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APPROXIMATING NUMBERS WITH MISSING DIGITS BY ALGEBRAIC NUMBERS

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Abstract. We show that for a given base b and a proper subset $E \subset \{0,\dots, b-1\}$, $\#E < b-1$, the set of numbers $x \in [0,1]$ which have no digits from E in their expansion to base b consists almost exclusively of S^*-numbers of type ≤ 2. We also give upper bounds on the Hausdorff dimension of some exceptional sets.

1. Introduction

Let $K \subseteq [0,1]$ be a compact set and suppose that K supports a measure μ such that for constants $c_1, c_2 > 0$,

$$c_1 r^\delta \leq \mu([c-r; c+r]) \leq c_2 r^\delta \tag{1}$$

for $c \in K$ and $r > 0$ small enough. It is easy to see that any non-atomic measure supported on K satisfying hypothesis (1) must also satisfy

$$\mu([c-\epsilon r; c+\epsilon r]) \leq c_3 \epsilon^\delta \mu([c-r; c+r]), \tag{2}$$

for some $c_3 > 0$, whenever r and ϵ are small and $c \in \mathcal{R}$. This is the appropriate one-dimensional specialisation of the notion of an *absolutely δ-decaying measure*, used in [5, 9]. Here and subsequently, we will assume that μ has been normalised so that $\mu(K) = 1$.

Let $n \in \mathbb{N}$ and let $\mathbb{A}_n$ denote the set of algebraic numbers of degree $\leq n$. For an algebraic number α, we denote its height by $H(\alpha)$, *i.e.*, the maximum modulus of the coefficients of the minimal polynomial of α. We are concerned with the approximation of elements of K by elements of $\mathbb{A}_n$, where the quality of approximation is measured in terms of the height of the approximating number. Let $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$. We define the set

$$\mathcal{K}_n^*(\psi; K) = \{x \in K : |x - \alpha| < \psi(H(\alpha)) \text{ for } \infty \text{ many } \alpha \in \mathbb{A}_n\}. \tag{3}$$

Properties (1) and (2) are important because of the following theorem, which combines specialisations of a theorem by Hutcheson [4] and of Kleinbock, Lindenstrauss and Weiss [5] with [3, Theorem 9.3].

Theorem. *Let $\{h_1, \dots, h_t\}$ be a family of affine contractions of $\mathbb{R}$, such that for some open set $U \subseteq \mathbb{R}$,*

$$h_i(U) \subseteq U \text{ for all } i = 1, \dots t, \text{ and } i \neq j \Rightarrow h_i(U) \cap h_j(U) = \emptyset. \tag{4}$$

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Suppose further that no finite set $\{x_1, \ldots, x_M\}$ *is invariant under the full family* $\{h_1, \ldots, h_t\}$. *Then, there is a unique non-empty compact set* K *such that*

$$K = \bigcup_{i=1}^{t} h_i(K), \tag{5}$$

which supports a non-atomic measure satisfying condition (1) *(and consequently condition* (2)*). Furthermore,* δ *is the positive real number satisfying*

$$\sum_{i=1}^{t} \rho_i^{\delta} = 1,$$

where $\rho_i > 0$ *is the ratio with which* h_i *contracts (*i.e., $h_i(x) = \pm\rho_i x + \tau_i$*).*

Condition (4) is known as the *open set condition*, and is sufficient to ensure the existence of the set K. The additional restriction that no finite set is invariant under the action of the family is to ensure that the limiting measure is non-atomic. The theorem has a higher dimensional generalisation, but this is not relevant for the purposes of this paper.

Let $b \in \mathbb{N}$, $b > 2$ and let $E \subset \{0, \ldots, b-1\}$. Consider the set $C_{b,E} \subseteq [0,1]$ of numbers whose expansion to base b does not contain any of the digits in E. This generalises the well-known ternary Cantor set, which is obtained when $b = 3$ and $E = \{1\}$. Of course, if $\#E = b - 1$, the set $C_{b,E}$ consists of a single point and $\delta = 0$. We will disregard this degenerate case and assume that $\#E < b - 1$. It is straightforward to construct a family of contractions having $C_{b,E}$ as their limit set (see *e.g.* [4]). One can easily show that this family satisfies the conditions of the above theorem. By that theorem, $C_{b,E}$ supports a measure satisfying (1). Consequently, in all results below, we may read $C_{b,E}$ for K to obtain statements about these sets. Also, we easily see that $\delta = \log(b - \#E)/\log b$.

The set $\mathcal{K}_n^*(\psi; [0,1])$ has been widely studied (see [2]). When $n = 1$, we are approximating elements of K by rationals, and further results on the measure and dimension are known [5, 9, 11]. In the present paper, we are interested in finding upper bounds for the measure and Hausdorff dimension of the sets $\mathcal{K}_n^*(\psi; K)$, where K is subject to condition (1). A particular example of such sets are the $C_{b,E}$, with the only restriction that $\#E < b-1$. This has some number theoretic consequences.

We briefly mention some related questions and results. Mahler asked [8] how close an element in the ternary Cantor set can be approximated by rationals (see also [7] and [6], where it is conjectured that the sets $C_{b,E}$ contain no algebraic irrationals). The partial answer by Weiss [11] was 'almost surely not better than expected'. The present paper gives a similar answer for approximation by algebraic numbers of bounded degree. Pollington [10] has calculated the Hausdorff dimension of $\mathcal{K}_n^*(r \mapsto r^{-(n+1)\lambda}; N)$, where N is the set of numbers which are non-normal to every base. This coincides with the Hausdorff dimension of the set $\mathcal{K}_n^*(r \mapsto r^{-(n+1)\lambda}; [0,1])$ which was shown by Baker and Schmidt [1] to be $1/\lambda$.

2. Statement of results

We will first find a criterion on the function ψ under which we are guaranteed that the set $\mathcal{K}_n^*(\psi; K)$ is null with respect to μ. We will obtain the following theorem:

Theorem 1. *Let $K \subseteq [0,1]$ be a compact set supporting a measure μ satisfying* (2)*. Suppose that $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ is non-increasing and such that either*

$$\sum_{r=1}^{\infty} r^{2n\delta-1}\psi(r)^{\delta} < \infty \quad \textit{or} \quad \sum_{r=1}^{\infty} r^{n}\psi(r)^{\delta} < \infty.$$

Then

$$\mu\left(\mathcal{K}_n^*(\psi; K)\right) = 0.$$

When $n = 1$ and $K = C_{3,\{1\}}$, this reduces to the theorem of [11]. Note that whenever $\delta \leq (n+1)/2n$, the first convergence condition is stronger than the second.

In Koksma's classification of transcendental numbers, we encounter the quantities

$$w_n^*(x) = \sup\{w > 0 : |x - \alpha| < H(\alpha)^{-w-1} \text{ for } \infty \text{ many } \alpha \in \mathbb{A}_n\}$$

and

$$w^*(x) = \limsup_{n\to\infty} \frac{w_n^*(x)}{n},$$

defined for any transcendental number x. If $w^*(x) < \infty$, x is said to be an S^*-number of type $w^*(x)$. Note that we are using the definitions from [2]. See the discussion in that book for alternative definitions of the quantities used. We have the following corollary to theorem 1.

Corollary 2. *For μ-almost every $x \in K$, $w_n^*(x) \leq \min\{2n-1, (n+1-\delta)/\delta\}$. Consequently, μ-almost every $x \in K$ is an S^*-number of type $\leq \min\{2, 1/\delta\}$.*

Note that we have lost some information by restricting to a Cantor set. Indeed, it is well-known that almost all real numbers are S^*-numbers of type ≤ 1 (see *e.g.* [2, Theorem 3.3]). At present, I do not know if the bound on the type can be improved for general sets satisfying (1).

In analogy with Koksma's classification, we have Mahler's classification (which actually predates Koksma's). In this classification, we have quantities

$$w_n(x) = \sup\{w > 0 : |P(x)| < H(P)^{-w} \text{ for } \infty \text{ many } P \in \mathbb{Z}[X], \deg(P) \leq n\}$$

and

$$w(x) = \limsup_{n\to\infty} \frac{w_n(x)}{n}.$$

If $0 < w(x) < \infty$, x is said to be an S-number of type $w(x)$. We now have a second corollary to theorem 1.

Corollary 3. *For μ-almost every $x \in K$, $w_n(x) \leq \min\{3n-2, ((1+\delta)n+1-2\delta)/\delta\}$. Consequently, μ-almost every $x \in K$ is an S-number of type $\leq \min\{3, (1+\delta)/\delta\}$.*

We now turn our attention to the Hausdorff dimension of the null sets from theorem 1. Denote by $\mathcal{H}^s(E)$ the s-dimensional Hausdorff measure of the set E and by $\dim_{\mathrm{H}}(E)$ the Hausdorff dimension of E (see *e.g.* [3] for the definitions). If K supports a measure satisfying (1), it follows directly that $\dim_{\mathrm{H}}(K) = \delta$. We will show the following:

Theorem 4. *Let $K \subseteq [0,1]$ be a compact set supporting a non-atomic measure μ satisfying* (1)*. Let $s \in [0,\delta]$ and let $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ be non-increasing. Suppose that either*

$$\sum_{r=1}^{\infty} r^{2n\delta-1}\psi(r)^{s} < \infty \quad \textit{or} \quad \sum_{r=1}^{\infty} r^{n}\psi(r)^{s} < \infty.$$

Then

$$\mathcal{H}^s(\mathcal{K}_n^*(\psi; K)) = 0.$$

From theorem 4, we may deduce an upper bound on the Hausdorff dimension of the sets $\mathcal{K}_n^*(\psi; K)$. For a function $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$, we define the *lower order at infinity of* $1/\psi$ to be

$$\lambda_\psi = \liminf_{r\to\infty} \frac{-\log\psi(r)}{\log(r)}.$$

Corollary 5. *Let* $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ *be non-increasing with* $\lambda_\psi \geq \min\{2n, (n+1)/\delta\}$. *Then*

$$\dim_{\mathrm{H}}(\mathcal{K}_n^*(\psi; K)) \leq \min\left\{\frac{2n\delta}{\lambda_\psi}, \frac{n+1}{\lambda_\psi}\right\}.$$

Of course, as $\mathcal{K}_n^*(\psi; K) \subseteq \mathcal{K}_n^*(\psi; [0,1])$ and as $\dim_{\mathrm{H}}(\mathcal{K}_n^*(\psi; [0,1])) = (n+1)/\lambda_\psi$ under the same assumptions as in corollary 5 (see [2, Theorem 6.7]), we recognise the second upper bound as the one of this theorem. The first estimate is stronger only if $\delta \leq (n+1)/2n$. This is certainly satisfied for all n if $\delta \leq 1/2$. For higher δ, new information is only gained for suitably small n.

The results obtained in the present paper are unlikely to be best possible. This is a consequence of the methods used in the proofs, and we will make conjectures on the best possible results in the final section. To prove stronger results of the type in this paper, information on the distribution of all algebraic numbers of bounded degree nearby K is needed. For the very general K studied here, we do not have sufficiently accurate information to obtain the conjectured results. Instead, we make do with weak distributional results which hold on all of $\mathbb{R}$, and use measure theoretical arguments to deduce distributional results for algebraic numbers nearby K.

3. Proof of Theorem 1

We first prove that the convergence of the first series ensures that the measure is zero. This is by fra the most difficult part of the proof. We will use a consequence of [2, Corollary A.2]. It is a consequence of this corollary, that if α and β are distinct algebraic numbers of degree at most n, then

$$|\alpha - \beta| \geq c_4 H(\alpha)^{-n} H(\beta)^{-n}, \tag{6}$$

where the constant $c_4 > 0$ depends solely on n. If for some $k \in \mathbb{N}$, $2^k \leq H(\alpha), H(\beta) < 2^{k-1}$, this implies that $|\alpha - \beta| > \frac{1}{2}c_4 2^{-2nk}$. Consequently, for distinct algebraic numbers α_i with $2^k \leq H(\alpha_i) < 2^{k+1}$, the intervals $[\alpha_i - \frac{1}{4}c_4 2^{-2kn}; \alpha_i + \frac{1}{4}c_4 2^{-2kn}]$ are disjoint.

Let $k \in \mathbb{N}$. We will show that as $k \to \infty$,

$$\max_{2^k \leq r < 2^{k+1}} \frac{\psi(r)}{4^{-1}c_4 2^{-2kn}} = o(1). \tag{7}$$

Indeed, suppose to the contrary that there is a $c_5 > 0$ and a strictly increasing sequence $\{k_i\}_{i=1}^\infty \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$

$$\max_{2^{k_i} \leq r < 2^{k_i+1}} \frac{\psi(r)}{4^{-1}c_4 2^{-2k_i n}} > c_5.$$

By the convergence assumption of the theorem together with Cauchy's condensation criterion,

$$\sum_{k=1}^{\infty} 2^{2kn\delta}\psi(2^k)^\delta = \sum_{k=1}^{\infty} \left(2^{2kn}\psi(2^k)\right)^\delta < \infty.$$

On the other hand, as ψ is non-increasing,

$$\begin{aligned}\sum_{k=1}^{\infty} \left(2^{2kn}\psi(2^k)\right)^\delta &\geq 4^{-\delta}c_4^\delta \sum_{i=1}^{\infty}\left(\max_{2^{k_i}\leq r<2^{k_i+1}} \frac{\psi(r)}{4^{-1}c_4 2^{-2k_i n}}\right)^\delta \\ &> 4^{-\delta}c_4^\delta c_5^\delta \sum_{i=1}^{\infty} 1 = \infty,\end{aligned}$$

which is the desired contradiction.

Consider the sets

$$E_k = \bigcup_{\substack{\alpha\in\mathbb{A}_n \\ 2^k\leq H(\alpha)<2^{k+1}}} [\alpha-\psi(H(\alpha));\alpha+\psi(H(\alpha))].$$

Clearly, for k large enough

$$\begin{aligned}\mu(E_k) &\leq \sum_{\substack{\alpha\in\mathbb{A}_n \\ 2^k\leq H(\alpha)<2^{k+1}}} \mu([\alpha-\psi(H(\alpha));\alpha+\psi(H(\alpha))]) \\ &\leq c_3 c_4^\delta 4^{-\delta} 2^{2kn\delta}\psi(2^k)^\delta \sum_{\substack{\alpha\in\mathbb{A}_n \\ 2^k\leq H(\alpha)<2^{k+1}}} \mu([\alpha-\tfrac{1}{4}c_4 2^{-2kn};\alpha+\tfrac{1}{4}c_4 2^{-2kn}]),\end{aligned}$$

where we have used (2) and (7). The intervals in the final sum are disjoint. Hence, the sum of their measure is bounded from above by the measure of K, which is equal to 1. We have shown that for $k \geq k_0$,

$$\mu(E_k) \leq c_3 c_4^\delta 4^{-\delta} 2^{2kn\delta}\psi(2^k)^\delta.$$

To complete the proof of this case, we note that $\mathcal{K}_n^*(\psi;K)$ is the set of points falling in infinitely many of the E_k. But

$$\sum_{k=k_0}^{\infty} \mu(E_k) \leq c_3 c_4^\delta 4^{-\delta} \sum_{k=k_0}^{\infty} 2^{2kn\delta}\psi(2^k)^\delta.$$

Using Cauchy's condensation criterion and the convergence assumption of the theorem, the latter converges. Hence, the Borel–Cantelli lemma implies the theorem.

To show that the convergence of the second series is sufficient to ensure zero measure, we note that

$$\#\{\alpha\in\mathbb{A}_n : \alpha\in[0,1], H(\alpha)=H\} \leq n(n+1)(2H+1)^n. \tag{8}$$

By (1), for any such α, we have $\mu([\alpha-\psi(H);\alpha+\psi(H)]) \leq c_6\psi(H)^\delta$ for some $c_6>0$. Elements of $\mathcal{K}_n^*(\psi;K)$ fall in infinitely many of these intervals, and as

$$\sum_{H=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{A}_n \\ \alpha\in[0,1] \\ H(\alpha)=H}} \mu([\alpha-\psi(H);\alpha+\psi(H)]) \leq n(n+1)c_6 \sum_{H=1}^{\infty}(2H+1)^n\psi(H)^\delta,$$

which converges by assumption, the measure of $\mathcal{K}_n^*(\psi;K)$ is zero by the Borel–Cantelli lemma. $\square$

We now prove corollary 2. Let n be fixed and suppose that $w_n^*(x) > \min\{2n-1, (n+1-\delta)/\delta\}$. Then, for some $\epsilon > 0$, $x \in \mathcal{K}_n^*(r \mapsto r^{-w_n^*(x)-1-\epsilon}; K)$. If $w_n^*(x) > 2n-1$, then

$$\sum_{r=1}^{\infty} r^{2n\delta-1} r^{(-2n+1-1-\epsilon)\delta} = \sum_{r=1}^{\infty} r^{-1-\delta\epsilon} < \infty,$$

so x is in some set of measure zero by theorem 1. If $w > (n+1-\delta)/\delta$, then

$$\sum_{r=1}^{\infty} r^{n} r^{((-n-1+\delta)/\delta-1-\epsilon)\delta} = \sum_{r=1}^{\infty} r^{-1-\delta\epsilon} < \infty,$$

so again x is in some set of measure zero by theorem 1. Hence, for fixed $n \in \mathbb{N}$, $w_n^*(x) \leq \min\{2n-1, (n+1-\delta)/\delta\}$ almost surely. This proves the first part of the corollary.

Suppose now that x is not an S^*-number of type $\leq \min\{2, 1/\delta\}$. Either x is algebraic, an S^*-number of higher type or a T^*- or U^*-number. The algebraic numbers may be disregarded, as there are only countably many of them, and so almost no such numbers (recall that μ is non-atomic). Consider now the set

$$\bigcup_{n=1}^{\infty} \{x \in K : w_n^*(x) > \min\{2n-1, (n+1-\delta)/\delta\}\}.$$

By the above argument, this is a countable union of null sets, and so a null set itself. Hence, almost all $x \in K$ are in the complement, and so satisfy for any $n \in \mathbb{N}$,

$$w_n^*(x) \leq \min\{2n-1, (n+1-\delta)/\delta\},$$

and consequently

$$w^*(x) = \limsup_{n\to\infty} \frac{w_n^*(x)}{n} \leq \limsup_{n\to\infty} \min\{(2n-1)/n, (n+1-\delta)/n\delta\} = \min\{2, 1/\delta\}.$$

This completes the proof. □

Finally, we prove corollary 3. From [2, Theorem 3.4] we know that

$$w_n(x) \leq w_n^*(x) + n - 1.$$

Inserting the bounds of corollary 2 into this inequality yeilds the first statement. The second statement is derived as in the proof of corollary 2. Alternatively, it can be deduced directly from this corollary together with [2, Theorem 3.6]. □

4. Proof of Theorem 4

We will define a family indexed by k_0 of covers of K with intervals and use these to obtain an upper bound on the Hausdorff s-measure of $\mathcal{K}_n^*(\psi; K)$ which tends to zero as k_0 tends to infinity. This will imply the theorem.

Suppose first that the first series converges. We define

$$\mathcal{D}_k = \left\{\alpha \in \mathbb{A}_n : 2^k \leq H(\alpha) < 2^{k+1}, [\alpha - \psi(2^k); \alpha + \psi(2^k)] \cap K \neq \emptyset\right\}.$$

With c_4 defined by (6), we see that by disjointness

$$\begin{aligned}\#\mathcal{D}_k c_1 4^{-\delta} c_4^{\delta} 2^{-2kn\delta} &\leq \mu\Big(\bigcup_{\alpha\in\mathcal{D}} [\alpha - \tfrac{1}{4}c_4 2^{-2kn}; \alpha + \tfrac{1}{4}c_4 2^{-2kn}]\Big) \\ &\leq \mu(K) = 1.\end{aligned}$$

Consequently,

$$\#\mathcal{D}_k \leq c_1^{-1} 4^{\delta} c_4^{-\delta} 2^{2kn\delta}. \tag{9}$$

For any $k_0 \in \mathbb{N}$, the family

$$\bigcup_{k=k_0}^{\infty} \bigcup_{\alpha \in \mathcal{D}} \{[\alpha - \psi(2^k); \alpha + \psi(2^k)]\}$$

covers $\mathcal{K}_n^*(\psi; K)$. Hence, for $s \in [0, \delta]$,

$$\mathcal{H}^s(\mathcal{K}_n^*(\psi; K)) \le \sum_{k=k_0}^{\infty} \sum_{\alpha \in \mathcal{D}} 2^s \psi(2^k)^s \le 2^s c_1^{-1} 4^\delta c_4^{-\delta} \sum_{k=k_0}^{\infty} 2^{2kn\delta} \psi(2^k)^s.$$

Using Cauchy's condensation criterion, we see that the latter tends to zero as k_0 tends to infinity by assumption.

Suppose now that the second series converges. In this case, we cover $\mathcal{K}_n^*(\psi; K)$ by

$$\bigcup_{H=H_0}^{\infty} \bigcup_{\substack{\alpha \in \mathbb{A}_n \\ \alpha \in [0,1] \\ H(\alpha)=H}} \{[\alpha - \psi(H); \alpha + \psi(H)]\},$$

for any $H_0 \in \mathbb{N}$. Using (8),

$$\mathcal{H}^s(\mathcal{K}_n^*(\psi; K)) \le \sum_{H=H_0}^{\infty} \sum_{\substack{\alpha \in \mathbb{A}_n \\ \alpha \in [0,1] \\ H(\alpha)=H}} 2^s \psi(H)^s \le 2^s n(n+1) \sum_{H=H_0}^{\infty} (2H+1)^n \psi(H)^s,$$

which tends to zero as H_0 tends to infinity by assumption. □

Finally, we prove corollary 5. Let $\eta > 0$ be fixed and let $s = 2n\delta/\lambda_\psi + \eta$. Choose $\epsilon < 2n\delta\eta(2n\delta/\lambda_\psi + \eta)^{-1}$ and let r_0 be sufficiently large that for $r \ge r_0$, $\log \psi(r)/\log r \ge -\lambda_\psi + \epsilon$. Then,

$$\sum_{r=r_0}^{\infty} r^{2n\delta-1} \psi(r)^s \le \sum_{r=r_0}^{\infty} r^{2n\delta-1-2n\delta-2n\delta\eta+\epsilon(2n\delta/\lambda_\psi+\eta)} = \sum_{r=r_0}^{\infty} r^{-1-\epsilon'},$$

where $\epsilon' = 2n\delta\eta - \epsilon(2n\delta/\lambda_\psi + \eta)^{-1} > 0$. Hence, the series converges, and

$$\dim_{\mathrm{H}}(\mathcal{K}_n^*(\psi; K)) \le \frac{2n\delta}{\lambda_\psi} + \eta.$$

As $\eta > 0$ was arbitrary, the first upper bound of the corollary follows. The second upper bound follows as $\mathcal{K}_n^*(\psi; K) \subseteq \mathcal{K}_n^*(\psi; [0,1])$, so that

$$\dim_{\mathrm{H}}(\mathcal{K}_n^*(\psi; K)) \le \dim_{\mathrm{H}}(\mathcal{K}_n^*(\psi; [0,1])) = (n+1)/\lambda_\psi$$

by [2, Theorem 6.7]. Of course, this could also be shown to follow from the convergence of the second series of theorem 4. □

5. Concluding remarks

The results of this paper are unlikely to be best possible, except possibly when $n = 1$ where approximation by rationals is considered. The reason for this is the use of inequality (6). When $n = 1$, this is sharp, since for $p/q, p'/q' \in \mathbb{Q}$ with $2^k \le q, q' < 2^{k+1}$, , $|p/q - p'/q'| \ge 1/(qq') \ge 2^{-2k}$. Under the assumption this is best possible, since if $(q, q') = 1$, we may choose p, p' such that $qp' - q'p = 1$. However, inequality (6) is not in general best possible when we consider algebraic numbers of bounded degree, which are not as regularly distributed as rationals. Indeed, if it was best possible, we would have to the order of 2^{2kn} algebraic numbers of degree $\leq n$ and height $H \in [2^k; 2^{k+1})$ in the unit interval, but by (8), there are only to the order of $2^{k(n+1)}$ such numbers. Hence, there must be larger gaps between at least some of these numbers. Identifying these gaps is a difficult problem, and at present we have no way of ensuring that the large gaps do not all fall outside of the sets K. Hence, our result is not as strong as could be desired.

In the light of [2, Theorem 6.7], the sharpest upper bound is likely to be obtained when the exponent $2n\delta - 1$ of r in the first series of theorems 1 and 4 is replaced by $(n+1)\delta - 1$. Such an upper bound would imply that for any $K \subseteq [0,1]$ supporting measure μ satisfying (1), μ-almost all numbers in K would be S^*-numbers of type ≤ 1. It would also remove the restrictions on δ under which corollary 5 improves upon what is known from [2, Theorem 6.7]. Better knowledge of the distribution of algebraic numbers than what is used here is clearly needed in order to prove this. It will be the subject of further study.

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