UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

A DILOGARITHMIC FORMULA FOR THE CHEEGER-CHERN-SIMONS CLASS

by Johan L. Dupont and Christian K. Zickert

Preprint Series No.: 13

August 2005

2005/08/31

Ny Munkegade, Bldg. 530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

A DILOGARITHMIC FORMULA FOR THE CHEEGER-CHERN-SIMONS CLASS

JOHAN L. DUPONT AND CHRISTIAN K. ZICKERT

Abstract. We present a simplification of Neumann's formula in [8] for the universal Cheeger-Chern-Simons class of the second Chern polynomial. Our approach is completely algebraic, and the final formula can be applied directly on a homology class in the bar complex.

Introduction

In the famous papers [1] and [2], J. Cheeger, S. Chern, and J. Simons define characteristic classes for flat G-bundles. Each such characteristic class is given by a corresponding universal cohomology class in $H^*(BG^\delta, \mathbb{C}/\mathbb{Z})$, where δ denotes discrete topology. The cohomology of the classifying space of a discrete group is isomorphic to the Eilenberg-Maclane group cohomology, and it has been a long standing problem to find explicit formulas for the universal classes directly in terms of the bar complex. In [3] it is proved that the universal Cheeger-Chern-Simons (C-C-S) class for the group $\mathrm{SL}(2,\mathbb{C})$ associated to the second Chern polynomial is given up to a $\mathbb{Q}/\mathbb{Z}$ indeterminacy by a dilogarithmic formula defined on the Bloch group $\mathcal{B}(\mathbb{C})$. In particular, for a compact hyperbolic 3-manifold, the imaginary part of the C-C-S class is just hyperbolic volume.

An element of $\mathcal{B}(\mathbb{C})$ is a formal sum of cross-ratios (see below), but the cross-ratio alone does not seem to carry enough information to get rid of the $\mathbb{Q}/\mathbb{Z}$ indeterminacy on the real part. Neumann constructs in [8] an extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, where elements, in addition to the cross-ratio, also contain information of two choices of logarithms. It follows from Neumann's article that this additional information is exactly what is needed to remove the $\mathbb{Q}/\mathbb{Z}$ indeterminacy. He shows that there is an isomorphism

$$\lambda\colon H_3(\mathrm{PSL}(2,\mathbb{C})) \approx \widehat{\mathcal{B}}(\mathbb{C}),$$

and furthermore that there is a natural extension of the dilogarithmic formula from [3] to $\widehat{\mathcal{B}}(\mathbb{C})$ such that the composition of λ with the dilogarithm is exactly the universal C-C-S class. The isomorphism λ is defined by representing an element of $H_3(\mathrm{PSL}(2,\mathbb{C}))$ by a "quasi-simplicial complex" and the appropriate choices of logarithms required to obtain an element in $\widehat{B}(\mathbb{C})$ are found by some rather complicated combinatorial topology.

We shall construct a map similar to Neumann's using $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{PSL}(2,\mathbb{C})$. The definition of this map uses only simple homological algebra, and we obtain a

2000 *Mathematics Subject Classification.* Primary 57M27;Secondary 57T30.

Key words and phrases. Extended Bloch group, Cheeger-Chern-Simons class.

This work was partially supported by the Danish Natural Science Research Council (Statens Naturvidenskabelige Forskningsråd), Denmark.

formula which enables us to calculate the universal C-C-S class directly from a representative of a homology class in the bar complex. All geometry is replaced by algebra which vastly simplifies the proofs.

We give a brief overview of the contents: In section 1 we review the basic theory of the C-C-S classes, group homology, and the Bloch group. Many details are included in order to make the paper self-contained. In section 2 we recall Neumann's definition of the extended Bloch group. This overlaps with Neumann's paper but for the sake of completeness, all details are included. In section 3, we construct a map $\widehat{\lambda}\colon H_3(\mathrm{SL}(2,\mathbb{C})) \to \widehat{\mathcal{B}}(\mathbb{C})$, by describing a way of detecting the appropriate two choices of logarithms directly from a tuple of group elements. The idea is that the extra information can be found in $\mathbb{C}^2\backslash\{0\}$ rather than S^2. In section 4 we show that our map actually calculates the C-C-S class and show that $\widehat{\lambda}$ is surjective with kernel of order 2. Finally, we show in the appendix that our definition of the extended Bloch group agrees with that of Neumann.

1. Preliminaries

In this section we review some basic theory and introduce our terminology. Throughout, $\mathbb{F}$ always denotes either $\mathbb{R}$ or $\mathbb{C}$.

1.1. **The Cheeger-Chern-Simons Classes.**

We here recall some facts about the C-C-S classes that we shall need. For their construction and basic properties, we refer to [1] or [2].

Let G be a Lie group with finitely many components and let $I^k(G)$ denote the group of invariant polynomials. Recall from classical Chern-Weil theory that there is a natural homomorphism

$$W\colon I^k(G) \to H^{2k}(BG,\mathbb{F}).$$

The C-C-S classes are defined from the following data:

(1) An invariant polynomial $P \in I^k(G)$.
(2) A class $u \in H^{2k}(BG,\mathbb{Z})$ satisfying $W(P) = ru$.

Let

$$K^k(G) = \big\{(P,u) \in I^k(G) \times H^{2k}(BG,\mathbb{Z}) \mid W(P) = ru\big\}$$

Let G^δ denote the underlying discrete group of G. In the articles [1] and [2], the authors describe a way of associating a cohomology class $\hat{P}(u)$ in $H^{2k-1}(BG^\delta,\mathbb{F}/\mathbb{Z})$ to an element (P,u) in $K^k(G)$. This association is natural in the following sense:

Theorem 1.1. *Let $\phi\colon G \to H$ be a Lie group homomorphisms between Lie groups with finitely many components. The diagram below is commutative.*

$$\begin{tikzcd} K^k(H) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{CCS}"] & K^k(G) \arrow[d, "\mathrm{CCS}"] \\ H^{2k-1}(BH^\delta,\mathbb{F}/\mathbb{Z}) \arrow[r, "\phi^*"] & H^{2k-1}(BG^\delta,\mathbb{F}/\mathbb{Z}) \end{tikzcd}$$

Remark 1.2. In the following we shall only be interested in the C-C-S classes corresponding to the second Chern polynomial and the first Pontrjagin polynomial. In both cases u is just the corresponding Chern class or Pontrjagin class, and we simply denote the associated C-C-S classes $\hat{C}_2$ and $\hat{P}_1$.

1.2. **The Homology of a Group.** Let G be a group. For a right G-module A, we let A_G denote the group $A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, where $\mathbb{Z}$ is regarded as a trivial G-module. The homology of G is by definition the homology of the complex $(P_*)_G$, where P_* is a projective resolution of $\mathbb{Z}$ by right G-modules.The following general construction of a projective resolution is of particular interest to us: For X a set, let $C_*(X)$ be the acyclic complex of free abelian groups, which in dimension n is generated by $(n+1)$-tuples of elements in X. The differential is given by

$$\partial(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_n)$$

In particular for $X = G$, the natural left G-action on $C_*(G)$ thus gives a free resolution of $\mathbb{Z}$ by G-modules (it is made into a right resolution in the standard way). The complex $C_*(G)_G$ thus calculates the homology of G.

There is another description of this complex. Consider the complex $B_*(G)$ of free abelian groups, which in dimension n is generated by symbols $[g_1|\cdots|g_n]$ and with differential given by

$$\begin{aligned}\partial[g_1|\cdots|g_n] = [g_2|\cdots|g_n] &+ \sum_{i=1}^{n-1} (-1)^i [g_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ &+ (-1)^n [g_1|\cdots|g_{n-1}].\end{aligned}$$

This complex is isomorphic to $C_*(G)_G$ via the map

$$[g_1|\cdots|g_n] \mapsto (1, g_1, g_1 g_2, \dots, g_1 \dots g_n)$$

with inverse

$$(g_0, \dots, g_n) \mapsto [g_0^{-1} g_1|\cdots|g_{n-1}^{-1} g_n].$$

Hence, we can represent a homology class in $H_n(G)$ either by a chain in $C_n(G)$ or by a cycle in $B_n(G)$. These two ways of representing homology classes are called the *homogenous* and the *inhomogenous* representation, respectively.

We will also be interested in certain subcomplexes of $C_*(G)$. For $g \in G$ there is a map $s_g \colon C_*(G) \to C_*(G)$ given by $s_g(g_0, \dots, g_n) = (g, g_0, \dots, g_n)$. We shall often use

Lemma 1.3. *Let $D_*(G)$ be a G-subcomplex of $C_*(G)$. Suppose that for each cycle σ in $D_*(G)$, there exists a point $g(\sigma)$ in G such that $s_{g(\sigma)}\sigma$ is in $D_{n+1}(G)$. Then $D_*(G)$ is acyclic and $D_*(G)_G$ calculates the homology of G.*

Proof. Note that $\partial s_g(g_0, \dots, g_n) = (g_0, \dots, g_n) - s_g(\partial(g_0, \dots, g_n))$. Let σ be a cycle in $D_*(G)$. Since $\partial\sigma = 0$ we have $\sigma = \partial s_{g(\sigma)}\sigma$, that is, σ is a boundary. □

Let M be a left G-module. The cohomology $H_*(G, M)$ is defined as the homology of the complex $\mathrm{Hom}_{\mathbb{Z}[G]}(P_*, M)$, where P_*, this time, is a projective resolution of $\mathbb{Z}$ by *left* G-modules. Regarding a divisible abelian group A as a trivial G-module, we have by the universal coefficient theorem a natural isomorphism

$$H^n(G, A) = \mathrm{Hom}(H_n(G), A).$$

It is well known that the homology of a group is isomorphic to the singular homology of its classifying space, and since the abelian group $\mathbb{F}/\mathbb{Z}$ is obviously divisible, we can regard the C-C-S classes as homomorphisms from $H_3(G)$ to $\mathbb{F}/\mathbb{Z}$. It is an interesting problem to try to find explicit formulas for the C-C-S classes directly in terms of the resolution $C_*(G)$ (or some subcomplex). We shall investigate this in the following sections.

1.3. **The Bloch Group.** In all the following, we let G denote the group $\mathrm{SL}(2,\mathbb{C})$.

Definition 1.4. The *pre-Bloch group* $\mathcal{P}(\mathbb{C})$ is an abelian group generated by symbols $[z]$, $z\in\mathbb{C}\backslash\{0,1\}$ subject to the relation

$$[x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right]=0. \tag{1.1}$$

This relation is called the *five term relation*.

In [3] and [7] the five term relation is different, but this is because of the different definition of the cross-ratio (see below).

Definition 1.5. The *Bloch group* $\mathcal{B}(\mathbb{C})$ is the kernel of the homomorphism

$$\nu\colon \mathcal{P}(\mathbb{C})\to\mathbb{C}^*\wedge\mathbb{C}^*$$

defined on generators by $[z]\mapsto z\wedge(1-z)$.

There is an important interpretation of the pre-Bloch group in terms of a homology group. Let $C_*^{\neq}(S^2)$ denote the subcomplex of $C_*(S^2)$ consisting of tuples of distinct elements. Recall that $G=\mathrm{SL}(2,\mathbb{C})$ acts on $S^2=\mathbb{C}\cup\{\infty\}$ by Möbius transformations, that is

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}z=\frac{az+b}{cz+d}.$$

This action is 3-transitive and four distinct points $z_0,\dots,z_3$ are determined up to the action by the *cross-ratio* (in [3] and [7] the cross-ratio is defined to be the reciprocal of (1.2)).

$$z=[z_0:z_1:z_2:z_3]:=\frac{(z_0-z_3)(z_1-z_2)}{(z_0-z_2)(z_1-z_3)}. \tag{1.2}$$

This means that $C_3^{\neq}(S^2)_G$ is just the free abelian group on $\mathbb{C}\backslash\{0,1\}$. Using (1.2), one easily checks that the five term relation is equivalent to the relation

$$\sum_{i=0}^{4}[[z_0:\dots:\hat{z}_i:\dots:z_4]]=0.$$

This means that the kernel of the cross-ratio map $\sigma\colon C_3^{\neq}(S^2)\to\mathcal{P}(\mathbb{C})$ is exactly the boundaries. Since $C_2^{\neq}(S^2)_G=\mathbb{Z}$ by 3-transitivity, $C_3^{\neq}(S^2)/G$ consists entirely of cycles, and σ induces an isomorphism

$$\sigma\colon H_3(C_*^{\neq}(S^2)_G)\to\mathcal{P}(\mathbb{C}).$$

We have the following relations in the pre-Bloch group (see [7])

$$[x]=\left[\frac{1}{1-x}\right]=\left[1-\frac{1}{x}\right]=-\left[\frac{1}{x}\right]=-\left[\frac{x}{x-1}\right]=-[1-x].$$

If we extend the cross-ratio by setting $[z_0:z_1:z_2:z_3]=0$ if there are equals among $z_0,\dots,z_3$, it follows from the above relations that σ can be extended to $H_3(C_*(S^2)_G)$. We omit the details. We can now define a map

$$\lambda\colon H_3(G)\to\mathcal{P}(\mathbb{C})$$

as the composition

$$H_3(G) \longrightarrow H_3(C_*(S^2)_G) \xrightarrow{\sigma} \mathcal{P}(\mathbb{C})$$

where the left map is induced by

$$C_3(G) \to C_3(S^2) \ , \ (g_0, \ldots, g_3) \mapsto (g_0\infty, g_1\infty, g_2\infty, g_3\infty).$$

In [7] it is shown that λ has image in the Bloch group, and that the following sequence, which is essentially due to Bloch and Wigner, is exact.

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(G) \xrightarrow{\lambda} \mathcal{B}(\mathbb{C}) \longrightarrow 0 \tag{1.3}$$

Using the isomorphism $\mathbb{Q}/\mathbb{Z} = \varinjlim \mathbb{Z}/n\mathbb{Z} = \varinjlim H_3(\mathbb{Z}/n\mathbb{Z})$, the left map is the limit map induced by the maps $\mathbb{Z}/n\mathbb{Z} \to G$ given by sending 1 to the matrix of a rotation by $2\pi/n$.

1.4. **Rogers' Dilogarithm.** We here review a result in [3] relating the C-C-S class $\hat{P}_1$ to a dilogarithm function via the Bloch group.

Rogers' dilogarithm is the following function defined on the open interval $(0, 1)$.

$$L(z) = -\frac{1}{2}\operatorname{Log}(z)\operatorname{Log}\Big(\frac{1}{1-z}\Big) + \operatorname{Li}_2(z) - \frac{\pi^2}{6} \tag{1.4}$$

Here $\operatorname{Li}_2(z) = -\int_0^z \frac{\operatorname{Log}(1-t)}{t}$ is the classical dilogarithm function. Note that we have subtracted $\frac{\pi^2}{6}$ from the original Rogers' dilogarithm. L is real analytic and satisfies the functional equations

$$L(x) + L(1-x) = -\frac{\pi^2}{6},$$

$$L(x) - L(y) + L\Big(\frac{y}{x}\Big) - L\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big) + L\Big(\frac{1-x}{1-y}\Big) = 0 \quad , \quad y < x. \tag{1.5}$$

We can extend L (discontinously) to $\mathbb{R}$ by setting

$$L(x) = \begin{cases} -L(1/x) \text{ for } x > 1 \\ -L(\frac{x}{x-1}) \text{ for } x < 0 \end{cases} \quad , \quad L(1) = 0, L(0) = -\frac{\pi^2}{6} \tag{1.6}$$

and define a map $L \colon C_3(\mathrm{SL}(2, \mathbb{R})) \to \mathbb{R}$ by

$$(g_0, \ldots, g_3) \to L([g_0\infty : \cdots : g_3\infty]). \tag{1.7}$$

This is clearly well defined (recall that cross-ratios are defined to be zero when there are equals) since all cross-ratios are real. Also, a few calculations using the functional equations show that the map takes boundaries to multiples of $\pi^2/6$, that is, it is a 3-cocycle modulo $\pi^2/6$. The theorem below can be found in [3].

Theorem 1.6. *$\frac{1}{4\pi^2}L$ equals the Cheeger-Chern-Simons class $\hat{P}_1$ modulo $1/24$.*

There is a minus sign in [3], but this is because of the different definition of the cross ratio.

Since the restriction of the second Chern polynomial to the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$ is minus the Pontrjagin polynomial, it follows from Theorem 1.1 that we have a commutative diagram

$$\begin{tikzcd} H_3(\mathrm{SL}(2,\mathbb{R})) \arrow[r, "-\hat{P}_1"] \arrow[d] & \mathbb{R}/\mathbb{Z} \arrow[d] \\ H_3(\mathrm{SL}(2,\mathbb{C})) \arrow[r, "\hat{C}_2"] & \mathbb{C}/\mathbb{Z}. \end{tikzcd}$$

By Theorem 1.6, $\hat{P}_1$ is (modulo 1/24) just a dilogarithm via the Bloch group. We wish to find a similar expression for $\hat{C}_2$ by extending L to $H_3(\mathrm{SL}(2,\mathbb{C}))$. This is partially solved in [3] by constructing a homomorphism $c\colon \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$ such that the composition below is $2\hat{C}_2$.

$$H_3(\mathrm{SL}(2,\mathbb{C})) \xrightarrow{\lambda} \mathcal{B}(\mathbb{C}) \xrightarrow{c} \mathbb{C}/\mathbb{Q}$$

We shall improve this by showing that there is a commutative diagram

$$\begin{tikzcd} H_3(\mathrm{SL}(2,\mathbb{C})) \arrow[r, "\widehat{\lambda}"] \arrow[d] & \widehat{\mathcal{B}}(\mathbb{C}) \arrow[r, "\widehat{L}"] \arrow[d] & \mathbb{C}/\mathbb{Z} \arrow[d] \\ H_3(\mathrm{SL}(2,\mathbb{C}))/(\mathbb{Q}/\mathbb{Z}) \arrow[r, "\lambda"] & \mathcal{B}(\mathbb{C}) \arrow[r, "c"] & \mathbb{C}/\mathbb{Q} \end{tikzcd}$$

so that the top composition is $2\hat{C}_2$. Here $\widehat{\mathcal{B}}(\mathbb{C})$ is Neumann's extended Bloch group (see [8] or Section 2 below). In other words, $\hat{C}_2$ is a dilogarithm via the extended Bloch group exactly as $\hat{P}_1$ is a dilogarithm via the Bloch group.

2. The Extended Bloch Group

In this section we review Neumann's definition of the extended Bloch group. The main reference is [8]. Our presentation resembles that of Neumann, but because of a few minor deviations, we give all the details for completeness and to avoid confusion.

We shall use the conventions that the argument $\operatorname{Arg} z$ of a complex number always denotes the main argument ($-\pi < \operatorname{Arg} z \leq \pi$) and the logarithm $\operatorname{Log} z$ always denotes the logarithm having $\operatorname{Arg} z$ as imaginary part.

The idea is to construct a Riemann surface $\widehat{\mathbb{C}}$ covering $\mathbb{C}\backslash\{0,1\}$ and then construct the extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ as in Definition 1.4, with an appropriate lift of the five term relation.

Let $\widehat{\mathbb{C}}$ denote the universal abelian cover of $\mathbb{C}\backslash\{0,1\}$. There is a nice way of representing points in $\widehat{\mathbb{C}}$. Let $\mathbb{C}_{\mathrm{cut}}$ denote $\mathbb{C}\backslash\{0,1\}$ cut open along each of the intervals $(-\infty,0)$ and $(1,\infty)$ so that each real number r outside of $[0,1]$ occurs twice in $\mathbb{C}_{\mathrm{cut}}$. We shall denote these two occurences of r by $r+0i$ and $r-0i$ respectively. It is now easy to see that $\widehat{\mathbb{C}}$ is isomorphic to the surface obtained from $\mathbb{C}_{\mathrm{cut}} \times 2\mathbb{Z} \times 2\mathbb{Z}$ by the following identifications:

$$(x+0i, 2p, 2q) \sim (x-0i, 2p+2, 2q) \text{ for } x \in (-\infty, 0)$$
$$(x+0i, 2p, 2q) \sim (x-0i, 2p, 2q+2) \text{ for } x \in (1, \infty).$$

This means that points in $\widehat{\mathbb{C}}$ are of the form (z,p,q) with $z \in \mathbb{C}\backslash\{0,1\}$ and p,q even integers. Note that $\widehat{\mathbb{C}}$ can be regarded as the Riemann surface for the function

$$\mathbb{C}\backslash\{0,1\} \to \mathbb{C}^2 \ , \ \ z \mapsto \Big(\operatorname{Log} z, \operatorname{Log}\Big(\frac{1}{1-z}\Big)\Big)$$

We shall show below that L can be extended holomorphically to be defined on $\widehat{\mathbb{C}}$, and then we shall simply define the extended five term relation to be the smallest possible extension of the relation (1.5).

Consider the set

$$\mathrm{FT} := \Big\{\Big(x, y, \frac{y}{x}, \frac{1-x^{-1}}{1-y^{-1}}, \frac{1-x}{1-y}\Big)\Big\} \subset (\mathbb{C}\backslash\{0,1\})^5$$

of five-tuples involved in the five term relation. Also let

$$\mathrm{FT}_0 = \big\{(x_0, \dots, x_4) \in \mathrm{FT} \mid 0 < x_1 < x_0 < 1\big\}$$

be the set of five-tuples involved in the functional equation (1.5). Define $\widehat{\mathrm{FT}} \subset \widehat{\mathbb{C}} \times \cdots \times \widehat{\mathbb{C}}$ to be the component of the preimage of FT that contains all points $\big((x_0; 0, 0), \dots, (x_4; 0, 0)\big)$ with $(x_0, \dots, x_4) \in \mathrm{FT}_0$.

Definition 2.1. The *extended pre-Bloch group* $\widehat{\mathcal{P}}(\mathbb{C})$ is the abelian group generated by symbols $[z; p, q]$, with $(z; p, q) \in \widehat{\mathbb{C}}$, subject to the relation

$$\sum_{i=0}^{4}(-1)^i[x_i; p_i, q_i] = 0 \text{ for } ((x_0; p_0, q_0), \dots, (x_4; p_4, q_4)) \in \widehat{\mathrm{FT}}.$$

This relation is called the *extended five term relation.*

Definition 2.2. The *extended Bloch group* $\widehat{\mathcal{B}}(\mathbb{C})$ is the kernel of the homomorphism (which is well defined by 2.3 below):

$$\widehat{\nu}\colon \widehat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C} \wedge \mathbb{C}$$

defined on generators by $[z; p, q] \mapsto (\operatorname{Log} z + p\pi i) \wedge (-\operatorname{Log}(1-z) + q\pi i)$.

Our definition of the Bloch group may look different from Neumann's, but in the appendix we show that our definition is equivalent to Neumann's definition of the more extended Bloch group ([8, Section 8]). We thus answer affirmatively a question raised in [8, page 443], about the relation of this group to $H_3(\mathrm{SL}(2,\mathbb{C}))$.

Proposition 2.3. *The map $\widehat{\nu}$ is well defined.*

Proof. If we apply ν to an element $\sum_{i=0}^{4}(-1)^i[x_i; p_i, q_i]$ with $(x_0, \dots, x_4) = (x, y, \dots) \in \mathrm{FT}_0$, we obtain after simplification:

$$\Big((q_0 - p_2 - q_2 + p_3 + q_3)\operatorname{Log} x \\ + (p_0 - q_3 + q_4)\operatorname{Log}(1-x) + (-q_1 + q_2 - q_3)\operatorname{Log} y \\ + (-p_1 + p_3 + q_3 - p_4 - q_4)\operatorname{Log}(1-y) + (p_2 - p_3 + p_4)\operatorname{Log}(x-y)\Big) \wedge \pi i.$$

An elementary linear algebra computation shows that this is zero if and only if $p_2 = p_1 - p_0$, $p_3 = p_1 - p_0 + q_1 - q_0$, $q_3 = q_2 - q_1$, $p_4 = q_1 - q_0$, and $q_4 = q_2 - q_1 - p_0$. The proposition now follows by analytic continuation from Lemma 4.17 in the appendix. □

We now extend L to $\widehat{\mathbb{C}}$. First note that the expression in (1.4) is well defined for all $z \in \mathbb{C}\backslash\{0,1\}$, and that L in this way becomes holomorphic except at real points outside the interval between 0 and 1.

Define

$$\widehat{L}(z;p,q) = L(z) + \frac{\pi i}{2}\Big(q\operatorname{Log}(z) - p\operatorname{Log}\Big(\frac{1}{1-z}\Big)\Big). \tag{2.1}$$

Remark 2.4. Neumann calls this map R (probably for Rogers), but in fact Rogers originally called his dilogarithm L. Also, the name $\widehat{L}$ is more consistent with our convention that all extended groups and maps be labelled with a hat.

Proposition 2.5 (Neumann). *$\frac{1}{2\pi^2}\widehat{L}$ gives a well defined and holomorphic map $\widehat{\mathbb{C}} \to \mathbb{C}/\mathbb{Z}$. Also the extended five term relation is a functional equation so that $\frac{1}{2\pi^2}\widehat{L}$ gives a homomorphism $\widehat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}$.*

Proof. If we follow a closed path from z going anti-clockwise around 0, then $\widehat{L}(z;p,q)$ is replaced by $\widehat{L}(z;p,q) - \pi i\operatorname{Log}(\frac{1}{1-z}) - q\pi^2 = \widehat{L}(z;p+2,q) - q\pi^2$. Similarly, if we follow a closed path from z going clockwise around 1, $\widehat{L}(z;p,q)$ is replaced by $\widehat{L}(z;p,q) - \pi i\operatorname{Log}(z) + 2\pi i\operatorname{Log}(z) + p\pi^2 = \widehat{L}(z;p,q+2) + p\pi^2$. This is because the discontinuity of Li_2 at a real number r bigger than one is $2\pi i\operatorname{Log} r$, which is easily verified. Since we are assuming that p and q are even, it follows that $\frac{1}{2\pi^2}\widehat{L}$ is well defined and holomorphic as a map to $\mathbb{C}/\mathbb{Z}$.

To prove the second assertion, note that if $((z_0;p_0,q_0),\dots,(z_4;p_4,q_4))$ is a point in $\widehat{\mathrm{FT}}$ then by analytic continuation

$$\sum_{i=0}^{4}(-1)^i\frac{1}{2\pi^2}\widehat{L}(x_i;p_i,q_i) = 0$$

as required. □

2.1. **Geometry of the Extended pre-Bloch Group.** We first recall some geometric properties of the cross-ratio. Let z_0, z_1, z_2, z_3 be four distinct ordered points in $\mathbb{C}\cup\{\infty\}$. Regarding $\mathbb{C}\cup\{\infty\}$ as the boundary of the standard compactification of hyperbolic 3-space $\mathbb{H}^3$, the four points define a unique ideal hyperbolic simplex $[z_0,\dots,z_3]$ which is determined up to orientation preserving congruence by the cross-ratio

$$z = [z_0 : z_1 : z_2 : z_3] := \frac{(z_0-z_3)(z_1-z_2)}{(z_0-z_2)(z_1-z_3)}. \tag{2.2}$$

Clearly $z \in \mathbb{C}\backslash\{0,1\}$ and since $[0:\infty:1:z] = z$, every $z \in \mathbb{C}\backslash\{0,1\}$ can be realized as the cross-ratio of an ideal hyperbolic simplex. It is well known that z is real if and only if the four points lie on a circle (that is circle or straight line) and in this case the simplex is called *flat*. For non-flat simplices, the imaginary part of z is positive if and only if the orientation of the simplex induced by the ordering of the z_i's agrees with the orientation of $\mathbb{H}^3$. There is a nice geometric interpretation of the argument of z. If the imaginary part of z is greater than or equal to zero then $\operatorname{Arg} z$ is the dihedral angle of the simplex corresponding to the edge $[z_0z_1]$. Otherwise, that is if the orientation disagrees with the orientation of $\mathbb{H}^3$, it is minus the dihedral angle.

It easily follows from (2.2) that an even permutation of the z_i's replaces z by one of three so-called *cross-ratio parameters*.

$$z\ ,\ z' = \frac{1}{1-z}\ ,\ z'' = 1 - \frac{1}{z}$$

In particular the dihedral angle corresponding to the edges $[z_1z_2]$ and $[z_1z_3]$ are $\mathrm{Arg}(z')$ and $\mathrm{Arg}(z'')$ respectively, (or their negatives if the vertex ordering does not agree with the orientation of $\mathbb{H}^3$). Since a product of two disjoint transpositions clearly keeps the cross-ratio fixed, we see that the dihedral angles of opposite edges are the same. Note that since $zz'z''=-1$ the sum of the dihedral angles is always π. This is not surprising since a horosphere at an ideal vertex of a hyperbolic simplex intersects the simplex in a euclidean triangle.

Definition 2.6. A *combinatorial flattening* of an ideal simplex with cross-ratio z is a triple (w_0, w_1, w_2) of complex numbers with $w_0 + w_1 + w_2 = 0$, where w_0 and w_1 are choices of logarithms of z and z'. We call w_0, w_1, and w_2 *log-parameters.*

Note that $w_2 - \pi i$ is a choice of logarithm of z''. Because of the relation to the dihedral angles, this explains the name combinatorial flattening. Note that the set of combinatorial flattenings of ideal simplices is in bijective correspondence with $\widehat{\mathbb{C}}$ by the map l given by

$$l(w_0, w_1, w_2) = \Big(z; \frac{w_0 - \mathrm{Log}\, z}{\pi i}, \frac{w_1 - \mathrm{Log}(\frac{1}{1-z})}{\pi i}\Big) \tag{2.3}$$

where $z = e^{w_0}$. This means that the extended pre-Bloch group can be regarded as being generated by combinatorial flattenings of ideal simplices, whereas the Bloch group can be regarded as being generated by congruence classes of ideal simplices. Let us discuss the five term relation in this geometric setup.

Suppose (w_0, w_1, w_2) is a combinatorial flattening of an ideal simplex $[z_0, \dots, z_3]$. Then we can assign log-parameters to each edge in such a way that w_0 is assigned to the edge $[z_0z_1]$, w_1 to the edge $[z_1z_2]$ and w_2 to the edge $[z_1z_3]$. The three remaining edges are assigned the same log-parameter as their opposite edge, see Figure 1. Let $z_0, \dots, z_4$ be five distinct points in $\mathbb{C} \cup \{\infty\}$ and let Δ_i denote the simplices

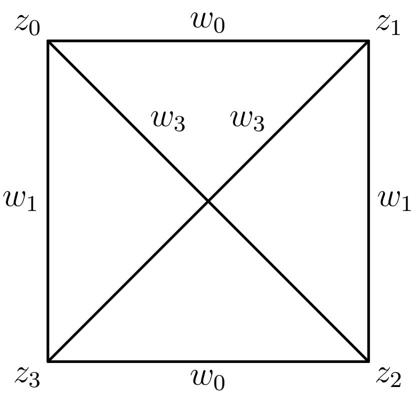

FIGURE 1. Assignment of log-parameters to edges of an ideal simplex.

$[z_0, \dots, \hat{z}_i, \dots, z_4]$. Using (2.2), it is easy to see that the cross-ratios x_i of Δ_i can be expressed in terms of $x := z_0$ and $y := z_1$ as follows:

$$\begin{aligned} x_0 &= [z_1 : z_2 : z_3 : z_4] := x \\ x_1 &= [z_0 : z_2 : z_3 : z_4] := y \\ x_2 &= [z_0 : z_1 : z_3 : z_4] \;= \frac{y}{x} \\ x_3 &= [z_0 : z_1 : z_2 : z_4] \;= \frac{1-x^{-1}}{1-y^{-1}} \\ x_4 &= [z_0 : z_1 : z_2 : z_3] \;= \frac{1-x}{1-y} \end{aligned}$$

Suppose (w_0^i, w_1^i, w_2^i) are combinatorial flattenings of the simplices Δ_i. Then every edge $[z_iz_j]$ belongs to exactly three of the Δ_i's and is therefore assigned three log-parameters.

Definition 2.7. Let (w_0^i, w_1^i, w_2^i) be combinatorial flattenings of five simplices $\Delta_i = [z_0, \dots, \hat{z}_i, \dots, z_4]$. The flattenings are said to satisfy the *flattening condition* if for each edge the signed sum of the three assigned log-parameters is zero (the sign is positive if and only if i is even).

It follows directly from the definition that the flattening condition is equivalent to the following ten equations.

$$\begin{array}{llll} [z_0z_1]: & w_0^2 - w_0^3 + w_0^4 = 0 & [z_0z_2]: & -w_0^1 - w_2^3 + w_2^4 = 0 \\ [z_1z_2]: & w_0^0 - w_1^3 + w_1^4 = 0 & [z_1z_3]: & w_2^0 + w_1^2 + w_2^4 = 0 \\ [z_2z_3]: & w_1^0 - w_1^1 + w_0^4 = 0 & [z_2z_4]: & w_2^0 - w_2^1 - w_0^3 = 0 \\ [z_3z_4]: & w_0^0 - w_0^1 + w_0^2 = 0 & [z_3z_0]: & -w_2^1 + w_2^2 + w_1^4 = 0 \\ [z_4z_0]: & -w_1^1 + w_1^2 - w_1^3 = 0 & [z_4z_1]: & w_1^0 - w_2^2 - w_2^3 = 0 \end{array}$$

Recall that combinatorial flattenings are in one to one correspondence with points in $\widehat{\mathbb{C}}$ via the map l in (2.3).

Theorem 2.8 (Neumann)**.** *Flattenings (w_0^i, w_1^i, w_2^i) satisfy the flattening condition if and only if $\sum_{i=0}^4 (-1)^i [l(w_0, w_1, w_2)] = 0$ in $\widehat{\mathcal{P}}(\mathbb{C})$.*

Proof. Let $l(w_0^i, w_1^i, w_2^i) = [x_i; p_i, q_i]$. By analytic continuation, it is enough to consider $(x_0, \dots, x_4) \in \mathrm{FT}_0$. Using the definition of the map l, we see that the ten equations above are equivalent to ten similar equations in the p_i's and q_i's, and by linear algebra, these are easily seen to be equivalent to the equations $p_2 = p_1 - p_0$, $p_3 = p_1 - p_0 + q_1 - q_0$, $q_3 = q_2 - q_1$, $p_4 = q_1 - q_0$, and $q_4 = q_2 - q_1 - p_0$. By Lemma 4.17, this is equivalent to $\sum_{i=0}^4 (-1)^i [x_i; p_i, q_i] = 0$. $\square$

This means that the flattening condition is actually equivalent to the extended five term relation.

3. Mappings via Configurations in $\mathbb{C}^2$

In this section we explore the idea that the extra information needed to remove the $\mathbb{Q}/\mathbb{Z}$ indeterminacy in Dupont's formula for the C-C-S class $\hat{C}_2$ can be detected by configurations in $\mathbb{C}^2$ instead of S^2. Let h denote the Hopf map $h\colon \mathbb{C}^2 \backslash \{0\} \to S^2 = \mathbb{C} \cup \{\infty\}$ given by

$$(z, w) \mapsto z/w.$$

We will show that for certain tuples $(v_0,\dots,v_3)$ of points in $\mathbb{C}^2$, there is a natural choice of combinatorial flattening of the ideal simplex $[hv_0,\dots,hv_3]$. This means that such a tuple gives an element in $\widehat{\mathcal{P}}(\mathbb{C})$. We also describe a way of associating such a tuple to a tuple of group elements in such a way that we obtain a map

$$\widehat{\lambda}\colon H_3(G)\to\widehat{\mathcal{P}}(\mathbb{C}).$$

Recall from Section 1.3 that there is a map $\sigma\colon C_*^{\neq}(S^2)_G\to\mathcal{P}(\mathbb{C})$. We saw that boundaries were mapped to zero and that the induced map

$$\sigma : H_3(C_*^{\neq}(S^2)_G)\to\mathcal{P}(\mathbb{C})$$

is an isomorphism. We shall elaborate on this and construct a G-complex $C_*^{h\neq}(\mathbb{C}^2)$ and a map $\widehat{\sigma}\colon C_3^{h\neq}(\mathbb{C}^2)_G\to\widehat{\mathcal{P}}(\mathbb{C})$ giving rise to a commutative diagram

$$\begin{tikzcd} H_3(C_*^{h\neq}(\mathbb{C}^2)_G) \arrow[r, "\widehat{\sigma}"] \arrow[d, "h"] & \widehat{\mathcal{P}}(\mathbb{C}) \arrow[d] \\ H_3(C_*^{\neq}(S^2)_G) \arrow[r, "\sigma"] & \mathcal{P}(\mathbb{C}). \end{tikzcd} \tag{3.1}$$

We define the complex $C_*^{h\neq}(\mathbb{C}^2)$ as the subcomplex of $C_*(\mathbb{C}^2\backslash\{0\})$ consisting of tuples mapping to different elements in S^2 by the Hopf map h. It is easy to see that the natural G-action on $\mathbb{C}^2$ is h-equivariant. This means that h induces a G-map $C_*^{h\neq}(\mathbb{C}^2)\to C_*^{\neq}(S^2)$ and hence a map

$$h\colon H_3(C_*^{h\neq}(\mathbb{C}^2)_G)\to H_3(C_*^{\neq}(S^2)_G).$$

3.1. **Mapping to the Extended pre-Bloch Group.** We now assign to each 4-tuple $(v_0,v_1,v_2,v_3)\in C_3^{h\neq}(\mathbb{C}^2)$ a combinatorial flattening of the ideal simplex $[hv_0,hv_1,hv_2,hv_3]$ in such a way that the combinatorial flattenings assigned to tuples $(v_0,\dots,\hat{v}_i,\dots v_4)$ satisfy the flattening condition. This will give us a map

$$\widehat{\sigma}\colon H_3(C_*^{h\neq}(\mathbb{C}^2)_G)\to\widehat{\mathcal{P}}(\mathbb{C}).$$

The key step is to observe that the cross-ratio parameters z and $\frac{1}{1-z}$ of a simplex $[hv_0,hv_1,hv_2,hv_3]$ can be expressed in terms of determinants.

$$z:=[hv_0:hv_1:hv_2:hv_3]=\frac{\frac{v_0^1}{v_0^2}-\frac{v_3^1}{v_3^2}}{\frac{v_0^1}{v_0^2}-\frac{v_2^1}{v_2^2}}\frac{\frac{v_1^1}{v_1^2}-\frac{v_2^1}{v_2^2}}{\frac{v_1^1}{v_1^2}-\frac{v_3^1}{v_3^2}}=\frac{\det(v_0,v_3)\det(v_1,v_2)}{\det(v_0,v_2)\det(v_1,v_3)}$$

where the upper indices refer to first or second coordinate in $\mathbb{C}^2$, respectively. Similarly,

$$\frac{1}{1-z}=[hv^1:hv^2:hv^0:hv^3]=\frac{\frac{v_1^1}{v_1^2}-\frac{v_3^1}{v_3^2}}{\frac{v_1^1}{v_1^2}-\frac{v_0^1}{v_0^2}}\frac{\frac{v_2^1}{v_2^2}-\frac{v_0^1}{v_0^2}}{\frac{v_2^1}{v_2^2}-\frac{v_3^1}{v_3^2}}=\frac{\det(v_1,v_3)\det(v_0,v_2)}{\det(v_0,v_1)\det(v_2,v_3)}$$

Since obviously $hv_i\neq hv_j$ if and only if $\det(v_i,v_j)\neq 0$, all these determinants are non-zero. This suggests that we can assign a flattening to (v_0,v_1,v_2,v_3) by setting

$$\begin{aligned} w_0 &= \operatorname{Log}\det(v_0,v_3)+\operatorname{Log}\det(v_1,v_2)-\operatorname{Log}\det(v_0,v_2)-\operatorname{Log}\det(v_1,v_3)\\ w_1 &= \operatorname{Log}\det(v_0,v_2)+\operatorname{Log}\det(v_1,v_3)-\operatorname{Log}\det(v_0,v_1)-\operatorname{Log}\det(v_2,v_3)\\ w_2 &= \operatorname{Log}\det(v_0,v_1)+\operatorname{Log}\det(v_2,v_3)-\operatorname{Log}\det(v_0,v_3)-\operatorname{Log}\det(v_1,v_2)\end{aligned}$$

This defines a map $\widehat{\sigma}\colon C_3^{h\neq}(\mathbb{C}^2)\to\widehat{\mathcal{P}}(\mathbb{C})$ by

$$(v_0,v_1,v_2,v_3)\mapsto[l(w_0,w_1,w_2)]. \tag{3.2}$$

Now suppose $(w_0^0,w_1^0,w_2^0),\dots,(w_0^4,w_1^4,w_2^4)$ are flattenings defined as above of simplices $[hv_0,\dots,\widehat{hv_i},\dots,hv_4]$. We must check that these flattenings satisfy the flattening condition. This is equivalent to checking that all the ten equations stated below Definition 2.7 are satisfied. We check the first of these and leave the others to the reader. Using the notation $(v,w):=\operatorname{Log}\det(v,w)$ we have:

$$w_0^2=(v_0,v_4)+(v_1,v_3)-(v_0,v_3)-(v_1,v_4)$$

$$w_0^3=(v_0,v_4)+(v_1,v_2)-(v_0,v_2)-(v_1,v_4)$$

$$w_0^4=(v_0,v_3)+(v_1,v_2)-(v_0,v_2)-(v_1,v_3)$$

and it follows that the equation $w_0^2-w_0^3+w_0^4=0$ is satisfied.

Having verified all the ten equations, it now follows from Theorem 2.8 that $\widehat{\sigma}$ sends boundaries to zero. Since $\widehat{\sigma}$ is obviously a G-map, we thus obtain a map $\widehat{\sigma}\colon H_3(C_*^{h\neq}(\mathbb{C}^2)_G)\to\widehat{\mathcal{P}}(\mathbb{C})$.

It is clear that the diagram below is commutative.

$$\begin{tikzcd} H_3(C_*^{h\neq}(\mathbb{C}^2)_G) \arrow[r, "\widehat{\sigma}"] \arrow[d, "h"] & \widehat{\mathcal{P}}(\mathbb{C}) \arrow[d] \\ H_3(C_*^{\neq}(S^2)_G) \arrow[r, "\sigma"] & \mathcal{P}(\mathbb{C}) \end{tikzcd} \tag{3.3}$$

Proposition 3.1. *The image of* $\widehat{\sigma}\colon H_3(C_*^{h\neq}(\mathbb{C}^2)_G)\to\widehat{\mathcal{P}}(\mathbb{C})$ *is in* $\widehat{\mathcal{B}}(\mathbb{C})$.

Proof. Define a map $C_2^{\neq}(\mathbb{C}^2)_G\to\mathbb{C}\wedge\mathbb{C}$ by

$$(v_0,v_1,v_2)\mapsto(v_0,v_1)\wedge(v_0,v_2)-(v_0,v_1)\wedge(v_1,v_2)+(v_0,v_2)\wedge(v_1,v_2)$$

where we still use the notation $(v,w):=\operatorname{Log}\det(v,w)$. A straightforward (but quite cumbersome) calculation shows that the diagram below is commutative.

$$\begin{tikzcd} C_3^{h\neq}(\mathbb{C}^2)_G \arrow[r, "\widehat{\sigma}"] \arrow[d, "\partial"] & \widehat{\mathcal{P}}(\mathbb{C}) \arrow[d, "\widehat{\nu}"] \\ C_2^{\neq}(\mathbb{C}^2)_G \arrow[r] & \mathbb{C}\wedge\mathbb{C} \end{tikzcd}$$

This means that cycles are mapped to $\widehat{\mathcal{B}}(\mathbb{C})$ as desired. □

3.2. **The Map from $H_3(G)$.** In this section we shall construct a map $\widehat{\lambda}\colon H_3(G)\to\widehat{\mathcal{P}}(\mathbb{C})$ via the group $H_3(C_*^{h\neq}(\mathbb{C}^2)_G)$. To define this map explicitly we need to restrict to a subcomplex of $C_*(G)$.

Definition 3.2. A chain in $C_*(G)$ is called *good* if all its tuples satisfy $g_i\neq\pm g_j$, and *v-good* ($v\in\mathbb{C}^2$) if all its tuples satisfy $\det(g_iv,g_jv)\neq 0$. The G-complexes of good and v-good chains are denoted $C_*^{\mathrm{good}}(G)$ and $C_*^{v}(G)$ respectively.

By Lemma 1.3, $C_*^{\mathrm{good}}(G)$ and $C_*^v(G)$ are both acyclic, and therefore both calculate the homology of G. We can therefore identify $H_3(G)$ with $H_3(C_*^{\mathrm{good}}(G)_G)$. Consider the G-maps

$$\begin{aligned} \Psi_v\colon C_n(G) &\to C_n(\mathbb{C}^2) & \qquad (g_0,\dots,g_n) &\mapsto (g_0v,\dots,g_nv) \\ \mathrm{conj}_g\colon C_n(G) &\to C_n(\mathbb{C}^2) & \qquad (g_0,\dots,g_n) &\mapsto (gg_0g^{-1},\dots,gg_ng^{-1}). \end{aligned}$$

Note that if σ is in $C_*^v(G)_G$ then $\mathrm{conj}_g(\sigma)$ is in $C_*^{gv}(G)_G$ and we have

$$\Psi_{gv}(\mathrm{conj}_g(\sigma)) = \Psi_v(\sigma). \tag{3.4}$$

It is clear that Ψ_v takes v-good chains to $C_n^{h\neq}(\mathbb{C}^2)$.

Lemma 3.3. *Let $g_1 \neq \pm g_2 \in G$. The subset*

$$\{v \in \mathbb{C}^2 \mid \det(g_1v, g_2v) \neq 0\} \subset \mathbb{C}^2$$

is open and dense.

For a good chain σ belonging to either $C_*^{\mathrm{good}}(G)$ or $C_*^{\mathrm{good}}(G)_G$ consider the set

$$S_\sigma = \{v \in \mathbb{C}^2 \mid \sigma \text{ is } v\text{-good}\}.$$

Since finite intersections of dense open subsets is dense open, it follows from Lemma 3.3 that S_σ is dense open. In other words, any good chain is also a v-good chain for almost all $v \in \mathbb{C}^2$. The following is a simple consequence of (3.4).

Theorem 3.4. *Let $\sigma \in C_*^{\mathrm{good}}(G)_G$ be a cycle. The cohomology class of $\Psi_v(\sigma)$ is independent of $v \in S_\sigma$.*

We can now define a map $H_3(G) \to H_3(C_*^{h\neq}(\mathbb{C}^2)_G)$ by

$$[\sigma] \mapsto [\Psi_v(\sigma)]\ ,\ \ v \in S_\sigma. \tag{3.5}$$

Theorem 3.5. *The diagram below is commutative.*

$$\begin{tikzcd} H_3(G) \arrow[r, "\Psi_v"] \arrow[d] & H_3(C_*^{h\neq}(\mathbb{C}^2)_G) \arrow[dr, "h"] & \\ H_3(G)/\mathbb{Q}/\mathbb{Z} \arrow[r, "\lambda"] & \mathcal{P}(\mathbb{C}) & H_3(C_*^{\neq}(S^2)_G) \arrow[l, "\sigma"'] \end{tikzcd}$$

Proof. The map in 3.5 obviously coincides with the map

$$H_3(G) \approx H_3(C_*^v(G)_G) \xrightarrow{\Psi_v} H_3(C_*^{h\neq}(\mathbb{C}^2)_G). \tag{3.6}$$

The theorem now follows from (3.6) with $v = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, since $h\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \infty$. $\square$

We can now define $\widehat{\lambda}$ as the composition

$$H_3(G) \longrightarrow H_3(C_*^{h\neq}(\mathbb{C}^2)_G) \xrightarrow{\widehat{\sigma}} \widehat{\mathcal{B}}(\mathbb{C}). \tag{3.7}$$

4. Relation with the Cheeger-Chern-Simons Class

In this section we relate the maps constructed above to the Cheeger-Chern-Simons class $\hat{C}_2$. Our goal is to prove

Theorem 4.1. $-\frac{1}{2\pi^2}\widehat{L}\circ\widehat{\lambda} = 2\hat{C}_2$.

Let $H_3(G)_\pm$ denote the subgroups $\{x \in H_3(G) \mid \tau x = \pm x\}$ where τ is the involution induced by complex conjugation. We shall refer to these subgroups as the real and the imaginary parts of $H_3(G)$.

The following is easy.

Proposition 4.2. $\frac{1}{2\pi^2}\widehat{L}\circ\widehat{\lambda}$ *is equivariant under complex conjugation.*

From Theorem 1.1, $\hat{C}_2$ is also equivariant under conjugation, and since $H_3(G)$ is divisible by a result in [7], it is enough to study the real and imaginary parts separately.

4.1. The Imaginary Part.

It is well known that the volume of an ideal simplex with cross-ratio z is given by

$$\operatorname{Vol}(z) = \operatorname{Arg}(1-z)\operatorname{Log}|z| - \operatorname{Im}\int_0^1 \frac{\operatorname{Log}(1-tz)}{t}dt.$$

Also, the five term relation (1.1) is a functional equation for Vol, which means that Vol is well defined on the pre-Bloch group. We recall from [3]:

Theorem 4.3. $\operatorname{Im}\hat{C}_2 = -\frac{1}{4\pi^2}\operatorname{Vol}\circ\lambda$.

Proposition 4.4. *The imaginary part of* $\widehat{L}\colon \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/2\pi^2\mathbb{Z}$ *gives volume.*

Proof. Let $\tau = \sum(-1)^{\varepsilon_i}[z_i;p_i,q_i] \in \mathcal{B}(\mathbb{C})$. Since

$$\begin{aligned}\operatorname{Im}\widehat{L}(z;p,q) = {} & \frac{1}{2}\big(\operatorname{Arg}(z)\operatorname{Log}|1-z| + \operatorname{Log}|z|\operatorname{Arg}(1-z)\big)\\ & - \operatorname{Im}\int_0^1\frac{\operatorname{Log}(1-tz)}{t}dt + \frac{\pi}{2}p\operatorname{Log}|1-z| + \frac{\pi}{2}q\operatorname{Log}|z|\end{aligned}$$

we have

$$\begin{aligned}\operatorname{Vol}(z) - \operatorname{Im}\widehat{L}(z;p,q) = {} & \frac{1}{2}\big(\operatorname{Log}|z|\operatorname{Arg}(1-z) - \operatorname{Arg}(z)\operatorname{Log}|1-z|\big)\\ & - \frac{\pi}{2}p\operatorname{Log}|1-z| - \frac{\pi}{2}q\operatorname{Log}|z|.\end{aligned}$$

Let ϕ denote the composition

$$\mathbb{C}\wedge\mathbb{C} = \mathbb{R}\wedge\mathbb{R}\oplus i\mathbb{R}\wedge i\mathbb{R}\oplus\mathbb{R}\otimes i\mathbb{R} \to \mathbb{R}\otimes i\mathbb{R} \to i\mathbb{R}$$

where the left map is projection and the right map is multiplication. A simple calculation shows that

$$\begin{aligned}\phi\big(\nu([z;p,q])\big) = {} & -\operatorname{Log}|z|\operatorname{Arg}(1-z) + \operatorname{Arg}(z)\operatorname{Log}|1-z|\\ & + p\pi\operatorname{Log}|1-z| + q\pi\operatorname{Log}|z| = -2\big(\operatorname{Vol}(z) - \operatorname{Im}\widehat{L}(z;p,q)\big).\end{aligned}$$

Since $\nu(\tau) = 0$, we have $\operatorname{Vol}(\tau) = \operatorname{Im}\widehat{L}(\tau)$ as desired. $\square$

4.2. **The Real Part.** Let $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. The key step is the following theorem by Dupont, Parry, and Sah (see [6] and [10]).

Theorem 4.5. *The inclusion $G_{\mathbb{R}} \to G$ induces an isomorphism*

$$H_3(G_{\mathbb{R}}) \approx H_3(G)_+.$$

This means that it is enough to study real cycles. The idea is that every homology class in $H_3(G_{\mathbb{R}})$ has a representative such that the image of $\widehat{L} \circ \widehat{\lambda}$ is the same as the image of the cocycle L from (1.7).

Definition 4.6. An element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_{\mathbb{R}}$ is called *positive* if c is positive and *non-zero* if c is non-zero. A chain in $B_n(G_{\mathbb{R}})$ is called *positive* if all its group elements are positive.

If (g_1, g_2, g_3) is a triple of positive elements that are so small (close to the identity) that also $g_1 g_2$, $g_2 g_3$ and $g_1 g_2 g_3$ are positive, we have

- $(v_0, v_1, v_2, v_3) := \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g_1\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g_1 g_2\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g_1 g_2 g_3\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$ is in $C_3^{h\neq}(\mathbb{C}^2)$.
- $\det(v_i, v_j) > 0$ for $i \neq j$.

This means that the log-parameters w_0, w_1, w_2 associated to $(v_0, \dots, v_3)$ are all positive, and $l(w_0, w_1, w_2) = (e^{w_0}; 0, 0)$. So if α is a positive representation of a class in $H_3(G_{\mathbb{R}})$ with all group elements sufficiently small then

$$\frac{1}{2\pi^2}\widehat{L} \circ \widehat{\lambda}(\alpha) = \frac{1}{2\pi^2} L(\alpha). \tag{4.1}$$

As we shall see below, every homology class in $H_3(G_{\mathbb{R}})$ can be represented by a cycle with all group elements arbitrarily small. The following is essentially just an application of barycentric subdivision, and we refer to [3] for a proof.

Lemma 4.7. *Let H be a contractible Lie group and U a neighborhood of the identity. Every cycle in $B_*(H)$ is homologous to a cycle consisting of elements in U.*

Let $\widetilde{G_{\mathbb{R}}}$ be the universal covering group of $G_{\mathbb{R}}$. Parry and Sah analyse the Hochshild-Serre spectral sequence for the exact sequence

$$0 \to \mathbb{Z} \to \widetilde{G_{\mathbb{R}}} \to G_{\mathbb{R}} \to 0$$

and obtain (see [9]):

Proposition 4.8. *$H_3(\widetilde{G_{\mathbb{R}}}) \to H_3(G_{\mathbb{R}})$ is surjective.*

Since $G_{\mathbb{R}}$ is homotopy equivalent to a circle, $\widetilde{G_{\mathbb{R}}}$ is contractible, and by Lemma 4.7 and Proposition 4.8, every homology class in $H_3(G_{\mathbb{R}})$ has a representative with all group elements arbitrarily small.

We now show that every small cycle in $B_3(G_{\mathbb{R}})$ is homologous to a small and positive cycle.

Define an ordering of elements in $G_{\mathbb{R}}$ by

$$g_1 < g_2 \iff g_1^{-1} g_2 \text{ is positive.}$$

This ordering is neither total nor transitive, but as we shall see, this can be fixed. The following is simple:

Lemma 4.9. *For every natural number n there exists an open neighborhood U_n of the identitity in $G_{\mathbb{R}}$ satisfying that any product of up to n positive elements in U_n is positive.*

Throughout we fix neighborhoods U_n as above.

Definition 4.10. Let $k \leq n$. A k-chain in $B_k(G_{\mathbb{R}})$ is called a U_n-k-chain if all its group elements are non-zero and lie in U_n. A k-chain in $C_n(G_{\mathbb{R}})$ is called a U_n-k-chain if it maps to a U_n-k-chain in $B_n(G_{\mathbb{R}})$. The set of U_n-k-chains in $C_n(G_{\mathbb{R}})$ is denoted $C_k(G_{\mathbb{R}})_{U_n}$.

Proposition 4.11. *Let $g_0, \dots, g_n$ be non-zero elements in U_n. There exists a* unique *permutation $\sigma \in S_{n+1}$ such that $g_{\sigma(0)} < \cdots < g_{\sigma(n)}$.*

Proof. The restriction of the ordering to $\{g_0, \dots, g_n\}$ is clearly transitive, and since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, we have either $g_i < g_j$ or $g_i > g_j$. This means that we can use the bubble sort algorithm to sort the g_i's and thereby produce the desired permutation. □

We thereby obtain $G_{\mathbb{R}}$-maps

$$\Psi_k \colon C_k(G_{\mathbb{R}})_{U_n} \to C_k(G_{\mathbb{R}})_{U_n}.$$

Proposition 4.12. *Let τ be a U_n-k-cycle, $k \leq n$. Then $\psi \circ \pi(\tau)$ and τ are homologous, that is, every U_n-k-cycle is homologous to a positive U_n-k-cycle.*

Proof. By the uniqueness in Proposition 4.11, the maps Ψ_k give rise to a chain map in dimensions up to n. By a standard argument, there exist $G_{\mathbb{R}}$-maps $S_k \colon C_k(G_{\mathbb{R}})_{U_n} \to C_{k+1}(G_{\mathbb{R}})$, $k = 0, \dots, n$ such that

$$\partial S_k + S_{k-1}\partial = \Psi_k - \mathrm{id}.$$

This proves the assertion. □

Definition 4.13. An element in $\mathrm{SL}(2, \mathbb{R})$ is called *small* if it lies in U_3.

Summing the above, we have proved

Proposition 4.14. *Every homology class in $H_3(\mathrm{SL}(2, \mathbb{R}))$ has an inhomogenous representative consisting of small, positive elements.*

Proof of Theorem 4.1. By equation (4.1), Proposition 4.4 and Theorems 4.3 and 1.6, we have that $-\frac{1}{2\pi^2}\widehat{L} \circ \widehat{\lambda} - 2\hat{C}_2$ has image in $\frac{1}{12}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}/12\mathbb{Z}$. As mentioned earlier $H_3(G)$ is divisible, which means that it has no finite quotient. Thus $2\hat{C}_2 = -\frac{1}{2\pi^2}\widehat{L} \circ \widehat{\lambda}$ as required. □

The rest of this section is devoted to a proof of Theorem 4.15 below, but in order to prove this theorem, we need to recall some properties of $\hat{C}_2$ and the relationship between $\widehat{\mathcal{B}}(\mathbb{C})$ and $\mathcal{B}(\mathbb{C})$.

Recall from (1.3) that $\mathbb{Q}/\mathbb{Z}$ can be regarded as a subgroup of $H_3(\mathrm{SL}(2, \mathbb{C}))$. The restriction of $\hat{C}_2$ to this subgroup is just the inclusion of $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{C}/\mathbb{Z}$. In other words, we have a commutative diagram

$$\begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r, hook] \arrow[dr, "i"'] & H_3(\mathrm{SL}(2,\mathbb{C})) \arrow[d, "\hat{C}_2"] \\ & \mathbb{C}/\mathbb{Z}. \end{tikzcd} \tag{4.2}$$

For a proof of this see [5, Theorem 10.2 and the remarks on page 60].

Neumann shows in [8] that $\widehat{\mathcal{B}}(\mathbb{C})$ and $\mathcal{B}(\mathbb{C})$ are related by an exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\widehat{\chi}} \widehat{\mathcal{B}}(\mathbb{C}) \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0 \tag{4.3}$$

where $\widehat{\chi}$ is the map given by

$$\widehat{\chi}(z) = [e^{2\pi i z}; 0, 2] - [e^{2\pi i z}; 0, 0].$$

Theorem 4.15. *The map* $\widehat{\lambda}\colon H_3(\mathrm{SL}(2,\mathbb{C})) \to \widehat{\mathcal{B}}(\mathbb{C})$ *is surjective with kernel* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Suppose $\widehat{\lambda}(\alpha) = 0$. By composing with the map to $\mathcal{B}(\mathbb{C})$, we see from (1.3) that α is in $\mathbb{Q}/\mathbb{Z}$. By (4.2) and Theorem 4.1, we have

$$0 = -\frac{1}{2\pi^2}\widehat{L} \circ \widehat{\lambda}(\alpha) = 2\hat{C}_2(\alpha) = 2\alpha.$$

Hence, α is either zero or the unique element in $\mathbb{Q}/\mathbb{Z}$ of order 2.

Let $\alpha \in \widehat{\mathcal{B}}(\mathbb{C})$. A simple calculation shows that we have

$$-\frac{1}{2\pi^2}\widehat{L} \circ \widehat{\chi} = i,$$

and using (4.2) we get a commutative diagram

$$\begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[dr, "2\widehat{\chi}"'] & H_3(\mathrm{SL}(2,\mathbb{C})) \arrow[d, "\widehat{\lambda}"] \\ & \widehat{\mathcal{B}}(\mathbb{C}). \end{tikzcd} \tag{4.4}$$

Let π denote the natural map $\widehat{\mathcal{B}}(\mathbb{C}) \to \mathcal{B}(\mathbb{C})$, and let x be an element in $H_3(\mathrm{SL}(2,\mathbb{C}))$ satisfying $\pi(\alpha) = \lambda(x)$. By (4.3), there exists z in $\mathbb{Q}/\mathbb{Z}$ such that

$$\widehat{\lambda}(x) - \alpha = \widehat{\chi}(z),$$

and by (4.4), we have $\widehat{\lambda}(x - \frac{1}{2}z) = \alpha$. □

Appendix

We conclude by proving that our definition of $\widehat{\mathcal{B}}(\mathbb{C})$ is equivalent to Neumann's definition of the more extended Bloch group $\mathcal{EB}(\mathbb{C})$. Recall the definition of FT from Section 2. Neumann defines

$$\mathrm{FT}^+ := \{(x_0, \ldots, x_4) \in \mathrm{FT} \mid \operatorname{Im} x_i > 0\}$$

and defines $\widehat{\mathrm{FT}}$ (which we will denote $\widehat{\mathrm{FT}}_{\mathrm{Neu}}$ to avoid confusion) to be the component of the preimage of FT that contains all points

$$\begin{aligned}\big((x_0; p_0, q_0), (x_1; p_1, q_1), (x_2; p_1 - p_2, q_2),\\ (x_3; p_1 - p_0 + q_1 - q_0, q_2 - q_1), (x_4; q_1 - q_0, q_2 - q_1 - p_0)\big)\end{aligned} \tag{4.5}$$

with $(x_0, \ldots, x_4) \in \mathrm{FT}^+$ and the p_i's and q_i's even integers. He then defines the more extended Bloch group $\mathcal{EB}(\mathbb{C})$, as in Definition 2.1, to be the abelian group generated by symbols $[z; p, q]$, subject to the relation

$$\sum_{i=0}^{4}(-1)^i[x_i; p_i, q_i] = 0 \text{ for } \big((x_0; p_0, q_0), \ldots, (x_4; p_4, q_4)\big) \in \widehat{\mathrm{FT}}_{\mathrm{Neu}}.$$

Proposition 4.16. $\widehat{\mathcal{B}}(\mathbb{C}) = \mathcal{EB}(\mathbb{C})$.

This follows immediately from

Lemma 4.17. $\widehat{\mathrm{FT}}_{\mathrm{Neu}} = \widehat{\mathrm{FT}}$.

Proof. Let $(x_0, \dots, x_4)$ be a fixed point in FT^+ and let

$$P = ((x_0; 0, 0), \dots, (x_4; 0, 0)) \in \widehat{\mathrm{FT}}_{\mathrm{Neu}}.$$

Consider the curve in $\widehat{\mathrm{FT}}_{\mathrm{Neu}}$ starting in P obtained by keeping x_1 fixed and letting x_0 move along a closed curve in $\mathbb{C} - \{0, 1, x_1\}$. By a simple analysis of the five term relation, we can examine exactly how the values of the p_i's and q_i's change when x_0 moves around. This is indicated in Figure 2. We see that if x_0 traverses a closed curve going p_0 times anticlockwise around the origin, followed by q_0 times clockwise around 1, followed by r times clockwise around x_1, then the curve in $\widehat{\mathrm{FT}}_{\mathrm{Neu}}$ ends in

$$\begin{aligned}\big((x_0; 2p_0, 2q_0), (x_1; 0, 0), (x_2; -2p_0, 2p_0 + 2r),& \\ (x_3; -2p_0 - 2q_0, 2p_0 + 2r), (x_4; -2q_0, 2r)\big)&\end{aligned}$$

If we start in this point and then follow the curve in $\widehat{\mathrm{FT}}_{\mathrm{Neu}}$ obtained by keeping x_0 fixed and letting x_1 traverse a curve going p_1 times anticlockwise around the origin followed by q_1 times clockwise around 1, a similar study shows that we end up at the point

$$\begin{aligned}Q = \big((x_0; 2p_0, 2q_0), (x_1; 2p_1, 2q_1), (x_2; -2p_0 + 2p_1, 2p_0 + 2r),& \\ (x_3; -2p_0 - 2q_0 + 2p_1 + 2q_1, 2p_0 - 2q_1 + 2r), (x_4; -2q_0 + 2q_1, -2q_1 + 2r)\big)&\end{aligned}$$

By letting $q_2 = 2r + 2p_0$ we see that Q is of the form (4.5). Since we can connect P to a point in $\widehat{\mathrm{FT}}$ by first sliding x_0 down to the interval $(0, 1)$ and then doing the same with x_1, the lemma follows. $\square$

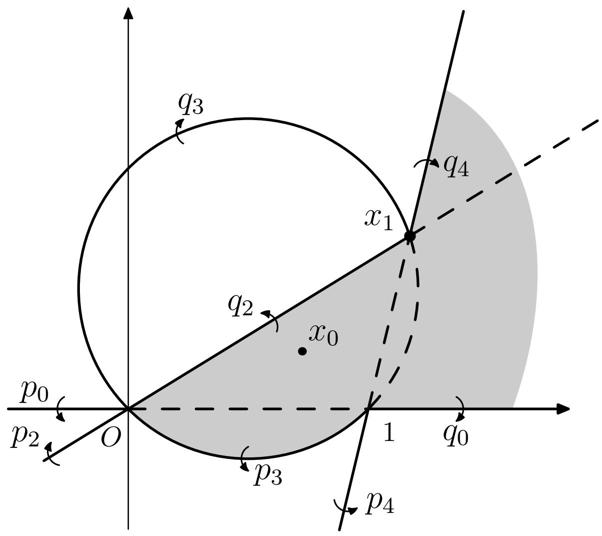

FIGURE 2. The relevant values of p_i and q_i increase by 2 whenever x_0 crosses the relevant line in the direction indicated by the arrows.

References

[1] J. Cheeger, J. Simons, *Differential characters and geometric invariants*, Springer Lecture Notes in Mathematics **1167** (1985) 50–80

[2] S.-S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974) 48–69

[3] J. Dupont, *The dilogarithm as a characteristic class for flat bundles*, J. Pure and Applied Algebra **44** (1987) 137–164

[4] J. Dupont, *Curvature and Characteristic Classes*, Lecture Notes in Mathematics **640**, Springer-Verlag (1978)

[5] J. Dupont, *Scissors Congruences, Group Homology and Characteristic classes*, Nankai Tracts in Mathematics **1**, World Scientific (2001)

[6] J. Dupont, W. Parry, C.-H. Sah, *Homology of classical Lie groups made discrete, II*, J. Pure and Applied Algebra **25** (1982), 215–260

[7] J. Dupont, C.-H. Sah, *Scissors congruence, II*, J. Pure and Applied Algebra **25** (1982) 159–195

[8] W. D. Neumann, *Extended Bloch group and the Cheeger-Chern-Simons class*, Geometry and Topology **8** (2004) 413–474

[9] W. Parry, C.-H. Sah, *Third homology of* $\mathrm{SL}(2,\mathbb{R})$ *made discrete*, J. Pure and Applied Algebra **30** (1983) 181–209

[10] C.-H. Sah, *Homology of Lie groups made discrete, III*, J. Pure and Applied Algebra **56** (1989), 269–312

Department of Mathematics, University of Aarhus, DK-8000 Århus, Denmark
E-mail address, J. L. Dupont: `dupont@imf.au.dk`

Department of Mathematics, University of Aarhus, DK-8000 Århus, Denmark
E-mail address, C. K. Zickert: `zickert@imf.au.dk`