UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

ON OPERATOR-VALUED SPHERICAL FUNCTIONS

Henrik Stetkær

Preprint Series No.: 9

July 2004

Ny Munkegade, Bldg. 530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

On operator-valued spherical functions

Henrik Stetkær

Abstract

Cauchy's equation and the cosine equation on an abelian group G are both particular cases of the general equation,

$$\int_K f(x+k\cdot y)dk = f(x)f(y), \quad x,y\in G,$$

in which a compact group K acts on G, viz. the cases $K=\{e\}$ and $K=\mathbb{Z}_2$, respectively. We extend a result due to Chojnacki on operator-valued solutions of the cosine equation to this general equation: We prove that if f takes its values in the normal operators on a Hilbert space $\mathcal{H}$, then $f(x)=\int_K U(k\cdot x)dk$, $x\in G$, where U is a unitary representation of G on H, and dk denotes the normalized Haar measure on K. We show that normality may not be needed if K is finite, thereby generalizing a result by Kurepa on the cosine equation.

Mathematics Subject Classification (2000): Primary 47D09. Secondary 22D10, 39B42.
Keywords: Cosine equation, transformation group, locally compact, unitary representation, spherical function.

1 Introduction

Let $(G,+)$ be an abelian group with neutral element 0. *The cosine equation*, also called d'Alembert's equation, on G is the equation

$$\frac{f(x+y)+f(x-y)}{2} = f(x)f(y), \quad x,y\in G, \tag{1.1}$$

where $f: G\to\mathbb{C}$ is the unknown.

The present paper deals with an extension of (1.1), both with respect to the form of (1.1) where a transformation group will enter, and to the range of f which will be in the bounded operators on a Hilbert space. The non-zero solutions of (1.1) are the functions of the form $f(x)=(\gamma(x)+\gamma(-x))/2$, $x\in G$, where γ is a homomorphism af G into the multiplicative group of non-zero complex numbers ([14, Theorem 2]). Our extension will generalize this fact.

Let G be a topological group and let K be a compact topological transformation group of G, acting by automorphisms of G. Writing the action by $k \in K$ on $x \in G$ as $k \cdot x$ and letting dk denote the normalized Haar measure on K a generalization of the cosine equation (1.1) is

$$\int_K f(x + k \cdot y)dk = f(x)f(y), \quad x, y \in G, \tag{1.2}$$

where $f \in C(G)$ is the unknown. The equation (1.2) is studied in the theory of group representations, being the relation defining K-spherical functions (for the terminology see [1, p. 88]).

The equation (1.1) is the case of $K = \mathbb{Z}_2 = \{\pm 1\}$, while the Cauchy equation has $K = \{I\}$. Another example of K is $\mathbb{Z}_N = \{\omega^n \mid n = 0, \dots, N-1\}$, where $\omega = \exp(2\pi i/N)$, acting on $\mathbb{R}^2 = \mathbb{C}$ by multiplication. A third one is $O(n)$ acting on $\mathbb{R}^n$ by rotations.

The first part of the following general Theorem 1.1 is due to Shin'ya [21, Corollary 3.12], and the second part to Chojnacki [4, Theorem 1.1]. As mentioned above it was derived by Kannappan [14, Theorem 2] in the special case of $K = \mathbb{Z}_2$.

Theorem 1.1. *Let G be a locally compact abelian group. If $f \in C(G)$ is a non-zero solution of* (1.2) *then there exists a continuous homomorphism $\gamma : G \to \mathbb{C}^*$ such that $f(x) = \int_K \gamma(k \cdot x)dk$ for all $x \in G$. If f is bounded, then γ may be chosen in the dual group $\widehat{G}$ of G.*

Theorem 1.1 is of a very general nature, giving no information about γ. We mention in passing that explicit expressions for γ are known from the theory of spherical functions on Euclidean-type symmetric spaces (see [12, Proposition IV.4.8]).

A natural generalization of the equation (1.2) is to consider solutions f that take their values in an algebra, and not just in $\mathbb{C}$. If f is a complex-valued non-zero solution of (1.1) then $f(0) = 1$, but in the algebra case this need no longer be true. So we shall impose the standard condition $f(0) = I$, where I denotes the identity of the algebra.

The purpose of the present paper is to extend the last part of Theorem 1.1 to solutions taking values in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space H.

A point of departure is the following fundamental result by Chojnacki [3, Théorème 1.1 and 3.1] for $K = \mathbb{Z}_2$:

Theorem 1.2. *Let G be a locally compact abelian group. Let $\mathcal{H}$ be a Hilbert space and let $f : G \to \mathcal{B}(\mathcal{H})$ be a solution of* (1.1) *such that $f(0) = I$. Assume furthermore that f is bounded and strongly continuous.*

Then there exists a bounded and strongly continuous representation Γ of G on $\mathcal{H}$ such that

$$f(x) = \frac{\Gamma(x) + \Gamma(-x)}{2} \quad \textit{for all } x \in G. \tag{1.3}$$

If f takes its values in the normal operators, then Γ may be chosen unitary.

In Theorem 3.1 below we find for any Hilbert space $\mathcal{H}$ the uniformly bounded, $\mathcal{B}(\mathcal{H})$-valued solutions Φ of the equation

$$\int_K \Phi(x + k \cdot y)dk = \Phi(x)\Phi(y), \; x, y \in G, \text{ such that } \Phi(e) = I, \tag{1.4}$$

thereby giving a simultaneous extension of the results by Chojnacki mentioned above in Theorems 1.1 and 1.2. More precisely we show that the weakly continuous, normal-operator-valued solutions are the functions of the form $\Phi(x) = \int_K U(k \cdot x)dk$, $x \in G$, where U is a unitary representation of G on $\mathcal{H}$, a result which is quite analogous to the one of the scalar case of Theorem 1.1. Our main tools are from harmonic analysis.

We refer to [3] for literature on the classical cosine equation (1.1), but let us for the sake of completeness mention that Székelyhidi in [22, Theorem 4.2] described the matrix-valued continuous solutions of the cosine equation (1.1) on an infinite topological group in which division by 2 is defined, without assuming that the matrices are normal. And that Elqorachi and Akkouchi in [8, Section 5] studied an integral equation related to (1.4), viz.

$$\int_G \Gamma(xty)d\mu(t) + \int_G \Gamma(xty^{-1})d\mu(t) = 2\Gamma(x)\Gamma(y), \quad x, y \in G,$$

where μ is a generalized Gelfand measure and where $\Gamma : G \to \mathcal{B}(\mathcal{H})$ takes its values in the normal operators on a Hilbert space $\mathcal{H}$.

2 Set-up and notation

Throughout the paper we let $(G, +)$ be an abelian, locally compact Hausdorff topological group with neutral element 0. $C(G)$ denotes the algebra of all continuous, complex-valued functions on G, $C_0(G)$ the subalgebra of functions vanishing at infinity and $C_c(G)$ the subalgebra of compactly supported functions. The dual group of G is denoted $\widehat{G}$. We fix a Haar measure dx on G and let $*$ denote the corresponding convolution of functions. If F is a function on G we define $\check{F}(x) = F(-x)$, $x \in G$.

K is a compact, Hausdorff topological group with neutral element I and normalized Haar measure dk. We assume that it acts as a topological transformation group on G (as defined in, e.g. [11, II §3]) and that the action is by automorphisms. The action of $k \in K$ on $x \in G$ is denoted $k \cdot x$. If f is a function on G we write $k \cdot f$ for the function $[k \cdot f](x) := f(k^{-1} \cdot x)$, $x \in G$.

When $\mathcal{H}$ is a complex Hilbert space, we let $\mathcal{B}(\mathcal{H})$ denote the algebra of all linear continuous maps of $\mathcal{H}$ into $\mathcal{H}$.

3 The main result

Theorem 3.1 below characterizes uniformly bounded, K-spherical, normal operator-valued functions on abelian groups, because its converse is also true as is easy to check. Theorem 3.1 is essentially [3, Théorème 1.1] for $K = \mathbb{Z}_2$.

Theorem 3.1. *Let G satisfy the second axiom of countability. Let $\mathcal{H}$ be a complex Hilbert space. Let $\Phi : G \to \mathcal{B}(\mathcal{H})$ be a weakly continuous mapping such that*

$$\int_K \Phi(x + k \cdot y)dk = \Phi(x)\Phi(y) \text{ for all } x, y \in G, \text{ and } \Phi(e) = I, \tag{3.1}$$

$\Phi(x)$ is normal for each $x \in G$, and finally $\sup\{\|\Phi(x)\| \mid x \in G\} < \infty$.

Then there exists a strongly continuous unitary representation U of G on $\mathcal{H}$ such that $\Phi = \int_K k \cdot U dk$.

In particular $\Phi(x)^* = \Phi(x^{-1})$ for all $x \in G$ and furthermore Φ is strongly continuous.

If $\mathcal{H}$ is finite-dimensional, then Theorem 3.1 can be proved simply by diagonalization. In this case the countability assumption on G is not used.

Remark 3.2. (a) We have in Theorem 3.1 assumed that Φ is bounded. An unbounded matrix-valued cosine function, even on $G = \mathbb{R}$, need not be of the form (1.3) up to similarity (Kisyński [15, Example 1]).

(b) On the other hand, Niechwiej has in [19] extended Chojnacki's results [3, 4] to solutions that are majorized by a calibrating function, but that need not be uniformly bounded.

(c) The Hilbert space can in general not be replaced by a Banach space (see Kisyński [15, 16] for examples). Even more is true: Chojnacki [5, Theorem 2.5] has shown the following result for any locally compact abelian group G: If for every Banach space E, any $\mathcal{B}(E)$-valued bounded strongly continuous solution of (1.1) may be written in the form (1.3), where Γ is a bounded group representation, then $\{2x \mid x \in G\}$ is finite.

Some of the technical details in our proof of Theorem 3.1 will be handled by the following Proposition 3.3 and Lemma 3.4. The proposition takes care of certain measure theoretical questions and the lemma of some algebraic ones.

Proposition 3.3. *Let G satisfy the second axiom of countability.*

Then K acts as a group of topological automorphisms of $\widehat{G}$ by $(k, \gamma) \mapsto k \cdot \gamma$, and the orbit space $\widehat{G}/K$ is with respect to the quotient topology a locally compact Hausdorff space. Furthermore there exists a Borel measurable cross section $s : \widehat{G}/K \to \widehat{G}$.

Proof. The orbit space $\widehat{G}/K$ is according to [2, III.4.1] locally compact, which in the Bourbaki terminology in particular means Hausdorff. $\widehat{G}$ is is a completely regular Hausdorff space [13, Theorem II.8.4] satisfying the second axiom of countability [20, Satz 57], so [10, Theorem 2] gives the existence of the desired cross section. $\square$

We need some notation: We let $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ denote the Fourier transform and put $A(\widehat{G}) := \mathcal{F}(L^1(G)) \subseteq C_0(\widehat{G})$. Let $A(\widehat{G})^K = \{f \in A(\widehat{G}) \mid k \cdot f = f \text{ for all } k \in K\}$.

If $f \in C(\widehat{G})$ we let $f^\natural := \int_K k \cdot f dk \in C(\widehat{G})$. Similarly we put $\psi^\natural := \int_K k \cdot \psi dk$ for $\psi \in L^1(G)$. We let $C_0(\widehat{G})^K = \{f \in C_0(\widehat{G}) \mid k \cdot f = f \text{ for all } k \in K\}$.

Lemma 3.4. *Let $\Phi : G \to \mathcal{B}(\mathcal{H})$ be a weakly continuous, uniformly bounded mapping satisfying* (3.1) *and define $\widehat{\Phi} : A(\widehat{G}) \to \mathcal{B}(\mathcal{H})$ by*

$$\widehat{\Phi}(\mathcal{F}\psi) := \int_G \Phi(x)\psi(x^{-1})dx \text{ for } \psi \in L^1(G).$$

Then

(a) $(\psi^\natural)^\natural = \psi^\natural$ *for all* $\psi \in L^1(G)$.

(b) $C_0(\widehat{G})^\natural = C_0(\widehat{G})^K$.

(c) $(\mathcal{F}\psi)^\natural = \mathcal{F}(\psi^\natural)$ *for all* $\psi \in L^1(G)$.

(d) $\widehat{\Phi}(f^\natural) = \widehat{\Phi}(f)$ *for all* $f \in A(\widehat{G})$.

(e) $\widehat{\Phi}(\mathcal{F}\psi_1)\widehat{\Phi}(\mathcal{F}\psi_2) = \widehat{\Phi}(\mathcal{F}\psi_1\mathcal{F}\psi_2)$ *for* $\psi_1, \psi_2 \in L^1(G)$, *such that* $\psi_2^\natural = \psi_2$.

Proof. (a) and (b): Trivial.
(c): The compactness of K implies that $\int_G \phi(k \cdot x)dx = \int_G \phi(x)dx$ for all $\phi \in L^1(G)$ and $k \in K$. Using that we find for any $\gamma \in \widehat{G}$ that

$$\begin{aligned}
(\mathcal{F}\psi)^\natural(\gamma) &= \int_K (\mathcal{F}\psi)(k^{-1} \cdot \gamma)dk = \int_K \int_G \psi(x)\overline{(k^{-1} \cdot \gamma)(x)}dxdk \\
&= \int_K \int_G \psi(x)\overline{\gamma(k \cdot x)}dxdk = \int_K \int_G \psi(k^{-1} \cdot x)\overline{\gamma(x)}dxdk \\
&= \int_G \psi^\natural(x)\overline{\gamma(x)}dx = \mathcal{F}(\psi^\natural)(\gamma).
\end{aligned}$$

(d): Putting $x = 0$ in (3.1) we find that $\Phi(k \cdot x) = \Phi(x)$ for all $k \in K$ and $x \in G$. Now, we find for any $\psi \in L^1(G)$ that

$$\begin{aligned}
\widehat{\Phi}((\mathcal{F}\psi)^\natural) &= \widehat{\Phi}(\mathcal{F}(\psi^\natural)) = \int_G \Phi(x)\psi^\natural(-x)dx \\
&= \int_G \Phi(x) \int_K \psi(-k^{-1} \cdot x)dkdx = \int_K \int_G \Phi(x)\psi(-k^{-1} \cdot x)dxdk \\
&= \int_K \int_G \Phi(k \cdot x)\psi(-x)dxdk = \int_G \Phi(x)\psi(-x)dx = \widehat{\Phi}(\mathcal{F}\psi).
\end{aligned}$$

(e): The computation

$$\begin{aligned}
\widehat{\Phi}(\mathcal{F}\psi_1)\widehat{\Phi}(\mathcal{F}\psi_2) &= \Big(\int_G \Phi(x)\check{\psi}_1(x)dx\Big)\Big(\int_G \Phi(y)\check{\psi}_2(y)dy\Big) \\
&= \int_G\int_G \Phi(x)\Phi(y)\check{\psi}_1(x)\check{\psi}_2(y)dxdy \\
&= \int_G\int_G\int_K \Phi(x+k\cdot y)dk\check{\psi}_1(x)\check{\psi}_2(y)dxdy \\
&= \int_K\int_G\int_G \Phi(x+k\cdot y)\check{\psi}_1(x)\check{\psi}_2(y)dxdydk \\
&= \int_K\int_G\int_G \Phi(x+y)\check{\psi}_1(x)\check{\psi}_2(k^{-1}\cdot y)dxdydk \\
&= \int_G\int_G \Phi(x+y)\check{\psi}_1(x)\check{\psi}_2(y)dxdy \\
&= \int_G\int_G \Phi(x)\check{\psi}_1(x-y)\check{\psi}_2(y)dxdy \\
&= \int_G \Phi(x)(\check{\psi}_2 * \check{\psi}_1)(x)dx = \int_G \Phi(x)(\psi_1 * \psi_2)^\vee(x)dx \\
&= \widehat{\Phi}(\mathcal{F}(\psi_1 * \psi_2)) = \widehat{\Phi}(\mathcal{F}(\psi_1)\mathcal{F}(\psi_2))
\end{aligned}$$

proves (e). □

Proof of Theorem 3.1.

Claim. $\widehat{\Phi} : A(\widehat{G}) \to \mathcal{B}(\mathcal{H})$ extends to a continuous linear map $\widehat{\Phi} : C_0(\widehat{G}) \to \mathcal{B}(\mathcal{H})$ with norm $\|\widehat{\Phi}\| \le 1$.

We start by proving the claim: Let $f = \mathcal{F}\psi$ where $\psi \in L^1(G)$, so that $f^\natural = (\mathcal{F}\psi)^\natural = \mathcal{F}(\psi^\natural)$.

G being abelian we get from the functional equation (3.1) that Φ is K-invariant and from this that $\Phi(x)\Phi(y) = \Phi(y)\Phi(x)$ for all $x, y \in G$, i.e. that $\Phi(x)$ and $\Phi(y)$ commute. It follows that $\Phi(x)$ and $\widehat{\Phi}(f) := \int_G \Phi(y)\check{\psi}(y)dy$ commute. We infer from Fuglede's theorem that $\Phi(x)$ also commutes with $\widehat{\Phi}(f)^*$, from which it follows that $\widehat{\Phi}(f)$ and $\widehat{\Phi}(f)^*$ commute, i.e. that $\widehat{\Phi}(f)$ is a normal operator.

$\widehat{\Phi}(f^\natural)$ is normal, so its norm equals its spectral radius. Combining that with (d) and (e) of Lemma 3.4 we get

$$\begin{aligned}
\|\widehat{\Phi}(f)\| &= \|\widehat{\Phi}(f^\natural)\| \\
&= \lim_{n\to\infty} \|[\widehat{\Phi}(f^\natural)]^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|\widehat{\Phi}((f^\natural)^n)\|^{\frac{1}{n}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \|\widehat{\Phi}(\mathcal{F}([\psi^\natural]^{*n}))\|^{\frac{1}{n}} = \lim_{n\to\infty} \Big\| \int_G \Phi(x)(\psi^\natural)^{*n}(-x)dx \Big\|^{\frac{1}{n}} \\
&\leq \limsup_{n\to\infty} \big\{ [\sup\{\|\Phi(x)\| \mid x \in G\}]^{\frac{1}{n}} [\int_G |(\psi^\natural)^{*n}(x)|dx]^{\frac{1}{n}} \big\} \\
&\leq \limsup_{n\to\infty} \big\{ [\int_G |(\psi^\natural)^{*n}(x)|dx]^{\frac{1}{n}} \big\}.
\end{aligned}
$$

Since $\|\alpha^{*n}\|_1^{1/n} \to \|\mathcal{F}\alpha\|_\infty$ as $n \to \infty$ for any $\alpha \in L^1(G)$, we find that

$$
\|\widehat{\Phi}(f)\| \leq \|\mathcal{F}(\psi^\natural)\|_\infty = \|(\mathcal{F}\psi)^\natural\|_\infty \leq \|\mathcal{F}\psi\|_\infty = \|f\|_\infty.
$$

This proves the claim because $A(\widehat{G})$ is dense in $C_0(\widehat{G})$.

Combining the above results we have that the restriction to the algebra $A(\widehat{G})^K$ of the extension $\widehat{\Phi} : C_0(\widehat{G}) \to \mathcal{B}(\mathcal{H})$ is a representation of $A(\widehat{G})^K$ on $\mathcal{H}$ (Here we do not view $A(\widehat{G})^K$ as a $*$-algebra, but just as an algebra).

Claim. This representation of $A(\widehat{G})^K$ on $\mathcal{H}$ is non-degenerate.

Proof. We shall prove that if $\xi \in \mathcal{H}$ is orthogonal to $\widehat{\Phi}(A(\widehat{G})^K)\eta$ for all $\eta \in \mathcal{H}$, then $\xi = 0$. From Lemma 3.4(d) we see that $\xi \perp \widehat{\Phi}(\mathcal{F}\psi)\eta$ for all $\psi \in L^1(G)$ and all $\eta \in \mathcal{H}$. This means that

$$
\int_G \langle \Phi(x)\eta, \xi \rangle \check{\psi}(x)dx = 0 \quad \text{ for all } \eta \in \mathcal{H} \text{ and } \psi \in L^1(G).
$$

It follows that $\langle \Phi(x)\eta, \xi \rangle = 0$ for all $x \in G$ and $\eta \in \mathcal{H}$. Since $\Phi(0) = I$ we get by choosing $x = 0$ that $\xi = 0$. □

Let $q : \widehat{G} \to \widehat{G}/K$ denote the quotient map and $q^t : C_0(\widehat{G}/K) \to C_0(\widehat{G})^K$ its transpose, which is an isomorphism. $\widehat{\Phi} \circ q^t$ is a non-degenerate representation of the abelian and hence nuclear C*-algebra $C_0(\widehat{G}/K)$ on the Hilbert space $\mathcal{H}$, so there exists a positive bounded operator h on $\mathcal{H}$ such that $\|\widehat{\Phi}\|^{-1} I \leq h \leq \|\widehat{\Phi}\| I$ and $\mathrm{Ad}(h) \circ (\widehat{\Phi} \circ q^t)$ is a $*$-representation [6, Theorem 4.1]. As we saw above $\|\widehat{\Phi}\| \leq 1$, so $h = I$, and hence $\widehat{\Phi} \circ q^t$ is a non-degenerate $*$-representation of $C_0(\widehat{G}/K)$ on $\mathcal{H}$.

By commutative representation theory there exists a spectral measure E on the Borel sets of $\widehat{G}/K$ with values in $\mathcal{B}(\mathcal{H})$ such that

$$
\widehat{\Phi}(f) = \int_{\widehat{G}/K} [(q^t)^{-1}(f)](\sigma)dE(\sigma) \quad \text{ for all } f \in C_0(\widehat{G})^K.
$$

Let $s : \widehat{G}/K \to \widehat{G}$ be a Borel measurable cross section for the quotient map $q : \widehat{G} \to \widehat{G}/K$. Such one exists by Proposition 3.3. It follows from the properties of a spectral measure that the formula

$$
U(x) := \int_{\widehat{G}/K} s(\sigma)(x)dE(\sigma), \quad x \in G, \tag{3.2}
$$

defines a strongly continuous unitary representation U of G on $\mathcal{H}$. For any $x \in G$ and $\psi \in C_c(G)^K$ we get, using the formula

$$[(q^t)^{-1}\mathcal{F}\psi](K \cdot \chi) = \int_G \psi(x) \int_K \overline{\chi(k^{-1} \cdot x)} dk dx,$$

that

$$\begin{aligned}
&\int_G \psi(x) \Big(\int_K k \cdot U dk \Big)(x) dx \\
&\quad = \int_{\widehat{G}/K} \Big\{ \int_G \check{\psi}(x) \Big[\int_K \overline{s(K \cdot \gamma)(k^{-1} \cdot x)} dk \Big] dx \Big\} dE(K \cdot \gamma) \\
&\quad = \int_{\widehat{G}/k} [(q^t)^{-1}(\mathcal{F}\check{\psi})](K \cdot s(K \cdot \gamma)) dE(K \cdot \gamma) \\
&\quad = \int_{\widehat{G}/k} [(q^t)^{-1}(\mathcal{F}\check{\psi})](K \cdot \gamma) dE(K \cdot \gamma) \\
&\quad = \widehat{\Phi}(\mathcal{F}\check{\psi}) = \int_G \Phi(x)\psi(x) dx.
\end{aligned}$$

Since Φ is K-invariant we conclude that $\Phi(x) = (\int_K k \cdot U dk)(x)$. $\square$

Remark 3.5. Substituting the formula (3.1) into the expression for Φ just found we get that

$$\Phi = \int_{\widehat{G}/K} \Big\{ \int_K k \cdot (s(\sigma)) dk \Big\} dE(\sigma)$$

which for $K = \{I\}$ reduces to the Stone-Naimark-Ambrose-Godement formula for unitary representations of locally compact abelian groups.

4 On the assumption of the operators being normal

For some time it was an open problem whether a uniformly bounded representation of a group G on a Hilbert space had to be similar to a unitary representation. Although it was solved in the negative for $G = SL(2, \mathbb{R})$ by Kunze and Stein [17], it is true for G abelian [7, Théorème 6]. This means that the assumption about normality in Theorem 3.1 is inessential if $K = \{I\}$. The same holds for $K = \mathbb{Z}_2$, i.e. for cosine functions, by [3, Théorème 2.1] and the earlier works by Fattorini [9] (when $G = \mathbb{R}$) and Kurepa [18] (when $G = \{2x \mid x \in G\}$). Our final result (Theorem 4.1), in which we replace $K = \{I\}$ and $K = \mathbb{Z}_2$ by a finite group, is a continuation of these investigations. Our condition in Theorem 4.1 reduces to Kurepa's, i.e. to $G = \{2x \mid x \in G\}$, when $K = \mathbb{Z}_2$. The proof is inspired by [3, Théorème 2.1] and as in [3] we do not need any topology on G here.

Theorem 4.1. *Let G be abelian. Let K be finite and such that the map $x \mapsto x - k \cdot x$ of G into G is surjective for each $k \in K \setminus \{I\}$. Let $\mathcal{H}$ be a Hilbert space and $\Phi : G \to \mathcal{B}(\mathcal{H})$ a uniformly bounded map such that $\Phi(e) = I$ and*

$$\frac{1}{|K|} \sum_{k \in K} \Phi(x + k \cdot y) = \Phi(x)\Phi(y) \quad \text{for all } x, y \in G. \tag{4.1}$$

Then there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$, equivalent with the original one on $\mathcal{H}$, such that

$$\langle \Phi(x)u, v \rangle = \langle u, \Phi(-x)v \rangle \quad \text{for all } x \in G, \ u, v \in \mathcal{H}. \tag{4.2}$$

In particular $\Phi(x)$ is for each $x \in G$ a normal operator with respect to the new inner product $\langle \cdot, \cdot \rangle$.

Proof. Let $(\cdot, \cdot)$ denote the inner product on $\mathcal{H}$, $\|\cdot\|$ the corresponding norm and $C := \sup\{\|\Phi(x)\| \mid x \in G\} < \infty$. Let m be an invariant mean on the bounded functions on G. Such one exists [13, Theorem 17.5]. We use the notation $m_x\{f(x)\}$ instead of $m(f)$ to tell that the mean of the bounded function f is taken with respect to the variable $x \in G$.

We can write an explicit formula for the desired new inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ down: For $u, v \in \mathcal{H}$ we put

$$\langle u, v \rangle := m_x\{(\Phi(x)u, \Phi(x)v) + \sum_{k \in K \setminus \{I\}} (\Phi(k \cdot x - x)u, \Phi(k \cdot x - x)v)\}. \tag{4.3}$$

It is obvious that $\langle \cdot, \cdot \rangle$ is sesquilinear, that $\langle u, u \rangle \geq 0$ and that $\langle u, u \rangle \leq |K| C^2 \|u\|^2$ for all $u \in \mathcal{H}$. To get an estimate the other way we let $u \in \mathcal{H}$ and $x \in G$ and compute

$$\|u\| \leq \|u + \sum_{k \neq I} \Phi(-x + k \cdot x)u\| + \sum_{k \neq I} \|\Phi(-x + k \cdot x)u\| \tag{4.4}$$

$$= |K| \|\Phi(-x)\Phi(x)u\| + \sum_{k \neq I} \|\Phi(-x + k \cdot x)u\| \tag{4.5}$$

$$\leq |K| C \|\Phi(x)u\| + \sum_{k \neq I} \|\Phi(-x + k \cdot x)u\|, \tag{4.6}$$

so by the Cauchy-Schwarz inequality $\|u\|^2 \leq C^2 |K|^3 \langle u, u \rangle$.

We show that (4.2) holds by proving the relation for each of the terms defining the new inner product (4.3). The first term is an easy consequence of the invariance of the mean m. The typical term of the remaining sum is the expression

$$\begin{aligned}&(\Phi(k \cdot x - x)\Phi(y)u, \Phi(k \cdot x - x)v) \\ &\quad = \frac{1}{|K|} \sum_{k' \in K} (\Phi(k \cdot x - x + k' \cdot y)u, \Phi(k \cdot x - x)v),\end{aligned}$$

where $k \in K \setminus \{I\}$. By assumption $k' \cdot y = k \cdot z' - z'$ for some $z' = z'(k', y) \in G$, so that we may rewrite the term to the following expression

$$
\begin{aligned}
&\frac{1}{|K|} \sum_{k' \in K} (\Phi(k \cdot x - x + k \cdot z' - z')u, \Phi(k \cdot x - x)v) \\
=&\frac{1}{|K|} \sum_{k' \in K} (\Phi(k \cdot [x + z'] - [x + z'])u, \Phi(k \cdot x - x)v).
\end{aligned}
$$

Now (4.2) follows from the invariance of the mean m by a simple computation.
$\square$

References

[1] Benson, C., Jenkins, J. and Ratcliff, G., *On Gelfand pairs associated with solvable Lie groups.* Trans. Amer. Math. Soc. **321** (1990), 85–116.

[2] Bourbaki, N.: "*Elements of Mathematics. General Topology*", Chapters 1–4. Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo. 1989.

[3] Chojnacki, W., *Fonctions cosinus hilbertiennes bornées dans les groupes commutatifs localement compacts.* Compositio Math. **57** (1986), 15–60.

[4] W. Chojnacki, *On some functional equation generalizing Cauchy's and d'Alembert's functional equations.* Colloq. Math. **55** (1988), 169–178.

[5] W. Chojnacki, *Group representations of bounded cosine functions.* J. reine angew. Math. **478** (1996), 61–84.

[6] Christensen, E., *On non self-adjoint representations of C*-algebras.* Amer. J. Math. **103** (1981), 817–833.

[7] Dixmier, J., *Les moyennes invariantes dans les semi-groupes et leurs applications.* Acta Sci. Math. (Szeged) **12 A** (1950), 213–227.

[8] Elqorachi, E. and Akkouchi, M., *Functional equations and Gelfand measures.* Publ. Math. Debrecen **63** (2003), 643–666.

[9] Fattorini, H. O., *Uniformly bounded cosine functions in Hilbert space.* Indiana Univ. Math. J. **20** (1970/1971), 411–425.

[10] Graf, S., *A measurable selection theorem for compact-valued maps.* Manuscripta Math. **27** (1979), 341–352.

[11] S. Helgason: *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press. New York - San Francisco - London. 1978.

[12] S. Helgason: *Groups and geometric analysis* Academic Press. Orlando - San Diego - San Francisco - New York - London - Toronto - Montreal - Sydney - Tokyo - São Paulo. 1984.

[13] Hewitt, E. and Ross, K. A.: "*Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations.*" Die Grundlehren der mathematischen Wissenschaften, Bd. 115 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. viii+519 pp.

[14] Kannappan, Pl., *The functional equation* $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ *for groups.* Proc. Amer. Math. Soc. **19** (1968), 69–74.

[15] Kisyński, J., *On operator-valued solutions of d'Alembert's functional equation. I.* Colloq. Math. **23** (1971), 107–114.

[16] Kisyński, J., *On operator-valued solutions of d'Alembert's functional equation. II.* Studia Math. **42** (1972), 43–66.

[17] Kunze, R. A. and Stein, E. M., *Uniformly bounded representations and harmonic analysis of the* 2×2 *real unimodular group.* Amer. J. Math. **82** (1960), 1–62.

[18] Kurepa, S., *Uniformly bounded cosine function in a Banach space.* Math. Balkanica **2** (1972), 109–115.

[19] Niechwiej, J., *The spectral representation of certain operator cosine functions on locally compact groups.* Univ. Iagel. Acta Math. No. 33, (1996), 101–107.

[20] Pontrjagin, L. S.: "*Topologische Gruppen. 2*". (German) B. G. Teubner Verlagsgesellschaft, Leipzig 1958. 308 pp.

[21] H. Shin'ya, *Spherical matrix functions and Banach representability for locally compact motion groups.* Japan. J. Math. **28** (2002), 163–201.

[22] Székelyhidi, L., *Functional equations on abelian groups.* Acta Math. Acad. Sci. Hungar. 37 (1981), 235–243.

Henrik Stetkær
Department of Mathematical Sciences
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: stetkaer@imf.au.dk