UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

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Preprint Series No.: 8

June 2005

2005/06/08

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ON THE CLOSURE OF STEINBERG FIBERS IN THE WONDERFUL COMPACTIFICATION

XUHUA HE AND JESPER FUNCH THOMSEN

Abstract. By a case-free approach we give a precise description of the closure of Steinberg fibers within the wonderful compactification of a not necessarily connected semisimple algebraic group. For connected groups this description was earlier obtained by the first author.

1. Introduction

Let G be a connected, simple algebraic group over an algebraically closed field k. Let T denote a maximal torus of G and let W denote the associated Weyl group. Fix a set of simple reflection s_i, $i \in I$, in W. For each subset $J \subset I$ we let W_J denote the subgroup of W generated by s_i for $i \in J$, and W^J be the set of minimal length coset representatives of W/W_J.

The wonderful compactification X of G (see e.g. [DP], [Str]), is a smooth projective $(G \times G)$-variety containing G as an open subset. The $G \times G$-orbits Z_J of X are indexed by the subsets J of I. We fix certain base points h_J of Z_J (see 1.2 for the precise definition of h_J) and define for each $w \in W^J$ a subset $Z_J^w = \text{diag}(G)(Bw, 1)h_J$ of Z_J, where $\text{diag}(G)$ denotes the diagonal in $G \times G$. Then Z_J^w is a locally closed subvariety of X and $X = \bigsqcup_{J \subset I, w \in W^J} Z_J^w$ (see [L3]). The subset Z_J^w of X is called a G-stable piece.

The G-stable pieces were first introduced by Lusztig to study the G-orbits and parabolic character sheaves. However, his original definition was based on some inductive method. The (equivalent) definition that we used above was due to the first author in [H1]. What we need in this paper is that the dimension of Z_J^w is equal to $\dim(G) - l(w) - |I - J|$, where $l(w)$ is the length of w and $|I - J|$ is the cardinality of the set $I - J$. More properties about the G-stable pieces can be found in [L3] and [H2]. The G-stable pieces were also used by Evens and Lu in [EL] to study the Poisson structure and symplectic leaves.

Let F be a Steinberg fiber in G, i.e. the set of elements whose semisimple part lies in a fixed conjugacy class. Some examples are the unipotent variety and the regular semisimple conjugacy classes. It is of some interest to study the closure of F in X.

In [L3], Lusztig gave an explicit description for the closure of the unipotent variety in the group compactification when $G = PGL_2$ or PGL_3. In [Spr2], Springer studied the closure of arbitrary Steinberg fiber for any connected, simple algebraic group and obtained some partial results. Based on their result, the first author got an explicit description of the closure using the G-stable pieces in [H1]. For $w \in W$ let supp(w) denote the minimal subset of I such that w is contained in W_J. The precise statement of [H1] is as follows.

2000 *Mathematics Subject Classification.* 14M17, 20G15.

Theorem. *Let F be a Steinberg fibers of G and $\bar{F}$ be its closure in X. Then*

$$\bar{F} - F = \bigsqcup_{J \subset I} \bigsqcup_{\substack{w \in W^J \\ \operatorname{supp}(w) = I}} Z_J^w.$$

As a consequence, the boundary of the closure is independent of the choice of the Steinberg fibers. More generally, the second author has proved in [T] that the boundary of the closure of F within any equivariant embedding of G is independent of the choice of F.

The proof in [H1] was based on a case-by-case checking. The main purpose of this paper is to generalize the result to the disconnected group case with a more conceptual (and easier) proof. We will also prove some properties about the "nilpotent cone" of X.

We thank Lusztig and Springer for some useful discussions and comments. We also thank Jantzen for pointing out the results by Mohrdieck.

2. Wonderful compactifications and G-stable pieces

2.1. Let G denote a connected semisimple linear algebraic group of adjoint type over an algebraically closed field k. Let B be a Borel subgroup of G, B^- be the opposite Borel subgroup and $T = B \cap B^-$. Let R denote the set of roots defined by T and let R^+ denote the set of positive roots defined by B. Let $(\alpha_i)_{i \in I}$ be the set of simple roots. For $i \in I$, we denote by ω_i and s_i the fundamental weight and the simple reflection corresponding to α_i.

We denote by W the Weyl group associated to T. For any subset J of I, let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and W^J be the set of minimal length coset representatives of W/W_J.

For $J \subset I$, let $P_J \supset B$ be the standard parabolic subgroup defined by J and $P_J^- \supset B^-$ be the opposite of P_J. Set $L_J = P_J \cap P_J^-$. Then L_J is a Levi subgroup of P_J and P_J^-. The semisimple quotient of L_J of adjoint type will be denoted by G_J. We denote by π_{P_J} (resp. $\pi_{P_J^-}$) the projection of P_J (resp. P_J^-) onto G_J.

2.2. Assume that G is of adjoint type and let X denote the wonderful compactification of G. It is known that X is an irreducible, smooth projective $(G \times G)$-variety with finitely many $G \times G$-orbits Z_J indexed by the subsets J of I. As a $(G \times G)$-variety the orbit Z_J is uniquely isomorphic to $(G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on the right on $G \times G$ and on the left on G_J by $(q, p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)^{-1}$. Let h_J be the image of $(1, 1, 1)$ in Z_J under this isomorphism.

We denote by $\operatorname{diag}(G)$ the image of the diagonal embedding of G in $G \times G$. For $J \subset I$ and $w \in W^J$, set $Z_J^w = \operatorname{diag}(G)(Bw, 1)h_J$. Then Z_J^w is a locally closed subvariety of X and $X = \bigsqcup_{J \subset I, w \in W^J} Z_J^w$ (see [L3]). We call Z_J^w a G-stable piece.

3. Preliminaries on disconnected groups

In this section G denotes a connected semisimple linear algebraic group over an algebraically closed field k. We assume furthermore that G is either simply connected or of adjoint type.

3.1. Let $\hat{G}$ be a possibly disconnected linear algebraic group with identity component G. An element $g \in G^1$ is called quasi-semisimple if g normalizes a Borel subgroup of G and a maximal torus contained in the Borel subgroup (see [Ste2, 9]). We have the following properties.

(a) *If g is semisimple, then it is quasi-semisimple.* See [Ste2, 7.5, 7.6].

(b) *Let $g \in G^1$ be a quasi-semisimple element and T_1 be a maximal torus of $Z_G(g)^0$, where $Z_G(g)^0$ is the identity component of $\{x \in G \mid xg = gx\}$. Then any quasi-semisimple element in gG is G-conjugate to some element of gT_1.* See [L1, 1.14].

(c) *g is quasi-semisimple if and only if the G-conjugacy class of g is closed in G^1.* The if-part was due to [Spa, 1.15(f)], the only-if-part was due to Lusztig in an unpublished note (see [H1, 4.1]).

3.2. Let G^1 be a connected component of $\hat{G}$. By the conjugacy of Borel subgroups and maximal tori we may find an element $g_0 \in G^1$ such that ${}^{g_0}B = B$ and ${}^{g_0}T = T$. Let δ be the automorphism of G given by conjugation with g_0. The induced automorphism of T is then independent of the choice of g_0. Consequently, also the induced automorphism of the weight lattice $\Lambda(R)$ of the root system R is independent of the choice of g_0. By abuse of notation we denote the latter automorphism by δ. Then R, R^+ and the set of simple roots $(\alpha_i)_{i \in I}$ are all invariant under δ. Thus δ generates a finite group of permutations of R and I and orbits under this action will be called δ-orbits.

For each simple root α choose an associated root homomorphism $x_\alpha : k \to G$. By substituting g_0 with $g_0 t$, for some $t \in T$, we may obtain that g_0 satisfies the relation $g_0 x_\alpha(z) g_0^{-1} = x_{\delta(\alpha)}(z)$ for all simple roots α and $z \in k$. In the following we will assume that x_α and g_0 has been fixed in this way. In particular, the order of δ regarded as an automorphism of G coincides with the order of δ regarded as a permutation of I.

Note that if $G^1 = G$, then δ acts as the identity map on the weight lattice $\Lambda(R)$ and thus also on R and I.

3.3. Let T^δ be the set of fixed points of the map $\delta : T \to T$. It is easily seen that T^δ is a torus of rank equal to the number l of δ-orbits in I. Moreover, T^δ is easily seen to contain regular semisimple elements and consequently $Z_G(T^\delta) = T$. Thus any maximal torus of $Z_G(g_0)^0$ containing T^δ is contained in $T \cap Z_G(g_0) = T^\delta$. Therefore T^δ is a maximal torus of $Z_G(g_0)^0$. By 3.1(b), any quasi-semisimple element in G^1 is G-conjugate to some element in $T^\delta g_0$.

Let $\mathrm{Spec}(k[G^1]^G) = G^1//G$ be the quotient of G^1 by the group G acting by conjugation. By invariant theory we may identify $G^1//G$ with the set of closed G-orbits within G^1. Furthermore, by 3.1(c) we may identify the latter set with the set of conjugacy classes of quasi-semisimple elements. The quotient morphism $\mathrm{St} : G^1 \to G^1//G$ then sends $g \in G^1$ to the unique G-conjugacy class of a quasi-semisimple element contained in the closure of the G-conjugacy class of g. If $G^1 = G$, then St is just the Steinberg morphism of G. Hence for arbitrary G^1, we call St the Steinberg morphism of G^1 and the fibers the Steinberg fibers of G^1.

To each $g \in G^1$ we may consider the automorphism of G induced by conjugation with g. By [Ste2, 7.2] this automorphism fixes some Borel subgroup of G and hence g is G-conjugate to some element of G^1 fixing B. In particular, g is G-conjugate to an element of the form bg_0 for some $b \in B$. Write $b = tu$ where $t \in T$ and u is an element of the unipotent radical U of B. It is then easily seen that there exists an element $t_1 \in T$, such that $t_1 t g_0 t_1^{-1} \in T^\delta g_0$. Hence, g is G-conjugate to some element in $T^\delta U g_0$, i.e. we may assume that $t \in T^\delta$. Notice now that the quasi-semisimple element tg_0 is contained in the closure of the G-conjugacy class of tug_0. In particular, the the image $\mathrm{St}(g) = \mathrm{St}(tug_0)$ of the Steinberg morphism at g is the G-conjugacy class of tg_0. We conclude that any Steinberg fiber of G^1 is of the form $\cup_{g \in G} g(tUg_0)g^{-1}$ for some $t \in T^\delta$. In particular, any Steinberg fiber is irreducible.

3.4. From now on assume that G is a simple linear algebraic group (of adjoint type) and that $\hat{G}$ and G^1 has been fixed as above. The wonderful compactification X_δ of G^1 is the $(G \times G)$-variety which as a variety is isomorphic to the wonderful compactification X of G and where the $G \times G$-action is twisted by the morphism $G \times G \to G \times G$, $(g, h) \mapsto (g, \delta(h))$ for $g, h \in G$. The $G \times G$-orbits in X_δ then coincide with the associated orbits in X and we let $Z_{J,\delta}$ denote the orbit coinciding with $Z_{\delta(J)}$. Accordingly we let $h_{J,\delta}$ denote the point in $Z_{J,\delta}$ identified with the base point $h_{\delta(J)}$ of $Z_{\delta(J)}$. We consider $G^1 = Gg_0$ as an open subset of X_δ by identifying gg_0, $g \in G$, with $(g, 1)h_{I,\delta}$. For $J \subset I$ and $w \in W^{\delta(J)}$, set $Z^w_{J,\delta} = \mathrm{diag}(G)(Bw, 1)h_{J,\delta}$. Then

$$X_\delta = \bigsqcup_{J \subset I} \bigsqcup_{w \in W^{\delta(J)}} Z^w_{J,\delta}.$$

We call $(Z^w_{J,\delta})_{J \subset I, w \in W^{\delta(J)}}$ the G-stable pieces of X_δ. More details can be found in [H2, 2.4].

3.5. Let G_{sc} be the connected, simply connected group associated to G. Denote by Λ^δ_+ the δ-stable dominant weights of G_{sc}. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l$ be the δ-orbits on I. Set $\omega_{\mathcal{C}_k} = \sum_{i \in \mathcal{C}_k} \omega_i$, where ω_i is the fundamental weight of G_{sc} associated to $i \in I$. For any dominant weight λ with $\lambda = \sum_{i \in I} a_i \omega_i$, set $I(\lambda) = \{i \in I \mid a_i \neq 0\}$. For $w \in W$, let $\mathrm{supp}(w)$ be the set of $i \in I$ such that $w\omega_i \neq \omega_i$ and let $\mathrm{supp}_\delta(w) = \cup_{k \geqslant 0} \delta^k(\mathrm{supp}(w))$. Notice that when $\lambda \in \Lambda_+$ is a dominant weight and $w \in W$ then $w\lambda \neq \lambda$ if and only if $I(\lambda) \cap \mathrm{supp}(w) \neq \varnothing$. The following characterization of $\mathrm{supp}(w)$ is also useful.

Lemma 3.6. *Let $w \in W$ and $i \in I$. Fix a reduced expression $w = s_{i_1} \ldots s_{i_n}$ of w as a product of simple reflections. Then $i = i_j$ for some j if and only if $i \in \mathrm{supp}(w)$.*

Proof. The only-if-part is clear. Consider the if-part. If $i_n \neq i$, then we are done by induction in n. Hence, we may assume that $i_n = i$. But then $w\alpha_i$ is a negative root. Thus $1 = \langle \omega_i, \alpha_i^\vee \rangle = \langle w\omega_i, (w\alpha_i)^\vee \rangle$ and, in particular, we cannot have $w\omega_i = \omega_i$.

3.7. By [Ste2, 9.16] the automorphism δ of G may be lifted to an automorphism of G_{sc} which we denote by σ. For any dominant weight $\lambda \in \Lambda_+$ let $\mathrm{H}(\lambda)$ denote the dual Weyl module for G_{sc} with lowest weight $-\lambda$. We then define ${}^\delta\mathrm{H}(\lambda)$ to be the G_{sc}-module which as a vector space is $\mathrm{H}(\lambda)$ and with G_{sc}-action twisted by the automorphism σ of G_{sc}. Notice that up to a nonzero constant there exists a unique G_{sc}-isomorphism ${}^\delta\mathrm{H}(\lambda) \simeq \mathrm{H}(\delta(\lambda))$. In particular, when $\lambda \in \Lambda^\delta_+$ is δ-invariant there exists a G_{sc}-equivariant isomorphism $f_\lambda : \mathrm{H}(\lambda) \to {}^\delta\mathrm{H}(\lambda)$. Now fix such an f_λ for the rest of the paper.

4. The "nilpotent cone" of X

4.1. For any dominant weight λ there exists (see [DS]) a $G\times G$-equivariant morphism

$$\rho_\lambda : X \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$$

which extends the morphism $G \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ defined by $g \mapsto g[\mathrm{Id}_\lambda]$, where g acts by the left action and where $[\mathrm{Id}_\lambda]$ denotes the class representing the identity map on $\mathrm{H}(\lambda)$. By the definition of X_δ we obtain a $G \times G$-equivariant morphism

$$X_\delta \to \mathbb{P}\big(\mathrm{Hom}_k({}^\delta\mathrm{H}(\lambda), \mathrm{H}(\lambda))\big).$$

Applying f_λ, for $\lambda \in \Lambda_+^\delta$, this induces a map $\rho_{\lambda,\delta} : X_\delta \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$.

4.2. An element in $\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ is said to be nilpotent if it may be represented by a nilpotent endomorphism of $\mathrm{H}(\lambda)$. For $\lambda \in \Lambda_+^\delta$ we let

$$\mathcal{N}(\lambda)_\delta = \{z \in X_\delta \mid \rho_{\lambda,\delta}(z) \text{ is nilpotent}\},$$

and call $\mathcal{N}(\lambda)_\delta$ the nilpotent cone of X_δ associated to the dominant weight λ. In 4.4, we will give an explicit description of $\mathcal{N}(\lambda)_\delta$.

4.3. Define ht to be the height map on the root lattice, i.e., the linear map on the root lattice which maps all the simple roots to 1.

Now assume that $\lambda \in \Lambda_+$. Choose a basis $v_1, ..., v_m$ for $\mathrm{H}(\lambda)$ consisting of T-eigenvectors with eigenvalues $\lambda_1, ..., \lambda_m$ and satisfying $\mathrm{ht}(\lambda_j + \lambda) \geq \mathrm{ht}(\lambda_i + \lambda)$ whenever $j \leq i$. Then B is upper triangular with respect to this basis.

Let A_J be a representative of $\rho_\lambda(h_J)$ in $\mathrm{End}(\mathrm{H}(\lambda))$. Then when $\lambda_j + \lambda$ is a linear combination of the simple roots in J we have that $A_J v_j \in k^\times v_j$. If $\lambda_j+\lambda$ is not a linear combination of the simple roots in J then $A_J v_j = 0$. Assuming that λ is δ-invariant we obtain, by the definitions in 4.1, a similar description for a representative $A_{J,\delta}$ of $\rho_{\lambda,\delta}(h_{J,\delta})$: if $\lambda_j + \lambda$ is a linear combination of the simple roots in J then we have that $A_{J,\delta} v_j \in k^\times f_\lambda(v_j)$; otherwise $A_{J,\delta} v_j = 0$. Notice that we regard $f_\lambda(v_j)$ as an element of $\mathrm{H}(\lambda)$ and as such $f_\lambda(v_j)$ is a T-eigenvector of weight $\delta(\lambda_j)$.

We now obtain.

Proposition 4.4. *Let $\lambda \in \Lambda_+^\delta$, then*

$$\mathcal{N}(\lambda)_\delta = \bigsqcup_{J \subset I} \bigsqcup_{\substack{w \in W^{\delta(J)} \\ I(\lambda)\cap \mathrm{supp}(w) \neq \varnothing}} Z_{J,\delta}^w.$$

Proof. Let $w \in W^{\delta(J)}$. Assume that $w\lambda \neq \lambda$. Note that if x is a linear combination of the simple roots in J with nonnegative coefficients, then $\mathrm{ht}(w\delta(x)) \geqslant \mathrm{ht}(x)$. Hence, $\mathrm{ht}\big(w\delta(-\lambda+x)+\lambda\big) = \mathrm{ht}(w\delta(x))+\mathrm{ht}(-w\lambda+\lambda) > \mathrm{ht}(x)$. Therefore, $(w,1)h_{J,\delta}$ is represented by a strictly upper triangular matrix with respect to the chosen basis in 4.3 above. As a consequence for any $b \in B$, $(bw,1)h_{J,\delta}$ is also represented by a strictly upper triangular matrix. So $(Bw,1)h_{J,\delta} \subset \mathcal{N}(\lambda)_\delta$. Since $\mathcal{N}(\lambda)_\delta$ is G-stable, then $Z_{J,\delta}^w = \mathrm{diag}(G)(Bw,1)h_{J,\delta} \subset \mathcal{N}(\lambda)_\delta$.

Now assume that $w\lambda = \lambda$. Let $b \in B$ and $z = (bw,1)h_{J,\delta}$. Denote by A a representative of $\rho_{\lambda,\delta}(z)$ in $\mathrm{End}(\mathrm{H}(\lambda))$. Let V be the subspace of $\mathrm{H}(\lambda)$ spanned by $v_1, \ldots, v_{m-1}$. Then $Av_m \in k^\times v_m + V$ and $AV \subset V$. Hence, $A^n v_m \neq 0$ for all $n \in \mathbb{N}$. Thus $z \notin \mathcal{N}(\lambda)_\delta$.

Corollary 4.5. *Let* $\lambda, \mu \in \Lambda_+^\delta$, *then*

$$\mathcal{N}(\lambda+\mu)_\delta = \mathcal{N}(\lambda)_\delta \cup \mathcal{N}(\mu)_\delta.$$

Proof. This follows from the relation $I(\lambda+\mu) = I(\lambda) \cup I(\mu)$.

5. A compactification of simply connected group

5.1. Consider the morphism $\psi_i : G_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i)) \oplus k\big)$ defined by $g \mapsto [(g \cdot I_{\mathrm{H}(\omega_i)}, 1)]$, where $I_{\mathrm{H}(\omega_i)}$ denotes the identity map on $\mathrm{H}(\omega_i)$ and g acts on $\mathrm{End}(\mathrm{H}(\omega_i))$ by the left action. Let furthermore $i : G_{\mathrm{sc}} \to X$ denote the the natural $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant morphism and consider the product map

$$\epsilon = \Big(i, \prod_{i \in I} \psi_i\Big) : G_{\mathrm{sc}} \to X \times \prod_{i \in I} \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i) \oplus k\big),$$

Let X_{sc} denote the closure of $\epsilon(G_{\mathrm{sc}})$. Then X_{sc} is an $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant variety containing G_{sc} as an open subset. Notice that unlike X the variety X_{sc} need not be smooth and in general it is not even normal. Still X_{sc} is closely related to X as seen by the following result.

Lemma 5.2. *The projection morphism* $\pi : X_{\mathrm{sc}} \to X$ *defines a bijection between* $X_{\mathrm{sc}} - G_{\mathrm{sc}}$ *and* $X - G$*. In particular,* π *is a finite morphism.*

Proof. As π is dominant and projective it follows that π is surjective. Let x denote an element of X_{sc} and consider its image $\psi_i(x) = [(f_i, a_i)]$. Notice that the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-invariant homogeneous polynomial function on $\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i)) \oplus k\big)$ defined by $[(f,a)] \mapsto \det(f) - a^{\dim_k(\mathrm{H}(\omega_i))}$, vanishes on G_{sc} and hence also on X_{sc}. As a consequence we have a commutative diagram

$$\begin{tikzcd} X_{\mathrm{sc}} \arrow[r, "\epsilon"] \arrow[d, "\pi"] & X \times \prod_{i\in I}\Big(\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i) \oplus k\big) - \mathbb{P}(0 \oplus k)\big)\Big) \arrow[d] \\ X \arrow[r, "\mathrm{id}_X \times \prod_{i \in I} \rho_{\omega_i}"] & X \times \prod_{i \in I} \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i))\big) \end{tikzcd}$$

where the right vertical morphism is the defined via the natural projection maps $\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i)) \oplus k\big) - \mathbb{P}(0 \oplus k) \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i))\big)$. Assume now that x is an element of the boundary $X_{\mathrm{sc}} - G_{\mathrm{sc}}$. As the dimension of G_{sc} and X_{sc} coincide the $(G,1)$-stabilizer of x has strictly positive dimension. In particular, the image $[(f_i, a_i)]$ has the same property. Thus, the endomorphism f_i is not invertible and thus $a_i = 0$. This proves that

$$X_{\mathrm{sc}} - G_{\mathrm{sc}} \subset X \times \prod_{i=1}^{l_0} \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_i))\big)$$

and hence π maps $X_{\mathrm{sc}} - G_{\mathrm{sc}}$ injectively to the boundary $X - G$. This proves the first assertion. That π is a finite morphism now follows as π is quasifinite and projective.

5.3. Let λ be any dominant weight and consider the map

$$\psi_\lambda : G_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big)$$

defined by $g \mapsto [(g \cdot I_{\mathrm{H}(\lambda)}, 1)]$, where $I_{\mathrm{H}(\lambda)}$ denotes the identity map on $\mathrm{H}(\lambda)$. Let X_{sc}^λ denote the closure of the image of the product map

$$(\epsilon, \lambda) : G_{\mathrm{sc}} \to X_{\mathrm{sc}} \times \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big).$$

Then the projection map from $X_{\mathrm{sc}}^{\lambda}$ to X_{sc} is an isomorphism. In particular, we obtain an extension $X_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big)$ of the morphism ψ_λ to X_{sc} which we also denote by ψ_λ. As in the proof of Lemma 5.2 we may prove that

$$\psi_\lambda(X_{\mathrm{sc}}) \subset \Big(\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda)) \oplus k\big) - \mathbb{P}\big(0 \oplus k\big)\Big)$$

and that the induced map $X_{\mathrm{sc}} \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ is compatibly with $\pi : X_{\mathrm{sc}} \to X$ and the map $\rho_\lambda : X \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$.

5.4. The variety X_{sc} is a compactification of G_{sc} with the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$ action defined in the natural way. Let $X_{\mathrm{sc},\delta}$ be the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-variety which as a variety is isomorphic to X_{sc} and where the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-action is twisted by the morphism $G_{\mathrm{sc}} \times G_{\mathrm{sc}} \to G_{\mathrm{sc}} \times G_{\mathrm{sc}}$, $(g, h) \mapsto (g, \sigma(g))$ for $g, h \in G_{\mathrm{sc}}$.

Let $G_{\mathrm{sc}}\sigma$ be the connected component $(G_{\mathrm{sc}}, \sigma)$ of the disconnected group $G_{\mathrm{sc}} \rtimes \langle\sigma\rangle$. Then $X_{\mathrm{sc},\delta}$ is a compactification of $G_{\mathrm{sc}}\sigma$ and the morphism $G_{\mathrm{sc}}\sigma \to G^1$ extends to a finite morphism $X_{\mathrm{sc},\delta} \to X_\delta$. Notice that by Lemma 5.2 we may identify the boundaries of $X_{\mathrm{sc},\delta}$ and X_δ and we may therefore also regard $Z_{J,\delta}^w$, for $J \neq I$, as subsets of $X_{\mathrm{sc},\delta}$.

5.5. Let Tr_i denote the trace function on $\mathrm{End}(\mathrm{H}(\omega_{\mathcal{C}_i}))$. To each $a_i \in k$ we may associate a global section (Tr_i, a_i) of the line bundle $\mathcal{O}_i(1) := \mathcal{O}_{\mathbb{P}(\mathrm{End}(\mathrm{H}(\omega_{\mathcal{C}_i}))\oplus k)}(1)$ on $\mathbb{P}\big(\mathrm{End}(\mathrm{H}(\omega_{\mathcal{C}_i})) \oplus k\big)$. The pull back of (Tr_i, a_i) to $X_{\mathrm{sc},\delta}$ is then a global section f_{i,a_i}^δ of a line bundle on $X_{\mathrm{sc},\delta}$. In the following, we will study the common zero set $Z(a_1, \dots, a_l)$ of the sections f_{i,a_i}^δ, for varying $a_i \in k$. By choosing a trivialization of the pull back of $\mathcal{O}_i(1)$ to $G_{\mathrm{sc}}\sigma$ we may think of f_{i,a_i}^δ as a function on $G_{\mathrm{sc}}\sigma$, and by abuse of notation we also denote this function by f_{i,a_i}^δ. Notice that the function f_{i,a_i}^δ on $G_{\mathrm{sc}}\sigma$ is determined up to a nonzero constant.

6. A generalization of a result by Mohrdieck

The following section gives a presentation of a results by Mohrdieck [M]. The original results by Mohrdieck assumes the characteristic of k to be different from 2 and not to divide the order of σ. However, only small modifications of the approach by Mohrdieck is needed in order to obtain the characteristic independent Corollary 6.6.

6.1. Let T_{sc} (resp. B_{sc}) denote the inverse image of T (resp. B) under the canonical map $G_{\mathrm{sc}} \to G$. Then T_{sc} is a σ-stable maximal torus of G_{sc} and we let T_{sc}^σ denote the set of σ-invariant elements within T_{sc}. We identify T_{sc} with $(k^*)^{l_0}$ in such a way that value of the fundamental weight ω_i on $(t_1, \dots, t_{l_0})$ is equal to t_i. Then $(t_1, \dots, t_{l_0})$ is an element of T_{sc}^σ exactly when $t_i = t_j$ for i and j in the same δ-orbit in I. The δ-invariant elements $\Lambda(R)^\delta$ of the character group $\Lambda(R)$ of T_{sc} is freely generated by the characters $\omega_{\mathcal{C}_i}$, $i = 1, \dots, l$ and defines a quotient torus $T' \simeq (k^*)^l$ of T_{sc}. The induced map

$$T_{\mathrm{sc}}^\sigma \to T' \simeq (k^*)^l$$

is then given by

$$(t_1, \dots, t_{l_0}) \mapsto (s_1, \dots, s_l),$$

where $s_i = t_j^{|\mathcal{C}_i|}$ for any $j \in \mathcal{C}_i$. Consider the set $(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$ of σ-invariant elements in $T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma$. Then $(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$ is a finite group which acts on T_{sc}^σ by letting tT_{sc}^σ act on $s \in T_{\mathrm{sc}}^\sigma$ by $ts\sigma(t^{-1})$. It is easily seen that the effect of the action of $(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$

on a element $(t_1, \dots, t_{l_0}) \in T_{\mathrm{sc}}$ is that the coordinates of $(t_1, \dots, t_{l_0})$ within a single δ-orbit of I is multiplied with a $|\mathcal{C}_i|$-th root of 1. In particular, we obtain

Lemma 6.2. *The morphism $T_{\mathrm{sc}}^\sigma \to T'$ respects the action of $(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$ on T_{sc}^σ and the induced map $\eta : T_{\mathrm{sc}}^\sigma/(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma \to T'$ is bijective. Moreover, the group of σ-invariant elements W^σ within the Weyl group W acts naturally on both $T_{\mathrm{sc}}^\sigma/(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$ and T' and under these actions η is W^σ-equivariant.*

6.3. For a root α we let $\alpha' \in \Lambda(R)$ denote the sum of the roots within the δ-orbit of α. When G_{sc} is not of type A_{2n} we then let $R' \subset \Lambda(R)$ consists of the elements α' for $\alpha \in R$. If G_{sc} is of type A_{2n} we instead let R' be the union of the element α' for $\alpha \in R$ satisfying $\langle \delta(\alpha), \alpha^\vee \rangle = 0$, and $2\alpha'$ for $\alpha \in R$ satisfying $\langle \delta(\alpha), \alpha^\vee \rangle \neq 0$ and $\alpha \neq \delta(\alpha)$. The set R' together with the σ-invariant Weyl group W^σ defines an irreducible root system (cf. [M, Sect. 2]). We let G' denote the associated connected and simply connected linear algebraic group. As the weight lattice of R' coincides with the δ-invariant elements in $\Lambda(R)$ we may consider T' as a maximal torus of G'. The set of positive roots in R defines in a natural way a choice of positive roots in R'. The associated Borel subgroup of G' containing T' will be denoted by B'.

6.4. Let $\lambda \in \Lambda_+^\delta$ be a δ-invariant dominant weight. We may then regard λ as a dominant T'-weight. The associated dual Weyl G'-module is denoted by $\mathrm{H}'(\lambda)$. Let χ_i' denote the G'-character associated to the G'-module $\mathrm{H}'(\omega_{\mathcal{C}_i})$. The following result is then essentially due to Jantzen (cf. proof of Prop. 3.15 in [M]).

Theorem 6.5. *There exists a nonzero constant $c_i \in k^*$ such that $\chi_i'(t') = c_i f_{i,0}^\delta(t\sigma)$ for all $t \in T_{\mathrm{sc}}$ and with t' denoting the image of t under the natural quotient map $T \to T'$.*

Proof. Chose $t_0 \in T$ such that the composition $\sigma' := \mathrm{int}(t_0) \circ \sigma$ of σ with the interior automorphism of G_{sc} defined by t_0, is a graph automorphism of G_{sc} of the form considered in Section 9 of [J]. Applying [J, Satz 9] we obtain that $\chi_i'(t') = c_i f_{i,0}^\delta(tt_0\sigma)$ for $t \in T_{\mathrm{sc}}$ and some nonzero constant c_i. We claim that t_0' is a central element in G' which will prove the statement. To see this notice that by [J, Sect. 9] the element σ' satisfies $\sigma'(x_\alpha'(z)) = x_{\delta(\alpha)}'(z)$ for α simple and some specific chosen root homomorphisms $x_\alpha' : k \to G_{\mathrm{sc}}$ defined from a Chevalley basis of the Lie algebra of G_{sc}. Similarly σ satisfies by 3.2 that $\sigma(x_\alpha(z)) = x_{\delta(\alpha)}(z)$ for $z \in k$. Fix nonzero constants c_α such that $x_\alpha(z) = x_\alpha'(c_\alpha z)$. Then $c_{\delta(\alpha)}\delta(\alpha)(t_0) = c_\alpha$ for all simple roots α and hence $\prod_{\alpha \in \mathcal{C}_i} \alpha(t_0) = 1$ for all $i = 1, \dots, l$. In particular, for each $\alpha' \in R'$ we have $\alpha'(t_0) = 1$ and hence t_0' is central in G'.

Notice that $f_{i,0}^\delta$ is an G_{sc}-invariant function on $G_{\mathrm{sc}}\sigma$. Hence $f_{i,0}^\delta$ induces a morphism $G_{\mathrm{sc}}\sigma//G_{\mathrm{sc}} \to k$ which we denote by $\bar{f}_{i,0}^\delta$.

Corollary 6.6. *The product morphism $\prod_{i=1}^l \bar{f}_{i,0}^\delta : G_{\mathrm{sc}}\sigma//G_{\mathrm{sc}} \to \mathbb{A}^l$ is bijective.*

Proof. By the considerations in 3.3 we first of all have an injective morphism

$$g^* : k[G_{\mathrm{sc}}\sigma]^{G_{\mathrm{sc}}} \to k[T_{\mathrm{sc}}^\sigma]^N,$$

where $N := W^\sigma \ltimes (T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$ denotes the semidirect product of the finite groups W^σ and $(T_{\mathrm{sc}}/T_{\mathrm{sc}}^\sigma)^\sigma$. Furthermore, by the considerations in 6.1 above we also have an injective morphism

$$f^* : k[T']^{W^\sigma} \to k[T_{\mathrm{sc}}^\sigma]^N.$$

By [Ste2, Lemma 7.3] the ring $k[T']^{W^\sigma}$ is a polynomial ring generated by the restriction of $\chi_1', \ldots, \chi_l'$ to T'. In particular, Theorem 6.5 implies that the image of g^* contains the image of f^* and thus there exists an induced injective map

$$h^* : k[T']^{W^\sigma} \to k[G_{\mathrm{sc}}\sigma]^{G_{\mathrm{sc}}}.$$

such that $g^* \circ h^* = f^*$. By the description of the map $T_{\mathrm{sc}}^\sigma \to T'$ the map f^* is integral and hence the same is true for h^* and g^*. Consider now that induced map of affine varieties

$$\begin{tikzcd} T_{\mathrm{sc}}^\sigma/N \arrow[r, "g"] \arrow[d, "f"] & G_{\mathrm{sc}}\sigma//G \arrow[dl, "h"] \\ \mathbb{A}^l \simeq T'/W^\sigma & \end{tikzcd}$$

By Lemma 6.2 the morphism f is bijective. Hence, g is injective. As g^* is injective and integral we conclude that g, and thus also h, is bijective. Finally notice that by definition h is the product of $c_i \bar{f}_{i,0}^\delta$, $i = 1, \ldots, l$, for certain nonzero constants c_i.

Remark. 1. For connected groups this corollary is just an easy consequence of a classical result by Steinberg [Ste1, Thm.6.1]. For disconnected groups and characteristics of k different from 2 and not dividing the order of σ the corollary is a consequence of a result by Mohrdieck [M, Thm.3.16]. In fact, the result by Mohrdieck, with the mentioned restrictions on the characteristic, shows that the map $\prod_{i=1}^l \bar{f}_{i,0}^\delta$ is even an isomorphism. It is not clear to us whether this remains valid for arbitrary characteristics.

2. In fact, one can show that $N = N_G(T_{sc}^\sigma \sigma)/T_{sc}^\sigma$, where $N_G(T_{sc}^\sigma \sigma) = \{g \in G \mid gT_{sc}^\sigma \sigma g^{-1} = T_{sc}^\sigma \sigma\}$. The finite group N plays a similar role for the disconnected group as the Weyl group for the connected group.

7. Steinberg fibers and trace maps

Lemma 7.1. *The intersection of $Z(a_1, \ldots, a_l)$ with the boundary $X_{\mathrm{sc}} - G_{\mathrm{sc}}$ of X_{sc} is independent of $a_1, \ldots, a_l$. Moreover, the intersection $Z(a_1, \ldots, a_l) \cap G_{\mathrm{sc}}\sigma$ is a single Steinberg fiber.*

Proof. Similar to the proof of Lemma 5.2 it may be seen that x is an element of $X_{\mathrm{sc}} - G_{\mathrm{sc}}$ exactly when the image $\psi_\lambda(x)$ is of the form $[(f, 0)]$. Thus, the section f_{i,a_i}^δ coincides with $f_{i,0}^\delta$ on the boundary of X_{sc}. This proves the first statement. The latter statement follows by 6.6.

Lemma 7.2. *Let $J \subsetneq I$, $w \in W^{\delta(J)}$ and $b \in B$. If $f_{i,0}^\delta((bw, 1)h_{J,\delta}) = 0$, then either (1) $w\omega_{\mathcal{C}_i} \neq \omega_{\mathcal{C}_i}$ or (2) $\mathcal{C}_i \subset J$ and $w\alpha_j = \alpha_j$ for all $j \in \mathcal{C}_i$.*

Proof. Assume that $w\omega_{\mathcal{C}_i} = \omega_{\mathcal{C}_i}$. Then the diagonal entry of the representative A of $\rho_{\omega_{\mathcal{C}_i},\delta}((bw, 1)h_{J,\delta})$ associated to the lowest weight space is nonzero. In particular, the relation $f_{i,0}^\delta((bw, 1)h_{J,\delta})) = 0$ cannot be satisfied unless there exists a weight $x - \omega_{\mathcal{C}_i}$ of $\mathrm{H}(\omega_{\mathcal{C}_i})$ satisfying that $x = \sum_{j \in J} a_j \alpha_j$, with $a_j \in \mathbb{N} \cup \{0\}$, is nonzero and $w\delta(x) = x$.

Let $K \subseteq J$ denote the set of $j \in J$ such that $a_j \neq 0$. As $x - \omega_{\mathcal{C}_i}$ is a weight of $\mathrm{H}(\omega_{\mathcal{C}_i})$ we know that $\mathcal{C}_i \cap K$ is nonempty. Now $\sum_{j \in K} a_j w\alpha_{\delta(j)} = \sum_{j \in K} a_j \alpha_j$ and thus $\sum_{j \in K} a_j(\mathrm{ht}(w\alpha_{\delta(j)}) - \mathrm{ht}(\alpha_j)) = 0$. As $w \in W^{\delta(J)}$ we conclude that $\mathrm{ht}(w\alpha_{\delta(j)}) \geqslant 1$ and consequently $w\alpha_{\delta(j)}$ is a simple root for all $j \in K$. By the assumption $w\omega_{\mathcal{C}_i} = \omega_{\mathcal{C}_i}$

we know that $w\alpha_{\delta(j)} = \alpha_{\delta(j)}$ for each $j \in \mathcal{C}_i \cap K$. In particular, when $j \in \mathcal{C}_i \cap K$ then $a_{\delta(j)} = a_j$. Hence, $\mathcal{C}_i \cap K$ is invariant under δ and as $\mathcal{C}_i$ is a single δ-orbit we have $\mathcal{C}_i \cap K = \mathcal{C}_i$. This ends the proof.

Lemma 7.3. *Let $J \subsetneq I$. Then*

$$Z(a_1, \dots, a_l) \cap Z_{J,\delta} = \bigsqcup_{\substack{w \in W^{\delta(J)} \\ \mathrm{supp}_\delta(w) = I}} Z^w_{J,\delta}.$$

Proof. By Lemma 7.1 it is enough to consider the case when all a_i are zero. By 4.4, $\bigsqcup_{J \subset I} \bigsqcup_{w \in W^{\delta(J)}, \mathrm{supp}_\delta(w) = I} Z^w_{J,\delta} = \cap_i \mathcal{N}(\omega_{\mathcal{C}_i})_\delta \subset Z(0, \dots, 0)$. For $z \in Z(0, \dots, 0) \cap Z_{J,\delta}$, we have that $z = (g, g)(bw, 1)h_{J,\delta}$ for some $g \in G, b \in B, J \subset I$ and $w \in W^{\delta(J)}$. Then $f^\delta_{i,0}((bw, 1)h_{J,\delta})) = 0$ for all i. It suffices to prove that $\mathrm{supp}_\delta(w) = I$.

If $w = 1$, then by Lemma 7.2, $\mathcal{C}_i \subset J$ for each δ-orbit $\mathcal{C}_i$. Thus $I = J$, which contradicts our assumption. Now assume that $w \neq 1$ and that $\mathrm{supp}_\delta(w) \neq I$. Then there exist simple roots α_i and α_j with $n = -\langle \alpha_j, \alpha_i^\vee \rangle \neq 0$ satisfying that $i \in \mathrm{supp}_\delta(w)$ and $j \notin \mathrm{supp}_\delta(w)$. Let $\mathcal{C}_i$ and $\mathcal{C}_j$ denote the associated δ-orbits of α_i and α_j. As $\mathrm{supp}_\delta(w)$ is δ-stable it follows that $\mathcal{C}_i \subset \mathrm{supp}_\delta(w)$ and $\mathcal{C}_j \subset I - \mathrm{supp}_\delta(w)$.

Now there exists $m \in \mathbb{N}$, such that $\delta^m(i) \in \mathrm{supp}(w)$ and thus $w\omega_{\delta^m(i)} \neq \omega_{\delta^m(i)}$. Hence redefining, if necessary, α_i and α_j we may assume that $w\omega_i \neq \omega_i$. Consider then the relation $\alpha_j = 2\omega_j - n\omega_i - \lambda$ with λ denoting a dominant weight. Now Lemma 7.2 implies that $w\alpha_j = \alpha_j$ and $w\omega_j = \omega_j$ and thus $w(n\omega_i + \lambda) = n\omega_i + \lambda$. As both ω_i and λ are dominant we conclude that $w\omega_i = \omega_i$ which is a contradiction.

Now we will prove the main theorem.

Theorem 7.4. *Let F be a Steinberg fiber of G^1 and $\bar{F}$ its closure in X_δ. Then*

$$\bar{F} - F = \bigsqcup_{J \subset I} \bigsqcup_{\substack{w \in W^{\delta(J)} \\ \mathrm{supp}_\delta(w) = I}} Z^w_{J,\delta}$$

which also coincides with the set $Z(a_1, \dots, a_l) \cap (X - G)$ for all $a_1, \dots, a_l$.

Proof. By Lemma 7.1 the set $F(a_1, \dots, a_l) := Z(a_1, \dots, a_l) \cap G_{\mathrm{sc}}\sigma$ is a single Steinberg fiber. In particular, $F(a_1, \dots, a_l)$ is by 3.3 irreducible. Let C be an irreducible component of $Z(a_1, \dots, a_l)$. By Krull's principal ideal theorem, $\dim(C) \geqslant \dim(G_{\mathrm{sc}}) - l$. Note that

$$\dim(Z^w_{J,\delta}) = \dim(G^1) - l(w) - |I - J| < \dim(G_{\mathrm{sc}}) - l,$$

for $J \neq I$ and $w \in W^{\delta(J)}$ with $\mathrm{supp}_\delta(w) = I$. By Lemma 7.3,

$$\dim\big(C \cap (X_{\mathrm{sc},\delta} - G_{\mathrm{sc}}\sigma)\big) < \dim(G_{\mathrm{sc}}) - l \leq \dim(C).$$

Hence $C \cap G_{\mathrm{sc}}\sigma$ is dense in C and since $C \cap G_{\mathrm{sc}}\sigma \subset F(a_1, \dots, a_l)$, we conclude that C is contained in the closure of $F(a_1, \dots, a_l)$. Thus the closure of $F(a_1, \dots, a_l)$ is $Z(a_1, \dots, a_l)$. In particular, $Z(a_1, \dots, a_l)$ is irreducible.

Let F be a Steinberg fiber of G^1. Then $F = \pi(F(a_1, \dots, a_l))$ for some $a_1, \dots, a_l \in k$. Hence $\bar{F} = \pi(Z(a_1, \dots, a_l))$. The statement now follows from Lemma 7.3 and Lemma 5.2.

Remark. 1. We call an element $w \in W$ a δ-twisted Coxeter element if $l(w) = l$ and $\mathrm{supp}_\delta(w) = I$. (The notation of twisted Coxeter elements was first introduced by Springer in [Spr1]. Our definition is slightly different from his.)

It follows easily from the theorem that $\overline{Z^w_{I-\{i\},\delta}}$ are the irreducible components of $\bar{F}-F$, where $i \in I$ and w runs over all δ-twisted Coxeter elements that are contained in $W^{I-\{\delta(i)\}}$.

2. By the proof of Theorem 7.4 we may also deduce that the closure of a Steinberg fiber F within $X_{\mathrm{sc},\delta}$ coincides with $Z(a_1, \dots, a_l)$ for certain uniquely determined $a_1, \dots, a_l$ depending on F. This result may be considered as an extension of Corollary 6.6 to the compactification $X_{\mathrm{sc},\delta}$ of $G_{\mathrm{sc}}\sigma$. More precisely, notice that the statement of Corollary 6.6 is equivalent to saying that a Steinberg fiber F of $G_{\mathrm{sc}}\sigma$ is the common zero set of the functions f^δ_{i,a_i} for uniquely determined $a_1, \dots, a_l$. Here we think of f^δ_{i,a_i} as regular functions on $G_{\mathrm{sc}}\sigma$ as explained in 5.5. When generalizing to $X_{\mathrm{sc},\delta}$ the only difference is that we have to regard f^δ_{i,a_i} as sections of certain line bundles on $X_{\mathrm{sc},\delta}$.

Similar to [H1, 4.6], we have the following consequence.

Corollary 7.5. *Assume that G^1 is defined and split over $\mathbb{F}_q$, then for any Steinberg fiber F of G^1, the number of $\mathbb{F}_q$-rational points of $\bar{F}-F$ is*

$$\Big(\sum_{w \in W} q^{l(w)}\Big)\Big(\sum_{\mathrm{supp}_\delta(w)=I} q^{l(w_0 w)+L(w_0 w)}\Big),$$

where w_0 is the maximal element of W and for $w \in W$, $l(w)$ is its length and $L(w)$ is the number of simple roots α satisfying $w\alpha < 0$.

References

[DP] C. De Concini and C. Procesi, *Complete symmetry varieties*, in Invariant theory (Montecatini, 1982), 1–44, Lecture Notes in Math., 996, Springer, Berlin, 1983.

[DS] C. De Concini and T. A. Springer, *Compactification of symmetric varieties*, Transform. Groups **4** (1999), no. 2–3, 273–300

[EL] S. Evens and J.-H. Lu, *On the variety of Lagrangian subalgebras, II*, math.QA/0409236.

[H1] X. He, *Unipotent variety in the group compactification*, Adv. in Math., in press.

[H2] X. He, *The G-stable pieces of the wonderful compactification*, submitted.

[J] J.C Jantzen, *Darstellungen Halbeinfacher Algebraischer Gruppen*, Bonner Math. Schriften, Bd. 67, 1973.

[L1] G. Lusztig, *Character sheaves on disconnected groups. I*, Represent. Theory **7** (2003), 374–403 (electronic).

[L2] G. Lusztig, *Character sheaves on disconnected groups. II*, Represent. Theory **8** (2004), 72–124 (electronic).

[L3] G. Lusztig, *Parabolic character sheaves, II*, Mosc. Math. J. **4** (2004), no. 4, 869–896.

[M] S. Mohrdieck, *Conjugacy classes of non-connected algebraic groups*, Transform. groups **8** (2003), 377–395.

[Spa] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Math., 946, Springer, Berlin, 1982.

[Spr1] T. A. Springer, *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159–198.

[Spr2] T. A. Springer, *Some subvarieties of a group compactification*, proceedings of the Bombay conference on algebraic groups, to appear.

[Ste1] R. Steinberg, *Regular elements of semisimple algebraic groups*, Publ. Math. I.H.E.S. 25 (1965), 49–80.

[Ste2] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., 80, Amer. Math. Soc., Providence, R.I., 1968.

[Str] E. Strickland, *A vanishing theorem for group compactifications*, Math. Ann. **277** (1987), no. 1, 165–171

[T] J.F. Thomsen, *Frobenius splitting of equivariant closures of regular conjugacy classes*, math.AG/0502114

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