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Abstract. Let B be a Borel subgroup in a reductive algebraic group G over a field k. We study the cohomology $H^\bullet(B,\lambda)$ of 1-dimensional B-modules λ. When $\operatorname{char} k = 0$ there is an easy a well-known description of this cohomology whereas the corresponding problem in characteristic $p > 0$ is wide open. We develop some new techniques which enable us to calculate all such cohomology in degrees at most 3 when p is larger than the Coxeter number for G. Our methods also apply to the corresponding question for quantum groups at roots of unity.

1. Introduction

Let G be a reductive algebraic group over an algebraically closed field k and denote by B a Borel subgroup in G. In this paper we shall study the B-cohomology $H^\bullet(B,-) = \operatorname{Ext}^\bullet_B(k,-)$, i.e. the derived functors of the B-fixed point functor $H^0(B,-)$.

We are especially interested in the B-cohomology of simple (i.e. 1-dimensional) B-modules. If T is a maximal torus contained in B then $B = TU$ where U is the unipotent radical of B. Any character $\lambda \in X(T)$ of T extends uniquely to B (by $\lambda(U) = 1$). The corresponding 1-dimensional B-module is also denoted λ or sometimes k_λ. In particular, the trivial B-module k may also be written k_0.

We want to compute $H^\bullet(B,\lambda)$ for each $\lambda \in X(T)$. When $\operatorname{char} k = 0$ this is easy because we can compare with the corresponding G-cohomology and take advantage of the fact that $H^i(G,-) = 0$ for all $i > 0$ (G is reductive). Moreover, the Borel-Weil-Bott theorem describes completely the cohomology $H^\bullet(G/B,\lambda)$, see (2.3) below. But when $\operatorname{char} k = p > 0$ this approach fails completely: There are non-vanishing higher G-cohomology and the Borel-Weil-Bott theorem is no longer true. In fact, the problem of determining $H^\bullet(B,\lambda)$ is in this case wide open. Our contribution in this paper is to give a couple of general results on the behaviour of $H^\bullet(B,\lambda)$ and to calculate $H^2(B,\lambda)$ and $H^3(B,\lambda)$ explicitly (for p larger than the Coxeter number for G).

Our results are based on a combination of several methods, see Section 3 below. The main ingredient is the spectral sequence relating B-cohomology to the cohomology for the first Frobenius kernel B_1 of B. We take advantage of the fact that the cohomology $H^\bullet(B_1,\lambda)$ was completely determined in [5].

Our approach works equally well for quantum groups. Let U_q denote the quantum group corresponding to G with parameter $q \in k^*$ and let B_q be a Borel

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subalgebra in U_q. When q is not a root of unity we can determine $H^\bullet(B_q, \lambda)$ exactly as in the above characteristic 0 case. So we consider the case where q is an l-th root of unity. Then the problem of describing $H^\bullet(B_q, \lambda)$ is again wide open in general. But our methods allow us to obtain similar results as described above for B.

2. Known results

2.1. **Notation.** First we fix some notation used throughout this paper. We have already introduced G, B, T and U above. We set $X = X(T)$, the character group of T (and of B). Then we denote by $R \subset X$ the root system for (G, T) and we fix a set of positive roots $R^+ \subset R$ by requiring that the roots of B are in $-R^+$. The positive roots induce an ordering on X given by $\lambda \geq \mu$ if and only if $\lambda - \mu$ can be written as a sum of positive roots.

We let S denote the corresponding set of simple roots and W will be the Weyl group. A weight $\lambda \in X$ is called dominant if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. Here $\alpha^\vee$ denotes the coroot of α. For each root $\alpha \in R$ we let s_α denote the reflection associated to α. In addition to the usual action of W on X where $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for all $\alpha \in R, \lambda \in X$ we also consider the so-called 'dot action' given by $w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W, \lambda \in X$. As usual ρ denotes half the sum of the positive roots.

The set of dominant weights X^+ parametrizes the simple modules of G via highest weight. For $\lambda \in X^+$ we let $L(\lambda)$ be the simple G-module of highest weight λ. All modules we consider in this paper will be finite dimensional.

Finally, let h be the Coxeter number for G and let $\mathrm{ht} : X \to \mathbb{Z}$ denote the height function on X which takes value 1 on all simple roots.

2.2. **Characteristic zero.** If M is a B-module then we have a spectral sequence

$$H^r(G, H^s(G/B, M)) \implies H^{r+s}(B, M). \tag{2.1}$$

Here $H^\bullet(G/B, -)$ denotes the derived functors of induction from B to G or alternatively $H^\bullet(G/B, M)$ is the coherent sheaf cohomology of the vector bundle associated to M.

When $\operatorname{char} k = 0$ we have $H^r(G, -) = 0$ for all $r > 0$ because G is reductive. Hence the above spectral sequence degenerates and gives us isomorphisms of B-modules

$$H^r(B, M) \simeq H^0(G, H^r(G/B, M)) \text{ for all } r \geq 0. \tag{2.2}$$

Now, suppose M is the 1-dimensional B-module determined by $\lambda \in X$. If we choose $w \in W$ such that $w(\lambda + \rho) \in X^+$ then the Borel-Weil-Bott theorem [9] (cf. also [11]) says

$$H^r(G/B, \lambda) \simeq \begin{cases} H^0(G/B, w \cdot \lambda) & \text{if } r = l(w), \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Here $l(w)$ denotes the length of w. Since the only dominant weight μ for which there is a non-trivial G-fixed point in $H^0(G/B, \mu)$ is $\mu = 0$ we conclude that (cf [3] Proposition 2.1)

$$H^r(B,\lambda) \simeq \begin{cases} k & \text{if } \lambda = w\cdot 0 \text{ for some } w \in W \text{ with } l(w) = r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

2.3. **Characteristic p.** For the rest of this section we assume char $k = p > 0$. As mentioned in the introduction the determination of $H^\bullet(B,\lambda)$ is wide open in this case. In the following subsections we summarize some known results.

2.4. **The linkage principle.** The strong linkage principle [2] implies that all composition factors of $H^r(G/B,\lambda)$ have highest weights in $W\cdot\lambda + p\mathbb{Z}R$. Moreover, it also gives that for each simple G-module $L(\mu)$ we have $H^\bullet(G, L(\mu)) = 0$ unless $\mu \in W\cdot 0 + p\mathbb{Z}R$. Hence the spectral sequence (2.1) shows that

$$H^\bullet(B,\lambda) = 0 \text{ unless } \lambda \in W\cdot 0 + p\mathbb{Z}R. \tag{2.5}$$

Remark 2.1. As observed in [3] the strong linkage principle implies also that we have the following characteristic p-analogue of (2.4)

$$H^r(B, w\cdot 0) \simeq \begin{cases} k & \text{if } r = l(w), \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

2.5. Let $k[U]$ denote the coordinate ring of U. Tensoring the 'standard' injective resolution

$$k \to k[U] \to k[U]\otimes k[U] \to \cdots$$

of the trivial B-module k by a weight $\lambda \in X$ gives

$$H^\bullet(B,\lambda) = 0 \text{ unless } \lambda \leq 0. \tag{2.7}$$

In fact, the weights of each term in the resulting resolution of the B-module λ has weights $\leq \lambda$. Hence there are no T-fixed points (and so certainly no B-fixed points either) unless $\lambda \leq 0$.

Remark 2.2. A little more careful argument (see e.g. [10], Lemma 2.3) shows that in fact we have

$$H^i(B,\lambda) = 0 \text{ unless } \lambda \leq 0 \text{ and } i \leq -\operatorname{ht}(\lambda). \tag{2.8}$$

2.6. The first cohomology group. It is clear that $H^0(B,k) = k$ and that $H^0(B,\lambda) = 0$ for all $\lambda \neq 0$. The first cohomology group $H^1(B,\lambda)$ is also completely known, see [3]

$$H^1(B,\lambda) \simeq \begin{cases} k & \text{if } \lambda = -p^r\alpha \text{ for some } \alpha \in S, r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

This may be deduced from the spectral sequence (2.1) by using that the G-socle of $H^1(\lambda) = H^1(G/B,\lambda)$ is known, see [1]. In particular, $H^0(G, H^1(G/B,\lambda)) = 0$ unless $\lambda = -p^r\alpha$ for some $\alpha \in S, r \geq 0$.

2.7. The second cohomology group. One of the main results in [8] is a complete description of $H^2(B,\lambda)$. When $p > h$ we shall recover this result below (see Section 5) so we do not give the statement here. One of the features is that for any λ its second B-cohomology group is at most 1-dimensional (as was the case for H^1, see 2.6).

We emphasize that [8] describes $H^2(B,\lambda)$ for all p whereas we focus in this paper only on the case $p > h$.

2.8. SL_2 and SL_3. The only Borel subgroup B for which the full story about $H^\bullet(B,\lambda)$ is known is the Borel subgroup of SL_2. Since (in general) U is normal in B and T is reductive we have $H^i(B,\lambda) = H^i(U,k)_{-\lambda}$. Now, when U is 1-dimensional the cohomology $H^\bullet(U,k)$ is completely described in [13].

In the SL_3 case the cohomology $H^\bullet(B_q,\lambda)$ was calculated in [4]. Here B_q (cf. Section 7 below) denotes the Borel subalgebra of the quantum group corresponding to SL_3 and q is assumed to be a complex root of unity of odd order at least 3. Many of the calculations for this case can easily be carried over to the characteristic p situation giving a start for the determination of B-cohomology for $B < SL_3(k)$.

3. Methods

3.1. In this section we continue to assume that $\operatorname{char} k = p > 0$. Even though the spectral sequence (2.1) is not so effective in characteristic p it has the following very useful variant.

Note that we may replace G by any parabolic subgroup P containing B. In particular, we shall explore the case where $P = P_\alpha$ is the minimal parabolic subgroup corresponding to $\alpha \in S$. Writing $H^i_\alpha(-)$ short for $H^i_\alpha(P_\alpha/B, -)$ we get in this way for all $i \geq 0$

$$H^i(B,\lambda) \simeq H^i(P_\alpha, H^0_\alpha(\lambda)) \text{ if } \langle\lambda,\alpha^\vee\rangle \geq 0, \tag{3.1}$$

$$H^{i+1}(B,\lambda) \simeq H^i(P_\alpha, H^1_\alpha(\lambda)) \text{ if } \langle\lambda,\alpha^\vee\rangle \leq -2, \tag{3.2}$$

$$H^i(B,\lambda) = 0 \text{ if } \langle\lambda,\alpha^\vee\rangle = -1. \tag{3.3}$$

Note also that $H^i(P_\alpha, M) \simeq H^i(B, M)$ for all i when M is a P_α-module (this follows from the same spectral sequence argument by observing that for such M we have $H^0_\alpha(M) \simeq M$ and $H^1_\alpha(M) = 0$).

Recall that when $0 \leq \langle\lambda,\alpha^\vee\rangle < p$ then $H^0_\alpha(\lambda) \simeq H^1_\alpha(s_\alpha \cdot \lambda)$. Using this together with (3.1) and (3.2) we get that for all $i \geq 0$

$$H^i(B,\lambda) \simeq H^{i+1}(B, s_\alpha \cdot \lambda) \text{ whenever } 0 \leq \langle\lambda,\alpha^\vee\rangle < p. \tag{3.4}$$

3.2. Let B_1 denote the first Frobenius kernel in B. This means that B_1 is the subgroup scheme obtained as the kernel of the Frobenius homomorphism F on B. When M is a B-module we denote by $M^{(1)}$ the Frobenius twist of M, i.e. the same vector space M but with action composed with F. Similarly, if N is a B-module whose restriction to B_1 is trivial then $N^{(-1)}$ is the B-module such that $(N^{(-1)})^{(1)} = N$.

We have then for each B-module M the Lyndon-Hochschild-Serre spectral sequence

$$H^r(B, H^s(B_1, M)^{(-1)}) \implies H^{r+s}(B, M). \tag{3.5}$$

3.3. Consider now the case where $M = \lambda$ for some $\lambda \in X$. If p is larger than h then the cohomology $H^\bullet(B_1, \lambda)$ is completely known for all $\lambda \in X$. By (2.5), we need only consider λ's of the form $\lambda = w \cdot 0 + p\mu$ for some $w \in W$ and $\mu \in X$. Then we have (see [5])

$$H^r(B_1, w \cdot 0 + p\mu)^{(-1)} \simeq S^{(r-l(w))/2}(u^*) \otimes \mu. \tag{3.6}$$

Here u^* denotes the dual of the Lie algebra $u = \mathrm{Lie}(U)$ with the adjoint B-action, S^r denotes the r-symmetric power, and we interpret S^r to be 0 whenever $r \notin \mathbb{N}$.

3.4. When we combine (3.6) and the spectral sequence (3.5) we obtain (cf. [4] Theorem 4.3.ii)

Proposition 3.1. *Suppose $p > h$. Let $w \in W$, $\mu \in X$. Then we have for all i*

$$H^i(B, w \cdot 0 + p\mu) \simeq H^{i-l(w)}(B, p\mu).$$

This result reduces the problem of computing $H^\bullet(B, \lambda)$ to the case where $\lambda \in pX$.

Note also that this proposition reproves Remark 2.1 when $p > h$.

3.5. In order to effectively take advantage of the spectral sequence (3.5) we need by (3.6) to determine the B-cohomology of $S^n u^* \otimes \lambda$ for $\lambda \in X$. This we don't know how to do in general but the following short exact sequence will allow us to settle some cases.

Let $\alpha \in S$. Note that the line of weight α in u^* is a B-submodule and that the quotient $V_\alpha = u^*/\alpha$ is a P_α-module. This leads to an exact sequence of B-modules for each $n > 0$

$$0 \to S^{n-1}u^* \otimes \alpha \to S^n u^* \to S^n V_\alpha \to 0. \tag{3.7}$$

Tensoring by a weight $\lambda \in X$ we get

$$0 \to S^{n-1}u^* \otimes (\alpha + \lambda) \to S^n u^* \otimes \lambda \to S^n V_\alpha \otimes \lambda \to 0. \tag{3.8}$$

This gives $H^i(B, S^n u^* \otimes \lambda) = 0$ unless $H^i(B, S^{n-1}u^* \otimes (\lambda + \alpha)) \neq 0$ or $H^i(B, S^n V_\alpha \otimes \lambda) \neq 0$.

As an easy consequence of (3.4) we get that if λ satisfies $0 \leq \langle \lambda, \alpha^\vee \rangle < p$ then we have for all i, n

$$H^i(B, S^n V_\alpha \otimes \lambda) \simeq H^{i+1}(B, S^n V_\alpha \otimes s_\alpha \cdot \lambda). \tag{3.9}$$

Lemma 3.2. *Suppose $p > h$ and let $\lambda \in X$. Then we have*

$$H^0(B, V_\alpha \otimes -\lambda) = 0 \textit{ unless } \lambda \in \{R^+ \setminus \{\alpha\} \mid \lambda - \alpha \notin R^+\}.$$

Proof: Let L_α denote the Levi subgroup of P_α. Since V_α is a L_α-module and $p > h$ we get from the linkage principle that $V_\alpha \simeq \oplus L_\alpha(\gamma)$ as L_α-modules. Here γ runs through those roots in $R^+ \setminus \{\alpha\}$ for which $\gamma + \alpha \notin R^+$ and $L_\alpha(\gamma)$ denotes the simple L_α-module of highest weight γ. Note that if $B_\alpha = B \cap L_\alpha$ then $H^0(B_\alpha, L_\alpha(\gamma) \otimes -\lambda) = 0$ unless $\lambda = s_\alpha(\gamma)$. Then the lemma follows.

4. B-cohomology of $S^n u^* \otimes \lambda$

In this and the following two sections we assume that $\operatorname{char} k = p > 0$.

As mentioned before, in order to calculate $H^2(B, \lambda)$ and $H^3(B, \lambda)$ explicitly, we need to compute some low degree cohomology of $S^n u^* \otimes \lambda$. This is what we do in this section.

4.1. Degree zero.

Proposition 4.1. *Fix $n \in \mathbb{N}$ and $\lambda \in X$. Then*

$$H^0(B, S^n u^* \otimes \lambda) \simeq \begin{cases} k & \text{if } n = -\operatorname{ht}(\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Since the weights of $S^n u^*$ are all ≥ 0 we can apply (2.7) to conclude that $H^0(B, S^n u^* \otimes \lambda) = 0$ unless $\lambda \leq 0$. So we may assume λ is not dominant. Choose then $\alpha \in S$ such that $\langle \lambda, \alpha^\vee \rangle < 0$. The exact sequence (3.8) gives

$$H^0(B, S^n u^* \otimes \lambda) \simeq H^0(S^{n-1} u^* \otimes (\alpha + \lambda)).$$

Now an easy induction on n proves the proposition.

Remark 4.2. Proposition 4.1 remains true when $\operatorname{char} k = 0$.

4.2. Degree 1. First note that for each $\alpha, \beta \in S$ we have

$$\alpha + \beta \in R^+ \text{ if and only if } \langle \beta, \alpha^\vee \rangle < 0.$$

Proposition 4.3. *Assume $p > h$ and let $\lambda \in X$. Then*

$$H^1(B, u^* \otimes \lambda) \simeq \begin{cases} k & \text{if } \lambda = -\beta - p^n\alpha \text{ for } \alpha, \beta \in S \text{ and } n > 0, \\ k & \text{if } \lambda = -\beta - \alpha \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle < 0, \\ k & \text{if } \lambda = -2\alpha \text{ for } \alpha \in S, \\ k^2 & \text{if } \lambda = -\beta - \alpha \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle = 0, \\ k & \text{if } \lambda = s_\alpha s_\beta \cdot 0 \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We begin by checking each of the first five cases where the proposition claims that the cohomology is non-zero.

So consider first the case where $\lambda = -\beta - p^n\alpha$ for some $\alpha, \beta \in S$, $n > 0$. We have the following exact sequence

$$0 \to (\beta + \lambda) \to u^* \otimes \lambda \to V_\beta \otimes \lambda \to 0. \tag{4.1}$$

We note that $-\lambda$ is not a weight of V_β and that no weights of $V_\beta \otimes \lambda$ have the form $-p^m\gamma$ with $\gamma \in S$, $m \geq 0$. Hence (using (2.9)) we have

$$H^0(B, V_\beta \otimes \lambda) = H^1(B, V_\beta \otimes \lambda) = 0.$$

This together with the long exact sequence arising from (4.1) give

$$H^1(B, u^* \otimes \lambda) \simeq H^1(B, -p^n\alpha) \simeq k.$$

Consider now $\lambda = -\beta - \alpha$ for some $\alpha, \beta \in S$ with $\alpha + \beta \in R^+$. In this case we still have that $H^0(B, V_\beta \otimes \lambda) = 0$, see Lemma 3.2. We claim that $H^1(B, V_\beta \otimes -\alpha - \beta) = 0$. To see this we consider the sequence

$$0 \to \alpha \to V_\beta \to Q \to 0. \tag{4.2}$$

Noting that $\alpha + \beta$ is a minimal weight of Q (with multiplicity 1) it follows immediately that $H^0(B, Q\otimes(-\beta-\alpha)) \simeq k$. No weights of $Q\otimes(-\beta-\alpha)$ have the form $-p^m\mu$ with $\mu \in S,\ m \geq 0$. Therefore we get $H^1(B, Q\otimes(-\beta-\alpha)) = 0$ and then the long exact sequence coming from (4.2) gives $H^1(B, V_\beta \otimes -\beta - \alpha) = 0$. Combining this claim with the exact sequence (3.8) we get

$$H^1(B, u^* \otimes (-\beta - \alpha)) \simeq H^1(B, -\alpha) \simeq k.$$

Next consider $\lambda = -\beta - \alpha$ for some $\alpha, \beta \in S$ with $\alpha + \beta \notin R^+$. Arguing as before we have $H^0(B, V_\beta \otimes -\alpha - \beta) = 0$, but this time we have also $H^0(B, Q \otimes (-\beta-\alpha)) = 0$. Note that if $\beta = \alpha$ then 2α is not a weight of V_α. In this case we get $H^1(B, V_\alpha \otimes -2\alpha) = 0$. Weight considerations as before imply that if $\alpha \neq \beta$ then $H^1(B, V_\beta \otimes -\alpha - \beta) \simeq k$. Inserting in the long exact sequence arising from (3.8) we get the desired conclusions because $H^2(B, -\alpha) \simeq H^1(B, k_0) = 0$.

Finally, consider $\lambda = s_\alpha s_\beta \cdot 0$ for some $\alpha, \beta \in S$ with $\langle \beta, \alpha^\vee \rangle < 0$. Then $\langle \lambda, \alpha^\vee \rangle = \langle \beta, \alpha^\vee \rangle - 2 < 0$. By (3.9), the sequence (3.8) then gives

$$H^1(B, u^* \otimes \lambda) \simeq H^1(B, V_\alpha \otimes \lambda) \simeq H^0(B, V_\alpha \otimes s_\alpha \cdot \lambda)$$

because $H^1(B, \lambda + \alpha) = H^2(B, \lambda + \alpha) = 0$. Since $s_\alpha \cdot \lambda = -\beta$ we have $H^0(B, V_\alpha \otimes s_\alpha \cdot \lambda) \simeq k$ and we have thus checked the last of the non-vanishing cases.

Assume therefore now that $H^1(B, u^* \otimes \lambda) \neq 0$ for some $\lambda \in X$. To finish the proof we need to show that we are then in one of the above five cases.

Weight considerations show via (2.9) that $\lambda = -\beta - p^n\alpha$ for some $\beta \in R^+,\ \alpha \in S,\ n \geq 0$. We claim that if $n > 0$ then $\beta \in S$ (i.e. we are in the first case listed in the proposition). If $\beta \notin S$ then (2.9) gives $H^1(B, \lambda + \alpha) = 0$ and hence the sequence (3.7) implies

$$H^1(B, u^* \otimes \lambda) \subseteq H^1(B, V_\alpha \otimes \lambda) \simeq H^0(B, V_\alpha \otimes H^1_\alpha(\lambda)).$$

Here the claimed isomorphism comes from the fact that $\langle \lambda, \alpha^\vee \rangle = -\langle \beta, \alpha^\vee \rangle - 2p^n < 0$. On the other hand, if we tensor the sequence (3.7) by $H^1_\alpha(\lambda)$ we get the sequence

$$\to H^0(B, u^* \otimes H^1_\alpha(\lambda)) \to H^0(B, V_\alpha \otimes H^1_\alpha(\lambda)) \to H^1(B, \alpha \otimes H^1_\alpha(\lambda)).$$

Recall that the weights of $H^1_\alpha(\lambda)$ are $\lambda + \alpha, \ldots, s_\alpha \cdot \lambda$. Therefore, if μ is a weight of $H^1_\alpha(\lambda)$ then $H^0(B, u^* \otimes \mu) = 0$ unless $\beta \in S$, see Proposition 4.1. Also $H^1(B, \alpha + \mu) = 0$ unless $\beta \in S$. This proves the claim.

On the other hand, if $n = 0$ then we claim that we are in one of the remaining four cases. Since $-\beta - \alpha \notin X^+$ we may choose $\gamma \in S$ such that $\langle \lambda, \gamma^\vee \rangle < 0$.

As $\langle \lambda, \gamma^\vee \rangle > -p$ we get from (3.9)

$$H^1(B, V_\gamma \otimes \lambda) \simeq H^0(B, V_\gamma \otimes s_\gamma \cdot \lambda). \tag{4.3}$$

Using our assumption that $H^1(B, u^* \otimes \lambda) \neq 0$, the sequence (4.1) relative to γ gives that either $H^1(B, \lambda + \gamma) \neq 0$ or $H^1(B, V_\gamma \otimes \lambda) \neq 0$.

Suppose first that $H^1(B, \lambda + \gamma) \neq 0$. Then $\lambda = -\gamma - p^m \delta$ for some $\delta \in S$, $m \geq 0$. Since $\lambda = -\beta - \alpha$ we have $m = 0$ and $\beta \in \{\gamma, \delta\} \subseteq S$. This means that we are in one of the cases 2, 3 or 4 on the list.

Suppose $H^1(B, V_\gamma \otimes \lambda) \neq 0$. By (4.3), we get $H^0(B, V_\gamma \otimes s_\gamma \cdot \lambda) \neq 0$. Then the sequence

$$H^0(B, u^* \otimes s_\gamma \cdot \lambda) \to H^0(B, V_\gamma \otimes s_\gamma \cdot \lambda) \to H^1(B, \gamma + s_\gamma \cdot \lambda)$$

gives either $H^0(B, u^* \otimes s_\gamma \cdot \lambda) \neq 0$ or $H^1(B, \gamma + s_\gamma \cdot \lambda) \neq 0$. This means that either $s_\gamma \cdot \lambda = -\delta$ or $\gamma + s_\gamma \cdot \lambda = -p^m \delta$ for some $\delta \in S$, $m \geq 0$. The first possibility means that $\lambda = s_\gamma \cdot (-\delta) = s_\gamma s_\delta \cdot 0$, i.e. we are in case 4 or 5 on our list. The second possibility can only occur with $m = 0$ and then $s_\gamma \cdot \lambda = -\gamma - \delta$. But in that case

$$H^0(B, V_\gamma \otimes s_\gamma \cdot \lambda) = H^0(B, V_\gamma \otimes -\delta - \gamma)$$

and this is 0 according to Lemma 3.2. This completes the proof.

The same arguments as in Proposition 4.3 give

Proposition 4.4. *Let $\lambda \in X$. If $\operatorname{char} k = 0$ then*

$$H^1(B, u^* \otimes \lambda) \simeq \begin{cases} k & \text{if } \lambda = -2\alpha \text{ for } \alpha \in S, \\ k & \text{if } \lambda = -\beta - \alpha \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle < 0, \\ k^2 & \text{if } \lambda = -\beta - \alpha \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle = 0, \\ k & \text{if } \lambda = s_\alpha s_\beta \cdot 0 \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle < 0, \\ 0 & \text{otherwise.} \end{cases}$$

5. $H^\bullet(B, \lambda)$ in degrees 2 and 3

In this section we assume $p > h$ and then compute $H^2(B, \lambda)$ and $H^3(B, \lambda)$ for all $\lambda \in X$.

5.1. Degree 2.

Theorem 5.1. *Let $\lambda \in X$. Then*

$$H^2(B, \lambda) \simeq \begin{cases} k & \text{if } \lambda = p^n(-\alpha) \text{ for } \alpha \in S \text{ and } n > 0, \\ k & \text{if } \lambda = p^n(w \cdot 0) \text{ for } w \in W \text{ with } l(w) = 2, n \geq 0, \\ k & \text{if } \lambda = p^n(-\alpha - p^m \beta) \text{ for } \alpha, \beta \in S, n \geq 0, m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: If $\lambda \notin pX$ then we use Proposition 3.1 to reduce to a lower degree cohomology group. These are described in Section 2. So suppose $\lambda = p\mu$ for some $\mu \in X$. We then use the spectral sequence (3.5) to compute $H^2(B, \lambda)$. By (3.6), there are only two E_2-terms that may contribute, namely $H^2(B, \mu)$ and $H^0(B, u^* \otimes \mu)$. If $\mu \in -S$ then the first of these terms vanishes (by Proposition 3.1) whereas the second equals k. Hence $H^2(B, -p\alpha) = k$ for all $\alpha \in S$.

On the other hand, if $\mu \notin -S$ then we have that that the second term vanishes (according to Proposition 4.1) and $H^2(B,\lambda) \simeq H^2(B,\mu)$. We repeat this argument if $\mu \in pX$ (note that this gives $H^2(B,p\mu) \simeq H^2(B,p^2\mu) \simeq \cdots \simeq H^2(B,p^n\mu)$ for all $\mu \in X$ and all $n > 0$). Otherwise, $H^2(B,\mu)$ identifies with a lower degree cohomology group as before. It is now a matter of bookkeeping to see that this leads to the statement in the theorem.

5.2. Degree 3.

Theorem 5.2. *Let $\lambda \in X$. Then*

$$H^3(B,\lambda) \simeq \begin{cases} k & \text{if } \lambda = p^n(-2\alpha) \text{ for } \alpha \in S \text{ and } n > 0, \\ k^2 & \text{if } \lambda = p^n(-\beta - p^m\alpha) \text{ for } \alpha,\beta \in S \text{ and } n,m > 0, \\ k & \text{if } \lambda = p^n(-\beta-\alpha) \text{ for } \alpha,\beta \in S \text{ with} \\ & \langle \beta,\alpha^\vee\rangle < 0 \text{ and } n > 0, \\ k^2 & \text{if } \lambda = p^n(-\beta-\alpha) \text{ for } \alpha,\beta \in S \text{ with} \\ & \langle \beta,\alpha^\vee\rangle = 0 \text{ and } n > 0, \\ k & \text{if } \lambda = p^n(s_\alpha s_\beta \cdot 0) \text{ for } \alpha,\beta \in S \text{ with} \\ & \langle \beta,\alpha^\vee\rangle < 0 \text{ and } n > 0, \\ k & \text{if } \lambda = p^n(w \cdot 0) \text{ for } w \in W \text{ with} \\ & l(w) = 3 \text{ and } n \geq 0, \\ k & \text{if } \lambda = p^n(w\cdot 0 - p^m\alpha) \text{ for } \alpha \in S,\ w \in W \text{ with} \\ & l(w) = 2 \text{ and } n \geq 0, m > 0, \\ k & \text{if } \lambda = p^n(p^m w \cdot 0 - \alpha) \text{ for } \alpha \in S,\ w \in W \text{ with} \\ & l(w) = 2 \text{ and } n \geq 0, m > 0, \\ k & \text{if } \lambda = -\beta - p^n\alpha \text{ for } \alpha,\beta \in S, n > 0, \\ k & \text{if } \lambda = p^n(-\alpha - p^m\beta - p^l\gamma) \text{ for } \alpha,\beta,\gamma \in S \text{ and} \\ & n \geq 0, m > l > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Suppose that $\lambda = p\mu$ for some $\mu \in X$. Consider the spectral sequence (3.5). The only E_2-terms that contribute to $H^3(B,\lambda)$ are $H^3(B,\mu)$ and $H^1(B,u^* \otimes \mu)$. The latter vanishes if $\mu \in pX$. Hence we get $H^3(B,p\mu) \simeq H^3(B,p^2\mu) \simeq \cdots \simeq H^3(B,p^n\mu)$ for all $\mu \in X$ and all $n > 0$.

For those μ listed in Proposition 4.3, we have that unless $\mu = -\beta - p^n\alpha$ for some $\alpha,\beta \in S, n > 0$ then $H^3(B,\mu) = 0$ and hence $H^3(B,p\mu) \simeq H^1(B,u^*\otimes\mu)$.

Suppose now that $\mu = -\beta - p^n\alpha$ with $\alpha,\beta \in S$ and $n > 0$. Proposition 4.3 and Theorem 5.1 (combined with Proposition 3.1) yield that both of the above terms equal k. In this situation we have an exact sequence

$$0 \to H^3(B,\mu) \to H^3(B,p\mu) \to H^1(B,u^* \otimes \mu) \to 0$$

i.e. we have $H^3(B,p\mu) \simeq k^2$.

On the other hand, if μ is not one of those weights listed in Proposition 4.3 then the second term vanishes. In this case we have $H^3(B,p\mu) \simeq H^3(B,\mu)$. Arguing as in Theorem 5.1 the stated results follow.

6. Upper bound

In this section we determine for each $\lambda \in X$ an upper bound i for the degree in which the cohomology $H^i(B,\lambda)$ can be non-zero.

We consider first the case where $\lambda \in pX$.

Proposition 6.1. *Let $\lambda \in X$. Then*

$$H^i(B, p\lambda) = 0 \text{ for } i > -2\,\mathrm{ht}(\lambda).$$

Proof: We have $H^\bullet(B, p\lambda) = 0$ unless $\lambda \leq 0$. In particular, we may assume that $\mathrm{ht}(\lambda) \leq 0$. We then proceed by induction on $n = -2\,\mathrm{ht}(\lambda)$. If $n = 0$ we have $\lambda = 0$. In this case the claim is clearly true, see 2.5-6.

Suppose now that $i > n > 0$. Since $\lambda \notin X^+$ we can choose $\alpha \in S$ with $\langle \lambda, \alpha^\vee \rangle < 0$. Then we have for each $i \geq 1$

$$H^i(B, p\lambda) \simeq H^{i-1}(B, H^1_\alpha(p\lambda)).$$

Set $\mu = p\lambda$ and $a = -\langle \lambda, \alpha^\vee \rangle - 1$. Then the weights of $H^1_\alpha(\mu)$ are $\mu + \alpha, \mu + 2\alpha, \dots, s_\alpha \cdot \mu$. Note that the weights ν of $H^1_\alpha(\mu)$ which belong to $W \cdot 0 + p\mathbb{Z}R$ have the form $\nu = \mu + jp\alpha$ with $j \in \{1, \dots, a\}$, or $\nu = s_\alpha \cdot \mu - jp\alpha$ with $j \in \{0, \dots, a\}$.

Consider first $\nu = \mu + jp\alpha$ for some $j \in \{1, \dots, a\}$. Since $i - 1 > n - 1 \geq -2\,\mathrm{ht}(\lambda + j\alpha) = n - 2j$ we get by induction that $H^{i-1}(B, \nu) = 0$.

Consider now $\nu = s_\alpha \cdot \mu - jp\alpha$ for some $j \in \{0, \dots, a\}$. Then

$$\nu = s_\alpha \cdot 0 + p(s_\alpha(\lambda) - j\alpha) = s_\alpha \cdot 0 + p(\lambda - (\langle \lambda, \alpha^\vee \rangle + j)\alpha).$$

Note

$$\begin{aligned} -2\,\mathrm{ht}(\lambda - (\langle \lambda, \alpha^\vee \rangle + j)\alpha) &= -2\,\mathrm{ht}(\lambda) + 2(\langle \lambda, \alpha^\vee \rangle + j) \\ &\leq -2\,\mathrm{ht}(\lambda) + 2(\langle \lambda, \alpha^\vee \rangle + a) \\ &= n - 2. \end{aligned}$$

Then by induction we get from Proposition 3.1 that

$$H^{i-1}(B, \nu) \simeq H^{i-2}(B, p(\lambda - (\langle \lambda, \alpha^\vee \rangle + j)\alpha)) = 0.$$

We conclude that $H^{i-1}(B, H^1_\alpha(\mu)) = 0$. This completes the proof.

Combining Proposition 6.1 with Proposition 3.1 we find

Corollary 6.2. *Let $\lambda \in X$ and $w \in W$. Then*

$$H^i(B, w \cdot 0 + p\lambda) = 0 \quad \text{for} \quad i > l(w) - 2\,\mathrm{ht}(\lambda).$$

Remark 6.3. We believe that the bound in Corollary 6.2 is in fact the best possible. As evidence we point to the rank 1 computations in [13], and to the quantum case, see Remark 7.1 below.

7. The quantum case

In this section the field k will be arbitrary and we consider an element $q \in k^*$. We denote by U_q the quantum group with parameter q associated with our root system R. By this we mean more precisely the specialization at $q \in k$ of the Lusztig integral form of the quantized enveloping algebra attached to R. We denote by B_q the Borel subalgebra in U_q corresponding to the negative roots.

Here we shall demonstrate that the results in the previous sections have direct analogues for B_q. The proofs are almost identical and we therefore omit the details.

7.1. Just as for B above each $\lambda \in X$ (now identified with the set of integral weights in $\mathrm{span}_{\mathbb{R}} R$) defines a character of B_q, see e.g. [6]. Our aim is to study the cohomology $H^\bullet(B_q, \lambda)$, where λ denotes the 1-dimensional B_q-module obtained in this way. Note that $H^0(B_q, -)$ is now the fixed point functor for B_q in the Hopf algebra sense.

When q is not a root of unity then we can argue as in Section 2.2 using this time the quantized Borel-Weil-Bott theorem [6] and the complete reducibility of U_q valid in this case by [6] Corollary 7.7. In this way we obtain then the following complete description of $H^\bullet(B_q, \lambda)$ (in analogy with (2.4)):

$$H^r(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = w \cdot 0 \text{ for some } w \in W \text{ with } l(w) = r, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

7.2. We let from now on $q \in k^*$ denote a primitive l-th root of unity. We shall assume that l is odd, larger than the Coxeter number h, and prime to 3 if the root system R contains a component of type G_2.

For each $\alpha \in S$ we let $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ denote the standard generators. Then U_q is generated by $K_\alpha^{\pm 1}$ together with the divided powers of all the E_α and F_α. The small quantum u_q is the subalgebra of U_q generated by all $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ modulo the ideal generated by $K_\alpha^l - 1$. Moreover, b_q will denote the small quantum Borel subalgebra of u_q corresponding to B_q.

We have a quantum Frobenius homomorphism, see [7], Section 1, $Fr_q : U_q \to \bar{U}$. Here $\bar{U}$ denotes the specialisation at k of the Kostant $\mathbb{Z}$-form of the enveloping algebra of the Lie algebra for the semisimple group $\bar{G}$ corresponding to R. We identify the category of finite dimensional $\bar{U}$-modules with the category of finite dimensional rational $\bar{G}$-modules. We shall also need the restriction of F_q to B_q mapping into the enveloping algebra associated with the Borel subgroup $\bar{B}$ in $\bar{G}$.

7.3. We limit ourselves to finite dimensional modules for U_q and B_q of type **1**. So if M is a U_q (resp. B_q)-module whose restriction to u_q (resp. b_q) is trivial then we use the quantum Frobenius homomorphism F_q to make M into a $\bar{G}$ (resp. $\bar{B}$)-module that we denote by $M^{(-1)}$ in analogy with the notation in Section 3.2. Similarly, if N is a $\bar{G}$ (resp. $\bar{B}$)-module then $N^{(1)}$ denotes the U_q (resp. B_q)-module obtained via Fr_q.

As in Section 3.2 we have for each B_q-module the Lyndon-Hochschild-Serre spectral sequence

$$H^r(\bar{B}, H^s(b_q, M)^{(-1)}) \implies H^{r+s}(B_q, M). \tag{7.2}$$

The cohomology $H^r(b_q, \lambda)$ is completely known, see [12]

$$H^r(b_q, \lambda) = 0 \text{ for all r} \geq 0 \text{ unless } \lambda \in W \cdot 0 + l\mathbb{Z}R. \tag{7.3}$$

$$H^r(b_q, w \cdot 0 + l\lambda)^{(-1)} \simeq S^{(r-l(w))/2}\bar{u}^* \otimes \lambda \tag{7.4}$$

where $\bar{u}$ is the Lie algebra of the unipotent radical of $\bar{B}$. The same arguments as before (see Sections 2.4 and 3.4, respectively Corollary 6.2) give then

$$H^r(B_q, \lambda) = 0 \text{ for all r} \geq 0 \text{ unless } \lambda \in W \cdot 0 + l\mathbb{Z}R, \tag{7.5}$$

$$H^r(B_q, w \cdot 0 + l\lambda) \simeq H^{r-l(w)}(B_q, l\lambda) \text{ for all w} \in W, r \in \mathbb{N}, \tag{7.6}$$

$$H^r(B_q, w \cdot 0 + l\lambda) = 0 \text{ for all r} > l(w) - 2\,\mathrm{ht}(\lambda). \tag{7.7}$$

Remark 7.1. Suppose that $\alpha \in S$ and let $w \in W$. In characteristic zero weight considerations (for details we refer to [4] Corollary 4.6) give for each $m > 0$

$$H^r(B_q, w \cdot 0 - ml\alpha) \simeq \begin{cases} k & \text{if } r = l(w) + 2m, l(w) + 2m - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

This shows that there are cases where $H^{l(w)-2\,\mathrm{ht}(\lambda)}(B_q, w \cdot 0 + l\lambda)$ is non-zero.

7.4. Degrees 0 and 1. Using the Lyndon-Hochschild-Serre spectral sequence (7.2), the cohomology for B_q can be related to that for $\bar{B}$. Combining this with the results in the previous sections, we are now able to completely determine some of the Hochschild cohomology of 1-dimensional B_q-modules.

It is clear that

$$H^0(B_q, k) \simeq k \text{ and } H^0(B_q, \lambda) \neq 0 \text{ if and only if } \lambda = 0.$$

Noting that the only E_2-term in (7.2) that contributes to $H^1(B_q, l\lambda)$ is $H^1(\bar{B}, \lambda)$, we have

$$H^1(B_q, l\lambda) \simeq H^1(\bar{B}, \lambda).$$

Therefore the description of the first cohomology $H^1(B_q, \lambda)$ depends on whether k is a field of characteristic 0 or of characteristic $p > 0$. If $\operatorname{char} k = 0$ then we obtain from (2.4)

$$H^1(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = -\alpha \text{ or } -l\alpha \text{ for } \alpha \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{7.9}$$

On the other hand, if $\operatorname{char} k = p > 0$ then we have (using this time (2.9))

$$H^1(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = -p^n\alpha \text{ or } -lp^n\alpha \text{ for } \alpha \in S, n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.10}$$

7.5. Degree 2. The only terms in (7.2) that contribute to $H^2(B_q, l\lambda)$ are $H^2(\bar{B}, \lambda)$ and $H^0(\bar{B}, \bar{u}^* \otimes \lambda)$. Hence by (2.4) and Proposition 4.1 we get

Theorem 7.2. *Let $\lambda \in X$. If* $\operatorname{char} k = 0$ *then*

$$H^2(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = -l\alpha \text{ for } \alpha \in S, \\ k & \text{if } \lambda = lw \cdot 0 \text{ for } w \in W \text{ with } l(w) = 2, \\ k & \text{if } \lambda = -\beta - l\alpha \text{ for } \alpha, \beta \in S, \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W \text{ with } l(w) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

When $p > 0$ we replace (2.4) in the above argument by Theorem 5.1. Then we find

Theorem 7.3. *Let $\lambda \in X$. If* $\operatorname{char} k = p > 0$ *then*

$$H^2(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = lp^n(-\alpha) \text{ for } \alpha \in S, n \geq 0, \\ k & \text{if } \lambda = lp^n(w \cdot 0) \text{ for } w \in W \text{ with } l(w) = 2, n \geq 0, \\ k & \text{if } \lambda = lp^n(-\alpha - p^m\beta) \text{ for } \alpha, \beta \in S, n \geq 0, m > 0 \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W \text{ with } l(w) = 2, \\ k & \text{if } \lambda = -\beta - lp^n\alpha \text{ for } \alpha, \beta \in S, n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

7.6. Degree 3. We now turn to $H^3(B_q, \lambda)$. The only E_2-terms in (7.2) that contribute to $H^3(B_q, l\lambda)$ are $H^3(\bar{B}, \lambda)$ and $H^1(\bar{B}, \bar{u}^* \otimes \lambda)$. As in the modular case we get

Theorem 7.4. *Let $\lambda \in X$. If* $\operatorname{char} k = 0$ *then*

$$H^3(B, \lambda) \simeq \begin{cases} k & \text{if } \lambda = l(-2\alpha) \text{ for } \alpha \in S, \\ k & \text{if } \lambda = l(-\beta - \alpha) \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle < 0, \\ k^2 & \text{if } \lambda = l(-\beta - \alpha) \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle = 0, \\ k & \text{if } \lambda = l(s_\alpha s_\beta \cdot 0) \text{ for } \alpha, \beta \in S \text{ with } \langle \beta, \alpha^\vee \rangle \neq 0, \\ k & \text{if } \lambda = l(w \cdot 0) \text{ for } w \in W \text{ with } l(w) = 3, \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W \text{ with } l(w) = 3, \\ k & \text{if } \lambda = w \cdot 0 - l\alpha \text{ for } \alpha \in S \text{ and } w \in W \text{ with} \\ & l(w) = 2, \\ k & \text{if } \lambda = lw \cdot 0 - \alpha \text{ for } \alpha \in S \text{ and } w \in W \text{ with} \\ & l(w) = 2, \\ k & \text{if } \lambda = -\beta - l\alpha \text{ for } \alpha, \beta \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.5. *Suppose that* $\operatorname{char} k = p > 0$. *If* $\lambda \in X$ *then*

$$H^3(B,\lambda) \simeq \begin{cases} k & \textit{if } \lambda = lp^n(-2\alpha) \textit{ for } \alpha \in S \textit{ and } n > 0, \\ k^2 & \textit{if } \lambda = lp^n(-\beta - p^m\alpha) \textit{ for } \alpha,\beta \in S \textit{ and} \\ & n,m > 0, \\ k & \textit{if } \lambda = lp^n(-\beta-\alpha) \textit{ for } \alpha,\beta \in S \textit{ with} \\ & \langle \beta, \alpha^\vee\rangle < 0 \textit{ and } n > 0, \\ k^2 & \textit{if } \lambda = lp^n(-\beta-\alpha) \textit{ for } \alpha,\beta \in S \textit{ with} \\ & \langle \beta, \alpha^\vee\rangle = 0 \textit{ and } n > 0, \\ k & \textit{if } \lambda = lp^n(s_\alpha s_\beta \cdot 0) \textit{ for } \alpha,\beta \in S \textit{ with} \\ & \langle \beta, \alpha^\vee\rangle \neq 0 \textit{ and } n > 0, \\ k & \textit{if } \lambda = lp^n(w\cdot 0) \textit{ for } w \in W \textit{ with} \\ & l(w) = 3 \textit{ and } n \geq 0, \\ k & \textit{if } \lambda = lp^n(w\cdot 0 - p^m\alpha) \textit{ for } \alpha \in S,\ w \in W \\ & \textit{with } l(w) = 2 \textit{ and } n \geq 0, m > 0 \\ k & \textit{if } \lambda = lp^n(p^m w\cdot 0 - \alpha) \textit{ for } \alpha \in S,\ w \in W \\ & \textit{with } l(w) = 2 \textit{ and } n \geq 0, m > 0 \\ k & \textit{if } \lambda = lp^n(-\alpha - p^m\beta - p^v\gamma) \textit{ for } \alpha,\beta,\gamma \in S \\ & \textit{and } n \geq 0, m > v > 0, \\ k & \textit{if } \lambda = l(-\beta - p^n\alpha) \textit{ for } \alpha,\beta \in S \textit{ and } n > 0, \\ k & \textit{if } \lambda = w\cdot 0 \textit{ for } w \in W \textit{ with } l(w) = 3, \\ k & \textit{if } \lambda = w\cdot 0 - lp^n\alpha \textit{ for } \alpha \in S,\, w \in W \textit{ with} \\ & l(w) = 2 \textit{ and } n \geq 0, \\ k & \textit{if } \lambda = -\beta - lp^n\alpha \textit{ for } \alpha,\beta \in S \textit{ and } n \geq 0, \\ k & \textit{if } \lambda = -\beta - lp^n w\cdot 0 \textit{ for } \alpha \in S,\ w \in W \\ & \textit{with } l(w) = 2 \textit{ and } n \geq 0, \\ k & \textit{if } \lambda = -\alpha + lp^n(-\beta - p^m\gamma) \textit{ for } \alpha,\beta,\gamma \in S \\ & \textit{and } n \geq 0, m > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

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