THE ERDÖS-PÓSA PROPERTY FOR MATROID CIRCUITS

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Abstract. The number of disjoint co-circuits in a matroid is bounded by its rank. There are, however, matroids with arbitrarily large rank that do not contain two disjoint co-circuits; consider, for example, $M(K_n)$ and $U_{n,2n}$. Also the bicircular matroids $B(K_n)$ have arbitrarily large rank and have no 3 disjoint co-circuits. We prove that for each $k$ and $n$ there exists a constant $c$ such that, if $M$ is a matroid with no $U_{n,2n}$-, $M(K_n)$-, or $B(K_n)$-minor, then either $M$ has $k$ disjoint co-circuits or $r(M) \leq c$.

1. Introduction

We prove the following theorem.

Theorem 1.1. There exists a function $\gamma : \mathbb{N}^3 \to \mathbb{N}$ such that, if $M$ is a matroid with no $U_{a,2n}$-, $M(K_n)$-, or $B(K_n)$-minor and $r(M) \geq \gamma(k,a,n)$, then $M$ has $k$ disjoint co-circuits.

Here $M(K_n)$ is the cycle matroid of $K_n$ and $B(K_n)$ is the bicircular matroid of $K_n$ (to be defined below).

A circuit-cover of a graph $G$ is a set $X \subseteq E(G)$ such that $G - X$ has no circuits. Thus the maximum number of (edge-) disjoint circuits in a graph is bounded by the minimum size of a circuit cover. This bound is not tight (consider $K_4$), but Erdős and Pósa in [3] proved that the maximum number of disjoint circuits is qualitatively related to the minimum size of a circuit cover.

Erdős-Pósa Theorem 1.2. There is a function $c : \mathbb{N} \to \mathbb{N}$ such that, if the size of a minimal circuit-cover of $G$ is at least $c(k)$, then $G$ has $k$ disjoint circuits.

Let $M$ be a matroid. A set $X \subseteq E(M)$ intersects each circuit of $M$ if and only if $E(M) - X$ is independent. So, a minimal circuit-cover of $M$ is a basis of $M^*$. The Erdős-Pósa Theorem was generalized to matroids by Geelen, Gerards, and Whittle [4] who proved:

Theorem 1.3. There exists a function $c : \mathbb{N}^3 \to \mathbb{N}$ such that, if $M$ is a matroid with no $U_{2,q+2}$- or $M(K_n)$-minor and $r(M) \geq c(k,q,n)$, then $M$ has $k$ disjoint co-circuits.

The result does not extend to all matroids. A matroid is round if it has no two disjoint co-circuits. Equivalently, $M$ is round if each co-circuit in $M$ is a spanning set of $M$.

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The matroid $U_{r,n}$, where $n \geq 2r - 1$ is round. Also, for any positive integer $n$, $M(K_n)$ is a round matroid. Generally, for a graph $G$, a co-circuit of $M(G)$ is a minimal edge-cut of $G$. If $G$ is simple, then it is easily seen, that $G$ has no two disjoint edge-cuts if and only if $G$ is complete.

Let $G = (V, E)$ be a graph. Define a matroid $\tilde{B}(G)$ on $V \cup E$ where $V$ is a basis of $\tilde{B}(G)$ and, for each edge $e = uv$ of $G$, place $e$ freely on the line spanned by $\{u, v\}$. Now $B(G) := \tilde{B}(G) \setminus V$ is the bicircular matroid of $G$. A different characterization of $B(G)$ is the following, which gives rise to the name bicircular matroid. It is easily verified (see [8, Prop. 12.1.6]).

Remark 1.4. Let $G$ be a graph. $C$ is a circuit of $B(G)$ if and only if $G[C]$ is a subdivision of one of the graphs below.

![Graphs](image)

The matroid $\tilde{B}(K_n)$ is also round, which is easily verified. The bicircular matroid $B(K_n)$ is not round, but it has no three disjoint co-circuits (for $n \neq 3$).

Our main theorem, Theorem 1.1, is a generalization of Theorem 1.3 and is, in some sense, best possible. Note that each of the classes
\[
\{M(K_n) : n \geq 1\}, \{B(K_n) : n \geq 1\}, \text{ and } \{U_{a,2a} : a \geq 1\}
\]
have unbounded rank but they have a bounded number of disjoint co-circuits.

We follow the notation of Oxley [8], and the reader is assumed familiar with standard matroid theory as described therein.

2. Covering number

We shall work with dense matroids in the proof. This section develops tools for measuring the size and density of a matroid.

A simple $GF(q)$-representable rank-$r$ matroid can be realized as a restriction of the projective geometry $PG(r-1, q)$. Thus, it has at most $\frac{q^r-1}{q-1}$ elements. Kung [5] extended this bound to the class of matroids with no $U_{2,q+2}$-minor (the shortest line not representable over $GF(q)$).

Theorem (Kung) 2.1. Let $q > 1$ be an integer, and let $M$ be a simple rank-$r$ matroid with no $U_{2,q+2}$-minor. Then
\[
|E(M)| \leq \frac{q^r-1}{q-1}.
\]

Projective geometries show, that the bound is sharp if $q$ is a prime-power. To bound the size of rank-$r$ matroids, it is necessary to restrict the length of lines, or there can be arbitrarily many elements in a rank-2 matroid. As we shall be excluding a uniform matroid of higher rank, we need a new measure of size, for an analogue of Kung’s Theorem to hold.

Definition 2.2. Let $a$ be a positive integer. An $a$-covering of a matroid $M$ is a collection $(X_1, \ldots, X_m)$ of subsets of $E(M)$, with $E(M) = \bigcup X_i$ and $r_M(X_i) \leq a$ for all $i$. The size of the covering is $m$. The $a$-covering number of $M$, $\tau_a(M)$ is the minimum size of an $a$-covering of $M$. If $r(M) = 0$, then we define $\tau_a(M) = 0$. 

Note that for a matroid $M$, $\tau_1(M) = |E(si(M))|$, where $si(M)$ denotes the simplification of $M$. If $M$ has non-zero rank $r(M) \leq a$, then $\tau_a(M) = 1$. Our first lemma bounds $\tau_a(M)$ in the case $r(M) = a + 1$.

**Lemma 2.3.** Let $b > a \geq 1$. If $M$ is a matroid of rank $a + 1$ with no $U_{a+1,b}$-minor, then

$$\tau_a(M) \leq \binom{b-1}{a}.$$

**Proof.** Let $X \subseteq E(M)$ be maximal with $M|X \simeq U_{a+1,b}$. Then $l \leq b - 1$. For an $x \not\in X$, by the maximality of $X$, there exists $Y \subseteq X$ with $|Y| = a$ such that $Y \cup x$ is dependent, and thus $x \in cl_M(Y)$.

It follows that $(cl_M(Y)|Y \subseteq X, |Y| = a)$ is an $a$-covering of $M$. It has size $\binom{b}{a} \leq \binom{b-1}{a}$.

**Lemma 2.4.** Let $b > a \geq 1$. If $M$ is a matroid with no $U_{a+1,b}$-restriction, then

$$\tau_a(M) \leq \binom{b-1}{a} \tau_{a+1}(M).$$

**Proof.** Let $(X^1, \ldots, X^k)$ be a minimal $(a + 1)$-covering of $M$. By Lemma 2.3 each $M|X^i$ has an $a$-covering $(X^i_1, \ldots, X^i_{m_i})$ of size $m_i \leq \binom{b-1}{a}$. Combining these we get an $a$-covering $(X^j_j|j = 1, \ldots, m_i, i = 1, \ldots, k)$ of $M$. Thus $\tau_a(M) \leq \sum m_i \leq \binom{b-1}{a} k$.

The next result extends Kung’s Theorem. The bound we obtain is not sharp, though.

**Lemma 2.5.** Let $b > a \geq 1$. If $M$ is a matroid of rank $r \geq a$ with no $U_{a+1,b}$-minor, then

$$\tau_a(M) \leq \binom{b-1}{a}^{r-a}.$$
Proof. Define $\sigma$ by

$$\sigma(a) = \prod_{k=2}^{a} \left( \frac{2k-1}{k-1} \right).$$

Since $M$ has no $U_{k,2k}$-restriction for $k = 2, \ldots, a$, Lemma 2.4 gives

$$\tau_{k-1}(M) \leq \left( \frac{2k-1}{k-1} \right)^{r_k(M)}, \quad k = 2, \ldots, a.$$

Putting these together, we get $|E(M)| = \tau_1(M) \leq \sigma(a)\tau_a(M)$.

We shall need one more specialized result, which is completely similar to the previous Lemma.

**Lemma 2.8.** There exists an integer-valued function $\sigma_2(a,b)$ such that, if $b \geq a \geq 1$ and $M$ is loop-less and has no $U_{k,b}$-restriction for $k = 1, \ldots, a$, then $|E(M)| \leq \sigma_2(a,b)\tau_a(M)$.

**Proof.** Define $\sigma_2$ by

$$\sigma_2(a,b) = (b-1) \prod_{k=2}^{a} \left( \frac{b-1}{k-1} \right).$$

Now use $|E(M)| \leq (b-1)\tau_1(M)$ and apply Lemma 2.4.

\[\square\]

3. Approaching Roundness

The first step in the proof of the main theorem is to show, that a matroid of large enough rank has either $k$ disjoint co-circuits or a large minor which is “nearly round”.

**Definition 3.1.** Let $M$ be a matroid. The rank-deficiency of a set of elements $X \subseteq E(M)$ is $r_M(X) = r(M) - r_M(X)$. Denote by $\Gamma(M)$ the maximum rank-deficiency among the co-circuits of $M$. For $t \in \mathbb{N}$ we say that $M$ is $t$-round if $\Gamma(M) \leq t$.

Notice that a matroid $M$ is round if and only if $\Gamma(M) = 0$, that is, $M$ is 0-round. The condition of being $t$-round is easily seen to be preserved under contractions. When we cannot obtain a $t$-round matroid, we shall sometimes work with the even weaker property: $\Gamma(M) \leq \frac{1}{2} r(M)$.

**Lemma 3.2.** Let $g : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. There exists a function $f_g : \mathbb{N} \to \mathbb{N}$ such that for any $k \in \mathbb{N}$, if $M$ is a matroid with $r(M) \geq f_g(k)$, then either

(a) $M$ has $k$ disjoint co-circuits or

(b) $M$ has a minor $N = M/Y$ with $r(N) \geq g(\Gamma(N))$.

**Proof.** Let $g$ be given and define $f_g$ as follows: $f_g(0) = f_g(1) = 1$ and

$$f_g(k) = g(f_g(k-1)), \quad k \geq 2.$$

The proof is by induction on $k$. If $r(M) \geq 1$, then $M$ has a co-circuit, so the result holds for $k = 0, 1$. Now let $k \geq 2$ and $r(M) \geq f_g(k) = g(f_g(k-1))$.

If $\Gamma(M) \geq f_g(k-1)$, then pick a co-circuit $C$ of $M$ with $r_M(C) = \Gamma(M)$. Then $r(M/C) = r_M(C) \geq f_g(k-1)$. If $M/C$ has the desired contraction minor, then we are done. If not, then by induction $M/C$ has $k-1$ disjoint co-circuits. These, together with $C$, give $k$ disjoint co-circuits of $M$.

If $\Gamma(M) \leq f_g(k-1)$, then as $g$ is non-decreasing, we have $r(M) \geq g(\Gamma(M))$. \[\square\]
4. Building density

The goal of this section is to prove, that a high-rank nearly round matroid with no $U_{a+1,b}$-minor contains a dense minor.

**Lemma 4.1.** Let $b > a \geq 1$. Let $M$ be a matroid with no $U_{a+1,b}$-minor and let $C$ be a co-circuit of $M$ of minimal size. If $C_1, \ldots, C_k$ are disjoint co-circuits of $M \setminus C$ with $|C_1| \leq \cdots \leq |C_k|$, then $|C_i| \geq |C|/(a^{(b-1)})$ for $i = a, \ldots, k$.

**Proof.** Let $C$ and $C_1, \ldots, C_k$ be given and let $i \in \{a, \ldots, k\}$.

$C_i$ is co-dependent in $M \setminus C \setminus C_i$. So there exists a co-circuit $C'_i \subseteq C_i$ of $M \setminus (C \cup C_i)$.

Now, $C_2$ is co-dependent in $M \setminus C \setminus (C_i \cup C'_i)$. So there is a co-circuit $C'_2 \subseteq C_2$ of $M \setminus (C \cup C_i \cup C'_i)$.

Continuing in this fashion, for each $j = 2, \ldots, a-1$ we pick a co-circuit $C'_j \subseteq C_j$ of $M \setminus (C \cup C_i \cup C'_1 \cup \cdots \cup C'_{a-1})$.

Denote by $F$ the set $E(M) - (C \cup C_i \cup C'_1 \cup \cdots \cup C'_{a-1})$. Deleting a co-circuit of a matroid drops its rank by 1, so we get $r_M(F) = a + 1$. Hence $N = M/F$ has rank $r(N) = a + 1$. Since $C$ is a co-circuit of $N$ of minimal size, $E(N) - C$ must be a rank-$a$ set of $N$ of maximal size. We now have

$$|C| \leq |E(N)| \leq \tau_a(N) |E(N) - C|$$

$$= \tau_a(N) |C_i \cup C'_1 \cup \cdots \cup C'_{a-1}|$$

$$\leq \left(\frac{b-1}{a}\right) a |C_i|$$

using Lemma 2.3. The result now follows. \qed

**Lemma 4.2.** There exists an integer-valued function $\kappa(\lambda, a, b)$ such that the following holds: Let $b > a \geq 1$ and $\lambda \in \mathbb{N}$. Let $M$ be an $a$-simple matroid with no $U_{a+1,b}$-minor, satisfying $\Gamma(M) \leq \frac{1}{2} r(M)$. Let $C$ be a minimal sized co-circuit of $M$. If $M \setminus C$ has $\kappa(\lambda, a, b)$ disjoint co-circuits, then $\tau_a(M) > \lambda r(M)$.

**Proof.** Let $a, b$ and $\lambda$ be given and define

$$\kappa(\lambda, a, b) = \kappa = 2a \left(\frac{b-1}{a}\right) \sigma(a) \lambda + a - 1.$$  

Let $M$ and $C$ be given and let $C_1, \ldots, C_\kappa$ be disjoint co-circuits of $M \setminus C$ of non-decreasing size. Note that

$$|C| \geq r_M(C) \geq r(M) - \Gamma(M) \geq r(M)/2.$$  

By Lemma 2.7 and the above lemma we have

$$\sigma(a) \tau_a(M) \geq |E(M)|$$

$$> |C_a| + \cdots + |C_\kappa| \geq (\kappa - a + 1) \frac{r(M)}{2a \left(\frac{b-1}{a}\right)} = \sigma(a) \lambda r(M),$$

and the result follows. \qed

For a matroid $M$ denote by $\Theta(M)$ the maximum number of disjoint co-circuits in $M$. So, $M$ is round if and only if $\Theta(M) = 1$. The two parameters $\Gamma(M)$ and $\Theta(M)$ are related by

$$\Theta(M) \leq \Gamma(M) + 1.$$
This follows from the observation, that if $C_1,\ldots,C_k$ are disjoint co-circuits of $M$, then $r(M\setminus(C_1 \cup \cdots \cup C_k)) \leq r(M) - k$. Equality does not hold (consider $U_{4,5}$). The following result lists hereditary properties of the two parameters.

**Lemma 4.3.** Let $M$ be a matroid and let $X,Y \subseteq E(M)$. Then

(i) $\Theta(M/Y) \leq \Theta(M)$ and $\Gamma(M/Y) \leq \Gamma(M)$.

(ii) $\Theta(M \setminus X) \geq \Theta(M)$ and $\Gamma(M \setminus X) \geq \Gamma(M)$, if $X$ is co-independent.

(iii) $\Theta(M \setminus X) = \Theta(M)$ and $\Gamma(M \setminus X) = \Gamma(M)$, if for some number $a \in \mathbb{N}$, $X$ is minimal with respect to inclusion, such that $M \setminus X$ is $a$-simple.

Proof. Every co-circuit $C$ of $M/Y$ is a co-circuit of $M$. A short calculation shows that $r_{M/Y}(C) \leq r_M(C)$, so the first assertion of the lemma holds.

To prove the second and third assertions, it is enough to consider $X = \{x\}$, where $x$ is not a co-loop of $M$. If $C$ is a co-circuit in $M$, then $C - x$ contains a co-circuit in $M\setminus x$. Thus $\Theta(M \setminus x) \geq \Theta(M)$ and also $\Gamma(M \setminus x) \geq \Gamma(M)$.

We turn to the third assertion. Assume that $x \in W$, where $M|W \simeq U_{k,2k}$, for a $k \in \mathbb{N}$. If $C$ is a co-circuit of $M\setminus x$, then $C = C' - x$ for a co-circuit $C'$ of $M$, that is either $C$ or $C \cup x$ is a co-circuit of $M$. We look at two cases:

- If $C \cap (W - x) = \emptyset$, then $C$ is a co-circuit of $M$, since the complement of a co-circuit is closed and $x \in c_M(W - x)$.
- If $C \cap (W - x) \neq \emptyset$, then we must have $|(W - x) - C| < k$; since $M|(W - x) \simeq U_{k,2k-1}$ and the complement of $C$ is closed. Hence, $|C \cap (W - x)| \geq k$.

Note that the second case can happen at most once in a collection of disjoint co-circuits. So given a collection of disjoint co-circuits of $M\setminus x$, by adding $x$ to at most one of them, we get a collection of disjoint co-circuits of $M$. Thus $\Theta(M) \geq \Theta(M\setminus x)$.

Note also, that for a co-circuit $C$ of $M\setminus x$, if $C \cup x$ is a co-circuit of $M$, then we are in the second case, and $r_M(C \cup x) = r_{M\setminus x}(C)$. Thus $\Gamma(M) \geq \Gamma(M\setminus x)$. Finally, since no co-circuit can contain a loop, deleting loops also preserves $\Theta$ and $\Gamma$. □

**Lemma 4.4.** There exists an integer-valued function $\delta(\lambda,a,b)$ such that the following holds: Let $b > a \geq 1$ and $\lambda \in \mathbb{N}$. If $M$ is a matroid with no $U_{a+1,k}$-minor, such that $\Gamma(M) \leq \frac{1}{2}r(M)$ and $r(M) \geq \delta(\lambda,a,b)$, then $M$ has a minor $N$ with $\tau_N(N) > \lambda r(N)$.

Proof. Let $a, b$ and $\lambda$ be given and fixed, and let us define $\delta(\lambda,a,b)$. First, we define a sequence of functions $g_n : \mathbb{N} \rightarrow \mathbb{N}$. Let $g_0(m) = 0$, and for $n \geq 1$ define $g_n$ recursively by

$$g_n(m) = \max(2m, \delta_n),$$

where $\delta_n = 2(f_{2n-1}(\kappa(\lambda,a,b)) + 1) \in \mathbb{N}$.

Finally, let $\delta(\lambda,a,b) = \delta_{n_0}$, where $n_0 = 2\sigma(a)\lambda$. The functions $\sigma, \kappa$ and $f_{n_0}$ are defined in previous lemmas. We first prove a partial result.

**Claim.** Let $n \geq 0$. If $M$ is a matroid with no $U_{a+1,k}$-minor, such that $r(M) \geq g_n(\Gamma(M))$, then either

- $M$ has a minor $N$ with $\tau_N(N) > \lambda r(N)$ or
- there exists a sequence of matroids $M = M_0, M_1, \ldots, M_n$, such that for $i = 0,\ldots,n - 1$, $M_{i+1} = M_i \setminus C_i/Y_i$, where $C_i$ is a co-circuit of $M_i$ that spans $M_i/Y_i$. 


We prove the claim by induction on $n$. The case $n = 0$ is trivial, so assume $n \geq 1$ and that the result holds for $n - 1$.

Let $X \subseteq E(M)$ be minimal such that $M \setminus X$ is $a$-simple. Pick a co-circuit $C_0$ of $M$ with $|C_0 - X|$ minimal. Note, that $C = C_0 - X$ is a co-circuit of $M \setminus X$ of minimal size.

Choose a basis $Z$ of $M/C_0$ and let $M' = M/Z$. Then $C_0$ spans $M'$, and since $r(M) \geq g_n(\Gamma(M))$, $r(M') = r_M(C_0) \geq r(M) - \Gamma(M) \geq \frac{1}{2} r(M) \geq \frac{1}{2} \delta_n$.

Now $r(M' \setminus C_0) = r(M') - 1 \geq f_{g_{n-1}}(\kappa(\lambda, a, b))$, so by Lemma 3.2 one of the following holds:

(a) $M' \setminus C_0$ has $\kappa(\lambda, a, b)$ disjoint co-circuits.

(b) $M' \setminus C_0$ has a minor $M_1 = M' \setminus C_0/Y$ with $r(M_1) \geq g_{n-1}(\Gamma(M_1))$.

Assume first that (a) holds. Since $M' \setminus C_0 = M \setminus C_0/Z$, by Lemma 4.3(i), $M \setminus C_0$ has $\kappa(\lambda, a, b)$ disjoint co-circuits. We claim, that $X - C_0$ is co-independent in $M \setminus C_0$. If not, then there exists a co-circuit $D \subseteq X \cup C_0$ of $M$ with $D \cap (X - C_0) \neq \emptyset$, contradicting our choice of $C_0$. Now, by Lemma 4.3(ii),

$$\Theta((M \setminus X) \setminus C) = \Theta(M \setminus (C_0 \cup X)) \geq \Theta(M \setminus C_0) \geq \kappa(\lambda, a, b).$$

The lemma also gives $\Gamma(M \setminus X) \leq \frac{1}{2} r(M \setminus X)$. We can now apply Lemma 4.2 to $N = M \setminus X$, and get the desired result.

Assume now that (b) holds. Letting $Y_0 = Z \cup Y$, we have $M_1 = M \setminus C_0/Y_0$ and $C_0$ spans $M/Y_0$. Applying the induction hypothesis to $M_1$ now gives the claim.

Let $M$ be given as in the lemma, and note that $r(M) \geq g_n(\Gamma(M))$, where $n = 2\sigma(a)\lambda$. By the claim, either we are done or there is a sequence $M = M_0, \ldots, M_n$, such that for $i = 0, \ldots, n - 1$, $M_{i+1} = M_i \setminus C_i/Y_i$, where $C_i$ is a co-circuit of $M_i$ that spans $M_i/Y_i$.

Let $M' = M/(Y_0 \cup \cdots \cup Y_{n-1})$. Notice, that for $i = 0, \ldots, n - 1$, $C_i$ is a spanning co-circuit of $M' \setminus (C_0 \cup \cdots \cup C_{i-1})$. Thus $r_{M'}(C_i) = r - i$, where $r = r(M')$. For all $i$, choose a basis $B_i$ for $M'/C_i$, and define $N = M'|(\cup B_i)$. Then

$$|E(N)| = \sum_{i=0}^{n-1} (r - i) > \frac{nr}{2}.$$ We claim that $N$ is $a$-simple. Suppose $N|W \simeq U_{k,2k}$ for a $W \subseteq E(N)$ and $k \in \{1, 2, 3, \ldots\}$. Then $|W \cap B_0| \leq k$, as $B_0$ is independent. So $|W \cap (E(N) - B_0)| \geq k$, and since $E(N) - B_0$ is closed, $W \cap B_0 = \emptyset$. Repeat this argument in $N \setminus B_0$ to see, that $W \cap B_1 = \emptyset$ etc. We end up with $W \subseteq B_{n-1}$, a contradiction.

Finally, by Lemma 2.7,

$$\sigma(a)r_a(N) \geq |E(N)| > \frac{nr}{2} = \sigma(a)\lambda r(N),$$

and the result follows. \hfill $\square$

5. Arranging circuits

We wish to identify some more concrete structure in a dense matroid. To do this, we need to be able to disentangle some of the many low-rank sets in the matroid.

For a matroid $M$, we call sets $A_1, \ldots, A_n \subseteq E(M)$ skew if $r_M(\cup_i A_i) = \sum_i r_M(A_i)$. This is analogous to subspaces of a vector-space forming a direct sum. The first
result of this section is a tool for finding sets in a matroid, that are close to being skew.

We define a function $\mu_M$ on collections of subsets of $E(M)$ as follows. For sets $A_1, \ldots, A_n \subseteq E(M)$, let

$$\mu_M(A_1, \ldots, A_n) = r_M(\bigcup_j A_j) - \sum_i \left( r_M(\bigcup_j A_j) - r_M(\bigcup_{j \neq i} A_j) \right)$$

$$= r_M(\bigcup_j A_j) - \sum_i r_M(\bigcup_{j \neq i} A_j)(A_i - \cup_{j \neq i} A_j)).$$

This function can be thought of as a generalized connectivity function. For $n = 2$, $\mu_M$ equals the connectivity function $\lambda_M(A_1, A_2) = r_M(A_1) + r_M(A_2) - r_M(A_1 \cup A_2)$. For $n \geq 2$ a recursive formula holds,

$$\mu_M(A_1, \ldots, A_n) = \lambda_M(A_1, A_2 \cup \cdots \cup A_n) + \mu_{M/A_1}(A_2, \ldots, A_n).$$

The function $\mu_M$ measures in a way the rank of the “overlap” of the sets, though this may not be an actual set in the matroid. Notice, that $\mu_M(A_1, \ldots, A_n) = 0$ if and only if $A_1, \ldots, A_n$ are skew. More generally, if there is a set $W \subseteq E(M)$ such that $A_1 - W, \ldots, A_n - W$ are skew in $M/W$, then $\mu_M(A_1, \ldots, A_n) \leq r_M(W)$.

Lemma 5.1. There exists an integer-valued function $\alpha_1(n, r, a, b)$ such that the following holds: Let $b > a \geq 1$, and let $r$ and $n$ be positive integers. If $M$ is a matroid with no $U_{a+1,b}$-minor, and $\mathcal{F}$ is a collection of rank-$r$ subsets of $E(M)$ with $r_M(\bigcup_{X \in \mathcal{F}} X) \geq \alpha_1(n, r, a, b)$, then there exist $X_1, \ldots, X_n \in \mathcal{F}$ satisfying

(a) $X_i \not\subseteq \text{cl}_M(\bigcup_{j \neq i} X_j)$ for $i = 1, \ldots, n$ and

(b) $\mu_M(X_1, \ldots, X_n) \leq (r - 1)a$.

Proof. For any positive integers $n, c, k$, we let $R(n, c, k)$ denote the following Ramsey number: The minimal $R$, such that if $X$ is a set with $|X| = R$, then for any $c$-coloring of $[X]^n$, $X$ has a monochromatic subset of size $k$. Here $[X]^n$ denotes the set of all subsets of $X$ of size $n$. By a monochromatic subset of $X$, we mean a subset $Y \subseteq X$ such that the sets in $[Y]^n$ all have the same color. This number exists by Ramsey’s Theorem (see [9] or [1, 9.1.4]).

Let $n, r, a, b$ be given and let us define $\alpha_1(n, r, a, b)$. First we define numbers $s_i, l_i$ for $i = 1, \ldots, r$. Let $s_r = 0, l_r = n$, and for $i = r - 1, r - 2, \ldots, 1$ define recursively:

$$s_i = s_{i+1} + l_{i+1}, \quad u_i = \binom{b-1}{a}^{s_i-a}, \quad l_i = n \binom{u_i}{r-i},$$

Let $m = s_1 + l_1$. So, we have $0 = s_r < s_{r-1} < \cdots < s_1 < m$. Next, define numbers $k_1, \ldots, k_m$ as follows. Let $k_m = m$ and define recursively:

$$k_{i-1} = R(i, r, k_i), \quad \text{for } i = m, m-1, \ldots, 2.$$ Finally, let $\alpha_1(n, r, a, b) = rk_1$.

In the following, for a set of subsets $X \subseteq 2^{E(M)}$, we use the shorthand notation $r_M(X) = r_M(\bigcup_{X \in X} X)$.

Let $M$ and $\mathcal{F}$ be given, with $r_M(\mathcal{F}) \geq \alpha_1(n, r, a, b) = rk_1$. We can choose sets $Y_1, \ldots, Y_{k_1} \in \mathcal{F}$, such that $Y_i \not\subseteq \text{cl}_M(Y_1 \cup \cdots \cup Y_{i-1})$. Let $\mathcal{F}_1 = \{Y_1, \ldots, Y_{k_1}\}$ and put $a_0 = 0, a_1 = r$. We shall iteratively construct sequences:

$$\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_m, \quad a_0 < a_1 < a_2 < \cdots < a_m,$$
such that for $i = 1, \ldots, m$, $|F_i| = k_i$, and if $F' \subseteq F_i$ with $|F'| = i$, then $r_M(F') = a_i$. This clearly holds for $F_1$. Let $i \geq 2$, assume that $F_{i-1}$ and $a_{i-1}$ satisfy the above, and let us find $F_i$ and $a_i$.

Note, that $r_M(F') \in \{a_{i-1} + 1, \ldots, a_i + r\}$, for $F' \subseteq F_{i-1}$ with $|F'| = i$. This defines an $r$-coloring of $[F_{i-1}]^l$. Since $|F_{i-1}| = k_{i-1} = R(i, r, k_i)$, there exists $F_i \subseteq F_{i-1}$ such that, every set in $[F_i]^l$ has the same rank, and we let $a_i$ be that number.

For $i = 1, \ldots, m$ let $b_i = a_i - a_{i-1}$. Notice that, by submodularity, this gives a decreasing sequence $(a_{i+1} + a_{i-1} \leq a_i + a_i)$,

$$r = b_1 \geq b_2 \geq \cdots \geq b_m \geq 1.$$ 

Hence, by definition of the pairs $(s_i, l_i)$, there exists an $r' \in \{1, \ldots, r\}$, such that

$$b_{s+1} = \cdots = b_{s+l} = r',$$

where $s = s_{r'}$ and $l = l_{r'}$. If $r' = r$, then we get $b_1 = \cdots = b_n = r$. Thus, if we choose any $n$ members $X_1, \ldots, X_n$, then they are skew and we are done.

Assume $r' < r$. Choose $s$ sets $Z_1, \ldots, Z_s \in F_m$, and let $F = \cup_{i=1}^s Z_i$. Choose another $l$ sets $X_1, \ldots, X_l \in F_m - \{Z_1, \ldots, Z_s\}$. Since $b_{s+1} = b_{s+l} = r'$, the sets $X_1 - F', \ldots, X_l - F$ are skew of rank $r'$ in $M/F$. For $i = 1, \ldots, l$, choose an independent set $B_i \subseteq X_i$ of size $r'$, skew from $F$. Expand this set to a basis $B_i \cup \overline{B_i}$ of $X_i$, so $|B_i| = r_0 = r - r'$.

Let $M' = M/(\cup \overline{B_i})$ and $B = \cup_i B_i$. Then $B_i \subseteq c_{M'}(F)$, and thus $r_{M'}(B) \leq r_{M'}(F) \leq sr$. Let $(W_1, \ldots, W_u)$ be a minimal $a$-covering of $M'|B$. By Lemma 2.5, we have

$$u = r_a(M'|B) \leq \left(\frac{b-1}{a}\right)^{sr-a} = u_{r'}.$$ 

For each $B_i$, we can find a set of indices $I_i \subseteq \{1, \ldots, u\}$ of size $r_0$, such that $B_i \subseteq \cup_{j \in I_i} W_j$. There are $(u^{r_0}) \leq \left(\binom{u_{r'}}{r'-1}\right)$ possible choices for $I_i$, and $l = n\binom{u_{r'}}{r'-1}$. By a majority argument, there must exist $I \subseteq \{1, \ldots, u\}$, such that $I_i = I$ for all $i \in J$, where $J \subseteq \{1, \ldots, l\}$ has size $n$. By possibly re-ordering the $X_i$'s and the $W_j$'s we can assume, that $B_1, \ldots, B_n \subseteq W_1 \cup \cdots \cup W_{r_0}$.

Let $W = W_1 \cup \cdots \cup W_{r_0}$. Then the sets $X_1 - W, \ldots, X_n - W$ are skew in $M/W$. It follows, that $\mu_M(X_1, \ldots, X_n) \leq r_{M}(W) \leq ar_0 \leq a(r - 1)$, and we are done.

The next lemma shows how, by doing suitable contractions, a large collection of nearly (but not completely) skew circuits, can yield a set of nearly skew triangles containing a common element. The idea is to put points in the “overlap” by contracting some of the circuits. The overlap can then be contracted to a point.

**Lemma 5.2.** There exists an integer-valued function $\alpha_2(l, r, m)$ such that the following holds: Let $r \geq 2$ and $l, m$ be positive integers. If $n \geq \alpha_2(l, r, m)$ and $C_1, \ldots, C_n$ are rank-$r$ circuits of a matroid $M$ satisfying

(a) $1 \leq r_M(\cup_j C_j) - r_M(\cup_{j \neq i} C_j) < r$ for all $i$, and
(b) $\mu_M(C_1, \ldots, C_n) \leq m$,

then $M$ has a minor $N = M/Y$ with an element $x \in E(N)$ and triangles $D_1, \ldots, D_l$ of $N$, such that

- $x \in D_i$ for all $i$, and $r_N(\cup_i D_i) = l + 1$,
- For all $i$, $D_i - x \subseteq C_j$ for some $j \in \{1, \ldots, n\}$. 


Proof. Let \( l \) and \( m \) be fixed. For \( r \geq 2 \), define \( \alpha_2(l, r, m) \) recursively as follows

\[
\alpha_2(l, r, m) = 2^m(q_r(r - 1) + 1),
\]

\[
q_2 = l, \quad q_r = \alpha_2(l, r - 1, r - 2), \quad \text{for } r > 2.
\]

To facilitate induction, the lemma is proved from the following weaker assumptions:

Let \( n \geq \alpha_2(l, r, m) \), and \( C_1, \ldots, C_n \) be circuits of \( M \) with \( 2 \leq r_M(C_i) \leq r \). Assume there is a set \( F \subseteq E(M) \), such that

1. \( 1 \leq r_M((\bigcup_j C_j) \cup F) - r_M((\bigcup_{j \neq i} C_j) \cup F) < r_M(C_i) - 1 \) for all \( i \).
2. \( \mu_M(C_1, \ldots, C_n, F) \leq m \).

These assumptions are indeed weaker, since the phrasing in the lemma is the case \( F = \emptyset \) and \( r_M(C_i) = r \) for all \( i \). The proof is by induction on \( r \). Let \( r = 2 \) or let \( r > 2 \) and assume the result holds for \( r - 1 \).

Let \( c_i = r_M((\bigcup_j C_j) \cup F) - r_M((\bigcup_{j \neq i} C_j) \cup F) \) for each \( i \). We first do an easy reduction. If not \( c_i = 1 \) for all \( i \), then for each \( i \) choose a set \( Y_i \subseteq C_i \) of size \( c_i - 1 \), which is skew from \((\bigcup_{j \neq i} C_j) \cup F \). We may then work with the circuits \( C_i - Y_i \) of \( M/\{Y_1, \ldots, Y_n\} \) instead. So without loss of generality, \( c_i = 1 \) for all \( i \).

Choose \( z_i \in C_i - c_M((\bigcup_{j \neq i} C_j)) \) for each \( i \), and let \( M = M/\{z_1, \ldots, z_n\} \). Letting \( W = \bigcup_i(C_i - z_i) \) we have

\[
r_{\overline{\pi}}(W) = r_M((\bigcup_j C_j) - r_M((\{z_1, \ldots, z_n\}) = \mu_M(C_1, \ldots, C_n, F) \leq m.
\]

Let \( B \) be a basis of \( M|W \) and choose a basis \( B_i \) of \( M|(C_i - z_i) \) for each \( i \). Now expand \( B_i \) to a basis \( B_i \cup X_i \) of \( M|W \) using elements of \( B \). For all \( i \) we have chosen \( X_i \subseteq B \) among the \( 2^{|B|} \leq 2^m \) subsets of \( B \). Hence, there exists an \( X_0 \subseteq B \), such that \( X_i = X_0 \) for \( i \in I \), where \( |I| = n' \geq n/2^m \). Let \( M_1 = M/X_0 \) and put \( r' = |B| - |X_0| + 1 \). Then \( r_{M_1}(C_i) = r' \) for all \( i \in I \), and \( \mu_{M_i}(C_i : i \in I) = r' - 1 \).

By possibly reordering the results, we can assume \( I = \{1, \ldots, n'\} \).

Pick an element of one circuit, \( z \in C_{n'} - c_M((\bigcup_{j<n} C_j) \) and let \( M_2 = M_1/z \). Define

\[
Z = c_{M_2}(C_{n'} - z) \subseteq c_{M_2}(C_i), \quad \text{for } i = 1, \ldots, n' - 1,
\]

so \( r_{M_2}(Z) = r' - 1 \). Choose a non-loop element \( x \in Z \) and elements \( y_i \in C_i - Z \) for \( i = 1, \ldots, n' - 1 \). Since \( x \in c_{M_2}(C_i) \), \( C_i \cup x \) is connected, so there is a circuit \( C_i' \) of \( M_2 \) with

\[
\{x, y_i\} \subseteq C_i' \subseteq C_i \cup x, \quad \text{and } \quad r_{M_2}(C_i') \in \{2, \ldots, r'\}.
\]

Notice that \( C_i' \not\subseteq c_{M_2}((\bigcup_{j<i,n} C_j) \), since \( y_i \in C_i' \).

By another majority argument, there exists \( s \in \{2, \ldots, r'\} \), such that \( r_{M_2}(C_i') = s \) for \( i \in J \), where \( |J| \geq (n'-1)/(r-1) \geq q_r \). We now have two cases:

1. \( s = 2 \): Since \( q_r \geq q_2 = l \) we can choose \( J' \subseteq J \) with \( |J'| = l \). We are now done with \( \{D_1, \ldots, D_l\} = \{C_i' : i \in J'\} \) and \( N = M_2 \).

2. \( 2 < s \leq r' \): Let \( M_3 = M_2/x \) and let \( C_i = C_i' - x \) for \( i \in J \). Then \( C_i \) is a rank-\((s-1)\) circuit of \( M_3 \), with \( C_i \subseteq C_i \). Letting \( F' = Z - x \) we have,

\[
\mu_{M_3}(C_i : i \in J, F') \leq r_{M_3}(F') = r' - 2 \leq r - 2.
\]

As \( |J| \geq \alpha_2(l, r - 1, r - 2) \) we get by induction the desired minor. \( \Box \)

The following result is just a corollary to Lemmas 5.1 and 5.2, that we state for easier reference.
Lemma 5.3. There exists an integer-valued function $\alpha_3(s,l,a,b)$ such that the following holds: Let $b > a \geq 1$ and let $s,l$ be positive integers. If $M$ is a matroid with no $U_{a+1,b}$-minor, and $C$ is a set of circuits of $M$ of rank at most $a + 1$, with $r_M(\cup_{C \in C}C) \geq \alpha_3(s,l,a,b)$, then either:

(i) There exist $s$ skew circuits $C_1, \ldots, C_s \in C$, or

(ii) $M$ has a minor $N = M/Y$ with an element $x \in E(N)$ and triangles $D_1, \ldots, D_l$ of $N$, such that

- $x \in D_i$ for all $i$, and $r_N(\cup_i D_i) = l + 1$, and
- For all $i$, $D_i - x \subseteq C$ for some $C \in C$.

Proof. Define $\alpha_3(s,l,a,b) = \alpha_3$ by

$$
\alpha_3 = \sum_{r=1}^{a+1} \alpha_1(n_r, r, a, b), \quad \text{where} \quad n_r = s + \alpha_2(l, r, (r-1)a),
$$

and let $M, \ C$ be given. By a majority argument, there exists a number $r \in \{1, \ldots, a+1\}$ and $C' \subseteq C$, such that $r_M(C) = r$ for all $C \in C'$, and $r_M(\cup_{C \in C'}C) \geq \alpha_1(n_r, r, a, b)$.

Now, by Lemma 5.1, there are $C_1, \ldots, C_n \in C'$, where $n = n_r = s + \alpha_2(l, r, (r-1)a)$, satisfying

$$
c_i = r_M(\cup_j C_j) - r_M(\cup_{j \neq i} C_j) \geq 1,
$$

for all $i$, and $\mu(C_1, \ldots, C_n) \leq (r-1)a$.

Let $I = \{i : c_i = r\}$ and $J = \{i : c_i < r\}$. If $|I| \geq s$, then case (i) holds, since the $C_i$ with $i \in I$ are skew. Otherwise, $|J| \geq \alpha_2(l, r, (r-1)a)$, and the $C_i$ with $i \in J$ still satisfy

$$
r_M(\cup_{j \in J} C_j) - r_M(\cup_{j \in J - \{i\}} C_j) < r.
$$

Lemma 5.2 now gives case (ii) of the result. \hfill \Box

6. Nests

By a long line in a matroid, we mean a rank-2 flat, that contains at least 3 rank-1 flats. So, a long line in a simple matroid is a rank-2 flat with at least 3 elements. Also, a line is long if and only if it contains a triangle. We need a lot of long lines to construct clique-like structures. We first aim to build an intermediate structure called a nest.

Definition 6.1. A matroid $M$ is a nest if $M$ has a basis $B = \{b_1, \ldots, b_n\}$ such that, for each pair of indices $i, j \in \{1, \ldots, n\}$, $i < j$, the set $\{b_i, b_j\}$ spans a long line in $M/\{b_1, \ldots, b_{i-1}\}$. The elements in $B$ are called the joints of the nest $M$.

A clique $M(K_n)$ is a nest, which is easily checked, taking the set of edges incident to a fixed vertex of $K_n$ as joints. The main result of this section is the following.

Lemma 6.2. There exists an integer-valued function $\nu(n,t,a,b)$ such that the following holds: Let $b > a \geq 1$ and let $n, t$ be positive integers. If $M$ is a $t$-round matroid with no $U_{a+1,b}$-minor and $r(M) \geq \nu(n,t,a,b)$, then $M$ has a rank-$n$ nest as a minor.

We obtain a nest by finding one joint at a time using the next lemma.
Lemma 6.3. There exists an integer-valued function \( \nu_1(m, t, a, b) \) such that the following holds: Let \( b > a \geq 1 \) and \( m, t \) be positive integers. If \( M \) is a \( t \)-round matroid with no \( U_{a+1,b} \)-minor, \( r(M) \geq \nu_1(m, t, a, b) \) and \( B \) is a basis of \( M \), then \( M \) has a rank-\( m \) minor \( N \), with a basis \( B' \subseteq B \cap E(N) \) and an element \( b_1 \in B' \), such that \( \{b_1, d\} \) spans a long line in \( N \) for each \( d \in B' - b_1 \).

Let us start by seeing how this result is used to prove Lemma 6.2.

Proof of Lemma 6.2. Let \( t \) be fixed. Let \( \nu(1, t, a, b) = 1 \) and for \( n \geq 2 \) define \( \nu \) recursively by
\[
\nu(n, t, a, b) = \nu_1(\nu(n-1, t, a, b) + 1, t, a, b).
\]
To facilitate induction, we prove the stronger statement:

If \( M \) is a \( t \)-round matroid with no \( U_{a+1,b} \)-minor, \( r(M) \geq \nu(n, t, a, b) \) and \( B \) is a basis of \( M \), then \( M \) has a rank-\( n \) nest \( M/Y \) as a minor, with joints contained in \( B \).

The proof is by induction on \( n \). For \( n = 1 \) the result is trivial, as any rank-1 matroid is a nest. Let \( n \geq 2 \) and assume the result holds for \( n - 1 \). Let \( M \) and \( B \) be given as above.

By Lemma 6.3, \( M \) has a minor \( N_1 \) of rank \( \nu(n-1, t, a, b) + 1 \), with a basis \( B_1 \subseteq B \) and \( b_1 \in B_1 \) such that \( \{b_1, d\} \) spans a long line in \( N_1 \) for \( d \in B_1 - b_1 \). We can assume \( N_1 = M/Y_1 \).

Let \( N'_1 = N_1/b_1 \). Since \( t \)-roundness is preserved under contractions, \( N'_1 \) is \( t \)-round. Now \( r(N'_1) = \nu(n-1, t, a, b) \) so by induction, \( N'_1 \) has a rank-(\( n-1 \)) nest \( N_2 = N'_1/Y_2 \) as a minor, with joints \( B_2 \subseteq B_1 - b_1 \).

Now, let \( Y = Y_1 \cup Y_2 \) and \( N = M/Y \), so we have \( N_2 = N/b_1 \). Then \( N \) satisfies the following

- \( b_1 \cup B_2 \subseteq B \) is a basis of \( N \),
- For each \( d \in B_2 \), \( \{b_1, d\} \) spans a long line in \( N \),
- \( N/b_1 = N_2 \) is a nest with joints \( B_2 \).

Thus, \( N \) is a nest with joints \( b_1 \cup B_2 \). \( \square \)

We shall consider coverings of matroids by connected sets. A loop is a trivial connected component of a matroid, that we wish to avoid counting. For a matroid \( M \) denote by \( \tau_a^c(M) \) the minimum size of an \( a \)-covering \( (X_1, \ldots, X_m) \) of \( M \setminus \{ \text{loops} \} \), where \( X_1, \ldots, X_m \) are connected sets. Clearly \( \tau_a^c(M) \geq \tau_a(M) \). Note also, that a loop-less rank-\( a \) matroid \( N \) has at most \( a \) connected components, so \( \tau_a^c(N) \leq a \). Thus, we have in general for a matroid \( M \):
\[
\tau_a(M) \leq \tau_a^c(M) \leq a\tau_a(M).
\]

We need a technical lemma before we prove Lemma 6.3.

Lemma 6.4. Let \( b > a \geq 1 \). Let \( M \) be a matroid with no \( U_{a+1,b} \)-minor, and let \( e \in E(M) \). Let \( \mathcal{F} \) be the collection of all connected rank-(\( a+1 \)) sets in \( M \) containing \( e \). If \( n = r_M(\cup_{X \in \mathcal{F}} X) \), then
\[
\tau_a^c(M) - \tau_a^c(M/e) \leq a^2\left(\frac{b-1}{a}\right)^{n-a} + 1.
\]

Proof. If \( e \) is a loop, then the result is trivially true, so let \( e \) be a non-loop element. We may assume, in fact, that \( M \) is loop-less.
Let \((X_1, \ldots, X_k)\) be a minimal \(a\)-covering of \(M/e\ \{\text{loops}\}\) by connected sets. We shall construct an \(a\)-covering of \(M\) by connected sets. Consider the following cases:

1. \(e \notin \text{cl}_M(X_i)\). Then \(X_i\) is connected already in \(M\), and \(r_M(X_i) = r_{M/e}(X_i) \leq a\).
2. \(e \in \text{cl}_M(X_i)\). In this case \(X_i \cup e\) is connected in \(M\), with rank \(r_M(X_i \cup e) = r_{M/e}(X_i) + 1 \leq a + 1\). Now either,

   - \((2a)\) \(r_M(X_i \cup e) \leq a\) or
   - \((2b)\) \(r_M(X_i \cup e) = a + 1\).

We can assume, after possibly reordering the sets, that \(X_1, \ldots, X_m\) satisfy \((2b)\), and \(X_{m+1}, \ldots, X_k\) satisfy \((1)\) or \((2a)\). For \(i = 1, \ldots, m\) we have

\[
\tau^c_a(M|(X_i \cup e)) \leq a \tau_a(M|(X_i \cup e)) \leq a \left( \frac{b-1}{a} \right).
\]

The elements of \(M\) destroyed when forming \(M/e\ \{\text{loops}\}\) is the connected set \(\text{cl}_M(\{e\})\). It is now clear, that we can get an \(a\)-covering of size \(s\) of \(M\) by connected sets, where

\[
\tau^c_a(M) \leq s \leq ma \left( \frac{b-1}{a} \right) + (k - m) + 1 \leq ma \left( \frac{b-1}{a} \right) + \tau^c_a(M/e) + 1.
\]

If \(m = 0\), we are done, so assume \(m \geq 1\). Define \(M' = (M/e)\bigcup_{i=1}^m X_i\) and note, that \((X_1, \ldots, X_m)\) is a minimal \(a\)-covering of \(M'\) by connected sets. Hence, by Lemma 2.5,

\[
m = \tau^c_a(M') \leq a \tau_a(M') \leq a \left( \frac{b-1}{a} \right)^{r(M')-a}.
\]

Also, \(r(M') = r_M(\bigcup_{i=1}^m X_i \cup e)) - 1 \leq n - 1\, \text{since}\, X_i \cup e \in \mathcal{F}\, \text{for}\, i = 1, \ldots, m\). Now, combining the inequalities gives the desired result. \(\square\)

Let \(M\) be a matroid, \(k \in \mathbb{N}\) and let \(B \subseteq E(M)\). We say that \(B\) \(k\)-dominates \(M\), if for any element \(x \in E(M)\) there is a set \(W \subseteq B\) with \(r_M(W) \leq k\), such that \(x \in \text{cl}_M(W)\). A \(k\)-dominating set clearly has to be spanning.

It is easily verifed, that \(k\)-domination is preserved under contractions in the following sense: If \(B, Y \subseteq E(M)\) and \(B\) \(k\)-dominates \(M\), then \(B - Y\) \(k\)-dominates \(M/Y\).

**Proof of Lemma 6.3.** Let \(m, t, a\) and \(b\) be given, and define the following constants,

\[
\begin{align*}
r_4 &= \alpha_3(m + 1, m, a, b), \quad l = m + r_4, \quad r_3 = \alpha_3(2, t, a, b), \\
\lambda = a^2 \left( \frac{b-1}{a} \right)^{r_3-a} + 1, \quad r_1 = \max(2t, \delta(\lambda, a, b)),
\end{align*}
\]

and let us define \(\nu_1(m, t, a, b) = \nu_1 = \sigma(a)(\frac{b-1}{a})^{r_1-a}\). Let \(M\) and \(B\) be given. We start with a quick observation:

**Claim A.** It is enough to find a minor \(N'\) of \(M\), with an element \(z \in E(N')\) and an \(m\)-set \(B' \subseteq B \cap E(N')\), such that \(B' \cup z\) is independent in \(N'\) and \(\{z, d\}\) spans a long line in \(N'\) for each \(d \in B'\).
To see this, we may assume that \(B' \cup z\) is a basis of \(N'\) (otherwise, we restrict to \(\text{cl}_{N'}(B' \cup z)\)). Now choose \(b_1 \in B'\) and an element \(y\), such that \(\{z, b_1, y\}\) is a triangle in \(N'\). Let \(N = N'/y\), and note, that \(z\) and \(b_1\) are parallel in \(N\). So \(\{b_1, d\}\) spans a long line in \(N\) for \(d \in B' - d\). Since \(B'\) is a basis of \(N\) we are done.

**Claim B.** \(M\) has a \(t\)-round minor \(N_1\) with \(r(N_1) \geq r_1\) and \(B \subseteq E(N_1)\), such that \(B\) \((a+1)\)-dominates \(N_1\).

Let \(N_1\) be a minimal minor of \(M\) satisfying, that \(N_1\) is \(t\)-round and \(a\)-simple and \(B \subseteq E(N_1)\). Such a minor exists, since we can choose \(X \subseteq E(M)\) minimal, such that \(M \setminus X\) is \(a\)-simple, and as \(B\) is independent we can take \(X\) with \(X \cap B = \emptyset\). We then have \(\Gamma(M \setminus X) = \Gamma(M) \leq t\).

To see that \(B\) \((a+1)\)-dominates \(N_1\), let \(f \in E(N_1) - B\). \(N_1/f\) is \(t\)-round, as \(N_1\) is \(t\)-round. Now \((N_1/f)B\) cannot be \(a\)-simple: If it is, then we may choose \(X \subseteq E(N_1/f) - B\) minimal, such that \(N_1/f\setminus X\) is \(a\)-simple. But then \(N_1/f\setminus X\) is \(t\)-round by Lemma 4.3, contradicting the minimality of \(N_1\). \(N_1\) is simple, so \(N_1/f\) is loop-less. Since \((N_1/f)B\) is not \(a\)-simple, there must be a \(W \subseteq B\), with

\[\frac{r(N_1/f)W \simeq U_{k,2k}}{a \in \{1, \ldots, a\}}\]

for a \(k \in \{1, \ldots, a\}\). Then \(r_{N_1}(W \cup f) = k + 1\), and we must have \(r_{N_1}(W) = k + 1\).

If not, then \(N_1W \simeq U_{k,2k}\), but \(N_1\) is \(a\)-simple. Thus, \(f \in \text{cl}_{N_1}(W)\), and \(B\) \((a+1)\)-dominates \(N_1\).

By Lemma 2.7, we have

\[\sigma(a)\tau_a(N_1) \geq |E(N_1)| \geq |B| = r(M) \geq \nu_1,\]

and so, \(\tau_a(N_1) \geq \left(\frac{b-a}{a}\right)^{r_1-a} > 1\). Clearly, \(r(N_1) > a\), so we can apply Lemma 2.5, and get \(r(N_1) \geq r_1\). This proves the claim.

Let \(N_1\) be given. By definition of \(r_1\), we have \(\Gamma(N_1) \leq t \leq \frac{1}{2}r(N_1)\) and \(r(N_1) \geq \delta(\lambda, a, b)\). Lemma 4.4 gives a dense minor \(N_2\) of \(N_1\) with \(\tau_a(N_2) > \lambda r(N_2)\). We may assume, that \(N_2 = N_1/Y_1\). \(N_2\) satisfies \(\tau_a^c(N_2) > \lambda r(N_2)\). Let \(Y_2 \subseteq E(N_2)\) be maximal, with

\[\tau_a^c(N_2/Y_2) > \lambda r(N_2/Y_2),\]

and let \(N_3 = N_2/Y_2\). \(N_3\) must be loop-less, since \(Y_2\) is maximal. Pick an element \(e \in E(N_3)\). Then,

\[\tau_a^c(N_3) - \tau_a^c(N_3/e) > \lambda r(N_3) - \lambda r(N_3/e) = \lambda.\]

Let \(\mathcal{F}\) denote the collection of all connected rank-\((a+1)\) sets in \(N_3\) containing \(e\), and let \(n = r_{N_3}(\bigcup_{X \in \mathcal{F}} e)\). By Lemma 6.4, we then have \(\lambda < a^2(k-1)^{-n-a} + 1\), and by definition of \(\lambda\), this yields \(n \geq r_3\).

Denote by \(\mathcal{C}\) the collection of all circuits of \(N_3\) of rank at most \(a + 1\) containing \(e\). For each \(X \in \mathcal{F}\) and non-loop \(y \in X - e\), since \(X\) is connected, there exists a circuit \(C \subseteq X\) containing \(e\) and \(y\), so \(C \in \mathcal{C}\). Hence, \(r_{\bigcup \mathcal{C}}(\bigcup_{C \in \mathcal{C}} e) \geq n\).

Since \(n \geq r_3 = \alpha_3(2, l, a, b)\) we can apply Lemma 5.3. As no two circuits in \(\mathcal{C}\) are skew, we get case (ii): There is a minor \(N_4 = N_3/Y_3\), with \(x \in E(N_4)\) and triangles \(D_1, \ldots, D_l\) of \(N_4\), such that \(x \in D_i\) and \(r_{N_4}(\bigcup_{i} D_i) = l + 1\). Pick an element \(h_i \in D_i - x\) for \(i = 1, \ldots, l\).

Let \(I = \{i : h_i \in B\}\). If \(|I| \geq m\), then we can choose an \(m\)-set \(B' \subseteq \{h_i : h_i \in B\}\) and we are done by Claim A, taking \(N' = N_4\) and \(z = x\). So, assume \(|I| \leq m\). By possibly re-ordering the \(D_i\), we may assume \(h_1, \ldots, h_{r_4} \notin B\).
By the remark preceding the proof, \( B \cap E(N_4) \) \((a + 1)\)-dominates \( N_4 \). So, for each \( i \in \{1, \ldots, r\} \), \( h_i \) is in the closure of a subset of \( B \) of rank at most \( a + 1 \). Choose a circuit \( C_i \) of \( N_4 \) containing \( h_i \), with \( r_{N_4}(C_i) \leq a + 1 \) and \( C_i \subseteq B \cup h_i \). Since \( \{h_1, \ldots, h_r\} \) is independent, we have \( r_{N_4}(\cup C_i) \geq r_4 \). As \( r_4 = \alpha_3(m + 1, m, a, b) \) we can apply Lemma 5.3 again, and we get one of two cases.

Consider case (i): There are \( s = m + 1 \) skew circuits among \( C_1, \ldots, C_{r_4} \) in \( N_4 \). After possibly re-ordering we may assume \( C_1, \ldots, C_s \) are skew. By omitting one member of \( \{C_1, \ldots, C_s\} \), we can obtain that \( \{x\} \) is skew from the union of the rest. We assume that \( C_1, \ldots, C_m, \{x\} \) are skew sets in \( N_4 \).

For \( i = 1, \ldots, m \), pick an element \( b_i \in C_i - h_i \), and let \( K_i = C_i - \{h_i, b_i\} \). Define \( N_5 = N_4/(\cup K_i) \). Then \( h_i \) and \( b_i \) are parallel in \( N_5 \). Letting \( B' = \{b_1, \ldots, b_m\} \) we are done by Claim A, with \( N' = N_5 \) and \( z = x \).

Consider now case (ii): \( N_4 \) has a minor \( N_5 \), with \( z \in E(N_5) \) and triangles \( D_1', \ldots, D_m' \) in \( N_5 \), such that \( z \in D_i' \) and \( r_{N_5}(\cup D_i') = m + 1 \). Also, for each \( i \), \( D_i' - z \subseteq C_j \) for some \( j \). Thus, \( D_i' - z \subseteq B \cup h_j \) and we can pick an element \( b_i \in (D_i' - z) \cap B \). Taking \( B' = \{b_1, \ldots, b_m\} \) and \( N' = N_5 \), we are again done by Claim A. \( \square \)

7. Dowling Cliques

The goal of this section is to extract from nests a general kind of clique. In [2], Dowling introduced a class of combinatorial geometries (simple matroids). We use a special case of his construction, that we shall call a Dowling clique.

**Definition 7.1.** A Dowling clique is a matroid \( M \), with \( E(M) = B \cup X \), where \( B = \{b_1, \ldots, b_r\} \) is a basis of \( M \), and \( X = \{e_{ij} : 1 \leq i < j \leq n\} \) satisfies that \( \{b_i, b_j, e_{ij}\} \) is a triangle, for all \( i < j \). We call the elements in \( B \) the joints of \( M \).

Notice, that Dowling cliques are Nests. We shall first go through yet another intermediate structure on our way to obtaining cliques.

**Definition 7.2.** Let \( n \geq 1 \). A matroid \( M \) is an \( n \)-storm, if its ground set is the disjoint union \( E(M) = F \cup C_1 \cup \cdots \cup C_m \), where \( r_M(F) = n \) and each \( C_i \) is a size-\((n + 1)\) independent co-circuit of \( M \), with \( F \subseteq cl_M(C_i) \). We call the \( C_i \) clouds of \( M \).

In an \( n \)-storm, the set \( F \) must be closed, since it is an intersection of hyperplanes. Note also, that \( C_1, \ldots, C_m \) are skew in \( M/F \), and hence that \( \mu_{M/(C_1, \ldots, C_m)} = n \).

We shall first see that nests contain storms as restrictions.

**Lemma 7.3.** Let \( m \) and \( n \) be positive integers. If \( M \) is a nest of rank at least \( n + m \), then \( M \) has an \( n \)-storm \( N \) with \( m \) clouds as a minor.

**Proof.** Let \( M \) be a rank-\( r \) nest with joints \( B = \{b_1, \ldots, b_r\} \), where \( r = n + m \). For each pair \( (i, j), 1 \leq i < j \leq r \), pick an element \( e_{ij} \), such that \( \{b_i, b_j, e_{ij}\} \) is a triangle of \( M/\{b_1, \ldots, b_{i-1}\} \). We need two observations:

1. \( e_{1k}, e_{2k}, \ldots, e_{k-1,k} \notin cl_M(B - b_k) \), for \( k = 2, \ldots, r \).
2. \( cl_M(\{b_1, b_i, b_k\}) = cl_M(\{e_{1k}, \ldots, e_{ik}, b_k\}) \), for \( i < k \).

To see that (1) holds, let \( i, k \) be given, with \( 1 \leq i < k \). By definition of \( e_{ik} \) we have \( e_{ik} \notin cl_M(\{b_1, \ldots, b_k\}) \), but \( e_{ik} \in cl_M(\{b_1, \ldots, b_i, b_k\}) \). So the fundamental circuit of \( e_{ik} \) in \( M \) with respect to the basis \( B \), must contain \( b_k \). Hence, \( e_{ik} \notin cl_M(B - b_k) \).
We prove (2) by induction on \( i \) with \( k \) fixed. The case \( i = 1 \) is trivial, since \( \{b_1, b_k, e_1k\} \) is a circuit in \( M \). Suppose \( 1 < i < k \) and (2) holds for \( i - 1 \). Again, by definition of \( e_{ik} \) we have \( e_{ik} \notin \text{cl}_M(\{b_1, \ldots, b_{i-1}, b_k\}) \), but \( e_{ik} \in \text{cl}_M(\{b_1, \ldots, b_i, b_k\}) \). Thus,

\[
\text{cl}_M(\{b_1, \ldots, b_i, b_k\}) = \text{cl}_M(\{b_1, \ldots, b_{i-1}, b_k, e_{ik}\}) = \text{cl}_M(\{e_{1k}, \ldots, e_{i-1,k}, b_k, e_{ik}\}),
\]

where the induction hypothesis is used in the second step.

Let \( S = \{b_1, \ldots, b_n\} \) and \( F = \text{cl}_M(S) \), and for each \( k = n + 1, \ldots, r \) define \( C_k = \{e_{1k}, \ldots, e_{nk}, b_k\} \).

Notice that by (1), \( C_k \cap F = \emptyset \) for all \( k \), and \( C_k \cap C_l = \emptyset \) for \( k \neq l \). From (2) we gather, that \( C_k \) is independent with \( F \subseteq \text{cl}_M(C_k) \) for all \( k \). Define \( N = M_1(F \cup C_{n+1} \cup \cdots \cup C_r) \). It is easily checked, that the \( C_k \)'s are co-circuits of \( N \).

For \( n = 2 \), the concept of an \( n \)-storm is similar to that of a “book”, used by Kung in [6], and an analogue of one of his ideas is part of the proof of the following result.

**Lemma 7.4.** There exist integer-valued functions \( \phi_1(n, a, b) \), \( \phi_2(n, a, b) \) such that the following holds: Let \( b > a \geq 1 \) and let \( n \) be a positive integer. If \( M \) is a \( \phi_1(n, a, b) \)-storm with \( \phi_2(n, a, b) \) clouds, and \( M \) has no \( U_{n+1,b} \)-minor, then \( M \) contains a rank-\( n \) Dowling clique \( N \) as a minor.

**Proof.** Let \( n, a, b \) be given and let us define \( \phi_1 \) and \( \phi_2 \). First, let \( l = na \). For \( r = 1, \ldots, a \), let \( s_r = \phi_1(n, r, a, b) \) and let \( s = \sum_{r=1}^{a} s_r \). Define a sequence of numbers \( m_0, \ldots, m_s \) recursively as follows: let \( m_s = l \), and

\[
m_k = \sigma_2(a, m_{k+1}) \left( \frac{b-1}{a} \right)^{s-a} + 1, \quad \text{for } k = s-1, \ldots, 1, 0.
\]

Finally, let \( \phi_1(n, a, b) = s \) and \( \phi_2(n, a, b) = m_0 \).

Let \( M \) be an \( s \)-storm with \( m = m_0 \) clouds. Denote the clouds by \( C_1, \ldots, C_m \) and their elements \( C_i = \{e_{1i}, e_{1i}, \ldots, e_{ai}\} \).

Define \( M' = M / \{e_{01}, e_{21}, \ldots, e_{0n}\} \). So \( r(M') = s \). Let \( I_0 = \{1, \ldots, m\} \). We wish to find a subcollection of the clouds, such that elements with the same index lie on a uniform restriction of \( M' \). We shall construct a sequence of subsets,

\[
I_0 \supseteq I_1 \supseteq \cdots \supseteq I_s, \quad \text{where } |I_k| = m_k,
\]

such that for \( k = 1, \ldots, s \), \( M' \{e_k^i : i \in I_k\} \cong U_{r_k,m_k} \), for some number \( r_k \in \{1, \ldots, a\} \), Let \( k \geq 1 \), suppose \( I_{k-1} \) has been defined and let us see how to find \( I_k \).

Let \( W = \{e_k^i : i \in I_{k-1}\} \) and suppose \( M'|W \) has no \( U_{r,m_k} \)-restriction for \( r = 1, \ldots, a \). Lemmas 2.8 and 2.5 then give

\[
m_{k-1} = |W| \leq \sigma_2(a, m_k) \tau_a(M'|W) \leq \sigma_2(a, m_k) \left( \frac{b-1}{a} \right)^{s-a},
\]

contradicting our definition of \( m_{k-1} \). So, take \( U \subseteq W \) such that \( M'|U \) is isomorphic to \( U_{r_k,m_k} \), and let \( I_k = \{i : e_k^i \in U\} \).

After possibly re-ordering the clouds in \( M \), we may assume that \( I_s = \{1, \ldots, l\} \).

Let then \( L_k = \{e_k^i, \ldots, e_k^l\} \) for \( k = 1, \ldots, s \), so \( M'|L_k \cong U_{r_k,l} \). Now, by a majority argument, there exist \( r \in \{1, \ldots, a\} \), and a subset \( J \subseteq \{1, \ldots, s\} \) with \( |J| \geq s_r \), such
that \( r_k = r \), for all \( k \in J \). Define \( \mathcal{L} = \{ L_k : k \in J \} \). Since \( C_1 - e_0^1 \) is independent in \( M' \), we have

\[
\gamma_m(\cup_{L \in \mathcal{L}} L) \geq |\mathcal{L}| = |J| \geq s_r = \alpha_1(n, r, a, b).
\]

We now apply Lemma 5.1 and get a subcollection \( \mathcal{L}' \subseteq \mathcal{L} \) of size \( n \), such that \( L \not\subseteq \cl_{M'}(\cup_{L' \in \mathcal{L}'-L'}) \), for each \( L \in \mathcal{L}' \). After possibly permuting the elements of each cloud, we can assume \( \mathcal{L}' = \{ L_1, \ldots, L_n \} \). Let \( D_i = \{ e_0^i, e_1^i, \ldots, e_n^i \} \subseteq C_i \).

By our arrangement of the \( L_k \)'s, we can find a set \( B \subseteq \cup_{i=1}^n L_k \) independent in \( M' \), such that \( r_{M'}(L_k \cup B) - r_{M'}(B) = 1 \), for each \( k \in \{1, \ldots, n\} \), and the sets \( L_1 - B, \ldots, L_n - B \) are skew in \( M'/B \).

Now \( L_k \not\subseteq \cl_{M'}(B) \), and since \( M'|L_k \) is rank-\( r \) uniform, we must have \( |L_k \cap \cl_{M'}(B)| \leq r - 1 \). Define \( I_B \subseteq \{1, \ldots, l\} \) by

\[
I_B = \{ i : D_i \cap \cl_{M'}(B) \neq \emptyset \}.
\]

Then \( B \subseteq \cup_{i \in I_B} D_i \) and \( |I_B| \leq n(r - 1) \leq n(a - 1) = l - n \). Notice, that for \( i \not\in I_B \), \( D_i \) and \( B \) are skew in \( M' \). We may assume, again after re-ordering the clouds, that \( \{1, \ldots, n\} \subseteq \{1, \ldots, l\} - I_B \). Let

\[
M_1 = M/\{ e_0^i : i \in I_B \} /B.
\]

Then, by construction, the elements in \( \{ e_k^i : i = 1, \ldots, n \} \) are in parallel in \( M_1/\{ e_0^i : i = 1, \ldots, n \} \), for each \( k = 1, \ldots, n \).

Let \( p_k = e_k^n \), for \( k = 1, \ldots, n \), and define

\[
M_2 = M_1/e_0^n((D_1 \cup \cdots \cup D_{n-1} \cup \{ p_1, \ldots, p_n \})).
\]

Now, \( M_2 \) is an \( n \)-storm with clouds \( D_1, \ldots, D_{n-1} \). It satisfies, for each \( i = 1, \ldots, n-1 \), and each \( k = 1, \ldots, n \), that \( p_k \) is on the line through \( e_0^i \) and \( e_k^i \).

We shall make \( \{ p_1, \ldots, p_n \} \) the joints of a Dowling clique. Let

\[
N = M_2/\{ e_1^1, e_2^2, \ldots, e_{n-1}^{n-1} \}.
\]

Then \( \{ p_1, \ldots, p_n \} \) is a basis for \( N \). Let \( (i, j) \) be given with \( 1 \leq i < j \leq n \). In \( N \), \( e_0^i \) and \( p_i \) are parallel, so \( e_j^i \in \cl_N(\{ e_0^i, p_i \}) = \cl_N(\{ p_i, p_j \}) \), and the set \( \{ p_i, p_j, e_j^i \} \) is a triangle in \( N \). So, \( N \) has a rank-\( n \) Dowling clique restriction.

\[
8. \text{Cliques}
\]

We need Mader’s Theorem [7] to extract graphic cliques.

\textbf{Mader’s Theorem 8.1.} \textit{Let} \( H \) \textit{be a simple graph. There exists} \( \lambda \in \mathbb{N} \) \textit{such that, if} \( G \) \textit{is a simple graph with no} \( H \)-

\textit{minor, then} \( |E(G)| \leq \lambda |V(G)| \).

An easy corollary is the following matroid version of the theorem. We take \( H \) to be a complete graph, and write the contrapositive statement.

\textbf{Corollary 8.2.} \textit{There exists an integer-valued function} \( \theta(n) \) \textit{such that, if} \( M \) \textit{is a simple graphic matroid with} \( |E(M)| > \theta(n)r(M) \), \textit{then} \( M \) \textit{has an} \( M(K_n) \)-

\textit{minor.}

Let \( M \) be a matroid and \( B \) a basis of \( M \), and let \( X = E(M) - B \). We call \( M \) a \textit{Dowling matroid} with \textit{joints} \( B \) if each \( x \in X \) is on a triangle with two elements of \( B \), and any two elements of \( B \) span at most one element of \( X \) (again, this is only a special case of Dowling’s combinatorial geometries). By the \textit{associated graph} of \( (M, B) \) we mean the graph \( G \) on the vertex set \( B \) with edge set labeled by \( X \), such that \( x \in X \) labels \( \{ b_1, b_2 \} \) if \( x \) is on the line through \( b_1 \) and \( b_2 \) in \( M \).

We shall use the following lemma to recognize graphic matroids.
Lemma 8.3. Let $M$ be a Dowling matroid with joints $B$, $X = E(M) - B$. If $B \cap \text{cl}_M(X) = \emptyset$, then $M|X$ is graphic.

Proof. Let $G$ be the associated graph of $(M, B)$. We claim that $M(G) = M|X$. It suffices to prove, that each circuit of $(M(G))$ is dependent in $M|X$, and that each independent set of $(M(G))$ is independent in $M|X$.

Let $C$ be a cycle of $G$ with vertex set $B'$ and edge set $X'$. Clearly $X' \subseteq \text{cl}_M(B')$ and by the assumption $B' \cap \text{cl}_M(X') = \emptyset$. Since $B'$ and $X'$ have equal size, $X'$ must be dependent in $M$.

Let $T$ be a forest in $G$. We prove by induction on $|E(T)|$ that $E(T)$ is independent in $M$. Let $e$ be a leaf edge in $T$ and assume $E(T) - e$ is independent in $M$. Let $b$ be a leaf of $T$ incident on $e$. Then $E(T) - e \subseteq \text{cl}_M(B - b)$, but $e \notin \text{cl}_M(B - b)$, so $E(T)$ is independent in $M$. \hfill \square

In a similar fashion, using Remark 1.4, the following lemma is easily proved.

Lemma 8.4. Let $M$ be a Dowling matroid with joints $B$, $X = E(M) - B$. Let $G$ be the associated graph of $(M, B)$. If for each cycle $C$ in $G$, $E(C)$ is independent in $M$, then $M|X = B(G)$ (in fact $M = \overline{B}(G)$).

We are ready for the final step in the proof of the main theorem.

Lemma 8.5. There exists an integer-valued function $\psi(n)$ such that, if $M$ is a Dowling clique with rank at least $\psi(n)$, then $M$ contains an $M(K_n)$- or $B(K_n)$-minor.

Proof. Let $n$ be given, and define $\psi(n) = nl$, where $l = 2m\theta(n) + 1$ and $m = 2^n n!$. Let $M$ be a Dowling clique and assume that $r(M) = nl$. Denote by $B$ the joints of $M$ and let $X = E(M) - B$.

Partition $B$ into $n$ sets, $B_1, \ldots, B_n$ of equal size, $|B_i| = l$. We shall contract each $B_i$ to a point. Let $M_i = M|\text{cl}_M(B_i)$. Choose $Y_i \subseteq E(M_i) \cap X$ such that $B_i$ is a set of parallel elements in $M_i/Y_i$ (take the edges of a spanning tree in the associated graph of $(M_i, B_i)$). Define $M' = M/(Y_1 \cup \cdots \cup Y_n)$ and pick a $b_i \in B_i$ for $i = 1, \ldots, n$. For each pair $i < j$, define

$$X_{ij} = \{ x \in X : x \in \text{cl}_M(b, d), b \in B_i, d \in B_j \}.$$

Note, that for each $x \in X_{ij}$, $\{b_i, b_j, x\}$ is a triangle in $M'$. We consider two cases.

(1). $r_M'(X_{ij}) > m$, for all pairs $i < j$. Put $B' = \{b_1, \ldots, b_n\}$. We shall choose a set $X' = \{x_{ij} : 1 \leq i < j \leq n\}$, where $x_{ij} \in X_{ij}$, such that $M'|B' \cup X'$ is the Dowling clique $\overline{B}(K_n)$. Let $X' \subseteq \cup X_{ij}$ be maximal, such that $|X' \cap X_{ij}| \leq 1$ for all $i < j$, and the cycles in the associated graph of $M'|B' \cup X'$ all have edge sets independent in $M'$. We claim, that $X' \cap X_{ij} \neq \emptyset$ for all $i < j$, and thus $M'|B' \cup X' \simeq \overline{B}(K_n)$ by Lemma 8.4.

Assume that $X' \cap X_{ij} = \emptyset$ for some $i < j$. Let $G$ be the associated graph of $M'|B' \cup X'$. If $Z \subseteq X'$ is the edge set of a path from $b_i$ to $b_j$ in $G$, then $r_M'(\text{cl}_{M'}(Z) \cap X_{ij}) \leq 1$. There can be at most $m$ such $Z$, since a simple graph on $n$ vertices has no more than $2^n n! = m$ cycles. Thus, we can pick $x_{ij} \in X_{ij}$ skew from each such $Z$. So, the cycles created in the associated graph, when adding $x_{ij}$ to $X'$ all have edge sets independent in $M'$, contradicting the maximality of $X'$.

(2). $r_M'(X_{ij}) \leq m$, for some pair $i < j$. As $|X_{ij}| = l^2$, there is a parallel class $P \subseteq X_{ij}$ of $M'$, with $|P| \geq l^2/m$. Now, since $B \cap \text{cl}_{M'}(P) = \emptyset$, also $B \cap \text{cl}_M(P) = \emptyset$,
and Lemma 8.3 gives, that $M|P$ is graphic. And $r(M|P) \leq |B_i \cup B_j| = 2l$. We have then

$$|E(M|P)| \geq \ell^2/m > 2\theta(n) \geq \theta(n)r(M|P),$$

and by Corollary 8.2, we get an $M(K_n)$-minor.

Finally, we restate and prove Theorem 1.1.

**Theorem 8.6.** There exists an integer-valued function $\gamma(k,a,n)$ such that, if $M$ is a matroid with $r(M) \geq \gamma(k,a,n)$, then either $M$ has $k$ disjoint co-circuits or $M$ has a minor isomorphic to $U_{1,2a}$, $M(K_n)$ or $B(K_n)$.

**Proof.** Let $k, a, n$ be positive integers. If $a = 1$, then we let $\gamma(k,a,n) = k$. If $a \geq 2$, then we define the following numbers: Put $a' = a - 1$ and $b = 2a$. Let $k = \psi(n)$ and let $m_1 = \phi_1(k, a', b)$ and $m_2 = \phi_2(k, a', b)$. Let $r = n + m$ and define $g : \mathbb{N} \to \mathbb{N}$ by $g(t) = \nu(r, t, a', b)$. Finally, $\gamma(k,a,n) = f_\phi(k)$.

Let $M$ be given with $r(M) \geq \gamma(k,a,n)$. If $a = 1$, the result is trivial, since in a matroid with no $U_{1,2}$-minor, every element is a loop or a co-loop.

If $a \geq 2$, then by Lemma 3.2, either $M$ has $k$ disjoint co-circuits or a minor $N$ with $r(N) \geq g(\Gamma(N))$. Assume the second case. Also, if $N$ has a $U_{a'+1,b}$-minor we are done, so assume this is not the case. Applying Lemmas 6.2, 7.3, 7.4 and 8.5 in succession, we obtain an $M(K_n)$- or a $B(K_n)$-minor of $N$. □

**REFERENCES**