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THE ERDÖS-PÓSA PROPERTY FOR MATROID CIRCUITS

JIM GEELEN AND KASPER KABELL

Abstract. The number of disjoint co-circuits in a matroid is bounded by its rank. There are, however, matroids with arbitrarily large rank that do not contain two disjoint co-circuits; consider, for example, $M(K_n)$ and $U_{n,2n}$. Also the bicircular matroids $B(K_n)$ have arbitrarily large rank and have no 3 disjoint co-circuits. We prove that for each k and n there exists a constant c such that, if M is a matroid with no $U_{n,2n}$-, $M(K_n)$-, or $B(K_n)$-minor, then either M has k disjoint co-circuits or $r(M) \leq c$.

1. Introduction

We prove the following theorem.

Theorem 1.1. *There exists a function $\gamma : \mathbb{N}^3 \to \mathbb{N}$ such that, if M is a matroid with no $U_{a,2a}$-, $M(K_n)$-, or $B(K_n)$-minor and $r(M) \geq \gamma(k, a, n)$, then M has k disjoint co-circuits.*

Here $M(K_n)$ is the cycle matroid of K_n and $B(K_n)$ is the bicircular matroid of K_n (to be defined below).

A *circuit-cover* of a graph G is a set $X \subseteq E(G)$ such that $G - X$ has no circuits. Thus the maximum number of (edge-) disjoint circuits in a graph is bounded by the minimum size of a circuit cover. This bound is not tight (consider K_4), but Erdös and Pósa in [3] proved that the maximum number of disjoint circuits is qualitatively related to the minimum size of a circuit cover.

Erdös-Pósa Theorem 1.2. *There is a function $c : \mathbb{N} \to \mathbb{N}$ such that, if the size of a minimal circuit-cover of G is at least $c(k)$, then G has k disjoint circuits.*

Let M be a matroid. A set $X \subseteq E(M)$ intersects each circuit of M if and only if $E(M) - X$ is independent. So, a minimal circuit-cover of M is a basis of M^*. The Erdös-Pósa Theorem was generalized to matroids by Geelen, Gerards, and Whittle [4] who proved:

Theorem 1.3. *There exists a function $c : \mathbb{N}^3 \to \mathbb{N}$ such that, if M is a matroid with no $U_{2,q+2}$- or $M(K_n)$-minor and $r(M) \geq c(k, q, n)$, then M has k disjoint co-circuits.*

The result does not extend to all matroids. A matroid is *round* if it has no two disjoint co-circuits. Equivalently, M is round if each co-circuit in M is a spanning set of M.

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The matroid $U_{r,n}$, where $n \geq 2r-1$ is round. Also, for any positive integer n, $M(K_n)$ is a round matroid. Generally, for a graph G, a co-circuit of $M(G)$ is a minimal edge-cut of G. If G is simple, then it is easily seen, that G has no two disjoint edge-cuts if and only if G is complete.

Let $G = (V, E)$ be a graph. Define a matroid $\tilde{B}(G)$ on $V \cup E$ where V is a basis of $\tilde{B}(G)$ and, for each edge $e = uv$ of G, place e freely on the line spanned by $\{u, v\}$. Now $B(G) := \tilde{B}(G)\backslash V$ is the *bicircular matroid* of G. A different characterization of $B(G)$ is the following, which gives rise to the name bicircular matroid. It is easily verified (see [8, Prop. 12.1.6]).

Remark 1.4. Let G be a graph. C is a circuit of $B(G)$ if and only if $G[C]$ is a subdivision of one of the graphs below.

The matroid $\tilde{B}(K_n)$ is also round, which is easily verified. The bicircular matroid $B(K_n)$ is not round, but it has no three disjoint co-circuits (for $n \neq 3$).

Our main theorem, Theorem 1.1, is a generalization of Theorem 1.3 and is, in some sense, best possible. Note that each of the classes

$$\{M(K_n) : n \geq 1\}, \ \{B(K_n) : n \geq 1\}, \text{ and } \{U_{a,2a} : a \geq 1\}$$

have unbounded rank but they have a bounded number of disjoint co-circuits.

We follow the notation of Oxley [8], and the reader is assumed familiar with standard matroid theory as described therein.

2. Covering number

We shall work with dense matroids in the proof. This section develops tools for measuring the size and density of a matroid.

A simple GF(q)-representable rank-r matroid can be realized as a restriction of the projective geometry PG$(r-1, q)$. Thus, it has at most $\frac{q^r-1}{q-1}$ elements. Kung [5] extended this bound to the class of matroids with no $U_{2,q+2}$-minor (the shortest line not representable over GF(q)).

Theorem (Kung) 2.1. *Let $q > 1$ be an integer, and let M be a simple rank-r matroid with no $U_{2,q+2}$-minor. Then*

$$|E(M)| \leq \frac{q^r - 1}{q - 1}.$$

Projective geometries show, that the bound is sharp if q is a prime-power. To bound the size of rank-r matroids, it is necessary to restrict the length of lines, or there can be arbitrarily many elements in a rank-2 matroid. As we shall be excluding a uniform matroid of higher rank, we need a new measure of size, for an analogue of Kung's Theorem to hold.

Definition 2.2. Let a be a positive integer. An *a-covering* of a matroid M is a collection $(X_1, \ldots, X_m)$ of subsets of $E(M)$, with $E(M) = \cup X_i$ and $r_M(X_i) \leq a$ for all i. The *size* of the covering is m. The *a-covering number* of M, $\tau_a(M)$ is the minimum size of an a-covering of M. If $r(M) = 0$, then we define $\tau_a(M) = 0$.

Note that for a matroid M, $\tau_1(M) = |E(\mathrm{si}(M))|$, where $\mathrm{si}(M)$ denotes the simplification of M. If M has non-zero rank $r(M) \leq a$, then $\tau_a(M) = 1$. Our first lemma bounds $\tau_a(M)$ in the case $r(M) = a + 1$.

Lemma 2.3. *Let $b > a \geq 1$. If M is a matroid of rank $a + 1$ with no $U_{a+1,b}$-minor, then*

$$\tau_a(M) \leq \binom{b-1}{a}.$$

Proof. Let $X \subseteq E(M)$ be maximal with $M|X \simeq U_{a+1,l}$. Then $l \leq b - 1$. For an $x \notin X$, by the maximality of X, there exists $Y \subseteq X$ with $|Y| = a$ such that $Y \cup x$ is dependent, and thus $x \in \mathrm{cl}_M(Y)$.

It follows that $(\mathrm{cl}_M(Y)|Y \subseteq X, |Y| = a)$ is an a-covering of M. It has size $\binom{l}{a} \leq \binom{b-1}{a}$. □

Lemma 2.4. *Let $b > a \geq 1$. If M is a matroid with no $U_{a+1,b}$-restriction, then*

$$\tau_a(M) \leq \binom{b-1}{a}\tau_{a+1}(M).$$

Proof. Let $(X^1, \ldots, X^k)$ be a minimal $(a+1)$-covering of M. By Lemma 2.3 each $M|X^i$ has an a-covering $(X^i_1, \ldots, X^i_{m_i})$ of size $m_i \leq \binom{b-1}{a}$. Combining these we get an a-covering $(X^i_j|j = 1, \ldots, m_i, i = 1, \ldots, k)$ of M. Thus $\tau_a(M) \leq \sum m_i \leq \binom{b-1}{a}k$. □

The next result extends Kung's Theorem. The bound we obtain is not sharp, though.

Lemma 2.5. *Let $b > a \geq 1$. If M is a matroid of rank $r \geq a$ with no $U_{a+1,b}$-minor, then*

$$\tau_a(M) \leq \binom{b-1}{a}^{r-a}.$$

Proof. The proof is by induction on r. The case $r = a$ is trivial since $(E(M))$ is an a-covering of size 1.

Let $r > a$ and assume that the result holds for rank $r - 1$. Let x be a non-loop element of M. Then $r(M/x) = r - 1$ and by induction $\tau_a(M/x) \leq \binom{b-1}{a}^{r-1-a}$.

Let $(X_1, \ldots, X_k)$ be a minimal a-covering of M/x, so $r_{M/x}(X_i) \leq a$ for all i. This implies $r_M(X_i \cup x) \leq a + 1$, and so $(X_i \cup x|i = 1, \ldots, k)$ is an $(a+1)$-covering of M. We conclude $\tau_{a+1}(M) \leq \tau_a(M/x)$.

Finally, by Lemma 2.4 we have $\tau_a(M) \leq \binom{b-1}{a}\tau_{a+1}(M)$ and combining inequalities we get the desired result. □

Definition 2.6. Let a be a positive integer. The matroid M is called *a-simple*, if M is simple and M has no $U_{k,2k}$-restriction for $k = 2, 3, \ldots, a$.

Equivalently, M is a-simple if it is loop-less and has no $U_{k,2k}$-restriction for $k = 1, 2, 3, \ldots, a$. This concept is just an abbreviation. We shall not define an "a-simplification" operation, since for $a \geq 2$ it would not be well-defined up to isomorphism. For a-simple matroids, the size is proportional to τ_a:

Lemma 2.7. *There exists an integer-valued function $\sigma(a)$ such that, if $a \geq 1$ and M is a-simple, then $|E(M)| \leq \sigma(a)\tau_a(M)$.*

Proof. Define σ by

$$\sigma(a) = \prod_{k=2}^{a} \binom{2k-1}{k-1}.$$

Since M has no $U_{k,2k}$-restriction for $k = 2, \dots, a$, Lemma 2.4 gives

$$\tau_{k-1}(M) \le \binom{2k-1}{k-1} \tau_k(M), \quad k = 2, \dots, a.$$

Putting these together, we get $|E(M)| = \tau_1(M) \le \sigma(a)\tau_a(M)$. $\square$

We shall need one more specialized result, which is completely similar to the previous Lemma.

Lemma 2.8. *There exists an integer-valued function $\sigma_2(a, b)$ such that, if $b \ge a \ge 1$ and M is loop-less and has no $U_{k,b}$-restriction for $k = 1, \dots, a$, then $|E(M)| \le \sigma_2(a, b)\tau_a(M)$.*

Proof. Define σ_2 by

$$\sigma_2(a, b) = (b-1) \prod_{k=2}^{a} \binom{b-1}{k-1}.$$

Now use $|E(M)| \le (b-1)\tau_1(M)$ and apply Lemma 2.4. $\square$

3. Approaching roundness

The first step in the proof of the main theorem is to show, that a matroid of large enough rank has either k disjoint co-circuits or a large minor which is "nearly round".

Definition 3.1. Let M be a matroid. The *rank-deficiency* of a set of elements $X \subseteq E(M)$ is $r_M^-(X) = r(M) - r_M(X)$. Denote by $\Gamma(M)$ the maximum rank-deficiency among the co-circuits of M. For $t \in \mathbb{N}$ we say that M is *t-round* if $\Gamma(M) \le t$.

Notice that a matroid M is round if and only if $\Gamma(M) = 0$, that is, M is 0-round. The condition of being t-round is easily seen to be preserved under contractions. When we cannot obtain a t-round matroid, we shall sometimes work with the even weaker property: $\Gamma(M) \le \frac{1}{2} r(M)$.

Lemma 3.2. *Let $g : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. There exists a function $f_g : \mathbb{N} \to \mathbb{N}$ such that for any $k \in \mathbb{N}$, if M is a matroid with $r(M) \ge f_g(k)$, then either*

(a) *M has k disjoint co-circuits or*
(b) *M has a minor $N = M/Y$ with $r(N) \ge g(\Gamma(N))$.*

Proof. Let g be given and define f_g as follows: $f_g(0) = f_g(1) = 1$ and

$$f_g(k) = g(f_g(k-1)), \quad k \ge 2.$$

The proof is by induction on k. If $r(M) \ge 1$, then M has a co-circuit, so the result holds for $k = 0, 1$. Now let $k \ge 2$ and $r(M) \ge f_g(k) = g(f_g(k-1))$.

If $\Gamma(M) \ge f_g(k-1)$, then pick a co-circuit C of M with $r_M^-(C) = \Gamma(M)$. Then $r(M/C) = r_M^-(C) \ge f_g(k-1)$. If M/C has the desired contraction minor, then we are done. If not, then by induction M/C has $k-1$ disjoint co-circuits. These, together with C, give k disjoint co-circuits of M.

If $\Gamma(M) \le f_g(k-1)$, then as g is non-decreasing, we have $r(M) \ge g(\Gamma(M))$. $\square$

4. Building density

The goal of this section is to prove, that a high-rank nearly round matroid with no $U_{a+1,b}$ minor contains a dense minor.

Lemma 4.1. *Let $b > a \geq 1$. Let M be a matroid with no $U_{a+1,b}$-minor and let C be a co-circuit of M of minimal size. If $C_1, \dots, C_k$ are disjoint co-circuits of $M \backslash C$ with $|C_1| \leq \dots \leq |C_k|$, then $|C_i| \geq |C| / (a\binom{b-1}{a})$ for $i = a, \dots, k$.*

Proof. Let C and $C_1 \dots, C_k$ be given and let $i \in \{a, \dots, k\}$.

C_1 is co-dependent in $M \backslash C \backslash C_i$. So there exists a co-circuit $C_1' \subseteq C_1$ of $M \backslash (C \cup C_i)$.

Now, C_2 is co-dependent in $M \backslash C \backslash (C_i \cup C_1')$. So there is a co-circuit $C_2' \subseteq C_2$ of $M \backslash (C \cup C_i \cup C_1')$.

Continuing in this fashion, for each $j = 2, \dots, a-1$ we pick a co-circuit $C_j' \subseteq C_j$ of $M \backslash (C \cup C_i \cup C_1' \cup \dots \cup C_{j-1}')$.

Denote by F the set $E(M) - (C \cup C_i \cup C_1' \cup \dots \cup C_{a-1}')$. Deleting a co-circuit of a matroid drops its rank by 1, so we get $r_M^-(F) = a + 1$. Hence $N = M/F$ has rank $r(N) = a + 1$. Since C is a co-circuit of N of minimal size, $E(N) - C$ must be a rank-a set of N of maximal size. We now have

$$\begin{aligned} |C| \leq |E(N)| &\leq \tau_a(N)\, |E(N) - C| \\ &= \tau_a(N) \left| C_i \cup C_1' \cup \dots \cup C_{a-1}' \right| \\ &\leq \binom{b-1}{a} a\, |C_i| \end{aligned}$$

using Lemma 2.3. The result now follows. □

Lemma 4.2. *There exists an integer-valued function $\kappa(\lambda, a, b)$ such that the following holds: Let $b > a \geq 1$ and $\lambda \in \mathbb{N}$. Let M be an a-simple matroid with no $U_{a+1,b}$-minor, satisfying $\Gamma(M) \leq \frac{1}{2} r(M)$. Let C be a minimal sized co-circuit of M. If $M \backslash C$ has $\kappa(\lambda, a, b)$ disjoint co-circuits, then $\tau_a(M) > \lambda r(M)$.*

Proof. Let a, b and λ be given and define

$$\kappa(\lambda, a, b) = \kappa = 2a \binom{b-1}{a} \sigma(a) \lambda + a - 1.$$

Let M and C be given and let $C_1, \dots, C_\kappa$ be disjoint co-circuits of $M \backslash C$ of non-decreasing size. Note that

$$|C| \geq r_M(C) \geq r(M) - \Gamma(M) \geq r(M)/2.$$

By Lemma 2.7 and the above lemma we have

$$\begin{aligned} \sigma(a)\tau_a(M) &\geq |E(M)| \\ &> |C_a| + \dots + |C_\kappa| \geq (\kappa - a + 1) \frac{r(M)}{2a\binom{b-1}{a}} = \sigma(a)\lambda r(M), \end{aligned}$$

and the result follows. □

For a matroid M denote by $\Theta(M)$ the maximum number of disjoint co-circuits in M. So, M is round if and only if $\Theta(M) = 1$. The two parameters $\Gamma(M)$ and $\Theta(M)$ are related by

$$\Theta(M) \leq \Gamma(M) + 1.$$

This follows from the observation, that if $C_1, \ldots, C_k$ are disjoint co-circuits of M, then $r(M\backslash(C_1 \cup \cdots \cup C_k)) \leq r(M) - k$. Equality does not hold (consider $U_{4,5}$). The following result lists hereditary properties of the two parameters.

Lemma 4.3. *Let M be a matroid and let $X, Y \subseteq E(M)$. Then*

(i) $\Theta(M/Y) \leq \Theta(M)$ *and* $\Gamma(M/Y) \leq \Gamma(M)$.
(ii) $\Theta(M\backslash X) \geq \Theta(M)$ *and* $\Gamma(M\backslash X) \geq \Gamma(M)$, *if X is co-independent.*
(iii) $\Theta(M\backslash X) = \Theta(M)$ *and* $\Gamma(M\backslash X) = \Gamma(M)$, *if for some number $a \in \mathbb{N}$, X is minimal with respect to inclusion, such that $M\backslash X$ is a-simple.*

Proof. Every co-circuit C of M/Y is a co-circuit of M. A short calculation shows that $r^-_{M/Y}(C) \leq r^-_M(C)$, so the first assertion of the lemma holds.

To prove the second and third assertions, it is enough to consider $X = \{x\}$, where x is not a co-loop of M. If C is a co-circuit in M, then $C - x$ contains a co-circuit in $M\backslash x$. Thus $\Theta(M\backslash x) \geq \Theta(M)$ and also $\Gamma(M\backslash x) \geq \Gamma(M)$.

We turn to the third assertion. Assume that $x \in W$, where $M|W \simeq U_{k,2k}$, for a $k \in \mathbb{N}$. If C is a co-circuit of $M\backslash x$, then $C = C' - x$ for a co-circuit C' of M, that is either C or $C \cup x$ is a co-circuit of M. We look at two cases:

- If $C \cap (W - x) = \emptyset$, then C is a co-circuit of M, since the complement of a co-circuit is closed and $x \in \mathrm{cl}_M(W - x)$.
- If $C \cap (W - x) \neq \emptyset$, then we must have $|(W - x) - C| < k$, since $M|(W - x) \simeq U_{k,2k-1}$ and the complement of C is closed. Hence, $|C \cap (W - x)| \geq k$.

Note that the second case can happen at most once in a collection of disjoint co-circuits. So given a collection of disjoint co-circuits of $M\backslash x$, by adding x to at most one of them, we get a collection of disjoint co-circuits of M. Thus $\Theta(M) \geq \Theta(M\backslash x)$. Note also, that for a co-circuit C of $M\backslash x$, if $C \cup x$ is a co-circuit of M, then we are in the second case, and $r_M(C \cup x) = r_{M\backslash x}(C)$. Thus $\Gamma(M) \geq \Gamma(M\backslash x)$. Finally, since no co-circuit can contain a loop, deleting loops also preserves Θ and Γ. $\square$

Lemma 4.4. *There exists an integer-valued function $\delta(\lambda, a, b)$ such that the following holds: Let $b > a \geq 1$ and $\lambda \in \mathbb{N}$. If M is a matroid with no $U_{a+1,b}$-minor, such that $\Gamma(M) \leq \frac{1}{2} r(M)$ and $r(M) \geq \delta(\lambda, a, b)$, then M has a minor N with $\tau_a(N) > \lambda r(N)$.*

Proof. Let a, b and λ be given and fixed, and let us define $\delta(\lambda, a, b)$. First, we define a sequence of functions $g_n : \mathbb{N} \to \mathbb{N}$. Let $g_0(m) = 0$, and for $n \geq 1$ define g_n recursively by

$$g_n(m) = \max(2m, \delta_n),$$
$$\text{where } \delta_n = 2(f_{g_{n-1}}(\kappa(\lambda, a, b)) + 1) \in \mathbb{N}.$$

Finally, let $\delta(\lambda, a, b) = \delta_{n_0}$, where $n_0 = 2\sigma(a)\lambda$. The functions σ, κ and f_{g_n} are defined in previous lemmas. We first prove a partial result.

Claim. *Let $n \geq 0$. If M is a matroid with no $U_{a+1,b}$-minor, such that $r(M) \geq g_n(\Gamma(M))$, then either*

- *M has a minor N with $\tau_a(N) > \lambda r(N)$ or*
- *there exists a sequence of matroids $M = M_0, M_1, \ldots, M_n$, such that for $i = 0, \ldots, n-1$, $M_{i+1} = M_i\backslash C_i/Y_i$, where C_i is a co-circuit of M_i that spans M_i/Y_i.*

We prove the claim by induction on n. The case $n = 0$ is trivial, so assume $n \geq 1$ and that the result holds for $n - 1$.

Let $X \subseteq E(M)$ be minimal such that $M\backslash X$ is a-simple. Pick a co-circuit C_0 of M with $|C_0 - X|$ minimal. Note, that $C = C_0 - X$ is a co-circuit of $M\backslash X$ of minimal size.

Choose a basis Z of M/C_0 and let $M' = M/Z$. Then C_0 spans M', and since $r(M) \geq g_n(\Gamma(M))$,

$$r(M') = r_M(C_0) \geq r(M) - \Gamma(M) \geq \tfrac{1}{2}r(M) \geq \tfrac{1}{2}\delta_n.$$

Now $r(M'\backslash C_0) = r(M') - 1 \geq f_{g_{n-1}}(\kappa(\lambda, a, b))$, so by Lemma 3.2 one of the following holds:

(a) $M'\backslash C_0$ has $\kappa(\lambda, a, b)$ disjoint co-circuits.
(b) $M'\backslash C_0$ has a minor $M_1 = M'\backslash C_0/Y$ with $r(M_1) \geq g_{n-1}(\Gamma(M_1))$.

Assume first that (a) holds. Since $M'\backslash C_0 = M\backslash C_0/Z$, by Lemma 4.3(i), $M\backslash C_0$ has $\kappa(\lambda, a, b)$ disjoint co-circuits. We claim, that $X - C_0$ is co-independent in $M\backslash C_0$. If not, then there exists a co-circuit $D \subseteq X \cup C_0$ of M with $D \cap (X - C_0) \neq \emptyset$, contradicting our choice of C_0. Now, by Lemma 4.3(ii),

$$\Theta((M\backslash X)\backslash C) = \Theta(M\backslash(C_0 \cup X)) \geq \Theta(M\backslash C_0) \geq \kappa(\lambda, a, b).$$

The lemma also gives $\Gamma(M\backslash X) \leq \frac{1}{2}r(M\backslash X)$. We can now apply Lemma 4.2 to $N = M\backslash X$, and get the desired result.

Assume now that (b) holds. Letting $Y_0 = Z \cup Y$, we have $M_1 = M\backslash C_0/Y_0$ and C_0 spans M/Y_0. Applying the induction hypothesis to M_1 now gives the claim.

Let M be given as in the lemma, and note that $r(M) \geq g_n(\Gamma(M))$, where $n = 2\sigma(a)\lambda$. By the claim, either we are done or there is a sequence $M = M_0, \ldots, M_n$, such that for $i = 0, \ldots, n-1$, $M_{i+1} = M_i\backslash C_i/Y_i$, where C_i is a co-circuit of M_i that spans M_i/Y_i.

Let $M' = M/(Y_0 \cup \cdots \cup Y_{n-1})$. Notice, that for $i = 0, \ldots, n-1$, C_i is a spanning co-circuit of $M'\backslash(C_0 \cup \cdots \cup C_{i-1})$. Thus $r_{M'}(C_i) = r - i$, where $r = r(M')$. For all i, choose a basis B_i for $M'|C_i$, and define $N = M'|(\cup B_i)$. Then

$$|E(N)| = \sum_{i=0}^{n-1}(r - i) > \frac{nr}{2}.$$

We claim that N is a-simple. Suppose $N|W \simeq U_{k,2k}$ for a $W \subseteq E(N)$ and $k \in \{1, 2, 3, \ldots\}$. Then $|W \cap B_0| \leq k$, as B_0 is independent. So $|W \cap (E(N) - B_0)| \geq k$, and since $E(N) - B_0$ is closed, $W \cap B_0 = \emptyset$. Repeat this argument in $N\backslash B_0$ to see, that $W \cap B_1 = \emptyset$ etc. We end up with $W \subseteq B_{n-1}$, a contradiction.

Finally, by Lemma 2.7,

$$\sigma(a)\tau_a(N) \geq |E(N)| > \frac{nr}{2} = \sigma(a)\lambda r(N),$$

and the result follows. □

5. Arranging circuits

We wish to identify some more concrete structure in a dense matroid. To do this, we need to be able to disentangle some of the many low-rank sets in the matroid.

For a matroid M, we call sets $A_1, \ldots, A_n \subseteq E(M)$ *skew* if $r_M(\cup_i A_i) = \sum_i r_M(A_i)$. This is analogous to subspaces of a vector-space forming a direct sum. The first result of this section is a tool for finding sets in a matroid, that are close to being skew.

We define a function μ_M on collections of subsets of $E(M)$ as follows. For sets $A_1, \dots, A_n \subseteq E(M)$, let

$$\begin{aligned}\mu_M(A_1, \dots, A_n) &= r_M(\bigcup_j A_j) - \sum_i \Big(r_M(\bigcup_j A_j) - r_M(\bigcup_{j \neq i} A_j) \Big) \\ &= r_M(\bigcup_j A_j) - \sum_i r_{M/(\cup_{j\neq i} A_j)}(A_i - \cup_{j \neq i} A_j)).\end{aligned}$$

This function can be thought of as a generalized connectivity function. For $n = 2$, μ_M equals the connectivity function $\lambda_M(A_1, A_2) = r_M(A_1) + r_M(A_2) - r_M(A_1 \cup A_2)$. For $n \geq 2$ a recursive formula holds,

$$\mu_M(A_1, \dots, A_n) = \lambda_M(A_1, A_2 \cup \dots \cup A_n) + \mu_{M/A_1}(A_2, \dots, A_n).$$

The function μ_M measures in a way the rank of the "overlap" of the sets, though this may not be an actual set in the matroid. Notice, that $\mu_M(A_1, \dots, A_n) = 0$ if and only if $A_1, \dots, A_n$ are skew. More generally, if there is a set $W \subseteq E(M)$ such that $A_1 - W, \dots, A_n - W$ are skew in M/W, then $\mu_M(A_1, \dots, A_n) \leq r_M(W)$.

Lemma 5.1. *There exists an integer-valued function $\alpha_1(n, r, a, b)$ such that the following holds: Let $b > a \geq 1$, and let r and n be positive integers. If M is a matroid with no $U_{a+1,b}$-minor, and $\mathcal{F}$ is a collection of rank-r subsets of $E(M)$ with $r_M(\cup_{X\in\mathcal{F}} X) \geq \alpha_1(n, r, a, b)$, then there exist $X_1, \dots, X_n \in \mathcal{F}$ satisfying*

(a) *$X_i \not\subseteq \mathrm{cl}_M(\cup_{j\neq i} X_j)$ for $i = 1, \dots, n$ and*
(b) *$\mu_M(X_1, \dots, X_n) \leq (r-1)a$.*

Proof. For any positive integers n, c, k, we let $R(n, c, k)$ denote the following Ramsey number: The minimal R, such that if X is a set with $|X| = R$, then for any c-coloring of $[X]^n$, X has a monochromatic subset of size k. Here $[X]^n$ denotes the set of all subsets of X of size n. By a monochromatic subset of X, we mean a subset $Y \subseteq X$ such that the sets in $[Y]^n$ all have the same color. This number exists by Ramsey's Theorem (see [9] or [1, 9.1.4]).

Let n, r, a, b be given and let us define $\alpha_1(n, r, a, b)$. First we define numbers s_i, l_i for $i = 1, \dots, r$. Let $s_r = 0, l_r = n$, and for $i = r-1, r-2, \dots, 1$ define recursively:

$$s_i = s_{i+1} + l_{i+1}, \quad u_i = \binom{b-1}{a}^{rs_i - a}, \quad l_i = n\binom{u_i}{r-i}.$$

Let $m = s_1 + l_1$. So, we have $0 = s_r < s_{r-1} < \dots < s_1 < m$. Next, define numbers $k_1, \dots, k_m$ as follows. Let $k_m = m$ and define recursively:

$$k_{i-1} = R(i, r, k_i), \qquad \text{for } i = m, m-1, \dots, 2.$$

Finally, let $\alpha_1(n, r, a, b) = rk_1$.

In the following, for a set of subsets $\mathcal{X} \subseteq 2^{E(M)}$, we use the shorthand notation $r_M(\mathcal{X}) = r_M(\cup_{X\in\mathcal{X}} X)$.

Let M and $\mathcal{F}$ be given, with $r_M(\mathcal{F}) \geq \alpha_1(n, r, a, b) = rk_1$. We can choose sets $Y_1, \dots, Y_{k_1} \in \mathcal{F}$, such that $Y_i \notin \mathrm{cl}_M(Y_1 \cup \dots \cup Y_{i-1})$. Let $\mathcal{F}_1 = \{Y_1, \dots, Y_{k_1}\}$ and put $a_0 = 0, a_1 = r$. We shall iteratively construct sequences:

$$\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \dots \supseteq \mathcal{F}_m, \quad a_0 < a_1 < a_2 < \dots < a_m,$$

such that for $i = 1, \dots, m$, $|\mathcal{F}_i| = k_i$, and if $\mathcal{F}' \subseteq \mathcal{F}_i$ with $|\mathcal{F}'| = i$, then $r_M(\mathcal{F}') = a_i$. This clearly holds for $\mathcal{F}_1$. Let $i \geq 2$, assume that $\mathcal{F}_{i-1}$ and a_{i-1} satisfy the above, and let us find $\mathcal{F}_i$ and a_i.

Note, that $r_M(\mathcal{F}') \in \{a_{i-1}+1, \dots, a_{i-1}+r\}$, for $\mathcal{F}' \subseteq \mathcal{F}_{i-1}$ with $|\mathcal{F}'| = i$. This defines an r-coloring of $[\mathcal{F}_{i-1}]^i$. Since $|\mathcal{F}_{i-1}| = k_{i-1} = R(i, r, k_i)$, there exists $\mathcal{F}_i \subseteq \mathcal{F}_{i-1}$ such that, every set in $[\mathcal{F}_i]^i$ has the same rank, and we let a_i be that number.

For $i = 1, \dots, m$ let $b_i = a_i - a_{i-1}$. Notice that, by submodularity, this gives a decreasing sequence $(a_{i+1} + a_{i-1} \leq a_i + a_i)$,

$$r = b_1 \geq b_2 \geq \cdots \geq b_m \geq 1.$$

Hence, by definition of the pairs (s_i, l_i), there exists an $r' \in \{1, \dots, r\}$, such that

$$b_{s+1} = \cdots = b_{s+l} = r', \qquad \text{where } s = s_{r'} \text{ and } l = l_{r'}.$$

If $r' = r$, then we get $b_1 = \cdots = b_n = r$. Thus, if we choose any n members $X_1, \dots, X_n \in \mathcal{F}_m$, then they are skew and we are done.

Assume $r' < r$. Choose s sets $Z_1, \dots, Z_s \in \mathcal{F}_m$, and let $F = \cup_{i=1}^s Z_i$. Choose another l sets $X_1, \dots, X_l \in \mathcal{F}_m - \{Z_1, \dots, Z_s\}$. Since $b_{s+1} = b_{s+l} = r'$, the sets $X_1 - F, \dots, X_l - F$ are skew of rank r' in M/F. For $i = 1, \dots, l$, choose an independent set $\overline{B}_i \subseteq X_i$ of size r', skew from F. Expand this set to a basis $B_i \cup \overline{B}_i$ of X_i, so $|B_i| = r_0 = r - r'$.

Let $M' = M/(\cup_i \overline{B}_i)$ and $B = \cup_i B_i$. Then $B_i \subseteq \mathrm{cl}_{M'}(F)$, and thus $r_{M'}(B) \leq r_{M'}(F) \leq sr$. Let $(W_1, \dots, W_u)$ be a minimal a-covering of $M'|B$. By Lemma 2.5, we have

$$u = \tau_a(M'|B) \leq \binom{b-1}{a}^{sr-a} = u_{r'}.$$

For each B_i, we can find a set of indices $I_i \subseteq \{1, \dots, u\}$ of size r_0, such that $B_i \subseteq \cup_{j \in I_i} W_j$. There are $\binom{u}{r_0} \leq \binom{u_{r'}}{r-r'}$ possible choices for I_i, and $l = n\binom{u_{r'}}{r-r'}$. By a majority argument, there must exist $I \subseteq \{1, \dots, u\}$, such that $I_i = I$ for all $i \in J$, where $J \subseteq \{1, \dots, l\}$ has size n. By possibly re-ordering the $X_i's$ and the $W_j's$ we can assume, that $B_1, \dots, B_n \subseteq W_1 \cup \cdots \cup W_{r_0}$.

Let $W = W_1 \cup \cdots \cup W_{r_0}$. Then the sets $X_1 - W, \dots, X_n - W$ are skew in M/W. It follows, that $\mu_M(X_1 \dots, X_n) \leq r_M(W) \leq ar_0 \leq a(r-1)$, and we are done. □

The next lemma shows how, by doing suitable contractions, a large collection of nearly (but not completely) skew circuits, can yield a set of nearly skew triangles containing a common element. The idea is to put points in the "overlap" by contracting some of the circuits. The overlap can then be contracted to a point.

Lemma 5.2. *There exists an integer-valued function $\alpha_2(l, r, m)$ such that the following holds: Let $r \geq 2$ and l, m be positive integers. If $n \geq \alpha_2(l, r, m)$ and $C_1 \dots, C_n$ are rank-r circuits of a matroid M satisfying*

(a) $1 \leq r_M(\cup_j C_j) - r_M(\cup_{j \neq i} C_j) < r$ *for all i, and*
(b) $\mu_M(C_1, \dots, C_n) \leq m$,

then M has a minor $N = M/Y$ with an element $x \in E(N)$ and triangles $D_1, \dots, D_l$ of N, such that

- *$x \in D_i$ for all i, and $r_N(\cup_i D_i) = l + 1$,*
- *For all i, $D_i - x \subseteq C_j$ for some $j \in \{1, \dots, n\}$.*

Proof. Let l and m be fixed. For $r \geq 2$, define $\alpha_2(l, r, m)$ recursively as follows

$$\alpha_2(l, r, m) = 2^m(q_r(r-1)+1),$$

$$q_2 = l, \qquad q_r = \alpha_2(l, r-1, r-2), \qquad \text{for } r > 2.$$

To facilitate induction, the lemma is proved from the following weaker assumptions:

Let $n \geq \alpha_2(l, r, m)$, and $C_1 \ldots, C_n$ be circuits of M with $2 \leq r_M(C_i) \leq r$. Assume there is a set $F \subseteq E(M)$, such that

(a) $1 \leq r_M((\cup_j C_j) \cup F) - r_M((\cup_{j \neq i} C_j) \cup F) < r_M(C_i) - 1$ *for all i.*
(b) $\mu_M(C_1, \ldots, C_n, F) \leq m$.

These assumptions are indeed weaker, since the phrasing in the lemma is the case $F = \emptyset$ and $r_M(C_i) = r$ for all i. The proof is by induction on r. Let $r = 2$ or let $r > 2$ and assume the result holds for $r - 1$.

Let $c_i = r_M((\cup_j C_j) \cup F) - r_M((\cup_{j \neq i} C_j) \cup F)$ for each i. We first do an easy reduction. If not $c_i = 1$ for all i, then for each i choose a set $Y_i \subseteq C_i$ of size $c_i - 1$, which is skew from $(\cup_{j \neq i} C_j) \cup F$. We may then work with the circuits $C_i - Y_i$ of $M/(Y_1 \cup \cdots \cup Y_n)$ instead. So without loss of generality, $c_i = 1$ for all i.

Choose $z_i \in C_i - \mathrm{cl}_M(\cup_{j \neq i} C_j)$ for each i, and let $\overline{M} = M/\{z_1, \ldots, z_n\}$. Letting $W = \cup_i (C_i - z_i)$ we have

$$r_{\overline{M}}(W) = r_M(\cup_i C_i) - r_M(\{z_1, \ldots, z_n\}) = \mu_M(C_1, \ldots, C_n, F) \leq m.$$

Let B be a basis of $\overline{M}|W$ and choose a basis B_i of $\overline{M}|(C_i - z_i)$ for each i. Now expand B_i to a basis $B_i \cup X_i$ of $\overline{M}|W$ using elements of B. For all i we have chosen $X_i \subseteq B$ among the $2^{|B|} \leq 2^m$ subsets of B. Hence, there exists an $X_0 \subseteq B$, such that $X_i = X_0$ for $i \in I$, where $|I| = n' \geq n/2^m$. Let $M_1 = M/X_0$ and put $r' = |B| - |X_0| + 1$. Then $r_{M_1}(C_i) = r'$ for all $i \in I$, and $\mu_{M_1}(C_i : i \in I) = r' - 1$. By possibly reordering the circuits, we can assume $I = \{1, \ldots, n'\}$.

Pick an element of one circuit, $z \in C_{n'} - \mathrm{cl}_{M_1}(\cup_{j<n'} C_j)$ and let $M_2 = M_1/z$. Define

$$Z = \mathrm{cl}_{M_2}(C_{n'} - z) \subseteq \mathrm{cl}_{M_2}(C_i), \quad \text{for } i = 1, \ldots, n'-1,$$

so $r_{M_2}(Z) = r' - 1$. Choose a non-loop element $x \in Z$ and elements $y_i \in C_i - Z$ for $i = 1, \ldots, n'-1$. Since $x \in \mathrm{cl}_{M_2}(C_i)$, $C_i \cup x$ is connected, so there is a circuit C'_i of M_2 with

$$\{x, y_i\} \subseteq C'_i \subseteq C_i \cup x, \quad \text{and} \quad r_{M_2}(C'_i) \in \{2, \ldots, r'\}.$$

Notice that $C'_i \nsubseteq \mathrm{cl}_{M_2}(\cup_{j \neq i, j<n'} C'_j)$, since $y_i \in C'_i$.

By another majority argument, there exists $s \in \{2, \ldots, r'\}$, such that $r_{M_2}(C'_i) = s$ for $i \in J$, where $|J| \geq (n'-1)/(r-1) \geq q_r$. We now have two cases:

$\underline{s = 2:}$ Since $q_r \geq q_2 = l$ we can choose $J' \subseteq J$ with $|J'| = l$. We are now done with $\{D_1, \ldots, D_l\} = \{C'_i : i \in J'\}$ and $N = M_2$.

$\underline{2 < s \leq r':}$ Let $M_3 = M_2/x$ and let $\overline{C}_i = C'_i - x$ for $i \in J$. Then $\overline{C}_i$ is a rank-$(s-1)$ circuit of M_3, with $\overline{C}_i \subseteq C_i$. Letting $F' = Z - x$ we have,

$$\mu_{M_3}(\overline{C}_i : i \in J, F') \leq r_{M_3}(F') = r' - 2 \leq r - 2.$$

As $|J| \geq \alpha_2(l, r-1, r-2)$ we get by induction the desired minor. □

The following result is just a corollary to Lemmas 5.1 and 5.2, that we state for easier reference.

Lemma 5.3. *There exists an integer-valued function $\alpha_3(s,l,a,b)$ such that the following holds: Let $b > a \geq 1$ and let s,l be positive integers. If M is a matroid with no $U_{a+1,b}$-minor, and $\mathcal{C}$ is a set of circuits of M of rank at most $a+1$, with $r_M(\cup_{C\in\mathcal{C}} C) \geq \alpha_3(s,l,a,b)$, then either:*

(i) *There exist s skew circuits $C_1,\dots,C_s \in \mathcal{C}$, or*
(ii) *M has a minor $N = M/Y$ with an element $x \in E(N)$ and triangles $D_1,\dots,D_l$ of N, such that*
- *$x \in D_i$ for all i, and $r_N(\cup_i D_i) = l+1$, and*
- *For all i, $D_i - x \subseteq C$ for some $C \in \mathcal{C}$.*

Proof. Define $\alpha_3(s,l,a,b) = \alpha_3$ by

$$\alpha_3 = \sum_{r=1}^{a+1} \alpha_1(n_r, r, a, b), \quad \text{where } \; n_r = s + \alpha_2(l, r, (r-1)a),$$

and let M, $\mathcal{C}$ be given. By a majority argument, there exists a number $r \in \{1,\dots,a+1\}$ and $\mathcal{C}' \subseteq \mathcal{C}$, such that $r_M(C) = r$ for all $C \in \mathcal{C}'$, and $r_M(\cup_{C\in\mathcal{C}'} C) \geq \alpha_1(n_r, r, a, b)$.

Now, by Lemma 5.1, there are $C_1,\dots,C_n \in \mathcal{C}'$, where $n = n_r = s + \alpha_2(l, r, (r-1)a)$, satisfying

$$c_i = r_M(\cup_j C_j) - r_M(\cup_{j\neq i} C_j) \geq 1,$$

for all i, and $\mu(C_1,\dots,C_n) \leq (r-1)a$.

Let $I = \{i : c_i = r\}$ and $J = \{i : c_i < r\}$. If $|I| \geq s$, then case (i) holds, since the C_i with $i \in I$ are skew. Otherwise, $|J| \geq \alpha_2(l, r, (r-1)a)$, and the C_i with $i \in J$ still satisfy

$$r_M(\cup_{j\in J} C_j) - r_M(\cup_{j\in J-\{i\}} C_j) < r.$$

Lemma 5.2 now gives case (ii) of the result. □

6. Nests

By a *long* line in a matroid, we mean a rank-2 flat, that contains at least 3 rank-1 flats. So, a long line in a simple matroid is a rank-2 flat with at least 3 elements. Also, a line is long if and only if it contains a triangle. We need a lot of long lines to construct clique-like structures. We first aim to build an intermediate structure called a nest.

Definition 6.1. A matroid M is a *nest* if M has a basis $B = \{b_1,\dots,b_n\}$ such that, for each pair of indices $i,j \in \{1,\dots,n\}$, $i < j$, the set $\{b_i, b_j\}$ spans a long line in $M/\{b_1,\dots,b_{i-1}\}$. The elements in B are called the *joints* of the nest M.

A clique $M(K_n)$ is a nest, which is easily checked, taking the set of edges incident to a fixed vertex of K_n as joints. The main result of this section is the following.

Lemma 6.2. *There exists an integer-valued function $\nu(n,t,a,b)$ such that the following holds: Let $b > a \geq 1$ and let n,t be positive integers. If M is a t-round matroid with no $U_{a+1,b}$-minor and $r(M) \geq \nu(n,t,a,b)$, then M has a rank-n nest as a minor.*

We obtain a nest by finding one joint at a time using the next lemma.

Lemma 6.3. *There exists an integer-valued function $\nu_1(m,t,a,b)$ such that the following holds: Let $b > a \geq 1$ and m, t be positive integers. If M is a t-round matroid with no $U_{a+1,b}$-minor, $r(M) \geq \nu_1(m,t,a,b)$ and B is a basis of M, then M has a rank-m minor N, with a basis $B' \subseteq B \cap E(N)$ and an element $b_1 \in B'$, such that $\{b_1, d\}$ spans a long line in N for each $d \in B' - b_1$.*

Let us start by seeing how this result is used to prove Lemma 6.2.

Proof of Lemma 6.2. Let t be fixed. Let $\nu(1,t,a,b) = 1$ and for $n \geq 2$ define ν recursively by

$$\nu(n,t,a,b) = \nu_1(\nu(n-1,t,a,b)+1,t,a,b).$$

To facilitate induction, we prove the stronger statement:

> *If M is a t-round matroid with no $U_{a+1,b}$-minor, $r(M) \geq \nu(n,t,a,b)$ and B is a basis of M, then M has a rank-n nest M/Y as a minor, with joints contained in B.*

The proof is by induction on n. For $n = 1$ the result is trivial, as any rank-1 matroid is a nest. Let $n \geq 2$ and assume the result holds for $n-1$. Let M and B be given as above.

By Lemma 6.3, M has a minor N_1 of rank $\nu(n-1,t,a,b)+1$, with a basis $B_1 \subseteq B$ and $b_1 \in B_1$ such that $\{b_1, d\}$ spans a long line in N_1 for $d \in B_1 - b_1$. We can assume $N_1 = M/Y_1$.

Let $N_1' = N_1/b_1$. Since t-roundness is preserved under contractions, N_1' is t-round. Now $r(N_1') = \nu(n-1,t,a,b)$ so by induction, N_1' has a rank-$(n-1)$ nest $N_2 = N_1'/Y_2$ as a minor, with joints $B_2 \subseteq B_1 - b_1$.

Now, let $Y = Y_1 \cup Y_2$ and $N = M/Y$, so we have $N_2 = N/b_1$. Then N satisfies the following

- $b_1 \cup B_2 \subseteq B$ is a basis of N,
- For each $d \in B_2$, $\{b_1, d\}$ spans a long line in N,
- $N/b_1 = N_2$ is a nest with joints B_2.

Thus, N is a nest with joints $b_1 \cup B_2$. □

We shall consider coverings of matroids by connected sets. A loop is a trivial connected component of a matroid, that we wish to avoid counting. For a matroid M denote by $\tau_a^c(M)$ the minimum size of an a-covering $(X_1, \ldots, X_m)$ of $M \setminus \{loops\}$, where $X_1, \ldots, X_m$ are connected sets. Clearly $\tau_a^c(M) \geq \tau_a(M)$. Note also, that a loop-less rank-a matroid N has at most a connected components, so $\tau_a^c(N) \leq a$. Thus, we have in general for a matroid M:

$$\tau_a(M) \leq \tau_a^c(M) \leq a\tau_a(M).$$

We need a technical lemma before we prove Lemma 6.3.

Lemma 6.4. *Let $b > a \geq 1$. Let M be a matroid with no $U_{a+1,b}$-minor, and let $e \in E(M)$. Let $\mathcal{F}$ be the collection of all connected rank-$(a+1)$ sets in M containing e. If $n = r_M(\cup_{X \in \mathcal{F}} X)$, then*

$$\tau_a^c(M) - \tau_a^c(M/e) \leq a^2 \binom{b-1}{a}^{n-a} + 1.$$

Proof. If e is a loop, then the result is trivially true, so let e be a non-loop element. We may assume, in fact, that M is loop-less.

Let $(X_1,\dots,X_k)$ be a minimal a-covering of $M/e\backslash\{loops\}$ by connected sets. We shall construct an a-covering of M by connected sets. Consider the following cases:

(1) $e\notin \mathrm{cl}_M(X_i)$. Then X_i is connected already in M, and $r_M(X_i)=r_{M/e}(X_i)\le a$.

(2) $e\in \mathrm{cl}_M(X_i)$. In this case $X_i\cup e$ is connected in M, with rank $r_M(X_i\cup e)=r_{M/e}(X_i)+1\le a+1$. Now either,

(2a) $r_M(X_i\cup e)\le a$ or

(2b) $r_M(X_i\cup e)=a+1$.

We can assume, after possibly reordering the sets, that $X_1,\dots,X_m$ satisfy (2b), and $X_{m+1},\dots,X_k$ satisfy (1) or (2a). For $i=1,\dots,m$ we have

$$\tau_a^c(M|(X_i\cup e))\le a\tau_a(M|(X_i\cup e))\le a\binom{b-1}{a}.$$

The elements of M destroyed when forming $M/e\backslash\{loops\}$ is the connected set $\mathrm{cl}_M(\{e\})$. It is now clear, that we can get an a-covering of size s of M by connected sets, where

$$\begin{aligned}\tau_a^c(M)\le s&\le ma\binom{b-1}{a}+(k-m)+1\\&\le ma\binom{b-1}{a}+\tau_a^c(M/e)+1.\end{aligned}$$

If $m=0$, we are done, so assume $m\ge 1$. Define $M'=(M/e)|(\cup_{i=1}^m X_i)$ and note, that $(X_1,\dots,X_m)$ is a minimal a-covering of M' by connected sets. Hence, by Lemma 2.5,

$$m=\tau_a^c(M')\le a\tau_a(M')\le a\binom{b-1}{a}^{r(M')-a}.$$

Also, $r(M')=r_M(\cup_{i=1}^m(X_i\cup e))-1\le n-1$, since $X_i\cup e\in\mathcal{F}$ for $i=1,\dots,m$. Now, combining the inequalities gives the desired result. $\square$

Let M be a matroid, $k\in\mathbb{N}$ and let $B\subseteq E(M)$. We say that B *k-dominates* M, if for any element $x\in E(M)$ there is a set $W\subseteq B$ with $r_M(W)\le k$, such that $x\in\mathrm{cl}_M(W)$. A k-dominating set clearly has to be spanning.

It is easily verified, that k-domination is preserved under contractions in the following sense: If $B,Y\subseteq E(M)$ and B k-dominates M, then $B-Y$ k-dominates M/Y.

Proof of Lemma 6.3. Let m,t,a and b be given, and define the following constants,

$$r_4=\alpha_3(m+1,m,a,b),\quad l=m+r_4,\quad r_3=\alpha_3(2,l,a,b),$$

$$\lambda=a^2\binom{b-1}{a}^{r_3-a}+1,\quad r_1=\max(2t,\delta(\lambda,a,b)),$$

and let us define $\nu_1(m,t,a,b)=\nu_1=\sigma(a)\binom{b-1}{a}^{r_1-a}$. Let M and B be given. We start with a quick observation:

Claim A. *It is enough to find a minor N' of M, with an element $z\in E(N')$ and an m-set $B'\subseteq B\cap E(N')$, such that $B'\cup z$ is independent in N' and $\{z,d\}$ spans a long line in N' for each $d\in B'$.*

To see this, we may assume that $B' \cup z$ is a basis of N' (otherwise, we restrict to $\mathrm{cl}_{N'}(B' \cup z)$). Now choose $b_1 \in B'$ and an element y, such that $\{z, b_1, y\}$ is a triangle in N'. Let $N = N'/y$, and note, that z and b_1 are parallel in N. So $\{b_1, d\}$ spans a long line in N for $d \in B' - d$. Since B' is a basis of N we are done.

Claim B. *M has a t-round minor N_1 with $r(N_1) \geq r_1$ and $B \subseteq E(N_1)$, such that B $(a+1)$-dominates N_1.*

Let N_1 be a minimal minor of M satisfying, that N_1 is t-round and a-simple and $B \subseteq E(N_1)$. Such a minor exists, since we can choose $X \subseteq E(M)$ minimal, such that $M \backslash X$ is a-simple, and as B is independent we can take X with $X \cap B = \emptyset$. We then have $\Gamma(M \backslash X) = \Gamma(M) \leq t$.

To see that B $(a+1)$-dominates N_1, let $f \in E(N_1) - B$. N_1/f is t-round, as N_1 is t-round. Now $(N_1/f)|B$ cannot be a-simple: If it is, then we may choose $X \subseteq E(N_1/f) - B$ minimal, such that $N_1/f\backslash X$ is a-simple. But then $N_1/f\backslash X$ is t-round by Lemma 4.3, contradicting the minimality of N_1. N_1 is simple, so N_1/f is loop-less. Since $(N_1/f)|B$ is not a-simple, there must be a $W \subseteq B$, with

$$(N_1/f)|W \simeq U_{k,2k},$$

for a $k \in \{1, \dots, a\}$. Then $r_{N_1}(W \cup f) = k+1$, and we must have $r_{N_1}(W) = k+1$. If not, then $N_1|W \simeq U_{k,2k}$, but N_1 is a-simple. Thus, $f \in \mathrm{cl}_{N_1}(W)$, and B $(a+1)$-dominates N_1.

By Lemma 2.7, we have

$$\sigma(a)\tau_a(N_1) \geq |E(N_1)| \geq |B| = r(M) \geq \nu_1,$$

and so, $\tau_a(N_1) \geq \binom{b-1}{a}^{r_1 - a} > 1$. Clearly, $r(N_1) > a$, so we can apply Lemma 2.5, and get $r(N_1) \geq r_1$. This proves the claim.

Let N_1 be given. By definition of r_1, we have $\Gamma(N_1) \leq t \leq \frac{1}{2} r(N_1)$ and $r(N_1) \geq \delta(\lambda, a, b)$. Lemma 4.4 gives a dense minor N_2 of N_1 with $\tau_a(N_2) > \lambda r(N_2)$. We may assume, that $N_2 = N_1/Y_1$. N_2 satisfies $\tau_a^c(N_2) > \lambda r(N_2)$. Let $Y_2 \subseteq E(N_2)$ be maximal, with

$$\tau_a^c(N_2/Y_2) > \lambda r(N_2/Y_2),$$

and let $N_3 = N_2/Y_2$. N_3 must be loop-less, since Y_2 is maximal. Pick an element $e \in E(N_3)$. Then,

$$\tau_a^c(N_3) - \tau_a^c(N_3/e) > \lambda r(N_3) - \lambda r(N_3/e) = \lambda.$$

Let $\mathcal{F}$ denote the collection of all connected rank-$(a+1)$ sets in N_3 containing e, and let $n = r_{N_3}(\cup_{X \in \mathcal{F}})$. By Lemma 6.4, we then have $\lambda < a^2 \binom{b-1}{a}^{n-a} + 1$, and by definition of λ, this yields $n \geq r_3$.

Denote by $\mathcal{C}$ the collection of all circuits of N_3 of rank at most $a+1$ containing e. For each $X \in \mathcal{F}$ and non-loop $y \in X - e$, since X is connected, there exists a circuit $C \subseteq X$ containing e and y, so $C \in \mathcal{C}$. Hence, $r_{N_3}(\cup_{C \in \mathcal{C}} C) \geq n$.

Since $n \geq r_3 = \alpha_3(2, l, a, b)$ we can apply Lemma 5.3. As no two circuits in $\mathcal{C}$ are skew, we get case (ii): There is a minor $N_4 = N_3/Y_3$, with $x \in E(N_4)$ and triangles $D_1, \dots, D_l$ of N_4, such that $x \in D_i$ and $r_{N_4}(\cup_i D_i) = l+1$. Pick an element $h_i \in D_i - x$ for $i = 1 \dots, l$.

Let $I = \{i : h_i \in B\}$. If $|I| \geq m$, then we can choose an m-set $B' \subseteq \{h_i : h_i \in B\}$ and we are done by Claim A, taking $N' = N_4$ and $z = x$. So, assume $|I| \leq m$. By possibly re-ordering the D_i, we may assume $h_1, \dots, h_{r_4} \notin B$.

By the remark preceding the proof, $B \cap E(N_4)$ $(a+1)$-dominates N_4. So, for each $i \in \{1, \dots, r_4\}$, h_i is in the closure of a subset of B of rank at most $a+1$. Choose a circuit C_i of N_4 containing h_i, with $r_{N_4}(C_i) \le a+1$ and $C_i \subseteq B \cup h_i$. Since $\{h_1, \dots, h_{r_4}\}$ is independent, we have $r_{N_4}(\cup_i C_i) \ge r_4$. As $r_4 = \alpha_3(m+1, m, a, b)$ we can apply Lemma 5.3 again, and we get one of two cases.

Consider case (i): There are $s = m+1$ skew circuits among $C_1 \dots, C_{r_4}$ in N_4. After possibly re-ordering we may assume $C_1, \dots, C_s$ are skew. By omitting one member of $\{C_1, \dots, C_s\}$, we can obtain that $\{x\}$ is skew from the union of the rest. We assume that $C_1, \dots, C_m, \{x\}$ are skew sets in N_4.

For $i = 1, \dots, m$, pick an element $b_i \in C_i - h_i$, and let $K_i = C_i - \{h_i, b_i\}$. Define $N_5 = N_4/(\cup_i K_i)$. Then h_i and b_i are parallel in N_5. Letting $B' = \{b_1, \dots, b_m\}$ we are done by Claim A, with $N' = N_5$ and $z = x$.

Consider now case (ii): N_4 has a minor N_5, with $z \in E(N_5)$ and triangles $D'_1, \dots, D'_m$ in N_5, such that $z \in D'_i$ and $r_{N_5}(\cup_i D'_i) = m+1$. Also, for each i, $D'_i - z \subseteq C_j$ for some j. Thus, $D'_i - z \subseteq B \cup h_j$ and we can pick an element $b_i \in (D'_i - z) \cap B$. Taking $B' = \{b_1, \dots, b_m\}$ and $N' = N_5$, we are again done by Claim A. $\square$

7. Dowling Cliques

The goal of this section is to extract from nests a general kind of clique. In [2], Dowling introduced a class of combinatorial geometries (simple matroids). We use a special case of his construction, that we shall call a Dowling clique.

Definition 7.1. A *Dowling clique* is a matroid M, with $E(M) = B \cup X$, where $B = \{b_1, \dots, b_n\}$ is a basis of M, and $X = \{e_{ij} : 1 \le i < j \le n\}$ satisfies that $\{b_i, b_j, e_{ij}\}$ is a triangle, for all $i < j$. We call the elements in B the *joints* of M.

Notice, that Dowling cliques are Nests. We shall first go through yet another intermediate structure on our way to obtaining cliques.

Definition 7.2. Let $n \ge 1$. A matroid M is an *n-storm*, if its ground set is the disjoint union $E(M) = F \cup C_1 \cup \dots \cup C_m$, where $r_M(F) = n$ and each C_i is a size-$(n+1)$ independent co-circuit of M, with $F \subseteq \mathrm{cl}_M(C_i)$. We call the C_i *clouds* of M.

In an n-storm, the set F must be closed, since it is an intersection of hyperplanes. Note also, that $C_1, \dots, C_m$ are skew in M/F, and hence that $\mu_M(C_1 \dots, C_m) = n$.

We shall first see that nests contain storms as restrictions.

Lemma 7.3. *Let m and n be positive integers. If M is a nest of rank at least $n+m$, then M has an n-storm N with m clouds as a minor.*

Proof. Let M be a rank-r nest with joints $B = \{b_1, \dots, b_r\}$, where $r = n+m$. For each pair (i,j), $1 \le i < j \le r$, pick an element e_{ij}, such that $\{b_i, b_j, e_{ij}\}$ is a triangle of $M/\{b_1, \dots, b_{i-1}\}$. We need two observations:

$$\text{(1)} \quad e_{1k}, e_{2k}, \dots, e_{k-1,k} \notin \mathrm{cl}_M(B - b_k), \quad \text{for } k = 2, \dots, r.$$

$$\text{(2)} \quad \mathrm{cl}_M(\{b_1, \dots, b_i, b_k\}) = \mathrm{cl}_M(\{e_{1k}, \dots, e_{ik}, b_k\}), \quad \text{for } i < k.$$

To see that (1) holds, let i, k be given, with $1 \le i < k$. By definition of e_{ik} we have $e_{ik} \notin \mathrm{cl}_M(\{b_1, \dots, b_i\})$, but $e_{ik} \in \mathrm{cl}_M(\{b_1, \dots, b_i, b_k\})$. So the fundamental circuit of e_{ik} in M with respect to the basis B, must contain b_k. Hence, $e_{ik} \notin \mathrm{cl}_M(B - b_k)$.

We prove (2) by induction on i with k fixed. The case $i = 1$ is trivial, since $\{b_1, b_k, e_{1k}\}$ is a circuit in M. Suppose $1 < i < k$ and (2) holds for $i - 1$. Again, by definition of e_{ik} we have $e_{ik} \notin \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}, b_k\})$, but $e_{ik} \in \mathrm{cl}_M(\{b_1, \ldots, b_i, b_k\})$. Thus,

$$\begin{aligned}\mathrm{cl}_M(\{b_1, \ldots, b_i, b_k\}) &= \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}, b_k, e_{ik}\}) \\ &= \mathrm{cl}_M(\{e_{1k}, \ldots, e_{i-1,k}, b_k, e_{ik}\}),\end{aligned}$$

where the induction hypothesis is used in the second step.

Let $S = \{b_1, \ldots, b_n\}$ and $F = \mathrm{cl}_M(S)$, and for each $k = n + 1, \ldots, r$ define $C_k = \{e_{1k}, \ldots, e_{nk}, b_k\}$.

Notice that by (1), $C_k \cap F = \emptyset$ for all k, and $C_k \cap C_l = \emptyset$ for $k \neq l$. From (2) we gather, that C_k is independent with $F \subseteq \mathrm{cl}_M(C_k)$ for all k. Define $N = M|(F \cup C_{n+1} \cup \cdots \cup C_r)$. It is easily checked, that the C_k's are co-circuits of N. □

For $n = 2$, the concept of an n-storm is similar to that of a "book", used by Kung in [6], and an analogue of one of his ideas is part of the proof of the following result.

Lemma 7.4. *There exist integer-valued functions $\phi_1(n, a, b)$, $\phi_2(n, a, b)$ such that the following holds: Let $b > a \geq 1$ and let n be a positive integer. If M is a $\phi_1(n, a, b)$-storm with $\phi_2(n, a, b)$ clouds, and M has no $U_{a+1,b}$-minor, then M contains a rank-n Dowling clique N as a minor.*

Proof. Let n, a, b be given and let us define ϕ_1 and ϕ_2. First, let $l = na$. For $r = 1, \ldots, a$, let $s_r = \alpha_1(n, r, a, b)$ and let $s = \sum_{r=1}^{a} s_r$. Define a sequence of numbers $m_0, \ldots, m_s$ recursively as follows: let $m_s = l$, and

$$m_k = \sigma_2(a, m_{k+1}) \binom{b-1}{a}^{s-a} + 1, \quad \text{for } k = s - 1, \ldots, 1, 0.$$

Finally, let $\phi_1(n, a, b) = s$ and $\phi_2(n, a, b) = m_0$.

Let M be an s-storm with $m = m_0$ clouds. Denote the clouds by $C_1, \ldots, C_m$ and their elements $C_i = \{e_0^i, e_1^i, \ldots, e_s^i\}$.

Define $M' = M / \{e_0^1, e_0^2, \ldots, e_0^m\}$. So $r(M') = s$. Let $I_0 = \{1, \ldots, m\}$. We wish to find a subcollection of the clouds, such that elements with the same index lie on a uniform restriction of M'. We shall construct a sequence of subsets,

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_s, \quad \text{where } |I_k| = m_k,$$

such that for $k = 1, \ldots, s$, $M'|\{e_k^i : i \in I_k\} \simeq U_{r_k, m_k}$, for some number $r_k \in \{1, \ldots, a\}$. Let $k \geq 1$, suppose I_{k-1} has been defined and let us see how to find I_k. Let $W = \{e_k^i : i \in I_{k-1}\}$ and suppose $M'|W$ has no U_{r,m_k}-restriction for $r = 1, \ldots, a$. Lemmas 2.8 and 2.5 then give

$$m_{k-1} = |W| \leq \sigma_2(a, m_k)\tau_a(M'|W) \leq \sigma_2(a, m_k)\binom{b-1}{a}^{s-a},$$

contradicting our definition of m_{k-1}. So, take $U \subseteq W$ such that $M'|U$ is isomorphic to U_{r_k, m_k}, and let $I_k = \{i : e_k^i \in U\}$.

After possibly re-ordering the clouds in M, we may assume that $I_s = \{1, \ldots, l\}$. Let then $L_k = \{e_k^1, \ldots, e_k^l\}$ for $k = 1, \ldots, s$, so $M'|L_k \simeq U_{r_k, l}$. Now, by a majority argument, there exist $r \in \{1, \ldots, a\}$, and a subset $J \subseteq \{1, \ldots, s\}$ with $|J| \geq s_r$, such that $r_k = r$, for all $k \in J$. Define $\mathcal{L} = \{L_k : k \in J\}$. Since $C_1 - e_0^1$ is independent in M', we have

$$r_{M'}(\cup_{L \in \mathcal{L}} L) \geq |\mathcal{L}| = |J| \geq s_r = \alpha_1(n, r, a, b).$$

We now apply Lemma 5.1 and get a subcollection $\mathcal{L}' \subseteq \mathcal{L}$ of size n, such that $L \not\subseteq \mathrm{cl}_{M'}(\cup_{L' \in \mathcal{L}' - L} L')$, for each $L \in \mathcal{L}'$. After possibly permuting the elements of each cloud, we can assume $\mathcal{L}' = \{L_1, \ldots, L_n\}$. Let $D_i = \{e_0^i, e_1^i, \ldots, e_n^i\} \subseteq C_i$.

By our arrangement of the L_k's, we can find a set $B \subseteq \cup_{k=1}^n L_k$ independent in M', such that $r_{M'}(L_k \cup B) - r_{M'}(B) = 1$, for each $k \in \{1, \ldots, n\}$, and the sets $L_1 - B, \ldots, L_n - B$ are skew in M'/B.

Now $L_k \not\subseteq \mathrm{cl}_{M'}(B)$, and since $M'|L_k$ is rank-r uniform, we must have $|L_k \cap \mathrm{cl}_{M'}(B)| \leq r - 1$. Define $I_B \subseteq \{1, \ldots, l\}$ by

$$I_B = \{i : D_i \cap \mathrm{cl}_{M'}(B) \neq \emptyset\}.$$

Then $B \subseteq \cup_{i \in I_B} D_i$ and $|I_B| \leq n(r-1) \leq n(a-1) = l - n$. Notice, that for $i \notin I_B$, D_i and B are skew in M'. We may assume, again after re-ordering the clouds, that $\{1, \ldots, n\} \subseteq \{1, \ldots, l\} - I_B$. Let

$$M_1 = M/\left\{e_0^i : i \in I_B\right\}/B.$$

Then, by construction, the elements in $\{e_k^i : i = 1, \ldots, n\}$ are in parallel in $M_1/\{e_0^i : i = 1, \ldots, n\}$, for each $k = 1, \ldots, n$.

Let $p_k = e_k^n$, for $k = 1, \ldots, n$, and define

$$M_2 = M_1/e_0^n|(D_1 \cup \cdots \cup D_{n-1} \cup \{p_1, \ldots, p_n\}).$$

Now, M_2 is an n-storm with clouds $D_1, \ldots, D_{n-1}$. It satisfies, for each $i = 1, \ldots, n-1$, and each $k = 1, \ldots, n$, that p_k is on the line through e_0^i and e_k^i.

We shall make $\{p_1, \ldots, p_n\}$ the joints of a Dowling clique. Let

$$N = M_2/\left\{e_1^1, e_2^2, \ldots, e_{n-1}^{n-1}\right\}.$$

Then $\{p_1, \ldots, p_n\}$ is a basis for N. Let (i, j) be given with $1 \leq i < j \leq n$. In N, e_0^i and p_i are parallel, so $e_j^i \in \mathrm{cl}_N(\{e_0^i, p_j\}) = \mathrm{cl}_N(\{p_i, p_j\})$, and the set $\left\{p_i, p_j, e_j^i\right\}$ is a triangle in N. So, N has a rank-n Dowling clique restriction. □

8. Cliques

We need Mader's Theorem [7] to extract graphic cliques.

Mader's Theorem 8.1. *Let H be a graph. There exists $\lambda \in \mathbb{N}$ such that, if G is a simple graph with no H-minor, then $|E(G)| \leq \lambda |V(G)|$.*

An easy corollary is the following matroid version of the theorem. We take H to be a complete graph, and write the contrapositive statement.

Corollary 8.2. *There exists an integer-valued function $\theta(n)$ such that, if M is a graphic and simple matroid with $|E(M)| > \theta(n)r(M)$, then M has an $M(K_n)$-minor.*

Let M be a matroid and B a basis of M, and let $X = E(M) - B$. We call M a *Dowling matroid* with *joints* B if each $x \in X$ is on a triangle with two elements of B, and any two elements of B span at most one element of X (again, this is only a special case of Dowling's combinatorial geometries). By the *associated graph* of (M, B) we mean the graph G on the vertex set B with edge set labeled by X, such that $x \in X$ labels $\{b_1, b_2\}$ if x is on the line through b_1 and b_2 in M.

We shall use the following lemma to recognize graphic matroids.

Lemma 8.3. *Let M be a Dowling matroid with joints B, $X = E(M) - B$. If $B \cap \mathrm{cl}_M(X) = \emptyset$, then $M|X$ is graphic.*

Proof. Let G be the associated graph of (M, B). We claim that $M(G) = M|X$. It suffices to prove, that each circuit of $M(G)$ is dependent in $M|X$, and that each independent set of $M(G)$ is independent in $M|X$.

Let C be a cycle of G with vertex set B' and edge set X'. Clearly $X' \subseteq \mathrm{cl}_M(B')$ and by the assumption $B' \cap \mathrm{cl}_M(X') = \emptyset$. Since B' and X' have equal size, X' must be dependent in M.

Let T be a forest in G. We prove by induction on $|E(T)|$ that $E(T)$ is independent in M. Let e be a leaf edge in T and assume $E(T) - e$ is independent in M. Let b be a leaf of T incident on e. Then $E(T) - e \subseteq \mathrm{cl}_M(B - b)$, but $e \notin \mathrm{cl}_M(B - b)$, so $E(T)$ is independent in M. □

In a similar fashion, using Remark 1.4, the following lemma is easily proved.

Lemma 8.4. *Let M be a Dowling matroid with joints B, $X = E(M) - B$. Let G be the associated graph of (M, B). If for each cycle C in G, $E(C)$ is independent in M, then $M|X = B(G)$ (in fact $M = \tilde{B}(G)$).*

We are ready for the final step in the proof of the main theorem.

Lemma 8.5. *There exists an integer-valued function $\psi(n)$ such that, if M is a Dowling clique with rank at least $\psi(n)$, then M contains an $M(K_n)$- or $B(K_n)$-minor.*

Proof. Let n be given, and define $\psi(n) = nl$, where $l = 2m\theta(n) + 1$ and $m = 2^n n!$. Let M be a Dowling clique and assume that $r(M) = nl$. Denote by B the joints of M and let $X = E(M) - B$.

Partition B into n sets, $B_1, \ldots, B_n$ of equal size, $|B_i| = l$. We shall contract each B_i to a point. Let $M_i = M|\,\mathrm{cl}_M(B_i)$. Choose $Y_i \subseteq E(M_i) \cap X$ such that B_i is a set of parallel elements in M_i/Y_i (take the edges of a spanning tree in the associated graph of (M_i, B_i)). Define $M' = M/(Y_1 \cup \cdots \cup Y_n)$ and pick a $b_i \in B_i$ for $i = 1, \ldots, n$. For each pair $i < j$, define

$$X_{ij} = \{x \in X : x \in \mathrm{cl}_M(b, d),\ b \in B_i,\ d \in B_j\}.$$

Note, that for each $x \in X_{ij}$, $\{b_i, b_j, x\}$ is a triangle in M'. We consider two cases.

(1). $\tau_1(M'|X_{ij}) > m$, for all pairs $i < j$. Put $B' = \{b_1, \ldots, b_n\}$. We shall choose a set $X' = \{x_{ij} : 1 \le i < j \le n\}$, where $x_{ij} \in X_{ij}$, such that $M'|(B' \cup X')$ is the Dowling clique $\tilde{B}(K_n)$. Let $X' \subseteq \cup X_{ij}$ be maximal, such that $|X' \cap X_{ij}| \le 1$ for all $i < j$, and the cycles in the associated graph of $M'|(V' \cup X')$ all have edge sets independent in M'. We claim, that $X' \cap X_{ij} \neq \emptyset$ for all $i < j$, and thus $M'|(V' \cup X') \simeq \tilde{B}(K_n)$ by Lemma 8.4.

Assume that $X' \cap X_{ij} = \emptyset$ for some $i < j$. Let G be the associated graph of $M'|(V' \cup X')$. If $Z \subseteq X'$ is the edge set of a path from b_i to b_j in G, then $r_{M'}(\mathrm{cl}_{M'}(Z) \cap X_{ij}) \le 1$. There can be at most m such Z, since a simple graph on n vertices has no more than $2^n n! = m$ cycles. Thus, we can pick $x_{ij} \in X_{ij}$ skew from each such Z. So, the cycles created in the associated graph, when adding x_{ij} to X' all have edge sets independent in M', contradicting the maximality of X'.

(2). $\tau_1(M'|X_{ij}) \le m$, for some pair $i < j$. As $|X_{ij}| = l^2$, there is a parallel class $P \subseteq X_{ij}$ of M', with $|P| \ge l^2/m$. Now, since $B \cap \mathrm{cl}_{M'}(P) = \emptyset$, also $B \cap \mathrm{cl}_M(P) = \emptyset$, and Lemma 8.3 gives, that $M|P$ is graphic. And $r(M|P) \le |B_i \cup B_j| = 2l$. We have then

$$|E(M|P)| \ge l^2/m > l2\theta(n) \ge \theta(n)r(M|P),$$

and by Corollary 8.2, we get an $M(K_n)$-minor. □

Finally, we restate and prove Theorem 1.1.

Theorem 8.6. *There exists an integer-valued function $\gamma(k,a,n)$ such that, if M is a matroid with $r(M) \ge \gamma(k,a,n)$, then either M has k disjoint co-circuits or M has a minor isomorphic to $U_{a,2a}$, $M(K_n)$ or $B(K_n)$.*

Proof. Let k,a,n be positive integers. If $a = 1$, then we let $\gamma(k,a,n) = k$. If $a \ge 2$, then we define the following numbers: Put $a' = a-1$ and $b = 2a$. Let $k = \psi(n)$ and let $m_1 = \phi_1(k,a',b)$ and $m_2 = \phi_2(k,a',b)$. Let $r = n+m$ and define $g : \mathbb{N} \to \mathbb{N}$ by $g(t) = \nu(r,t,a',b)$. Finally, $\gamma(k,a,n) = f_g(k)$.

Let M be given with $r(M) \ge \gamma(k,a,n)$. If $a = 1$, the result is trivial, since in a matroid with no $U_{1,2}$-minor, every element is a loop or a co-loop.

If $a \ge 2$, then by Lemma 3.2, either M has k disjoint co-circuits or a minor N with $r(N) \ge g(\Gamma(N))$. Assume the second case. Also, if N has a $U_{a'+1,b}$-minor we are done, so assume this is not the case. Applying Lemmas 6.2, 7.3, 7.4 and 8.5 in succession, we obtain an $M(K_n)$- or a $B(K_n)$-minor of N. □

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