

Proof. Suppose that $F(x, y) = F(y, x)$ for all $x, y \in G$. Then

$$f(xy) * f(x\sigma(y)) = f(yx) * f(y\sigma(x)) \text{ for all } x, y \in G. \tag{9.1}$$

Putting $x = e$ in (9.1) we get from the assumption about left cancellation that $f \circ \sigma = f$. Using this on the last term of (9.1) we get that $f(y\sigma(x)) = f(\sigma(y\sigma(x))) = f(x\sigma(y))$, so that (9.1) reads

$$f(xy) * f(x\sigma(y)) = f(yx) * f(x\sigma(y)) \text{ for all } x, y \in G. \tag{9.2}$$

By right cancellation we get that $f(xy) = f(yx)$.

Conversely, if (a) holds then we get for any $x, y \in G$ that

$$\begin{aligned} F(x, y) &= f(xy) * f(x\sigma(y)) = f(yx) * f(\sigma(x\sigma(y))) \\ &= f(yx) * f(y\sigma(x)) = F(y, x). \end{aligned}$$

□

Proposition 9.2. *Let $f : G \to H$ and $F : G \times G \to H$ satisfy*

$$f(xy) * f(\sigma(y)x) = F(x, y) \text{ for all } x, y \in G.$$

Then the two statements

(a) $f(xy) = f(yx)$ for all $x, y \in G$, and $f \circ \sigma = f$.

(b) $F(x, y) = F(y, x)$ for all $x, y \in G$.

are equivalent.

Proof. Suppose that $F(x,y) = F(y,x)$ for all $x, y \in G$. For any $x \in G$ we get

$$\begin{aligned} f(x) * f(x) &= f(xe) * f(\sigma(e)x) = F(x,e) = F(e,x) \\ &= f(ex) * f(\sigma(x)e) = f(x) * f(\sigma(x)), \end{aligned}$$

which by left cancellation implies that $f(x) = f(\sigma(x))$ for all $x \in G$.

Next we find for any $x, y \in G$ that

$$\begin{aligned} f(xy) * f(\sigma(y)x) &= F(x,y) = F(y,x) = f(yx) * f(\sigma(x)y) \\ &= f(yx) * f(\sigma(\sigma(x)y)) = f(yx) * f(\sigma(y)x), \end{aligned}$$

which by right cancellation implies that $f(xy) = f(yx)$ for all $x, y \in G$.

Conversely, if (a) holds then we get for any $x, y \in G$ that

$$\begin{aligned} F(x,y) &= f(xy) * f(\sigma(y)x) = f(yx) * f(\sigma(\sigma(y)x)) \\ &= f(yx) * f(\sigma(x)y) = F(y,x). \end{aligned}$$

□

Corollary 9.3. *Let $f : G \to H$ and $F : G \times G \to H$ and assume that $F(x,y) = F(y,x)$ for all $x, y \in G$. Then the pair $\{f, F\}$ is a solution of*

$$f(xy) * f(x\sigma(y)) = F(x,y) \text{ for all } x, y \in G,$$

if and only if $\{f, F\}$ is a solution of

$$f(xy) * f(\sigma(y)x) = F(x,y) \text{ for all } x, y \in G.$$

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