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ON COMPACTIFICATIONS OF THE STEINBERG ZERO-FIBER

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ABSTRACT. Let G be a connected semisimple linear algebraic group over an algebraically closed field k of positive characteristic and let X denote an equivariant embedding of G. We define a distinguished Steinberg fiber N in G, called the zero-fiber, and prove that the closure of N within X is normal and Cohen-Macaulay. Furthermore, when X is smooth we prove that the closure of N is a local complete intersection.

1. INTRODUCTION

Let G be a connected semisimple linear algebraic group over an algebraically closed field k of positive characteristic. The set of elements in G with semisimple part within a fixed G-conjugacy class is called a Steinberg fiber. Examples of Steinberg fibers include the unipotent variety and the conjugacy class of a regular semisimple element. Lately there has been some interest in describing the closure of Steinberg fibers within equivariant embeddings of the group G (see [He], [H-T], [Spr], [T]).

In this paper we study the closure of a distinguished Steinberg fiber N called the Steinberg zero-fiber (see Section 3 for the precise definition of N). We will prove that the closure $\bar{N}$ of N within any equivariant embedding X of G will be normal and Cohen-Macaulay. Moreover, when X is smooth we will prove that $\bar{N}$ is a local complete intersection. These results will all be proved by Frobenius splitting techniques. As a byproduct we find that $\bar{N}$ has a canonical Frobenius splitting and hence the set of global sections of any G-linearized line bundle on $\bar{N}$ will admit a good filtration.

The presentation in this paper is close to [T], but the setup is somehow opposite. More precisely, in loc.cit. the group G was fixed to be of simply connected type while the Steinberg fiber was arbitrary; in the present paper the Steinberg fiber is fixed but the semisimple group G is of arbitrary type.

It is worth noticing (see [T]) that for a fixed equivariant embedding X of G the boundary $\overline{F} - F$ of the closure of a Steinberg fiber F in G is independent of F. This shows that the results obtained in this paper for the rather special Steinberg zero-fiber will provide some knowledge about closures of Steinberg fibers in general. E.g. the boundary $\overline{F} - F$ will always be Frobenius split (see Theorem 8.1). This suggests, that the results in this paper may be generalized to arbitrary Steinberg fibers. However, we give an example (see Example 8.2) showing that this is not always the case.

2. NOTATION

Let G denote a connected semisimple linear algebraic group over an algebraically closed field k. The associated groups of simply connected and adjoint type will be denoted by G_{sc} and G_{ad} respectively. Let T denote a maximal torus in G and let B denote a Borel subgroup of G containing T. The associated maximal torus and

Borel subgroup in G_{ad} (resp. G_{sc}) will be denoted by T_{ad} and B_{ad} (resp. T_{sc} and B_{sc}).

The set of roots associated to T is denoted by R. We define the set of positive roots R^+ to be the nonzero T-weights of the Lie algebra of B. The set of simple roots $(\alpha_i)_{i \in I}$ will be indexed by I and will have cardinality l. To each simple root α_i we let s_i denote the associated simple reflection in the Weyl group W defined by T. The length of an element w in W is defined as the number of simple reflections in a reduced expression of w. The unique element in W of maximal length will be denoted by w_0.

The weight lattice $\Lambda(R)$ of the root system associated to G is identified with the character group $X^*(T_{\mathrm{sc}})$ of T_{sc} and contains the set of T-characters $X^*(T)$. We let $\alpha_i^\vee$, $i \in I$, be the set of simple coroots and let $\langle , \rangle$ denote the pairing between coweights and weights in the root system R. A weight λ is then dominant if $\langle \lambda, \alpha_i^\vee \rangle$ for all $i \in I$. The fundamental dominant weight associated to α_i, $i \in I$, will be denote by ω_i.

For a dominant weight $\lambda \in \Lambda(R)$ we let $\mathrm{H}(\lambda)$ denote the dual Weyl G_{sc}-module with heighest weight λ, i.e. containing a B-semiinvariant element v_λ^+ of weight λ. The Picard group of G/B may be identified with the weight lattice $\Lambda(R)$ and we let $\mathcal{L}(\lambda)$ denote the line bundle associated to $\lambda \in \Lambda(R)$. The line bundle $\mathcal{L}(\lambda)$ has a unique G_{sc}-linearization so we may regard the set of global sections of $\mathcal{L}(\lambda)$ as a G_{sc}-module. We assume that the notation is chosen such that the set of global sections of $\mathcal{L}(\lambda)$ is isomorphic to $\mathrm{H}(-w_0\lambda)$. For $\lambda, \mu \in \Lambda(R)$ we denote by $\mathcal{L}(\lambda, \mu)$ the line bundle $\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu)$ on the variety $G/B \times G/B$.

3. Steinberg fibers

The set of elements g in G with semisimple part g_s in a fixed G-conjugacy class is called a *Steinberg fiber*. Any Steinberg fiber is a closed irreducible subset of G of codimension l (see [St, Thm.6.11]). Examples of Steinberg fibers include the conjugacy classes of regular semisimple elements and the unipotent variety; i.e. the set of elements with g_s equal to the identity element e.

When $G = G_{\mathrm{sc}}$ is simply connected the Steinberg fibers may also be described as genuine fibers of a morphism $G_{\mathrm{sc}} \to k^l$. Here the i-th coordinate map is given by the G_{sc}-character of the representation $\mathrm{H}(\omega_i)$. In this formulation the fiber N_{sc} above $(0, 0, \ldots, 0)$ is called the *Steinberg zero-fiber*. When G is arbitrary we define the Steinberg zero-fiber N of G to be the image of N_{sc} under the natural morphism $\pi : G_{\mathrm{sc}} \to G$.

The structure of the Steinberg zero-fiber is very dependent on the characteristic of k. In most cases N is just the conjugacy class of a regular semisimple element. At the other extreme we can have that N coincides with the unipotent variety of G.

Remark 3.1. *In the following cases the Steinberg zero-fiber and the unipotent variety of G coincide :*

Type A_n: when $n = p^m - 1$ ($m \in \mathbb{N}$) and $p = \mathrm{char}(k) > 0$.
Type C_n: when $n = 2^m - 1$ ($m \in \mathbb{N}$) and $\mathrm{char}(k) = 2$.
Type D_n: when $n = 2^m$ ($m \in \mathbb{N}$) and $\mathrm{char}(k) = 2$.
Type E_6: when $\mathrm{char}(k) = 3$.
Type E_8: when $\mathrm{char}(k) = 31$.
Type F_4: when $\mathrm{char}(k) = 13$.
Type G_2: when $\mathrm{char}(k) = 7$.

4. Equivariant Embeddings

An *equivariant embedding* of G is a normal $G \times G$-variety containing a $G \times G$-invariant open dense subset isomorphic to G in such a way that the induced $G \times G$-action on G is by left and right translation.

4.1. **The wonderful compactification.** When $G = G_{\mathrm{ad}}$ is of adjoint type there exists a distinguished equivariant embedding $\boldsymbol{X}$ of G called the wonderful compactification (see e.g. [DC-S]). The wonderful compactification of G is a smooth projective variety with finitely many $G \times G$-obits O_J indexed by the subsets J of the simple roots I. We let $\boldsymbol{X}_J$ denote the closure of O_J and assume that the index set is chosen such that $\boldsymbol{X}_{J'} \cap \boldsymbol{X}_J = \boldsymbol{X}_{J' \cap J}$ for all $J, J' \subset I$. Then $\boldsymbol{Y} := \boldsymbol{X}_\emptyset$ is the unique closed orbit in $\boldsymbol{X}$. It is known that $\boldsymbol{Y}$ is isomorphic to $G/B \times G/B$ as a $G \times G$-variety.

To each dominant element λ in the weight lattice $\Lambda(R)$ we let

$$\rho_\lambda^{\mathrm{ad}} : G \to \mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda))),$$

denote the $G \times G$-equivariant morphism defined by letting $\rho_\lambda^{\mathrm{ad}}(e)$ be the element in $\mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda)))$ represented by the identity map on $\mathrm{H}(\lambda)$. Then it is known (see [DC-S]) that $\rho_\lambda^{\mathrm{ad}}$ extends to a morphism $\boldsymbol{X} \to \mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda)))$ which we also denote by $\rho_\lambda^{\mathrm{ad}}$.

Lemma 4.1. *Let v_λ^+ (resp. u_λ^+) denote a nonzero B-stable element in $\mathrm{H}(\lambda)$ (resp. $\mathrm{H}(\lambda)^*$) of weight λ (resp. $-w_0\lambda$). Identify $\mathrm{End}(\mathrm{H}(\lambda))$ with $\mathrm{H}(\lambda) \boxtimes \mathrm{H}(\lambda)^*$. Then the restriction of $\rho_\lambda^{\mathrm{ad}}$ to $\boldsymbol{Y} \simeq G/B \times G/B$ is given by*

$$\rho_\lambda^{\mathrm{ad}}(gB, g'B) = (g, g')[v_\lambda^+ \boxtimes u_\lambda^+].$$

Proof. It suffices to prove that the only $B \times B$-invariant element of the image of $\rho_\lambda^{\mathrm{ad}}$ is $[v_\lambda^+ \boxtimes u_\lambda^+]$. So let $x = [f]$ denote a $B \times B$-invariant element of the image of $\rho_\lambda^{\mathrm{ad}}$ represented by an element $f \in \mathrm{End}(\mathrm{H}(\lambda))$. Then f is $B \times B$-semiinvariant and thus when writing f as an element of $\mathrm{H}(\lambda) \boxtimes \mathrm{H}(\lambda)^*$ it will be equal to $v_\lambda^+ \boxtimes v$ for some B-semiinvariant element v of $\mathrm{H}(\lambda)^*$.

Let L denote the unique simple submodule of $\mathrm{H}(\lambda)$ and let M denote the kernel of the associated morphism $\mathrm{H}(\lambda)^* \to L^*$. Assume that v is contained in M. Then every $G \times G$-translate of f is contained in the subset $L \boxtimes M$ consisting of nilpotent endomorphisms. Now we apply [B-K, Lemma.6.1.4]. It follows that the closure C of the $T \times T$-orbit through $\rho_\lambda^{\mathrm{ad}}(e)$ will contain an element represented by a nilpotent endomorphism. But clearly every element in C will be represented by a semisimple endomorphism of $\mathrm{H}(\lambda)$. This is a contradiction.

As a consequence v is not contained in M and therefore its image in L^* will be a nonzero B-semiinvariant vector. As a consequence v must be a nonzero multiple of u_λ^+. $\square$

4.2. **Toroidal embeddings.** Now let G be an arbitrary connected semisimple group. An equivariant embedding X of G is called toroidal if the natural map $\pi_{\mathrm{ad}} : G \to G_{\mathrm{ad}}$ extends to a morphism $X \to \boldsymbol{X}$. In the present paper toroidal embeddings will play a central role. This is due to the following fact (see [Rit, Prop.3])

Theorem 4.2. *Let X be an arbitrary equivariant embedding of G. Then there exists a smooth toroidal embedding X' of G and a birational projective morphism $X' \to X$ extending the identity map on G.*

An important property of toroidal embeddings is that for each dominant weight λ there exists a $G \times G$-equivariant morphism

$$\rho_\lambda : X \to \mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda)))$$

induced by $\rho_\lambda^{\mathrm{ad}}$. When X is a complete toroidal embedding of G and Y is a closed $G \times G$-orbit in X we may describe the restricted mapping $Y \to \mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda)))$ using Lemma 4.1. Notice that Y maps surjectively to $\boldsymbol{Y} \simeq {}^G/_B \times {}^G/_B$ and that Y is a quotient of $G \times G$. Consequently Y maps bijectively onto $\boldsymbol{Y}$ and hence Y must be $G \times G$-equivariantly isomorphic to ${}^G/_B \times {}^G/_B$. Moreover

Lemma 4.3. *Let X be a complete toroidal embedding of a connected semisimple group G and let Y be a closed $G \times G$-orbit in X. Then Y is $G \times G$-equivariantly isomorphic to ${}^G/_B \times {}^G/_B$ and*

$$\rho_\lambda(gB, g'B) \mapsto (g, g')[v_\lambda^+ \boxtimes u_\lambda^+].$$

Consequently, the pull back of the ample generator $\mathcal{O}_\lambda(1)$ of the Picard group of $\mathbb{P}(\mathrm{End}(\mathrm{H}(\lambda)))$ to Y is isomorphic to $\mathcal{L}(\lambda, -w_0\lambda)$.

4.3. **The dualizing sheaf of equivariant embeddings.** Let X be a smooth equivariant embedding of G. As G is an affine variety the complement $X - G$ of G is of pure codimension 1 in X. Let $X_1, \ldots, X_n$ denote the irreducible components of $X - G$ which are then divisors in X. Let D_i, $i = 1, \ldots, l$, denote the closures of the Bruhat cells $Bw_0 s_i B$ within X. Then also D_i is a divisor in X. When X is the wonderful compactification $\boldsymbol{X}$ of a group of adjoint type we will also denote X_j and D_i by $\boldsymbol{X}_j$ and $\boldsymbol{D}_i$ respectively.

Proposition 4.4. [B-K, Prop.6.2.6] *The canonical divisor of the smooth equivariant embedding X is*

$$K_X = -2\sum_{i \in I} D_i - \sum_{j=1}^n X_j.$$

The line bundle $\mathcal{L}(D_i)$ associated to the divisor D_i is connected to the morphisms ρ_λ as explained by

Lemma 4.5. *Assume that X is a toroidal embedding. Then there exists an isomorphism*

$$\mathcal{L}(D_i) \simeq \rho_{\omega_i}^*(\mathcal{O}_{\omega_i}(1)),$$

such that $\rho_{\omega_i}^*(u_{\omega_i}^+ \boxtimes v_{\omega_i}^+)$ is the canonical section of $\mathcal{L}(D_i)$.

Proof. Consider first the case when X is the wonderful compactification of G_{ad}. Consider the pull back $s_{\mathrm{ad}} := (\rho_{\omega_i}^{\mathrm{ad}})^*(u_{\omega_i}^+ \boxtimes v_{\omega_i}^+)$ of the global section $u_{\omega_i}^+ \boxtimes v_{\omega_i}^+$ of $\mathcal{O}_{\omega_i}(1)$. Then the zero divisor $(s_{\mathrm{ad}})_0$ of s_{ad} is $B \times B$-invariant. Thus there exist nonnegative integers a_r and b_j, for $r, j = 1, \ldots, l$, such that

$$(s_{\mathrm{ad}})_0 = \sum_{r=1}^l a_r \boldsymbol{D}_r + \sum_{j=1}^l b_j \boldsymbol{X}_j.$$

If $b_j > 0$ for some j then s_{ad} vanishes on $\boldsymbol{Y}$ which by Lemma 4.1 is a contradiction. Hence, $(s_{\mathrm{ad}})_0 = \sum_{r=1}^l a_r \boldsymbol{D}_r$. It is known (see e.g. [B-K, Prop. 6.1.11]) that the restriction of $\mathcal{L}(\boldsymbol{D}_i)$ to $\boldsymbol{Y}$ is isomorphic to $\mathcal{L}(\omega_i, -w_0\omega_i)$, so using using Lemma 4.3 it follows that $a_i = 1$ and $a_r = 0$ for $r \neq i$. This proves the statement when X is the wonderful compactification of G_{ad}.

Consider now an arbitrary toroidal equivariant embedding X of G. Let $s :=$ $\rho_{\omega_i}^*(u_{\omega_i}^+ \boxtimes v_{\omega_i}^+)$ be the pull back of the the global section $u_{\omega_i}^+ \boxtimes v_{\omega_i}^+$ of $\mathcal{O}_{\omega_i}(1)$. Then the zero divisor $(s)_0$ of s is $B \times B$-invariant and by the already proved case above we conclude that

$$(s)_0 = cD_i + \sum_{j=1}^{n} d_j X_j,$$

for certain nonnegative integers $c > 0$ and d_j, $j = 1, \ldots, n$. Assume $d_j > 0$ for some j. Then s vanishes on a $G \times G$-stable subset and as s is the pull back of s_{ad} from $\mathbf{X}$ to X, we conclude that s_{ad} also vanishes on a $G \times G$-stable subset V. But then s_{ad} vanishes on a closed $G \times G$-orbit in the closure of V which can only be $\mathbf{Y}$. As above this is a contradiction and we conclude that $(s)_0 = cD_i$.

In order to prove that $c = 1$ we may assume that $G = G_{\mathrm{sc}}$ is simply connected and that $X = G$. In this case the statement is well known (cf. proof of 6.1.11 in [B-K]).   □

4.4. **Sections of the dualizing sheaf.** Let again X be a smooth equivariant embedding of G. For each $i = 1, \ldots, l$, there exists a unique $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-linearization of the line bundle $\mathcal{L}(D_i)$. The set of global sections of $\mathcal{L}(D_i)$ may then be regarded as a $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-module. We claim

Proposition 4.6. *There exists a $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant morphism*

$$\psi_i : \mathrm{H}(\omega_i)^* \boxtimes \mathrm{H}(\omega_i) \to \mathrm{H}^0\big(X, \mathcal{L}(D_i)\big),$$

such that $\psi_i(u_{\omega_i}^+ \boxtimes v_{\omega_j}^+)$ is the canonical section of $\mathcal{L}(D_i)$.

Proof. When X is toroidal this follows by Lemma 4.5. For a general smooth embedding X there exists by Zariski's main theorem (cf. proof of Prop.6.2.6 [B-K]) an open subset X' of X such that X' is a toroidal embedding of G and such that the complement $X - X'$ has codimension ≥ 2 in X. As the statement is invariant under replacing X with X' the result now follows.   □

4.4.1. *The complete toroidal case.* When X is a complete toroidal embedding of G we may even give more structure to the map ψ_i given in Proposition 4.6. To this end, let Y denote a closed $G \times G$-orbit in X and consider the restriction map

$$i_{|Y}^* : \mathrm{H}^0\big(X, \mathcal{L}(D_i)\big) \to \mathrm{H}^0\big(Y, \mathcal{L}(D_i)_{|Y}\big).$$

By Lemma 4.3 and Lemma 4.5 it follows that $\mathcal{L}(D_i)_{|Y} \simeq \mathcal{L}(\omega_i, -w_0\omega_i)$. Consequently, there exists a $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant isomorphism

$$\mathrm{H}^0\big(Y, \mathcal{L}(D_i)_{|Y}\big) \simeq \mathrm{H}(-w_0\omega_i) \boxtimes \mathrm{H}(\omega_i).$$

By Lemma 4.3 the composition of $i_{|Y}^*$ with ψ_i is nonzero and hence we obtain a commutative $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant diagram

$$\begin{array}{ccc}
\mathrm{H}(\omega_i)^* \boxtimes \mathrm{H}(\omega_i) & \xrightarrow{\ \psi_i\ } & \mathrm{H}^0(X, \mathcal{L}(D_i)) \\
\downarrow & & \downarrow{\scriptstyle i_{|Y}^*} \\
\mathrm{H}(-w_0\omega_i) \boxtimes \mathrm{H}(\omega_i) & \xrightarrow{\ \simeq\ } & \mathrm{H}(Y, \mathcal{L}(D_i)_{|Y})
\end{array}$$

where the left vertical map is defined by some nonzero G_{sc}-equivariant map $\mathrm{H}(\omega_i)^* \to \mathrm{H}(-w_0\omega_i)$. Notice that by Frobenius reciprocity the map $\mathrm{H}(\omega_i)^* \to \mathrm{H}(-w_0\omega_i)$ is defined uniquely up to a nonzero constant.

5. Frobenius Splitting

In this section we collect a number of facts from the theory of Frobenius splitting. The presentation will be sketchy only stating the results which we will need. For a more thorough, and closely related, presentation we refer to [T].

5.1. **Frobenius splitting.** Let X denote a scheme of finite type over an algebraically closed field k of characteristic $p > 0$. The *absolute Frobenius morphism* on X is the morphism $F : X \to X$ of schemes, which is the identity on the set of points and where the associated map of sheaves

$$F^{\sharp} : \mathcal{O}_X \to F_*\mathcal{O}_X$$

is the p-th power map. We say that X is *Frobenius split* (or just F-split) if there exists a morphism $s \in \operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ such that the composition $s \circ F^{\sharp}$ is the identity map on $\mathcal{O}_X$.

5.2. **Stable Frobenius splittings along divisors.** Let D denote an effective Cartier divisor on X with associated line bundle $\mathcal{O}_X(D)$ and canonical section σ_D. We say that X is *stably Frobenius split along D* if there exists a positive integer e and a morphism

$$s \in \operatorname{Hom}_{\mathcal{O}_X}(F_*^e\mathcal{O}_X(D), \mathcal{O}_X),$$

such that $s(\sigma_D) = 1$. In this case we say that s is a stable Frobenius splitting of X along D of degree e. Notice that X is Frobenius split exactly when there exists a stable Frobenius splitting of X along the zero divisor $D = 0$.

Remark 5.1. *Consider an element $s \in \operatorname{Hom}_{\mathcal{O}_X}(F_*^e\mathcal{O}_X(D), \mathcal{O}_X)$. Then the condition $s(\sigma_D) = 1$ on s for it to define a stable Frobenius splitting of X, may be checked on any open dense subset of X.*

5.3. **Subdivisors.** Let $D' \le D$ denote an effective Cartier subdivisor and let s be a stable Frobenius splitting of X along D of degree e. The composition of s with the map

$$F_*^e\mathcal{O}_X(D') \to F_*^e\mathcal{O}_X(D),$$

defined by the canonical section of the divisor $D - D'$, is then a stable Frobenius splitting of X along D' of degree e. Applying this to the case $D' = 0$ it follows that if X is stably Frobenius split along any effective divisor D then X is also Frobenius split.

5.4. **Compatibly split subschemes.** Let Y denote a closed subscheme of X with sheaf of ideals $\mathcal{I}_Y$. When

$$s \in \operatorname{Hom}_{\mathcal{O}_X}(F_*^e\mathcal{O}_X(D), \mathcal{O}_X)$$

is a stable Frobenius splitting of X along D we say that s *compatibly Frobenius splits Y* if the following conditions are satisfied

  (1) The support of D does not contain any of the irreducible components of Y.
  (2) $s\big(F_*^e(\mathcal{I}_Y \otimes \mathcal{O}_X(D))\big) \subseteq \mathcal{I}_Y$.

When s compatibly Frobenius splits Y there exists an induced stable Frobenius splitting of Y along $D \cap Y$ of degree e.

Lemma 5.2. *Let s denote a stable Frobenius splitting of X along D which compatibly Frobenius splits a closed subscheme Y of X. If $D' \leq D$ then the induced stable Frobenius splitting of X along D', defined in Section 5.3, compatibly Frobenius splits Y.*

Lemma 5.3. *Let D_1 and D_2 denote effective Cartier divisors. If s_1 (resp. s_2) is a stable Frobenius splitting of X along D_1 (resp. D_2) of degree e_1 (resp. e_2) which compatibly splits a closed subscheme Y of X, then there exists a stable Frobenius splitting of X along $D_1 + D_2$ of degree $e_1 + e_2$ which compatibly splits Y.*

Lemma 5.4. *Let s denote a stable Frobenius splitting of X along an effective divisor D. Then*

(1) *If s compatibly Frobenius splits a closed subscheme Y of X then Y is reduced and each irreducible component of Y is also compatibly Frobenius split by s.*

(2) *Assume that s compatibly Frobenius splits closed subschemes Y_1 and Y_2 and that the support of D does not contain any of the irreducible components of the scheme theoretic intersection $Y_1 \cap Y_2$. Then s compatibly Frobenius splits $Y_1 \cap Y_2$.*

The following statement relates stable Frobenius splitting along divisors to compatibly Frobenius splitting.

Lemma 5.5. *Let D and D' denote effective Cartier divisors and let s denote a stable Frobenius splitting of X along $(p-1)D + D'$ of degree 1. Then there exists a stable Frobenius splitting of X along D' of degree 1 which compatibly splits the closed subscheme defined by D.*

5.5. **Cohomology and Frobenius splitting.** The notion of Frobenius splitting is particular useful in connection with proving higher cohomology vanishing for line bundles. We will need

Lemma 5.6. *Let s denote a stable Frobenius splitting of X along D of degree e and let Y denote a closed compatibly Frobenius split subscheme of X. Then for every line bundle $\mathcal{L}$ on X and every integer i there exists an inclusion*

$$\mathrm{H}^i(X, \mathcal{I}_Y \otimes \mathcal{L}) \subseteq \mathrm{H}^i(X, \mathcal{I}_Y \otimes \mathcal{L}^{p^e} \otimes \mathcal{O}_X(D)),$$

of abelian groups. In particular, when X is projective, $\mathcal{L}$ is globally generated and D is ample then the group $\mathrm{H}^i(X, \mathcal{I}_Y \otimes \mathcal{L})$ is zero for $i > 0$.

5.6. **Push forward.** Let $f : X \to X'$ denote a proper morphism of schemes and assume that the induced map $\mathcal{O}_{X'} \to f_* \mathcal{O}_X$ is an isomorphism. Then every Frobenius splitting of X induces, by application of the functor f_*, a Frobenius splitting of X'. Moreover, when Y is a compatibly Frobenius split subscheme of X then the induced Frobenius splitting of X' compatibly splits the scheme theoretic image $f(Y)$ (see [M-R, Prop.4]). We will need the following connected statement.

Lemma 5.7. *Let $f : X \to X'$ denote a morphism of projective schemes such that $\mathcal{O}_{X'} \to f_* \mathcal{O}_X$ is an isomorphism. Let Y be a closed subscheme of X and denote by Y' the scheme theoretic image $f(Y)$. Assume that there exists a stable Frobenius splitting of X along an ample divisor D which compatibly splits Y. Then $f_* \mathcal{O}_Y = \mathcal{O}_{Y'}$ and $\mathrm{R}^i f_* \mathcal{O}_Y = 0$ for $i > 0$.*

5.7. **Frobenius splitting of smooth varieties.** When X is a smooth variety there exists a canonical $\mathcal{O}_X$-linear identification (see e.g. [B-K, §1.3.7])

$$F_* \omega_X^{1-p} \simeq \mathrm{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X).$$

Hence, a Frobenius splitting of X may be identified with a global section of ω_X^{1-p} with certain properties. A global section τ of ω_X^{1-p} which corresponds to a Frobenius splitting up to a nonzero constant will be called a *Frobenius splitting section*.

Lemma 5.8. *Let τ be a Frobenius splitting section of a smooth variety X. Then there exists a stable Frobenius splitting of X of degree 1 along the Cartier divisor defined by τ. In particular, if $\tau = \tilde{\tau}^{p-1}$ is a $(p-1)$-th power of a global section $\tilde{\tau}$ of ω_X^{-1}, then X is Frobenius split compatibly with the zero divisor of $\tilde{\tau}$.*

5.8. **Frobenius splitting of G/B.** The flag variety $X = G/B$ is a smooth variety with dualizing sheaf $\omega_X = \mathcal{L}(-2\rho)$ where ρ is a dominant weight defined as half of the sum of the positive roots. Let $\mathrm{St} := \mathrm{H}((p-1)\rho)$ denote the Steinberg module of G_{sc} and consider the multiplication map

$$m_{G/B} : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}(2(p-1)\rho) \simeq \mathrm{H}^0(G/B, \omega_{G/B}^{1-p}).$$

The Steinberg module St is an irreducible selfdual G_{sc}-module and hence there exists a unique (up to nonzero scalars) nondegenerate G_{sc}-invariant bilinear form

$$\phi_{G/B} : \mathrm{St} \otimes \mathrm{St} \to k.$$

Then (see [L-T, Thm.2.3])

Theorem 5.9. *Let t be an element in $\mathrm{St} \otimes \mathrm{St}$. Then $\phi_{G/B}(t)$ is a Frobenius splitting section of G/B if and only if $\phi_{G/B}(t)$ is nonzero.*

6. F-SPLITTING OF SMOOTH EQUIVARIANT EMBEDDINGS

Let X denote a smooth equivariant embedding of G. Define S to be the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-module

$$S = \bigotimes_{i=1}^{l} \left(\mathrm{H}(\omega_i)^* \boxtimes \mathrm{H}(\omega_i) \right)^{\otimes (p-1)}.$$

By Proposition 4.6 there exists a $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant morphism

$$\psi_X : S \to \mathrm{H}^0 \left(X, \mathcal{L}\left((p-1) \sum_{i=1}^{l} D_i \right) \right).$$

defined as the $(p-1)$-th product of the ψ_i's. Let σ_j denote the canonical section of $\mathcal{L}(X_j)$, for $j = 1, \ldots, n$, and define for $s, t \in S$ the section

$$\Psi_X(s, t) = \psi_X(s)\psi_X(t) \prod_{i=1}^{n} \sigma_i^{p-1},$$

of the line bundle ω_X^{1-p} on X. Notice that if X' is an equivariant embedding of G which moreover is an open subset of X, then the restriction of $\Psi_X(s, t)$ to X' is equal to $\Psi_{X'}(s, t)$. The main result Theorem 6.4 in this section describes when $\Psi_X(s, t)$ is a Frobenius splitting section of the smooth embedding X.

6.1. **F-splitting smooth complete toroidal embeddings.** Consider a smooth complete toroidal embedding X of G and choose a closed $G \times G$-orbit Y in X. By Lemma 4.3 we may identify Y with $G/B \times G/B$ and under this isomorphism the restriction of $\mathcal{L}(D_i)$ to Y corresponds to $\mathcal{L}(\omega_i, -w_0\omega_i)$. In particular, restricting to Y induces a map

$$i^*_{|Y} : \mathrm{H}^0\Big(X, \mathcal{L}\big(2(p-1)\sum_{i=1}^{l} D_i\big)\Big) \to \mathrm{H}^0\big(Y, \omega_Y^{1-p}\big).$$

This leads to the following result which also explains the standard way of Frobenius splitting X (cf. proof of Thm.6.2.7 [B-K])

Lemma 6.1. *Let X denote a smooth complete toroidal embedding of G and let Y denote a closed $G \times G$-orbit in X. Let s and t be elements of S. Then $\Psi_X(s,t)$ is a Frobenius splitting section of X if and only if the restriction of $\psi_X(s)\psi_X(t)$ to Y is a Frobenius splitting section of Y.*

In order to control the restriction of $\psi_X(s)\psi_X(t)$ to Y we use Section 4.4.1. It follows that we have a commutative $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-equivariant diagram with nonzero maps

$$\begin{array}{ccc}
S & \xrightarrow{\psi_X} & \mathrm{H}^0\big(X, \mathcal{L}\big((p-1)\sum_{i=1}^{l} D_i\big)\big) \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle i^*_{|Y}} \\
\mathrm{St} \boxtimes \mathrm{St} & \xrightarrow{\simeq} & \mathrm{H}^0\big(Y, \omega_Y^{(1-p)/2}\big)
\end{array}$$

where $\mathrm{St} = \mathrm{H}\big((p-1)\rho\big)$ denotes the Steinberg module of G_{sc}. Using the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-invariant form $\phi_{G/B \times G/B}$ on $\mathrm{St} \boxtimes \mathrm{St}$ we may define a similar form on S by

$$\phi : S \otimes S \to k,$$

$$s \otimes t \mapsto \phi_{G/B \times G/B}(\eta(s) \otimes \eta(t))$$

Notice that S and the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-invariant form ϕ is defined without the help of X. In particular, S and ϕ does not depend on X. Now by Lemma 6.1 and Theorem 5.9 we find

Proposition 6.2. *Let the notation be as above and let s and t be elements of S. Then $\Psi_X(s,t)$ is a Frobenius splitting section of X if and only if $\phi(s \otimes t)$ is nonzero.*

6.2. **Frobenius splitting G.** By restricting the statement of Proposition 6.2 to G we find

Corollary 6.3. *Let s and t be elements of S. Then*

$$\Psi_G(s,t) := \psi_G(s)\psi_G(t),$$

is a Frobenius splitting section of G if and only if $\phi(s \otimes t)$ is nonzero.

Proof. Choose a smooth complete toroidal embedding X of G and consider $\Psi_X(s,t)$. Remember that checking whether $\Psi_X(s,t)$ is a Frobenius splitting section of X may be done on the open subset G (see Remark 5.1). Now apply Proposition 6.2. $\qquad\square$

6.3. **Frobenius splitting smooth equivariant embeddings.** We can now prove that main result of this section.

Theorem 6.4. *Let X denote an arbitrary smooth embedding of G and let s and t be elements of S. Then $\Psi_X(s,t)$ is a Frobenius splitting section of X if and only if $\phi(s \otimes t)$ is nonzero.*

Proof. That $\Psi_X(s,t)$ is a Frobenius splitting section may be checked on the open subset G. Now apply Corollary 6.3. $\qquad\square$

In the following statement t_i, $i = 1, \dots, l$, denotes the identity map in $\mathrm{End}(\mathrm{H}(\omega_i)^*) \simeq \mathrm{H}(\omega_i)^* \boxtimes \mathrm{H}(\omega_i)$. Notice that as an element of $\mathrm{End}(\mathrm{H}(\omega_i))^*$ the element t_i is just the trace map on $\mathrm{End}(\mathrm{H}(\omega_i))$. We also fix a nonzero weight vector $u_{\omega_i}^-$ of $\mathrm{H}(\omega_i)^*$ of weight $-\omega_i$.

Corollary 6.5. *The global section*

$$\prod_{i=1}^{l} \psi_i(t_i)^{p-1} \prod_{i=1}^{l} \psi_i(u_{\omega_i}^- \boxtimes v_{\omega_i}^+)^{p-1} \prod_{j=1}^{n} \sigma_j^{p-1},$$

of ω_x^{1-p} is a Frobenius splitting section of X.

Proof. It suffices by Theorem 6.4 to prove that

$$\phi\Big(\bigotimes_{i=1}^{l} t_i^{\otimes(p-1)} \otimes \bigotimes_{i=1}^{l} (u_{\omega_i}^- \boxtimes v_{\omega_i}^+)^{\otimes(p-1)} \Big)$$

is nonzero. The image of $\bigotimes_{i=1}^{l} t_i^{\otimes(p-1)}$ in $\mathrm{St} \boxtimes \mathrm{St}$ coincides with a nonzero $\mathrm{diag}(G)$-invariant element v_Δ. Moreover, the image of the element $\bigotimes_{i=1}^{l} (u_{\omega_i}^- \boxtimes v_{\omega_i}^+)^{\otimes(p-1)}$ in $\mathrm{St} \boxtimes \mathrm{St}$ equals $v_- \boxtimes v_+$ for some nonzero weight vectors v_+ and v_- in St of weight $(p-1)\rho$ and $-(p-1)\rho$ respectively. Thus, we have to show that $\phi_{G/B \times G/B}\big(v_\Delta \otimes (v_- \boxtimes v_+)\big)$ is nonzero. But by weight consideration this is clearly the case. $\qquad\square$

7. Consequences in the smooth case

In this section we collect a number of consequences of the results in Section 6 and the following Lemma 7.1. Notice that when f is a global section of a line bundle $\mathcal{L}$ on a variety X, then we may regard f as an element in the local rings $\mathcal{O}_{X,x}$ at points $x \in X$. This identification is unique up to units in $\mathcal{O}_{X,x}$. Using this identification we may now state

Lemma 7.1. *Let X denote a smooth variety with dualizing sheaf ω_X and let $\mathcal{L}_1, \dots, \mathcal{L}_N$ denote a collection of line bundles on X such that $\otimes_{i=1}^{N} \mathcal{L}_i \simeq \omega_X^{-1}$. Let f_i, $i = 1, \dots, N$, denote a global section of $\mathcal{L}_i$ and assume that $\prod_{i=1}^{N} f_i^{p-1}$, considered as a global section of ω_X^{1-p}, is a Frobenius splitting section of X. Choose a sequence $1 \le i_1, \dots, i_r \le N$ of pairwise distinct integers. Then*

 (1) *The sequence $f_{i_1}, \dots, f_{i_r}$ forms a regular sequence in the local ring $\mathcal{O}_{X,x}$ at a point x contained in the common zero set of $f_{i_1}, \dots, f_{i_r}$.*
 (2) *The common zero set of $f_{i_1}, \dots, f_{i_r}$ has pure codimension r.*

Proof. As all statements are local we may assume that X is affine and that ω_X and $\mathcal{L}_1, \dots, \mathcal{L}_N$ are all trivial. Hence, the elements $f_1, \dots, f_N$ are just regular global

functions on X. Moreover, by assumption there exists a function $\tau : F_* k[X] \to k[X]$ such that $\tau\big(a^p(f_1 \cdots f_N)^{p-1}\big) = a$ for all global regular functions $a \in k[X]$.

Let x be a common zero of $f_{i_1}, \ldots, f_{i_r}$ and assume that we have a relation of the form $\sum_{s=1}^{j} a_s f_{i_s} = 0$ for certain elements a_s in $\mathcal{O}_{X,x}$. In particular, the product $a_j^p(f_1 \cdots f_N)^{p-1}$ is contained in the ideal $(f_{i_1}^p, \ldots, f_{i_{j-1}}^p)$ of $\mathcal{O}_{X,x}$ and hence

$$a_j = \tau\big(a_j^p(f_1 \cdots f_N)^{p-1}\big) \in (f_{i_1}, \ldots, f_{i_{j-1}}).$$

This proves (1). Now (2) follows as a direct consequence of (1). □

We can now prove the first of our main results

Corollary 7.2. *Let X be a smooth equivariant embedding of G and let $\bar{N}$ denote the closure of the Steinberg zero-fiber in X. Then*

(1) *$\bar{N}$ coincides with the scheme theoretic intersection of the zero sets of t_i, $i = 1, \ldots, l$. In particular, $\bar{N}$ is a local complete intersection.*
(2) *$\bar{N}$ is normal, Gorenstein and Cohen-Macaulay.*
(3) *The dualizing sheaf of $\bar{N}$ is isomorphic to the restriction of the line bundle $\mathcal{L}_{\bar{N}} := \mathcal{L}(-\sum_{i=1}^{l}(w_0, 1)D_i - \sum_{j=1}^{n} X_j)$ to $\bar{N}$.*

Proof. Consider the Frobenius splitting section of X defined in Corollary 6.5. By Lemma 5.8 and Lemma 5.4 the scheme theoretic intersection C of t_i, $i = 1, \ldots, l$, is a reduced scheme. Moreover, by Lemma 7.1 each component of C has codimension l and will intersect the open locus G (else, by Lemma 7.1, such a component would have codimension $\geq l + 1$). We conclude that $C \cap G$ is dense in C and that C is a local complete intersection. But clearly (see remark above Corollary 6.5) $C \cap G$ coincides with the Steinberg zero-fiber N, and thus C must be equal to the closure $\bar{N}$. This proves (1).

To prove (2) it then suffices to show that $\bar{N}$ is regular in codimension 1. Let Z denote a component of the singular locus of $\bar{N}$. If $Z \cap G \neq \emptyset$ then the codimension of Z is ≥ 2 as N is normal by [St, Thm.6.11]. So assume that Z is contained in a boundary component X_j. Now, by Lemma 5.8 the scheme theoretic intersection $\bar{N} \cap X_j$ is reduced. Hence, as X_j is a Cartier divisor, every smooth point of $\bar{N} \cap X_j$ is also a smooth point of $\bar{N}$. In particular, Z is properly contained in a component of $\bar{N} \cap X_j$. But the variety $\bar{N} \cap X_j$ has pure codimension 1 in $\bar{N}$ which ends the proof of (2).

Statement (3) follows by (1) and the description of the dualizing sheaf of X in Proposition 4.4. □

7.1. **Stable Frobenius splittings along divisors.**

Proposition 7.3. *Let X be a smooth equivariant embedding of G. Then there exists a stable Frobenius splitting of X along the divisor*

$$(p-1)\Big(\sum_{j=1}^{n} X_j + \sum_{i=1}^{\ell}(w_0, 1)D_i\Big)$$

of degree 1 which compatibly Frobenius splits the closure $\bar{N}$ of the Steinberg zero-fiber.

Proof. Let τ denote the Frobenius splitting section of Corollary 6.5. By Lemma 5.8, Lemma 5.5 and Lemma 4.6 we know that τ defines a degree 1 stable Frobenius

splitting of X along the divisor

$$(p-1)\Big(\sum_{j=1}^{n} X_j + \sum_{i=1}^{\ell}(w_0,1)D_i\Big),$$

which compatibly Frobenius splits the zero divisor of $\prod_{i=1}^{l}\psi_i(t_i)$. Now apply Lemma 5.4(2), Corollary 7.2 and Lemma 7.1. $\square$

Corollary 7.4. *Let X denote a projective smooth equivariant embedding of G. Then there exists a stable Frobenius splitting of X along an ample divisor with support $X - G$ which compatibly Frobenius splits the subvariety $\bar{N}$.*

Proof. By Proposition 7.3 and Lemma 5.2 there exists a stable Frobenius splitting of X along the divisor $\sum_{j=1}^{n} X_j$ which compatibly splits $\bar{N}$. Applying Lemma 5.2 and Lemma 5.3 it suffices to show that there exist positive integers $c_j > 0$ such that $\sum_{j=1}^{n} c_j X_j$ is ample. This follows from [B-T, Prop.4.1(2)]. $\square$

This has the following implications for resolutions

Corollary 7.5. *Let X be a projective equivariant embedding of G and let $f : X' \to X$ be a projective resolution of X by a smooth projective equivariant G-embedding X'. Denote by $\bar{N}'$ (resp. $\bar{N}$) the closure of the Steinberg zero-fiber within X' (resp. X). Then*
(i) $f_*\mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathrm{R}^i f_*\mathcal{O}_{X'} = 0$ for $i > 0$. (cf. [Rit, pf. of Cor.2])
(ii) $f_*\mathcal{O}_{\bar{N}'} = \mathcal{O}_{\bar{N}}$ and $\mathrm{R}^i f_*\mathcal{O}_{\bar{N}'} = 0$ for $i > 0$.

Proof. As X' is normal and f is birational it follows from Zariski's main theorem that $f_*\mathcal{O}_X = \mathcal{O}_{X'}$. Hence, by Lemma 5.7 it suffices to prove that there exists a stable Frobenius splitting of X along an ample divisor which compatibly Frobenius splits $\bar{N}$. Now apply Corollary 7.4. $\square$

8. Frobenius splitting $\bar{N}$ for general embeddings

In this section X will denote an arbitrary equivariant embedding of G and $\bar{N}$ will denote the closure of the Steinberg zero-fiber in X.

Theorem 8.1. *There exists a Frobenius splitting of X which simultaneously compatibly splits the closed subvarieties $\bar{N}, (w_0,1)D_i, X_j$, for $i = 1, \ldots, l$ and $j = 1, \ldots, n$.*

Proof. By Theorem 4.2 we may find a projective resolution $f : X' \to X$ by a smooth toroidal embedding X' of G. By Zariski's Main Theorem, $f_*\mathcal{O}_{X'} = \mathcal{O}_X$. Thus, by [M-R, Prop.4] (cf. section 5.6) we can reduce to the case where X is smooth. Now apply Corollary 6.5, Lemma 5.8, Lemma 5.4, Lemma 4.6 and Corollary 7.2 in the given order. $\square$

Example 8.2. Consider the group $G = \mathrm{PSL}_2(k)$ over a field k of positive characteristic different from 2. Then the wonderful compactification $\boldsymbol{X}$ of G may be identified with the projectivization of the set of 2×2-matrices with entries in k. Denote the homogeneous coordinates in $\boldsymbol{X}$ by a, b, c and d. Then the closure $\bar{\mathcal{U}}$ of the unipotent variety $\mathcal{U}$ of G within $\boldsymbol{X}$, is defined by the polynomial $f = (a + d)^2 - 4(ad - bc)$. Moreover, the boundary is defined by the polynomial $g = (ad - bc)$. In particular, the ideal generated by f and g is not reduced and, as a consequence, the boundary $\boldsymbol{X} - G$ and the closure $\bar{\mathcal{U}}$ cannot be compatibly Frobenius split at the same time. When k has characteristic 2 the unipotent variety $\mathcal{U}$ coincides with the Steinberg

zero-fiber. In this case the polynomial defining $\bar{\mathcal{U}}$ is given by $f = a + d$ and we do not see a similar problem.

Remark 8.3. *W. van der Kallen and T. Springer has informed us that they have proved Theorem 8.1 in case X is the wonderful compactification of a group of adjoint type. Their proof proceeds by descending the Frobenius splitting results in* [T] *to the wonderful compactification.*

We can also prove a vanishing result for line bundles on $\bar{N}$:

Proposition 8.4. *Let X denote a projective equivariant G-embedding and let $\mathcal{L}$ (resp. $\mathcal{M}$) denote a globally generated line bundle on X (resp. $\bar{N}$). Then*

$$\mathrm{H}^i(X, \mathcal{L}) = \mathrm{H}^i(\bar{N}, \mathcal{M}) = 0, \ i > 0.$$

Moreover, the restriction map

$$\mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(\bar{N}, \mathcal{L}),$$

is surjective.

Proof. By Corollary 7.5 we may assume that X is smooth. Now apply Corollary 7.4 and the "in particular" part of Lemma 5.6. $\square$

8.1. **Canonical Frobenius splittings of X.** A Frobenius splitting $s : F_*\mathcal{O}_Z \to \mathcal{O}_Z$ of a B-variety Z is a T-invariant Frobenius splitting such that the action of a root subgroup of G associated to the simple root α_i, is of the form

$$x_{\alpha_i}(c)s = \sum_{j=1}^{p-1} c^j s_j,$$

for certain morphisms $s_j : F_*\mathcal{O}_Z \to \mathcal{O}_Z$ and all $c \in k$.

As a subset of X the closure $\bar{N}$ is invariant under the diagonal action of G. In particular, $\bar{N}$ is invariant under $\mathrm{diag}(B)$ and we claim

Lemma 8.5. *The variety $\bar{N}$ is canonical Frobenius split with respect to the action of $\mathrm{diag}(B)$.*

Proof. It suffices to prove that X has a $\mathrm{diag}(B)$-canonical splitting which compatibly splits $\bar{N}$. Moreover, by Theorem 4.2 we may assume that X is smooth. By the proof of Corollary 7.2 it then suffices to prove that the Frobenius splitting section of Corollary 6.5 is canonical.

As $\psi_i^*(t_i)$ and σ_j are $\mathrm{diag}(G)$-invariant we may concentrate on the $\mathrm{diag}(T)$-invariant factors $\psi_i^*(u_{\omega_i}^- \otimes v_{\omega_i}^+)$. The statement follows now as

$$x_{\alpha_j}(c).v_{\omega_i}^+ = v_{\omega_i}^+,$$
$$x_{\alpha_j}(c).u_{\omega_i}^- = u_{\omega_i}^- + cu_{i,j} \ ,$$

for certain elements $u_{i,j} \in \mathrm{H}(-w_o\omega_i)^*$. $\square$

As a consequence we have (see [B-K, Thm.4.2.13])

Proposition 8.6. *Let $\mathcal{L}$ denote a G_{sc}-linearized line bundle on $\bar{N}$. Then the G_{sc}-module $\mathrm{H}^0(\bar{N}, \mathcal{L})$ admits a good filtration, i.e. there exists a filtration by G_{sc}-modules*

$$0 = M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots \subseteq \mathrm{H}^0(\bar{N}, \mathcal{L}),$$

such that $\mathrm{H}^0(\bar{N}, \mathcal{L}) = \cup_i M^i$ and satisfying that the successive quotients M^{j+1}/M^j are isomorphic to modules of the form $\mathrm{H}(\lambda_j)$ for certain dominant weights λ_j.

9. Geometric properties of $\bar{N}$

Let X be an arbitrary equivariant G-embedding. When X is smooth we have seen that $\bar{N}$ is normal and Cohen-Macaulay. In this section we will extend these two properties to arbitrary equivariant embeddings.

The following result is due to G. Kempf although the version stated here is taken from [B-P, §7] :

Lemma 9.1. *Let $f : Z' \to Z$ denote a proper map of algebraic schemes satisfying that $f_* \mathcal{O}_{Z'} \simeq \mathcal{O}_Z$ and $\mathrm{R}^i f_* \mathcal{O}_{Z'} = 0$ for $i > 0$. If Z' is Cohen-Macaulay with dualizing sheaf $\omega_{Z'}$ and if $\mathrm{R}^i f_* \omega_{Z'} = 0$ for $i > 0$, then Z is Cohen-Macaulay with dualizing sheaf $f_* \omega_{Z'}$.*

We will also need the following result due to V. Mehta and W. van der Kallen ([M-vdK, Thm.1.1]):

Lemma 9.2. *Let $f : Z' \to Z$ denote a proper morphism of schemes and let V' (resp. V) denote a closed subscheme of Z' (resp. Z). By $\mathcal{I}_{V'}$ we denote the sheaf of ideals of V'. Fix an integer i and assume*

   (1) $f^{-1}(V) \subseteq V'$.
   (2) $\mathrm{R}^i f_* \mathcal{I}_{V'}$ *vanishes outside V.*
   (3) V' *is compatibly F-split in Z'.*

Then $\mathrm{R}^i f_* \mathcal{I}_{V'} = 0$.

We are ready to prove

Theorem 9.3. *Let X denote an arbitrary equivariant G-embedding. Then the closure $\bar{N}$ of the Steinberg zero-fiber in X is normal and Cohen-Macaulay.*

Proof. Any equivariant embedding has an open cover by open equivariant subsets of projective equivariant embeddings (see e.g. proof of [B-K] Corollary 6.2.8). This reduces the statement to the case where X is projective. Choose a projective resolution $f : X' \to X$ of X by a smooth equivariant embedding X'. By Corollary 7.5 we know that $f_* \mathcal{O}_{\bar{N}'} = \mathcal{O}_{\bar{N}}$ and applying Corollary 7.2 this implies that $\bar{N}$ is normal.

In order to show that $\bar{N}$ is Cohen-Macaulay we apply the above Lemma 9.1 and Lemma 9.2. By Corollary 7.5 it suffices to prove that $\mathrm{R}^i f_* \omega_{\bar{N}'} = 0$, $i > 0$, where $\omega_{\bar{N}'}$ is the dualizing sheaf of $\bar{N}'$. By Corollary 7.2 the dualizing sheaf $\omega_{\bar{N}'}$ is isomorphic to the restriction of $\mathcal{L}_{\bar{N}'}$ to $\bar{N}'$. Let s' denote the canonical section of the line bundle $\mathcal{L}_{\bar{N}'}^{-1}$ on X' and let V' denote the intersection of $\bar{N}'$ with the zero divisor of s'. Combining Proposition 7.3 and Lemma 5.5 we find that $\bar{N}'$ is Frobenius split compatibly with the closed subscheme V'. Moreover, $f : \bar{N}' \to \bar{N}$ is an isomorphism above the open subset N and

$$f^{-1}(\bar{N} \setminus N) \subseteq V'.$$

Hence, by Lemma 9.2 we conclude $\mathrm{R}^i f_* \mathcal{I}_{V'} = 0$ for $i > 0$. But $\mathcal{I}_{V'}$ is isomorphic to the restriction of $\mathcal{L}_{\bar{N}'}$ to $\bar{N}'$. $\qquad\square$

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