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The polaron revisited

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Abstract

In recent years, the spectral properties of the translation invariant Nelson model has been studied. Some of the results obtained did not extend to the related polaron model for technical reasons related to the typical assumption of boundedness of the phonon dispersion relation in the polaron model. In this paper we work with a large class of linearly coupled translation invariant models which includes both the Nelson model and H. Fröhlich's polaron model. The problems considered are chosen based on relevance for the polaron model. A key input is an analysis of the behaviour of the bottom of the spectrum of the fiber Hamiltonians at large total momentum.

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1 Introduction

In this paper we study the operator on $L^2(\mathbb{R}^\nu)\otimes\mathcal{F}$ given by

$$H = H_0 + \Phi(e^{-\mathrm{i}k\cdot \mathrm{x}}v), \quad \text{where} \quad H_0 = \Omega(\tfrac{1}{\mathrm{i}}\nabla_{\mathrm{x}})\otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{L^2(\mathbb{R}^\nu)}\otimes d\Gamma(\omega), \tag{1.1}$$

and Ω and ω are dispersion relations for a non-relativistic particle and a scalar field respectively. The particle position is denoted by x and the phonon momentum by k. Throughout the paper we use the Γ functor to denote second quantization. We write $\mathcal{F} = \Gamma(L^2(\mathbb{R}^\nu))$ for the symmetric Fock space over $L^2(\mathbb{R}^\nu)$. The coupling function v is assumed to be an $L^2(\mathbb{R}^\nu)$ function and the field operator $\Phi(g)$ is defined by

$$\Phi(g) = \int_{\mathbb{R}^\nu}(g(k)a^*(k) + g(k)^*a(k))dk, \tag{1.2}$$

for $g\in L^2(\mathbb{R}^\nu;\mathcal{B}(L^2(\mathbb{R}^\nu)))$. Here $a^*(k)$ and $a(k)$ are phonon creation and annihilation operators. They satisfy the canonical commutation relations.

The operator H has an important symmetry. It is translation invariant, in the sense that it commutes with the operator of total momentum

$$P = \tfrac{1}{\mathrm{i}}\nabla_{\mathrm{x}}\otimes\mathbb{1}_{\mathcal{F}} + \mathbb{1}_{L^2(\mathbb{R}^\nu)}\otimes d\Gamma(k). \tag{1.3}$$

Using a unitary transform which goes back to Lee-Low-Pines [27] one can bring H on the form $\oint_{\mathbb{R}^\nu} H(\xi)d\xi$ on $L^2(\mathbb{R}^\nu;\mathcal{F})$, where the fiber Hamiltonians $H(\xi)$ are given by

$$H(\xi) = H_0(\xi) + \Phi(v), \quad \text{where} \quad H_0(\xi) = \Omega(\xi - d\Gamma(k)) + d\Gamma(\omega) \tag{1.4}$$

as operators on $\mathcal{F}$. The Lee-Low-Pines transform is given by

$$I_{\mathrm{LLP}} := (F\otimes\mathbb{1}_{\mathcal{F}})\circ\Gamma(e^{-\mathrm{i}k\cdot\mathrm{x}}), \tag{1.5}$$

where F denotes the Fourier transform in $L^2(\mathbb{R}^\nu)$. The field operator $\Phi(v)$ is defined as in (1.2). We refer to the set $\{(\xi,E)|\xi\in\mathbb{R}^\nu, E\in\sigma(H(\xi))\}$ as the energy-momentum spectrum of H. We are mainly interested in the bottom of this set, in particular the ground state as a function of total momentum

$$\Sigma_0(\xi) = \inf\sigma(H(\xi)), \tag{1.6}$$

and the bottom of the essential spectrum.

An important motivating example is H. Fröhlich's polaron model [13] of one electron (or hole) in a ionic crystal. Here $\Omega(\eta) = \eta^2/(2M_{\mathrm{eff}})$, $\omega(k)\equiv h\omega_0 > 0$ a constant, and v a coupling function which in 3 dimensions take the form

$$v(k) = \sqrt{\alpha}h\omega_0\left(\frac{h}{2M_{\mathrm{eff}}\omega_0}\right)^{\frac{1}{4}}(4\pi)^{\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\frac{1}{|k|} \tag{1.7}$$

We remark that in the literature v often comes with a factor i. This factor can be removed by the unitary transformation $\mathbb{1}_{L^2(\mathbb{R}^\nu)}\otimes\Gamma(\mathrm{i}\mathbb{1}_{L^2(\mathbb{R}^\nu)})$ (see also (4.2)). For general $\nu\geq 1$, the coupling function is a multiple of $|k|^{-(\nu-1)/2}$ (a constant in dimension 1), see [34]. The mass M_{eff} is an effective mass which comes from approximating an electron in a static periodic background potential by a free electron with an effective mass. The frequency ω_0 is that of long wavelength longitudinal optical phonons. The acoustic phonons as well as the transverse optical phonons are neglected. (The acoustic phonons do not contribute to the polarization of the crystal, since they model vibrations of neutral unit cells consisting of two oppositely charged ions.)

H. Fröhlich's model is often called the large polaron model, because in its derivation [12, 13, 26] it is assumed that the charged particle is smeared out over a region large compared to the lattice spacing. This permits a continuum approximation and explains that the entire momentum space is used, and not just a bounded Brillouin zone. (In [27] the charge distribution of the polaron, i.e. a ground state, is computed and the result is consistent with this assumption.) It should also be noted that a thermodynamic limit (infinite crystal) is implied by the choice of a continuum for the phonon momentum space. The model is ultraviolet singular and infrared regular (v is not square integrable at infinity in any dimension) and one has to add an ultraviolet cutoff to make v square integrable.

We remark that it is common in the literature to take a finite size continuous crystal. Let $\Lambda \subset \mathbb{R}^\nu$ be a box of side length L and write Δ for the Laplacian in $L^2(\Lambda)$, the electron (hole) Hilbert space, with periodic boundary conditions (to retain translation invariance). The momentum space for the phonons is now the dual lattice $\Lambda^* = [\mathbb{Z}/(2\pi L)]^\nu$ and the Fock-space for the phonons is $\Gamma(\ell^2(\Lambda^*))$. The total momentum (with $\partial/\partial x_i$ defined again using periodic boundary conditions) has discrete spectrum equal to Λ^*. Note that the integral in (1.2) should be replaced by $(2\pi L)^{-\nu}\sum_{k\in\Lambda^*}$, and hence an infrared singularity appears at $k = 0$, which one should remove by redefining v at 0. Our results, suitably reformulated, holds in this situation also. We make no further remarks on this.

Another model, the small polaron model, is used if the charged particle is localized on a length scale comparable to the lattice spacing. Here one can use the Holstein Hamiltonian [25] which we briefly explain. The crystal is kept as a cubic lattice, either infinite or periodic, with lattice spacing equal to 1, and the electron is confined to the lattice sites. The electron kinetic energy is modelled by a translation invariant hopping matrix, e.g. the discrete Laplacian, and the phonon momentum space is the dual lattice (the Brillouin zone $[-\pi,\pi]^\nu$ for an infinite crystal). The fiber Hamiltonians again take the form (1.4), where Ω is here the Fourier transform of the hopping matrix, e.g. $\sum_{j=1}^{\nu}(1-\cos(k_j))$ for the discrete Laplacian. For more material we refer the reader to [30].

For an overview of a number of polaron models and their properties see the review by Devreese [8].

Inspired by the Fröhlich polaron Nelson introduced in [33] (see also [6]) a phenomenological model for non-relativistic charged particles interacting with a scalar field, often referred to as the Nelson model. In the one-particle sector it has the form (1.1) but with $\Omega(\eta) = \eta^2/(2M)$, $\omega(k) = \sqrt{k^2+m^2}$ and $v(k) = g\omega(k)^{-\frac{1}{2}}$. Here M is the (bare) mass of the charged particle, m the mass of the field particle, and g a coupling constant. As for the Fröhlich polaron, the model has an ultraviolet singularity. Nelson noted that an ultraviolet cutoff can be removed by subtraction of an infinite self energy, without leaving the physical Hilbert space, hence defining the model rigorously. The same holds true for the Fröhlich polaron [19].

In 1973–74 J. Fröhlich's PhD thesis appeared in the form of two papers [15, 16]. They were concerned with the spectral and scattering theory for the massless translation invariant Nelson model (in the one-particle sector), in particular in the context of the infrared problem. These two papers are the starting point for most mathematical work on the operator (1.1) in the last 30 years. The methods employed are robust and the results obtained on the structure of the bottom of the energy-momentum spectrum extend to ω which are subadditive and satisfies $\omega(k) \to \infty$ for $|k| \to \infty$. The Nelson analogue of the polaron model is the massive translation invariant Nelson model.

The structure of the bottom of the energy-momentum spectrum of models of the form (1.1) has been studied by a number of authors [3, 19, 24, 31, 32, 39, 40]. Many of the results obtained were proved using the property $\omega(k) \to \infty$ for $|k| \to \infty$ and do for a particular technical reason not extend to H. Fröhlich's polaron model (with the exception of [3] where the situation is the excact opposite). This paper is devoted to overcoming this difficulty. In particular we make an effort to work with minimal assumptions, cf. Section 2, on Ω, ω and v, such as to encompass models of the form (1.1) which typically appear in the literature. When presenting our main theorems we have made an effort to focus on results pertinent to the Fröhlich polaron. Recently a relativistic electron, modelled by the Dirac operator, linearly coupled to a massless field was analyzed in [36]. For minimally coupled models we refer the reader to the monograph [41] and the references therein.

We list in telegraphic style the results obtained leaving comments on the literature to Section 2, where the precise statements are given. All results are concerned with the structure of the bottom of the energy-momentum spectrum and holds for all coupling strengths: An HVZ theorem determining the essential spectrum, uniqueness of groundstates, (strict) monotonicity of $\Sigma_0(\xi)$ and $\inf[\sigma_{\mathrm{ess}}(H(\xi))]$ in the coupling function, and existence/non-existence of groundstates for $H(\xi)$.

A central theme in the paper is a study of the spectral gap $\inf[\sigma_{\mathrm{ess}}(H(\xi))] - \Sigma_0(\xi)$ in the limit of large total momentum $|\xi| \to \infty$. We show that for a class of models including the Fröhlich polaron (but not the Nelson model) the spectral gap closes in the limit of large total momentum. This is the key new observation which we use to conclude a number of our results. We repeat that these results are already known under the assumption $\omega(k) \to \infty$. The contribution here is the extension to the polaron model, or more generally, models where $\omega(k) \not\to \infty$. Our central new results are not relevant for the Holstein model, because the phonon momentum space in that model is bounded.

We remark that the early literature on the Fröhlich polaron was focused on the groundstate energy and the effective mass (inverse curvature of $\xi \to \Sigma_0(\xi)$ at $\xi = 0$). This was a particulary challenging problem because the value of the coupling constant α for typical ionic crystals is not small (typically between 3 and 6, cf. [12, 27]), and perturbation theory is inadequate. In fact the main thrust was towards large coupling results, cf. the key papers [27, 28]. In [11] Feynman applied his newly invented path integral technique to get bounds on the groundstate energy valid for all α, an idea pursued and made mathematically rigorous in [9], see also [39] and the more recent paper [29] where the authors derive the large coupling asymptotics together with an error bound.

In Section 2 we formulate precise assumptions and state our main theorems. In Section 3 we prove the HVZ theorem and in Section 4 we discuss uniqueness of groundstates and derive some consequences. The core of the paper is Section 5 where we analyze the bottom of the spectrum at large total momentum, and prove the main new results. We have included here for completeness some considerations which are not so relevant for the particular case of the Fröhlich polaron. In Appendix A we recall a partition of unity in Fock-space, and in Appendix B we discuss how to extend all of our main results to models with a weak ultraviolet singularity.

2 Model and results

We begin with basic assumptions on the two dispersion relations.

Condition 2.1. (The particle dispersion relation) *Let $\Omega \in C^2(\mathbb{R}^\nu)$ satisfy that $\Omega \geq 0$ and*

i) There exists C_Ω such that for all $\eta \in \mathbb{R}^\nu$ we have $|\nabla\Omega(\eta)| \leq C_\Omega \Omega(\eta) + C_\Omega$.

ii) $\widetilde{C}_\Omega := \sup_{\eta\in\mathbb{R}^\nu} \|\nabla^2\Omega(\eta)\| < \infty$.

Condition 2.2. (The phonon dispersion relation) *We have $\omega \in C^0(\mathbb{R}^\nu)$ and $\omega_0 > 0$, the phonon mass, such that $\inf_{k\in\mathbb{R}^\nu} \omega(k) = \omega_0$.*

Assuming Conditions 2.1 i) and 2.2, we can construct the Hamiltonian's (1.1) and (1.4) as follows (for more details see e.g. [16, 31]). Let

$$\mathcal{C}_0^\infty = \Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^\nu)), \tag{2.1}$$

where $\Gamma_{\mathrm{fin}}(\mathcal{V})$ denotes the subspace of elements of the form $(u_0, \ldots, u_n, 0, \ldots)$, where the u_ℓ's are elements from ℓ-fold algebraic tensor products of $\mathcal{V}$. Clearly H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu) \otimes \mathcal{C}_0^\infty$ and the $H_0(\xi)$'s are essentially self-adjoint on $\mathcal{C}_0^\infty$.

As for H, the field operator $\Phi(e^{-\mathrm{i}k\cdot x}v)$ is $\mathbb{1}_{L^2(\mathbb{R}^\nu)} \otimes d\Gamma(\omega)$-bounded with relative bound zero. Hence, by Kato-Rellich, H is self-adjoint on $\mathcal{D}(H_0)$ and essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu) \otimes \mathcal{C}_0^\infty$. Another application of Kato-Rellich shows that $\mathcal{D}(H_0(\xi))$ is independent of ξ. We denote the common domain by $\mathcal{D}$. Secondly, as above, $\Phi(v)$ is $d\Gamma(\omega)$-bounded with relative bound zero so $H(\xi)$ is self-adjoint on $\mathcal{D}$ and essentially self-adjoint on $\mathcal{C}_0^\infty$.

Condition 2.3. *Let ω and Ω be continuous functions satisfying either that there exists $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^\nu$ with $\lim_{j\to\infty} |k_j| = \infty$ such that*

$$\lim_{j\to\infty} \omega(k_j)^{-1} = 0 \tag{2.2}$$

or

$$\sup_k \omega(k) < \infty \quad \text{and} \quad \lim_{|\eta|\to\infty} \Omega(\eta)^{-1} = 0. \tag{2.3}$$

For $n \geq 1$ and $\underline{k} = (k_1, \dots, k_n) \in \mathbb{R}^{n\nu}$ we write

$$k^{(n)} := k_1 + \cdots + k_n. \tag{2.4}$$

We now introduce the bottom of the spectrum for a composite system at total momentum ξ, consisting of an interacting system in the ground state at total momentum $\xi - k^{(n)}$ and n non-interacting phonons with momenta $\underline{k}$:

$$\Sigma_0^{(n)}(\xi; \underline{k}) := \Sigma_0\big(\xi - k^{(n)}\big) + \sum_{j=1}^{n} \omega(k_j). \tag{2.5}$$

The following functions are thresholds due to ground states dressed by n free phonons, at critical momenta:

$$\Sigma_0^{(n)}(\xi) := \inf_{\underline{k} \in \mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi; \underline{k}). \tag{2.6}$$

There may be other thresholds coming from other local extrema of $\underline{k} \to \Sigma_0^{(n)}(\xi; \underline{k})$ and likewise from exited bands of eigenvalues. The bottom of the essential spectrum (see Theorem 2.1 below)

$$\Sigma_{\mathrm{ess}}(\xi) := \inf_{n \geq 1} \Sigma_0^{(n)}(\xi). \tag{2.7}$$

The first result is the HVZ theorem, which goes back to J. Fröhlich [16] who proved i) below for the massless translation invariant Nelson model. His proof, which relies on a method of Glimm and Jaffe [20], extends to models where ω is subadditive and $\omega(k) \to \infty$ for $|k| \to \infty$. Spohn established in [40] that $\Sigma_{\mathrm{ess}}(\xi)$ is the bottom of the essential spectrum, again assuming ω to be subadditive. A result which shows up again in [19]. However in the latter two papers, the authors refer to J. Fröhlich for a crucial step where the property $\omega(k) \to \infty$ is used, and no mention is made of how to repair it. In [31], i) and ii) below were established (using the method of [7]) without the subadditivity assumption, but still with the assumption $\omega(k) \to \infty$. The proof we give here follows the Glimm-Jaffe approach and establishes i) in full detail, and we argue using the proof given in [31] how to verify ii). The remaining statement iii) is only non-trivial under the assumption (2.3), where the result is new. The proof relies on the vanishing of $\Sigma_{\mathrm{ess}}(\xi) - \Sigma_0(\xi)$ at large total momentum, cf. Theorem 2.4 below.

Theorem 2.1. *Suppose $v \in L^2(\mathbb{R}^\nu)$, Conditions 2.1 and 2.2. We have*

i) The spectrum of $H(\xi)$ below $\Sigma_{\mathrm{ess}}(\xi)$ consists at most of eigenvalues of finite multiplicity, with $\Sigma_{\mathrm{ess}}(\xi)$ as the only possible accumulation point.

ii) We have $\{\Sigma_0^{(n)}(\xi; \underline{k}) \,|\, n \in \mathbb{N} \text{ and } \underline{k} \in \mathbb{R}^{n\nu}\} \subset \sigma_{\mathrm{ess}}(H(\xi))$.

iii) If Condition 2.3 is also satisfied then $\sigma_{\mathrm{ess}}(H(\xi)) = [\Sigma_{\mathrm{ess}}(\xi), \infty)$.

Remark 2.2. 1) Note that there are no gaps in the essential spectrum of the uncoupled model $v \equiv 0$ if $\omega(k) = \omega_0 > 0$ a constant and $\Omega(\eta) \to \infty$, for $|\eta| \to \infty$. It should be noted that the choice $\Omega \equiv 0$ and $\omega(k) \equiv \omega_0 > 0$ yields $H(\xi) = \omega_0 N + \Phi(v)$

which is unitarily equivalent to $\omega_0 N - \omega_0^{-1}\|v\|_2^2$. Hence $H(\xi)$ has plenty of gaps in the essential spectrum in this case. Here $N = d\Gamma(\mathbb{1}_{L^2(\mathbb{R}^\nu)})$ is the phonon number operator.

2) In general the inclusion in Theorem 2.1 ii) may be strict, if there are gaps in the set on the left-hand side and the model has excited bands of eigenvalues. These bands could give rise to extra contributions to the essential spectrum which may narrow the gaps. The only result on existence/non-existence of excited bands of eigenvalues is due to [3]: For $\Omega(\eta) = \eta^2$, a class of ω's, which include the constant function (but exclude $\sqrt{k^2+m^2}$, $m > 0$), and v's which are smooth with all derivatives vanishing faster than a polynomial (implying a strong infrared reguralization as well as an ultraviolet one), they prove that for small values of a coupling constant there are no excited eigenstates for $H(\xi)$ below $\Sigma_2(\xi)$.

3) Note that $0 \le \Sigma_{\mathrm{ess}}(\xi) - \Sigma_0(\xi) \le \omega(0)$. We do not assume in this paper that $\omega(0) = \omega_0$, so the spectral gap may be bigger than ω_0.

We introduce the notation $\mathcal{I}_0 \subset \mathbb{R}^\nu$ for the set of total momenta where $H(\xi)$ admits an isolated groundstate

$$\mathcal{I}_0 = \{\xi \in \mathbb{R}^\nu | \Sigma_0(\xi) < \Sigma_{\mathrm{ess}}(\xi)\}. \tag{2.8}$$

Note that $\{\xi \in \mathbb{R}^\nu | \Sigma_0(\xi) < \inf_\eta \Sigma_0(\eta) + \omega_0\} \subset \mathcal{I}_0$. In particular $\mathcal{I}_0 \ne \emptyset$.

The following theorem is concerned with uniqueness of groundstates, a type of result which is usually proved by a suitable Perron-Frobenius theorem. It was established for $\xi = 0$ by L. Gross in [21] using the Schrödinger representation of the free field, but this method does not extend to $\xi \ne 0$ and assumes that

$$\eta \to \exp(-t\Omega(\eta)) \text{ is positive definite for all } t > 0. \tag{2.9}$$

(See also [19].) In [16] (cf. also [31] and [40]) an abstract Perron-Frobenius theorem of Faris was employed to establish uniquess of groundstates for all ξ provided $v \ne 0$ a.e. and real-valued. Here the property (2.9) is not needed. We improve this result in two directions. We avoid the assumption that v be real-valued (a trivial extension of the proof in [31]) and we show that isolated groundstates are unique without any assumption on v apart from it being square integrable. This is particularly useful since one often employs sharp ultraviolet cutoffs.

Theorem 2.3. *Let $\xi \in \mathbb{R}^\nu$. Assume $v \in L^2(\mathbb{R}^\nu)$ and Conditions 2.1 and 2.2. If $\Sigma_0(\xi)$ is an eigenvalue for $H(\xi)$ and either $v \ne 0$ a.e. or $\xi \in \mathcal{I}_0$, then $\Sigma_0(\xi)$ is non-degenerate. Furthermore, the groundstate $\psi_\xi = (\psi_\xi^{(0)}, \dots, \psi_\xi^{(n)}, \dots)$ can be chosen such that: $\psi_\xi^{(0)} > 0$ and for any $n \ge 1$*

$$(-1)^n \overline{v(k_1)} \cdots \overline{v(k_n)} \psi_\xi^{(n)}(\underline{k}) > 0 \text{ a.e. in } \{\underline{k} \in \mathbb{R}^{n\nu} | \forall j : \ v(k_j) \ne 0\} \tag{2.10}$$

$$\psi_\xi^{(n)}(\underline{k}) = 0 \ \text{ a.e. in } \{\underline{k} \in \mathbb{R}^{n\nu} | \exists j \text{ s.t. } v(k_j) = 0\}. \tag{2.11}$$

The proof is a combination of the method of J. Fröhlich and an application of the HVZ theorem.

The central new observation used to prove Theorem 2.1 iii) in the case of bounded ω's is the following result, which is a special case of the more general Theorem 5.2, which is formulated and proved in Section 5. Before stating the result we impose a condition which is slightly stronger than Condition 2.3.

Condition 2.4. *Let ω and Ω be continuous functions satisfying either*

$$\lim_{|k|\to\infty} \omega(k)^{-1} = 0 \tag{2.12}$$

or there exists a sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^\nu$, with $|k_j| \to \infty$ for $j \to \infty$, such that

$$\sup_j \omega(k_j) < \infty \quad \text{and} \quad \lim_{|\eta|\to\infty} \frac{\omega(\eta)}{\Omega(\eta)} = 0. \tag{2.13}$$

We have

Theorem 2.4. *Suppose $v \in L^2(\mathbb{R}^\nu)$, Conditions 2.1, 2.2, and (2.13). Then for any $\bar{\Sigma} \in \mathbb{R}$*

$$\lim_{|\xi|\to\infty, \Sigma_0(\xi)\le\bar{\Sigma}} \Sigma_{\mathrm{ess}}(\xi) - \Sigma_0(\xi) = 0.$$

The reader should of course keep in mind the Fröhlich polaron $\omega(k) \equiv \omega_0$, for which $\sup_\xi \Sigma_0(\xi) < \infty$. This theorem may seem trivial in light of the uncoupled model where the ground state disappears into the essential spectrum. It should be seen in connection with a surprising result of Spohn [40, Section 5], Theorem 2.6 i) below.

We pause to introduce some notation. Let $v_1, v_2 \in L^2(\mathbb{R}^\nu)$. We will distinguish between the interacting Hamiltonians $H_0(\xi) + \Phi(v_i)$ by adding an index, i.e. $H_1(\xi)$ and $H_2(\xi)$. Likewise we will distinguish spectral functions by adding an index as in $\Sigma_{0,i}(\xi) = \inf \sigma(H_i(\xi))$ and $\Sigma_{\mathrm{ess},i}(\xi) = \inf \sigma_{\mathrm{ess}}(H_i(\xi))$.

We derive the following monotonicity result from Theorems 2.3 and 2.4.

Corollary 2.5. *Suppose Conditions 2.1 and 2.2. Let $v_1, v_2 \in L^2(\mathbb{R}^\nu)$ be coupling functions satisfying*

$$v_1\bar{v}_2 \ge 0 \text{ a.e.} \quad \text{and} \quad |v_1| \ge |v_2| \text{ a.e.} \tag{2.14}$$

Then for all ξ we have

- *i)* $\Sigma_{0,1}(\xi) \le \Sigma_{0,2}(\xi)$ *and* $\Sigma_{\mathrm{ess},1}(\xi) \le \Sigma_{\mathrm{ess},2}(\xi)$.
- *ii) If in addition Condition 2.4 is satisfied, $v_1 \ne 0$ a.e. and $v_1 \ne v_2$, then* $\Sigma_{0,1}(\xi) < \Sigma_{0,2}(\xi)$ *and* $\Sigma_{\mathrm{ess},1}(\xi) < \Sigma_{\mathrm{ess},2}(\xi)$.

For the remaining theorem we impose the following condition

Condition 2.5. *The phonon dispersion relation ω is strictly subadditive, that is:*

$$\forall k_1, k_2 \in \mathbb{R}^\nu : \quad \omega(k_1 + k_2) < \omega(k_1) + \omega(k_2). \tag{2.15}$$

We furthermore require that either $\lim_{|k|\to\infty} \omega(k)^{-1} = 0$ or: ω is bounded,

$$\lim_{|\eta|\to\infty} \Omega(\eta)^{-1} = 0 \quad \text{and} \quad 2 \liminf_{|k|\to\infty} \omega(k) > \sup_k \omega(k). \tag{2.16}$$

See Remark 5.6 for a discussion of (2.16). Note that the last part of Condition 2.5 implies Condition 2.4. If ω is subadditive (inequality instead of strict inequality in (2.15)) then $\Sigma_{\mathrm{ess}}(\xi) = \Sigma_0^{(1)}(\xi)$.

We have

Theorem 2.6. *Suppose Conditions 2.1, 2.2 and 2.5,*

$$v \in L^2(\mathbb{R}^\nu) \quad \text{and} \quad \forall R > 0: \operatorname*{essinf}_{k:|k|\leq R} |v(k)| > 0. \tag{2.17}$$

Then

i) For $\nu \in \{1, 2\}$ and all $\xi \in \mathbb{R}^\nu$, the bottom of the spectrum $\Sigma_0(\xi)$ is an isolated eigenvalue.

ii) For $\nu \in \{3, 4\}$ and $\xi \notin \mathcal{I}_0$, the bottom of the spectrum $\Sigma_0(\xi)$ is not an eigenvalue.

In dimension $\nu = 3$ a formal calculation of Feynman, cf. [12], indicates that $\mathcal{I}_0$ should remain a bounded set when the coupling is turned on, but this has only been established for weak coupling in [3], under the assumptions outlined in Remark 2.2 2).

Spohn proved i) above, but under a technical condition which excluded the Nelson model, and under the implicit condition that the function $k \to \Sigma_0^{(1)}(\xi; k)$ attains its infimum. This condition was verified by Spohn in the following situation: Under the assumption (2.9), an argument of L. Gross [21] shows that $\xi = 0$ is a global minimum of $\xi \to \Sigma_0(\xi)$. If in addition ω attains its supremum at infinity (this is in particular true if ω is constant) then the implicit condition is clearly satisfied. Our condition (2.16) is more natural as is illustrated by Remark 5.6, and we do not require (2.9). We remark that it has been observed by Gerlach and Löwen [19] that $\xi = 0$ is a unique global minimum of Σ_0 under the assumption (2.9).

In [31] the theorem above was established for the Nelson model and a proof of i) simpler than Spohn's was given. We verify Spohn's implicit assumption on the mass-shell and hence prove the result for the wider class of polaron models considered here. The proof of ii) in [31] reduces in the case of the polaron model to a similar problem as for i). The property (2.10) and Theorem 2.4 are key ingredients in the proof.

In the presence of a weak ultraviolet singularity with $\omega^{-\frac{1}{2}} v \in L^2(\mathbb{R}^\nu)$ the Hamiltonian can be defined via the KLMN theorem. All the results in this section remain valid as formulated, except for Theorem 2.6 ii) for which we require the extra assumption (B.7). We refer the reader to Appendix B where this problem is addressed. Note that this is only of interest if ω is unbounded.

The class of UV singular models just discussed is not large enough to include physical models. In the litterature three different renormalization schemes have been used. The simplest approach is due to Nelson [33] and has been implemented for $\Omega(\eta) = \eta^2$ only. Here a dressing transformation is applied and in the new coordinate system a limiting renormalized operator can be defined via the KLMN theorem after subtraction of an infinite self energy. This idea has been pursued further by Cannon [6].

The second method, which goes back to lecture notes of Hepp [23], has only been applied for $\Omega(\eta) = \sqrt{\eta^2 + M^2}$ and consists of a systematic reordering of a perturbation expansion of the resolvent of the Hamiltonian (shifted by a self-energy). This method was implemented in the thesis of Eckmann [10], for an interaction resembling that of the Nelson model, (see also [1, 2]), and adapted to the massless

Nelson model by Fröhlich [16]. A full renormalization has not been achieved (in dimension 3) becaue of the condition $0 \le v(k) \le C(|k|+1)^{-\frac{1}{2}}$ imposed on the coupling function.

The third method is due to Gross [22] and Sloan [37], and as the second method it has also only been implemented for $\Omega(\eta) = \sqrt{\eta^2+M^2}$. Here a renormalized resolvent is constructed pointwise using a compactness argument, which gives existence, but not uniqueness, of a limiting resolvent in the strong resolvent sense. This result is weaker than the two others which give norm-resolvent convergence. Gross implemented this method by renormalizing the mass whereas Sloan renormalized the self-energy as was also done in the other two approaches. (Gross constructed a completely renormalized model in three dimensions but this requires a change of Hilbert space. Sloan worked in dimension two where this is not nescessary.)

Finally we discuss the type of results presented above in the context of renormalized Hamiltonians. Here only Hamiltonians constructed via the first method has been analyzed. The HVZ theorem has been established in [19, 40] (for subadditive ω's) but the proofs need elaboration, just as is explained in Section 2.1 for the non-singular case. (This can be done but we have elected to omit the details here.) The extension of the Perron-Frobenius argument to the renormalized Hamiltonian is claimed in [16] but the details were left to a preprint [14]. The proofs of the remaining results in this paper rely on groundstates having non-zero overlap with the vacuum, which is a consequence of the resolvent of the Hamiltonian being positivity improving. Without that information the arguments wont work. (This includes the proofs given in [19, 40].)

3 On the HVZ theorem

We will give a proof of Theorem 2.1 i) following the Glimm-Jaffe approach, [20], which was employed by Fröhlich in [16, Section 2.1]. As for Theorem 2.1 ii) we explain how to employ the proof given in [31] for the Nelson model. Throughout this section we assume Conditions 2.1 and 2.2.

The strategy of the proof is the same as the one outlined in [19], and consists of two steps. First we prove the result for compactly supported v's, and secondly we extend the result to general v's in $L^2(\mathbb{R}^\nu)$.

Let $\delta > 0$ and write as a disjoint union $\mathbb{R}^\nu = \cup_{\eta\in\delta\mathbb{Z}^\nu}\mathcal{K}_\delta(\eta)$, where $\mathcal{K}_\delta(\eta) := \times_{j=1}^{\nu}[\eta_j, \eta_j+\delta)$. We write η_k for the unique $\eta \in \delta\mathbb{Z}^\nu$ with $k \in \mathcal{K}_\delta(\eta)$. For $\underline{k} \in \mathbb{R}^{n\nu}$ we write $\underline{\eta}_{\underline{k}} \in \delta\mathbb{Z}^{n\nu}$ for the vector $(\eta_{k_1},\dots,\eta_{k_n})$. Furthermore, for $\underline{\eta} \in \mathbb{R}^{n\nu}$ we write $\mathcal{K}_\delta(\underline{\eta}) := \times_{i=1}^{n}\mathcal{K}_\delta(\eta_i)$.

For the following constructions we assume that v has compact support.

For $\delta > 0$ let n_δ be the smallest integer such that $\mathrm{supp}(v) \subset [-\delta n_\delta, \delta n_\delta)^\nu =: \Lambda_\delta$. Note that δn_δ is bounded uniformly in $0 < \delta < 1$.

We define a discretized dispersion relation and form factor,

$$\omega_\delta(k) := \begin{cases} \omega(\eta_k) \text{ if } k \in \Lambda_\delta \\ \omega(k) \text{ if } k \notin \Lambda_\delta \end{cases} \quad \text{and} \quad v_\delta = P_\delta v, \tag{3.1}$$

where $P_\delta : L^2(\mathbb{R}^\nu) \to L^2(\mathbb{R}^\nu)$ is an orthogonal projection defined by

$$(P_\delta v)(k) = \delta^{-\nu} \int_{\mathcal{K}_\delta(\eta_k)} v(k')\,dk'. \tag{3.2}$$

We note that $\mathrm{s} - \lim_{\delta\to 0} P_\delta = 1\!\!1_{L^2(\mathbb{R}^\nu)}$, as can be seen by computing the limit on the dense subspace of compactly supported continuous functions. This observation implies

$$\lim_{\delta\to 0} \|v - v_\delta\| = 0. \tag{3.3}$$

As approximating Hamiltonians we take

$$H_\delta(\xi) := H_{0,\delta}(\xi) + \Phi(v_\delta), \text{ where } H_{0,\delta}(\xi) := d\Gamma(\omega_\delta) + \Omega(\xi - d\Gamma(\eta_k)).$$

We leave the proof of the following to the reader (cf. the proof of [31, Proposition 1.1])

Lemma 3.1. *Fix $\delta > 0$. Let $v \in L^2(\mathbb{R}^\nu)$ and assume Ω and ω, satisfy Conditions 2.1 and 2.2 respectively. Then*

i) $\mathcal{D}(H_{0,\delta}(\xi))$ is independent of ξ and we denote it by $\mathcal{D}_\delta$.

ii) $\Phi(v_\delta)$ is $H_{0,\delta}(\xi)$-bounded with relative bound 0. $H_\delta(\xi)$ is bounded from below, self-adjoint on $\mathcal{D}(H_\delta(\xi)) = \mathcal{D}_\delta$, and essentially self-adjoint on $\mathcal{C}_0^\infty$.

iii) The bottom of the spectrum, $\xi \to \Sigma_{0,\delta}(\xi) := \inf \sigma(H_\delta(\xi))$, is Lipschitz continuous.

We introduce

$$\Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k}) := \Sigma_{0,\delta}(\xi - k^{(\ell)}) + \sum_{j=1}^{\ell} \omega_\delta(k_j)$$
$$\Sigma_{0,\delta}^{(\ell)}(\xi) := \inf_{\underline{k}\in\mathbb{R}^{\ell\nu}} \Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k}).$$

Lemma 3.2. *Fix $\delta > 0$. Then the spectrum of $H_\delta(\xi)$ below $\min_{\ell\geq 1} \Sigma_{0,\delta}^{(\ell)}(\xi)$ consists at most of isolated eigenvalues with finite multiplicity.*

Proof. Let $\mathfrak{h}_\infty = \{f \in L^2(\mathbb{R}^\nu) | \forall \eta \in \delta\mathbb{Z}^\nu \cap \Lambda_\delta : \int_{\mathcal{K}_\delta(\eta)} f(k)dk = 0\}$, and $\mathfrak{h}_0 = \mathfrak{h}_\infty^\perp$ is the finite dimensional subspace consisting of functions vanishing outside Λ_δ and constant on $\mathcal{K}_\delta(\eta)$, $\eta \in \delta\mathbb{Z}^\nu$. Let $j = (j_0, j_\infty)$, where $j_\#$ are the orthogonal projections onto $\mathfrak{h}_\#$. Then $\check{\Gamma}(j)$, the geometric partition of unity recalled in Appendix A, is a unitary map from $\Gamma(L^2(\mathbb{R}^\nu))$ to $\Gamma(\mathfrak{h}_0) \otimes \Gamma(\mathfrak{h}_\infty)$. Below we will consider the operator $H_\delta(\xi)$ on the Fock space $\Gamma(\mathfrak{h}_0)$, on which it is naturally defined. We use the same notation for the restricted operator. As in [31, Section 2.5], we write

$$H_\delta(\xi) = \check{\Gamma}(j)^* H_\delta^{\mathrm{ext}}(\xi) \check{\Gamma}(j),$$

where we split $\Gamma(\mathfrak{h}_0)\otimes\Gamma(\mathfrak{h}_\infty)=\Gamma(\mathfrak{h}_0)\oplus[\oplus_{\ell=1}^\infty\Gamma(\mathfrak{h}_0)\otimes\Gamma^{(\ell)}(\mathfrak{h}_\infty)]$ and identify $\Gamma(\mathfrak{h}_0)\otimes\Gamma^{(\ell)}(\mathfrak{h}_\infty)$ with a subspace of $L^2_{\mathrm{sym}}(\mathbb{R}^{\ell\nu};\mathfrak{h}_0)$. Here the subscript sym indicates that the functions are symmetric under interchange of the $\mathbb{R}^\nu$-valued variables. We get

$$H_\delta^{\mathrm{ext}}(\xi)=H_\delta(\xi)\oplus\Big[\bigoplus_{\ell=1}^\infty\oint_{\mathbb{R}^{\ell\nu}}H_\delta^{(\ell)}(\xi;\underline{k})\,d\underline{k}\Big],\tag{3.4}$$

$$H_\delta^{(\ell)}(\xi;\underline{k})=H_\delta(\xi-k^{(\ell)})+\Big[\sum_{j=1}^{\ell}\omega_\delta(k_j)\Big]\mathbb{1}_{\Gamma(\mathfrak{h}_0)}.\tag{3.5}$$

We thus find

$$H_\delta^{\mathrm{ext}}(\xi)\,\mathbb{1}_{\Gamma(\mathfrak{h}_0)}\otimes\mathbb{1}(N\ge1)\ge\min_{\ell\ge1}\Sigma_{0,\delta}^{(\ell)}(\xi)\mathbb{1}_{\Gamma(\mathfrak{h}_0)}\otimes\mathbb{1}(N\ge1).\tag{3.6}$$

Since $(H_\delta(\xi)+\mathrm{i})^{-1}\otimes\mathbb{1}(N=0)$ is compact we find for any $f\in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(f)\subset(-\infty,\min_{\ell\ge1}\Sigma_{0,\delta}^{(\ell)}(\xi))$, that $f(H_\delta(\xi))$ is compact. This completes the proof. $\square$

Lemma 3.3. *Let $v,\tilde v\in L^2(\mathbb{R}^\nu)$. Then for $\psi\in\mathcal{D}$ we have*

$$|\langle\psi,\Phi(v)\psi\rangle|\le2\omega_0^{-\frac12}\|v\|\|\psi\|\big(|\langle\psi,[H_0(\xi)+\Phi(\tilde v)]\psi\rangle|^{\frac12}+|\langle\psi,\Phi(\tilde v)\psi\rangle|^{\frac12}\big)\tag{3.7}$$

$$|\langle\psi,\Phi(v)\psi\rangle|\le4\omega_0^{-\frac12}\|v\|\|\psi\||\langle\psi,[H_0(\xi)+\Phi(v)]\psi\rangle|^{\frac12}+4\omega_0^{-1}\|v\|^2\|\psi\|^2.\tag{3.8}$$

The estimates (3.7) and (3.8) hold also with $H_0(\xi)$ replaced by $H_{0,\delta}(\xi)$.

Proof. Recall the formula

$$(\mathbf{a}(v)\psi)^{(n)}(k_1,\dots,k_n)=\sqrt{n+1}\int_{\mathbb{R}^\nu}\overline{v(k_{n+1})}\psi^{(n+1)}(k_1,\dots,k_{n+1})dk_{n+1}\ \text{a.e.},\tag{3.9}$$

which, for $\psi\in\mathcal{C}_0^\infty$, is meaningful for any distribution v. Using this identity we get

$$\begin{aligned}
&|\langle\varphi,\mathbf{a}(v)\psi\rangle|\\
&\quad\le\sum_{n=0}^\infty\int_{\mathbb{R}^{(n+1)\nu}}|\psi^{(n)}(k_1,\dots,k_n)|\sqrt{n+1}|v(k_{n+1})\psi^{(n+1)}(k_1,\dots,k_{n+1})|d\underline{k}\\
&\quad\le\|\omega^{-\frac12}v\|\sum_{n=0}^\infty\|\varphi^{(n)}\|\\
&\qquad\times\Big((n+1)\int_{\mathbb{R}^{(n+1)\nu}}\omega(k_{n+1})|\psi^{(n+1)}(k_1,\dots,k_{n+1})|^2dk_1\cdots dk_{n+1}\Big)^{\frac12}\\
&\quad\le\|\omega^{-\frac12}v\|\|\varphi\|\langle\psi,d\Gamma(\omega)\psi\rangle^{\frac12}.
\end{aligned}\tag{3.10}$$

This implies, for $\psi\in\mathcal{D}$, the bound

$$|\langle\psi,\Phi(v)\psi\rangle|\le2\omega_0^{-\frac12}\|v\|\|\psi\|\langle\psi,d\Gamma(\omega)\psi\rangle^{\frac12}.\tag{3.11}$$

To prove (3.7) we use (3.11) and estimate for $v,\tilde v\in L^2(\mathbb{R}^\nu)$ and $\psi\in\mathcal{D}$

$$\begin{aligned}
|\langle\psi,\Phi(v)\psi\rangle|&\le2\omega_0^{-\frac12}\|v\|\|\psi\|\langle\psi,H_0(\xi)\psi\rangle^{\frac12}\\
&\le2\omega_0^{-\frac12}\|v\|\|\psi\|\big(|\langle\psi,[H_0(\xi)+\Phi(\tilde v)]\psi\rangle|^{\frac12}+|\langle\psi,\Phi(\tilde v)\psi\rangle|^{\frac12}\big).
\end{aligned}\tag{3.12}$$

This proves (3.7). Taking $v = \tilde{v}$ yields (3.8) (after a small computation). Since we only used $\omega(k) \geq \omega_0 > 0$ and $\Omega(\eta) \geq 0$ we conclude the bounds also with $H_0(\xi)$ replaced by $H_{0,\delta}(\xi)$. □

An application of (3.7) with $\tilde{v} = 0$ yields the following useful a priori lower bound, which is valid for all ξ

$$H(\xi) \geq -\omega_0^{-1}\|v\|^2 \mathbb{1}_{\mathcal{F}}. \tag{3.13}$$

The same lower bound holds for $H_\delta(\xi)$ (note that $\|v_\delta\| = \|P_\delta v\| \leq \|v\|$).

Lemma 3.4. *Let $W_\delta(\xi) = H_\delta(\xi) - H(\xi)$. There exist a family of ξ-independent positive numbers $\{C_\delta\}_{\delta>0}$, with $\lim_{\delta\to 0} C_\delta = 0$, such that for any $\varphi, \psi \in \mathcal{D}$ we have*

$$|\langle \varphi, W_\delta(\xi)\psi\rangle| \leq C_\delta \big(\|H(\xi)\varphi\| + \|\varphi\|\big)\big(\|H_\delta(\xi)\psi\| + \|\psi\|\big).$$

Proof. As an operator on $\mathcal{C}_0^\infty$ we have

$$W_\delta(\xi) = d\Gamma(\omega_\delta - \omega) + \Omega(\xi - d\Gamma(\eta_k)) - \Omega(\xi - d\Gamma(k)) + \Phi(v_\delta - v). \tag{3.14}$$

We estimate the terms of $W_\delta(\xi)$ one by one beginning with $d\Gamma(\omega_\delta - \omega)$.

Since ω_δ is only discretized in a compact set (uniformly in $0 < \delta < \delta_0$, for any δ_0) we obtain a family of constants $\widetilde{C}_\delta > 0$ such that $\sup_{k\in\mathbb{R}^\nu} |\omega(k) - \omega_\delta(k)| \leq \widetilde{C}_\delta$ and $\lim_{\delta\to 0} \widetilde{C}_\delta = 0$. This implies (cf. the operator bound [31, (2.11)])

$$|\langle \varphi, d\Gamma(\omega_\delta - \omega)\psi\rangle| \leq \widetilde{C}_\delta \|N\varphi\|\|\psi\| \leq \widetilde{C}_\delta \omega_0^{-1}\|H_0(\xi)\varphi\|\|\psi\|. \tag{3.15}$$

Let $\tilde{v} \in L^2(\mathbb{R}^\nu)$. From (3.10) we get

$$\begin{aligned} |\langle \varphi, \Phi(\tilde{v})\,\psi\rangle| &\leq 2\omega_0^{-\frac{1}{2}}\|\tilde{v}\|\|\varphi\|\|d\Gamma(\omega)^{\frac{1}{2}}\psi\| \\ &\leq 2\omega_0^{-\frac{1}{2}}\|\tilde{v}\|\|\varphi\|\big(\min\{\|H_0(\xi)^{\frac{1}{2}}\psi\|, \|H_{0,\delta}(\xi)^{\frac{1}{2}}\psi\|\} + \|\psi\|\big), \end{aligned} \tag{3.16}$$

which implies the bound

$$|\langle \varphi, \Phi(\tilde{v})\,\psi\rangle| \leq 2\omega_0^{-\frac{1}{2}}\|\tilde{v}\|\|\varphi\|(\|H_{0,\delta}(\xi)^{\frac{1}{2}}\psi\| + \|\psi\|). \tag{3.17}$$

Finally we use Condition 2.1 to estimate, writing $z_t(k) = (1-t)\eta_k + tk$,

$$\begin{aligned} &|\langle \varphi, (\Omega(\xi - d\Gamma(\eta_k)) - \Omega(\xi - d\Gamma(k))\psi\rangle| \\ &\quad\leq |\langle \varphi, \nabla\Omega(\xi - d\Gamma(k)) \cdot d\Gamma(\eta_k - k)\psi\rangle| \\ &\qquad + \int_0^1 |\langle d\Gamma(\eta_k - k)\varphi, \nabla^2\Omega(\xi - d\Gamma(z_t(k)))d\Gamma(\eta_k - k)\psi\rangle| dt \\ &\quad\leq \delta\nu^{\frac{1}{2}}\|\nabla\Omega(\xi - d\Gamma(k))\varphi\|\|N\psi\| + \delta^2\nu\widetilde{C}_\Omega\|N\varphi\|\|N\psi\| \\ &\quad\leq \delta[\nu^{\frac{1}{2}}C_\Omega\omega_0^{-1} + \delta\nu\widetilde{C}_\Omega\omega_0^{-2}](\|H_0(\xi)\varphi\| + \|\varphi\|)\|H_{0,\delta}(\xi)\psi\|. \end{aligned} \tag{3.18}$$

Combining (3.3), (3.15), (3.17) applied with $\tilde{v} = v_\delta - v$, and (3.18) we get, for $\delta > 0$,

$$|\langle \varphi, W_\delta(\xi)\psi\rangle| \leq C'_\delta(\|H_0(\xi)\varphi\| + \|\varphi\|)(\|H_{0,\delta}(\xi)\psi\| + \|\psi\|), \tag{3.19}$$

where $\lim_{\delta\to 0} C'_\delta = 0$ and C'_δ does not depend on ξ. The bound extends by continuity to $\psi \in \mathcal{D}$.

Abbreviate $C = C(v,\omega_0) := 2\|v\|\omega_0^{-\frac{1}{2}}$. To pass to the interacting Hamiltonians we note that taking supremum over normalized φ's in (3.16), with $\tilde{v} = v$, yields

$$\begin{aligned}\|H_0(\xi)\psi\| &\le \|H(\xi)\psi\| + C(\langle \psi, H_0(\xi)\psi\rangle^{\frac{1}{2}} + \|\psi\|)\\ &\le \|H(\xi)\psi\| + \tfrac{1}{2}\|H_0(\xi)\psi\| + [\tfrac{1}{2}C^2 + C]\|\psi\|.\end{aligned}$$

Rearrangement and a repetition of the argument with v and $H_0(\xi)$ replaced by v_δ and $H_{0,\delta}(\xi)$ give for $\psi \in \mathcal{D}$

$$\begin{aligned}\|H_0(\xi)\psi\| &\le 2\|H(\xi)\psi\| + [C^2 + 2C]\|\psi\|\\ \|H_{0,\delta}(\xi)\psi\| &\le 2\|H_\delta(\xi)\psi\| + [C^2 + 2C]\|\psi\|.\end{aligned}$$

Recall that $\|v_\delta\| \le \|v\|$ so that the C's can be taken identical. Plugging these two bounds into (3.19) yields the lemma. □

We get in particular the following corollary.

Corollary 3.5. *Let $v \in L^2(\mathbb{R}^\nu)$. The family $H_\delta(\xi)$ converges, as $\delta \to 0$, in norm resolvent sense to $H(\xi)$.*

Proof of Theorem 2.1 i), for v's with compact support: Let $\psi \in \mathcal{F}$ and compute for $z \in \mathbb{C}\backslash[\sigma(H(\xi)) \cup \sigma(H_\delta(\xi))]$

$$\langle \psi, [(H(\xi)-z)^{-1} - (H_\delta(\xi)-z)^{-1}]\psi\rangle = \langle (H(\xi)-\bar{z})^{-1}\psi, W_\delta(\xi)(H_\delta - z)^{-1}\psi\rangle \quad (3.20)$$

We abbreviate

$$\lambda_0 := -\omega_0^{-1}\|v\|^2 - 1. \quad (3.21)$$

Then by (3.13) we find that for all $\xi \in \mathbb{R}^\nu$

$$H(\xi) \ge (\lambda_0 + 1)\mathbb{1}_{\mathcal{F}} \quad \text{and} \quad H_\delta(\xi) \ge (\lambda_0 + 1)\mathbb{1}_{\mathcal{F}}. \quad (3.22)$$

Let $\epsilon > 0$. Putting together Lemma 3.4, applied with a normalized $\psi_\epsilon \in \mathbb{1}(\Sigma_0(\xi) \le H(\xi) \le \Sigma_0(\xi) + \epsilon)\mathcal{F}$, (3.20), (3.22) and the Rayleigh-Ritz variational principle we estimate

$$\begin{aligned}&(\Sigma_0(\xi) - \lambda_0)^{-1} - (\Sigma_{0,\delta}(\xi) - \lambda_0)^{-1}\\ &\quad \le \langle \psi_\epsilon, [(H(\xi)-\lambda_0)^{-1} - (H_\delta(\xi)-\lambda_0)^{-1}]\psi_\epsilon\rangle + \epsilon\\ &\quad \le C_\delta\big(\|H(\xi)(H(\xi)-\lambda_0)^{-1}\psi_\epsilon\|\|H_\delta(\xi)(H_\delta(\xi)-\lambda_0)^{-1}\psi_\epsilon\| + 1\big) + \epsilon\\ &\quad \le C_\delta\big((1+|\lambda_0|)^2 + 1\big) + \epsilon.\end{aligned}$$

Taking ϵ to zero and repeating the argument for $(\Sigma_{0,\delta}(\xi) - \lambda_0)^{-1} - (\Sigma_0(\xi) - \lambda_0)^{-1}$, we get

$$\big|(\Sigma_0(\xi) - \lambda_0)^{-1} - (\Sigma_{0,\delta}(\xi) - \lambda_0)^{-1}\big| \le C_\delta\big((1+|\lambda_0|)^2 + 1\big). \quad (3.23)$$

Here it is important that $C_\delta \to 0$ and is independent of ξ.

Fix $\xi \in \mathbb{R}^\nu$, $\ell \in \mathbb{N}$ and let $\bar\Sigma \in \mathbb{R}$. The bound above implies the following statement. There exists $\bar\delta = \bar\delta(\bar\Sigma)$ such that for $0 < \delta < \bar\delta$:

$$\begin{aligned}\{\underline{k} \in \mathbb{R}^{\ell\nu} \mid \Sigma_0^{(\ell)}(\xi;\underline{k}) \le \bar\Sigma\} &\subset \{\underline{k} \in \mathbb{R}^{\ell\nu} \mid \Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k}) \le 2\bar\Sigma\}\\ &\subset \{\underline{k} \in \mathbb{R}^{\ell\nu} \mid \Sigma_0^{(\ell)}(\xi;\underline{k}) \le 3\bar\Sigma\}.\end{aligned} \tag{3.24}$$

From (3.23) and the first inclusion above we find that for any $\bar\Sigma \in \mathbb{R}$

$$\lim_{\delta\to 0} \inf_{\underline{k}\in\{\underline{k}'|\Sigma_0^{(\ell)}(\xi;\underline{k}')\le\bar\Sigma\}} \Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k}) = \inf_{\underline{k}\in\{\underline{k}'|\Sigma_0^{(\ell)}(\xi;\underline{k}')\le\bar\Sigma\}} \Sigma_0^{(\ell)}(\xi;\underline{k}).$$

Let $\bar\Sigma = \Sigma_0^{(\ell)}(\xi;\underline{0})$, such that the first set in (3.24) is non-empty. We get the bounds

$$\begin{aligned}\limsup_{\delta\to 0} \Sigma_{0,\delta}^{(\ell)}(\xi) &\le \lim_{\delta\to 0} \inf_{\underline{k}\in\{\underline{k}'|\Sigma_0^{(\ell)}(\xi;\underline{k}')\le\bar\Sigma\}} \Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k})\\ &= \inf_{\underline{k}\in\{\underline{k}'|\Sigma_0^{(\ell)}(\xi;\underline{k}')\le\bar\Sigma\}} \Sigma_0^{(\ell)}(\xi;\underline{k})\\ &= \Sigma_0^{(\ell)}(\xi)\\ &= \inf_{\underline{k}\in\{\underline{k}'|\Sigma_0^{(\ell)}(\xi;\underline{k}')\le 3\bar\Sigma\}} \Sigma_0^{(\ell)}(\xi;\underline{k})\\ &= \lim_{\delta\to 0} \inf_{\underline{k}\in\{\underline{k}'|\Sigma_0^{(\ell)}(\xi;\underline{k}')\le 3\bar\Sigma\}} \Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k})\\ &\le \liminf_{\delta\to 0} \inf_{\underline{k}\in\{\underline{k}'|\Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k}')\le 2\bar\Sigma\}} \Sigma_{0,\delta}^{(\ell)}(\xi;\underline{k})\\ &= \liminf_{\delta\to 0} \Sigma_{0,\delta}^{(\ell)}(\xi).\end{aligned}$$

Hence

$$\lim_{\delta\to 0} \Sigma_{0,\delta}^{(\ell)}(\xi) = \Sigma_0^{(\ell)}(\xi). \tag{3.25}$$

(Note that norm-resolvent convergence is not sufficient to get the above limit. In [16] the fact that $\omega(k) = |k| \to \infty$, $|k| \to \infty$, was used instead to ensure that only convergence for a compact subset of ξ's was needed.)

Now i) is an easy consequence of (3.25), Lemma 3.2 and Corollary 3.5.

The following lemma will be used to extend the result just proven to general $v \in L^2(\mathbb{R}^\nu)$. We use the notation introduced in the paragraph leading into Corollary 2.5.

Lemma 3.6. *Let $v_1, v_2 \in L^2(\mathbb{R}^\nu)$ and suppose Conditions 2.1 and 2.2. Then*

$$\Sigma_{0,2}(\xi) - \Sigma_{0,1}(\xi) \le \|v_1 - v_2\|\big(4\omega_0^{-\frac12}|\Sigma_{0,1}(\xi)|^{\frac12} + 6\omega_0^{-1}\|v_1\|\big).$$

Proof. Let $\epsilon > 0$. We apply (3.8) with $v = v_1$ and $\psi = \psi_{\epsilon,1} \in \mathbb{1}(\Sigma_{0,1}(\xi) \le H_1(\xi) \le \Sigma_{0,1}(\xi) + \epsilon)\mathcal{F}$, which we take to be normalized. We thus get, abbreviating $c = 2\omega_0^{-\frac12}$,

$$|\langle\psi_{\epsilon,1}, \Phi(v_1)\psi_{\epsilon,1}\rangle| \le 2c\|v_1\|(|\Sigma_{0,1}(\xi)| + \epsilon)^{\frac12} + c^2\|v_1\|^2.$$

Next we apply (3.7) with $v = v_2 - v_1$, $\tilde{v} = v_1$ and $\psi = \psi_{\epsilon,1} \in \mathbb{1}(\Sigma_{0,1}(\xi) \leq H_1(\xi) \leq \Sigma_{0,1}(\xi) + \epsilon)\mathcal{F}$ in conjunction with the Rayleigh-Ritz variational principle to obtain (with c as above)

$$\begin{aligned}
&\Sigma_{0,2}(\xi) - \Sigma_{0,1}(\xi) - \epsilon \\
&\quad \leq |\langle \psi_{\epsilon,1}, \Phi(v_2 - v_1)\psi_{\epsilon,1}\rangle| \\
&\quad \leq c\|v_1 - v_2\|\big((|\Sigma_{0,1}(\xi)| + \epsilon)^{\frac{1}{2}} + (2c)^{\frac{1}{2}}\|v_1\|^{\frac{1}{2}}(|\Sigma_{0,1}(\xi)| + \epsilon)^{\frac{1}{4}} + c\|v_1\|\big) \\
&\quad \leq \|v_1 - v_2\|\big(2c(|\Sigma_{0,1}(\xi)| + \epsilon)^{\frac{1}{2}} + \tfrac{3}{2}c^2\|v_1\|\big).
\end{aligned}$$

Taking ϵ to zero concludes the proof. □

Proof of Theorem 2.1 i) for general v: Let $v \in L^2(\mathbb{R}^\nu)$ and define for $\Lambda > 0$ a cutoff coupling function $v_\Lambda := \mathbb{1}(|k| \leq \Lambda)v$.

Denote by $\Sigma_{0,\Lambda}(\xi)$ the bottom of the spectrum of $H_\Lambda(\xi) = H_0(\xi) + \Phi(v_\Lambda)$, and by $\Sigma_{\mathrm{ess},\Lambda}(\xi)$ the usual function constructed from $\Sigma_{0,\Lambda}$, cf. (2.7).

Applying Lemma 3.6 twice we find that

$$|\Sigma_0(\xi) - \Sigma_{0,\Lambda}(\xi)| \leq C\|\mathbb{1}(|k| > \Lambda)v\|(\max\{|\Sigma_0(\xi)|^{\frac{1}{2}}, |\Sigma_{0,\Lambda}(\xi)|^{\frac{1}{2}}\} + \|v\|^{\frac{1}{2}}),$$

where C does not depend on ξ. This estimate implies

$$|(\Sigma_0(\xi) - \lambda_0)^{-1} - (\Sigma_{0,\Lambda}(\xi) - \lambda_0)^{-1}| \leq C_\Lambda,$$

where $\lim_{\Lambda\to\infty} C_\Lambda = 0$, and C_Λ does not depend on ξ. Here λ_0 is as in (3.21). This estimate replaces (3.23) in the proof for the compactly supported case, and the rest of the proof is identical, except for the last line where the compactness now comes from Theorem 2.1 i), with compactly supported v, and not from Lemma 3.2.

To end this section we explain how to verify Theorem 2.1 ii). Expanding on the notation (2.8), we introduce for $n \geq 1$

$$\mathcal{I}_0^{(n)}(\xi) := \{\underline{k} \in \mathbb{R}^{n\nu} | \xi - k^{(n)} \in \mathcal{I}_0\}. \tag{3.26}$$

See (2.4) for the notation $k^{(n)}$. The following crucial lemma was proved in [31].

Lemma 3.7. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 2.1 and 2.2. Let $\xi \in \mathbb{R}^\nu$, $n \geq 1$ and $\underline{k} \in \mathbb{R}^{n\nu}$. If $\Sigma_0^{(n)}(\xi;\underline{k}) < \inf_{n'>n} \Sigma_0^{(n')}(\xi)$, then $\underline{k} \in \mathcal{I}_0^{(n)}(\xi)$.*

Proof of Theorem 2.1 ii): First we note that the construction of Weyl sequences in the proof of the HVZ theorem in [31, Section 3.2] goes through in exactly the same way, and implies that for all $\xi \in \mathbb{R}^\nu$

$$\big\{\Sigma_0^{(n)}(\xi;\underline{k}) \big| n \geq 1 \text{ and } \underline{k} \in \mathcal{I}_0^{(n)}(\xi)\big\} \subset \sigma_{\mathrm{ess}}(H(\xi)). \tag{3.27}$$

Let n_0 be the largest n such that $\Sigma_0^{(n)}(\xi) = \Sigma_{\mathrm{ess}}(\xi)$. Then, by Lemma 3.7, $\Sigma_0^{(n_0)}(\xi)$, and hence $\Sigma_{\mathrm{ess}}(\xi)$, can be approximated by a sequence $\{\Sigma_0^{(n_0)}(\xi;\underline{k}_\ell)\}_{\ell\in\mathbb{N}}$, with $\underline{k}_\ell \in \mathcal{I}_0^{(n_0)}(\xi)$. This observation together with (3.27) implies the inequality $\Sigma_{\mathrm{ess}}(\xi) \geq \inf \sigma_{\mathrm{ess}}(H(\xi))$. In conjunction with Theorem 2.1 i) we find that

$$\inf \sigma_{\mathrm{ess}}(H(\xi)) = \Sigma_{\mathrm{ess}}(\xi). \tag{3.28}$$

It remains to prove that $\Sigma_0^{(n)}(\xi;\underline{k}) \in \sigma_{\mathrm{ess}}(H(\xi))$ if $\underline{k} \notin \mathcal{I}_0^{(n)}(\xi)$. For $\underline{k} \notin \mathcal{I}_0^{(n)}(\xi)$ we have, by (3.28), that $\Sigma_0(\xi - k^{(n)}) = \Sigma_{\mathrm{ess}}(\xi - k^{(n)})$. By definition of Σ_{ess}, there exists n' and a sequence $\{\underline{k}'_j\}_{j\in\mathbb{N}} \subset \mathcal{I}_0^{(n')}(\xi - k^{(n)})$ such that

$$\begin{aligned}
\Sigma_0^{(n)}(\xi;\underline{k}) &= \Sigma_0(\xi - k^{(n)}) + \sum_{i=1}^{n} \omega(k_i) \\
&= \Sigma_{\mathrm{ess}}(\xi - k^{(n)}) + \sum_{i=1}^{n} \omega(k_i) \\
&= \lim_{j\to\infty} \Sigma_0^{(n')}(\xi - k^{(n)};\underline{k}'_j) + \sum_{i=1}^{n} \omega(k_i) \\
&= \lim_{j\to\infty} \Sigma_0^{(n+n')}(\xi;(\underline{k},\underline{k}'_j)).
\end{aligned}$$

Since $(\underline{k},\underline{k}_j) \in \mathcal{I}_0^{(n+n')}(\xi)$ and the essential spectrum is a closed set, this together with (3.27) concludes the proof.

In the last paragraph of the proof above we demonstrated the following

Corollary 3.8. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 2.1 and 2.2. The closure of the set $\{\Sigma_0^{(n)}(\xi;\underline{k}) | n \in \mathbb{N}, \underline{k} \in \mathcal{I}_0^{(n)}(\xi)\}$ equals $\{\Sigma_0^{(n)}(\xi;\underline{k}) | n \in \mathbb{N}, \underline{k} \in \mathbb{R}^{n\nu}\}$.*

4 On uniqueness of ground states

In this section we prove Theorem 2.3 and Corollary 2.5 i).

As for Theorem 2.3, we can assume that v is not the zero function, since the theorem is trivially satisfied if the system is uncoupled. Let $A \subset \mathbb{R}^\nu$ be the Lebesgue measurable set

$$A := \{k | v(k) \neq 0\}, \tag{4.1}$$

which has non-zero measure.

We define $d\Gamma(\omega) + \Omega(\xi - d\Gamma(k))$ directly on $\Gamma(L^2(A))$ and denote it by $H_0(\xi;A)$. The operator $H(\xi;A) = H_0(\xi;A) + \int_A \{v(k)\mathbf{a}^*(k) + \overline{v(k)}\mathbf{a}(k)\}dk$ is defined on $\Gamma(L^2(A))$ by the Kato-Rellich Theorem. We abbreviate in the following $\Phi_A(v) = \int_A \{v(k)\mathbf{a}^*(k) + \overline{v(k)}\mathbf{a}(k)\}dk$.

The Perron-Frobenius argument given in [31, Section 3.3] (cf. also [16, Section 3.2]) yields that a groundstate of $H(\xi)$, if it exists, is non-degenerate and the eigenfunction $\psi_{\xi;A} = (\psi_{\xi;A}^{(0)}, \ldots, \psi_{\xi;A}^{(n)}, \ldots)$ can be chosen such that

$$(-1)^n \overline{v(k_1)} \cdots \overline{v(k_n)} \psi_{\xi;A}^{(n)}(k_1, \ldots, k_n) > 0,$$

for almost every $(k_1, \ldots, k_n) \in A^n$. We note that in [16] and [31] only real-valued v was considered. The argument however works also for complex valued v. The important observation is that $-\mathbf{a}^*(v)$ and $-\mathbf{a}(v)$ preserve the Hilbert cone. Alternatively, one can reduce the problem to the case $v \geq 0$, which was treated in [16], using the following observation: Let $\varphi : \mathbb{R}^\nu \to \mathbb{R}$ be measurable. Then the unitary transform $\Gamma(e^{\mathrm{i}\varphi})$ satisfies $\Gamma(e^{\mathrm{i}\varphi})^* = \Gamma(e^{-\mathrm{i}\varphi})$ and

$$\Gamma(e^{\mathrm{i}\varphi})(H_0(\xi) + \Phi(v))\Gamma(e^{-\mathrm{i}\varphi}) = H_0(\xi) + \Phi(e^{i\varphi}v). \tag{4.2}$$

Let j_0 and j_∞ be the restriction maps from $L^2(\mathbb{R}^\nu)$ to $L^2(A)$ and $L^2(A^c)$ respectively. Then

$$\check{\Gamma}(j) : \Gamma(L^2(\mathbb{R}^\nu)) \to \Gamma(L^2(A)) \otimes \Gamma(L^2(A^c)) = \Gamma(L^2(A)) \oplus [\oplus_{\ell=1}^\infty \Gamma^{(\ell)}(L^2(A^c))]$$

is unitary and

$$H(\xi) = \check{\Gamma}(j)^* \Big\{ H(\xi; A) \oplus \Big[\bigoplus_{\ell=1}^\infty \oint_{(A^c)^\ell} H^{(\ell)}(\xi; \underline{k}, A)\, d\underline{k} \Big] \Big\} \check{\Gamma}(j). \tag{4.3}$$

From the Rayleigh-Ritz variational principle we find

$$\inf \sigma(H^{(\ell)}(\xi; \underline{k}, A)) \geq \inf \sigma(H^{(\ell)}(\xi; \underline{k})).$$

From this estimate we get

$$\begin{aligned}
&\inf \sigma\Big(\oint_{(A^c)^\ell} H^{(\ell)}(\xi; \underline{k}, A)\, d\underline{k} \Big) \\
&\quad \geq \inf_{\underline{k} \in (A^c)^\ell} \inf \sigma(H^{(\ell)}(\xi; \underline{k}, A)) \\
&\quad \geq \inf_{\underline{k} \in (A^c)^\ell} \inf \sigma(H^{(\ell)}(\xi; \underline{k})) \\
&\quad = \inf_{\underline{k} \in (A^c)^\ell} \Sigma_0^{(\ell)}(\xi; \underline{k}) \geq \Sigma_0^{(\ell)}(\xi).
\end{aligned}$$

We thus get

$$\inf \sigma\Big(\bigoplus_{\ell=1}^\infty \oint_{(A^c)^\ell} H^{(\ell)}(\xi - k^{(\ell)}; A)\, d\underline{k} \Big) \;\geq\; \Sigma_{\mathrm{ess}}(\xi). \tag{4.4}$$

Hence, if $\Sigma_0(\xi)$ is an isolated eigenvalue, i.e. $\Sigma_0(\xi) < \Sigma_{\mathrm{ess}}(\xi)$, then a corresponding eigenfunction must have the form $\psi_\xi = (\psi_{\xi;A}, 0)$, where $\psi_{\xi;A}$ is a groundstate of $H(\xi; A)$. The result now follows from the earlier discussion.
Proof of Corollary 2.5 i): First suppose $\Sigma_{0,2}(\xi)$ is an isolated groundstate and let $\psi_{0,2}$ denote the unique groundstate satisfying (2.10) and (2.11). Then

$$\Sigma_{0,1}(\xi) \leq \langle \psi_{0,2}, H_1(\xi)\psi_{0,2} \rangle = \Sigma_{0,2}(\xi) + \langle \psi_{0,2}, \phi(v_1 - v_2)\psi_{0,2} \rangle. \tag{4.5}$$

The last term on the right-hand side is non-positive under the hypothesis (2.14).

It remains to treat the case $\Sigma_{0,2}(\xi) = \Sigma_{\mathrm{ess},2}(\xi)$. For this we study the bottom of the essential spectrum. Use Theorem 2.1 ii), Corollary 3.8, and the result just proved to estimate

$$\begin{aligned}
\Sigma_{\mathrm{ess},2}(\xi) &= \min_n \inf_{\underline{k} \in \mathbb{R}^{n\nu}} \Sigma_{0,2}^{(n)}(\xi; \underline{k}) \\
&= \min_n \inf_{\underline{k} \in \mathcal{I}_{0,2}^{(n)}(\xi)} \Sigma_{0,2}(\xi - k^{(n)}) + \sum_{j=1}^n \omega(k_j) \\
&\geq \min_n \inf_{\underline{k} \in \mathcal{I}_{0,2}^{(n)}(\xi)} \Sigma_{0,1}(\xi - k^{(n)}) + \sum_{j=1}^n \omega(k_j) \\
&\geq \Sigma_{\mathrm{ess},1}(\xi).
\end{aligned} \tag{4.6}$$

Hence, $\Sigma_{0,1}(\xi) \leq \Sigma_{\mathrm{ess},1}(\xi) \leq \Sigma_{\mathrm{ess},2}(\xi) = \Sigma_{0,2}(\xi)$.

Another consequence of Theorem 2.3 is that if Ω is real analytic, then Σ_0 restricted to $\mathcal{I}_0$ is a real analytic function, cf. [15, Lemma 1.6].

5 Large total momenta

We begin with

Proposition 5.1. *Suppose $v \in L^2(\mathbb{R}^\nu)$ has compact support, Conditions 2.1 and 2.2, and*

$$\lim_{|k|\to\infty} \frac{\omega(k)}{|k|} = \lim_{|k|\to\infty} \frac{\omega(k)}{\Omega(k)+1} = 0 \text{ and } \gamma := \inf_{\xi} \inf_{\eta:|\eta|\geq|\xi|/2} \frac{\omega(\eta)}{\omega(\xi)} > 0. \tag{5.1}$$

Then $\mathcal{I}_0$ is a bounded set.

Proof. Let Λ be such that $v(k) = 0$ for $|k| > \Lambda$. Put $A_\Lambda = \{k \in \mathbb{R}^\nu | |k| \leq \Lambda\}$. Then $H(\xi)$ partitions as in (4.3), with A replaced by A_Λ. (Note that as opposed to the A in (4.1), v may vanish in A_Λ.)

Since 0 is the ground state energy at $\xi = 0$ for the uncoupled model, we have from Corollary 2.5 i) that

$$\Sigma_0(0) \leq 0. \tag{5.2}$$

We have the basic bound which follows from Theorem 2.1 ii) and (5.2)

$$\Sigma_{\mathrm{ess}}(\xi) \leq \Sigma_0^{(1)}(\xi;\xi) = \Sigma_0(0) + \omega(\xi) \leq \omega(\xi). \tag{5.3}$$

Secondly we write

$$\kappa(\xi) := \inf_{\eta:|\eta|\geq|\xi|/2} \frac{\Omega(\eta)}{\omega(\eta)}. \tag{5.4}$$

Abbreviate $c = \omega_0/(2\Lambda)$ and estimate from below

$$\begin{aligned} H_0(\xi;A_\Lambda) &= 1\!\!1(N \leq |\xi|/(2R))H_0(\xi;A_\Lambda) + 1\!\!1(N > |\xi|/(2R))H_0(\xi;A_\Lambda) \\ &\geq 1\!\!1(N \leq |\xi|/(2\Lambda))\Omega(\xi - d\Gamma(k)) + 1\!\!1(N > |\xi|/(2\Lambda))d\Gamma(\omega) \\ &\geq 1\!\!1(N \leq |\xi|/(2\Lambda))\kappa(\xi)\omega(\xi - d\Gamma(k)) + 1\!\!1(N > |\xi|/(2\Lambda))c|\xi| \\ &\geq \min\bigl\{\gamma\kappa(\xi), c|\xi|/\omega(\xi)\bigr\}\omega(\xi). \end{aligned} \tag{5.5}$$

Here we used γ from (5.1) and that $|\sum_{j=1}^n k_i| \leq |\xi|/2$, for $n \leq |\xi|/(2\Lambda)$. Since the constant in front of $\omega(\xi)$ goes to $+\infty$ as $|\xi| \to \infty$, we find in conjunction with (5.3) that there exists $C = C(\Lambda) > 0$ such that for $|\xi| \geq C$:

$$\Sigma_{\mathrm{ess}}(\xi) \leq \inf \sigma(H_0(\xi;A_\Lambda)). \tag{5.6}$$

But since $\Sigma_0(\xi) \leq \Sigma_{\mathrm{ess}}(\xi)$, this implies by the decomposition (4.3) and the estimate (4.4) that

$$\forall \xi \text{ s.t. } |\xi| \geq C: \quad \Sigma_0(\xi) = \Sigma_{\mathrm{ess}}(\xi). \tag{5.7}$$

□

In the following proposition we make use of a function $f : [0,\infty) \to (0,\infty)$ which is monotone non-decreasing and satisfies

$$\lim_{r\to\infty} \frac{f(r)}{r} = 0 \quad\text{and}\quad \rho := \inf_{r\geq 0} \frac{f(\frac{1}{2}r)}{f(r)} > 0. \tag{5.8}$$

Furthermore the function $g(r) := r/f(r)$ is assumed to be strictly monotone increasing. Below we will use the inverse g^{-1} of g, which is also strictly monotone increasing and satisfies $s/g^{-1}(s) \to 0$ as $s \to \infty$.

Theorem 5.2. *Let $\mathcal{U} \subset \mathbb{R}^\nu$ be an unbounded set and $v \in L^2(\mathbb{R}^\nu)$. Suppose Condition 2.1 and 2.2, and that there exists $C_\omega \geq 1$ such that*

$$\forall k \in \mathbb{R}^\nu : \omega(k) \geq C_\omega^{-1} f(|k|),\ \forall k \in \mathcal{U} :\ \omega(k) \leq C_\omega f(|k|) \tag{5.9}$$

$$\lim_{|k|\to\infty} \frac{\omega(k)}{\Omega(k)+1} = 0 \tag{5.10}$$

$$\|1\!\!1(|k| > \Lambda)v\| = o\big([\Lambda/g^{-1}(\Lambda)]^{\frac{1}{2}}\big). \tag{5.11}$$

Then

$$\lim_{|\xi|\to\infty, \xi\in\mathcal{U}} \Sigma_{\mathrm{ess}}(\xi) - \Sigma_0(\xi) = 0.$$

Remark 5.3. A choice of f above is $f(r) = (1+r)^t$ with $0 \leq t < 1$. The particular choice $t = 0$ applies to the Fröhlich polaron, and in this case the assumption on v is automatically satisfied for any $v \in L^2(\mathbb{R}^\nu)$ since $g(s) = g^{-1}(s) = s$. For general t the assumption on v becomes $\|1\!\!1(|k| > \Lambda)v\| = o(\Lambda^{-\frac{t}{2(1-t)}})$. One could also multiply $(1+r)^t$ by powers of logarithms or by iterated logarithms.

Proof. Abbreviate $A_\Lambda = \{k \in \mathbb{R}^\nu | |k| \leq \Lambda\}$. We repeat the estimate (5.5), taking into account the bounds in (5.8), (5.9) and (5.10), and obtain for $\xi \in \mathcal{U}$

$$\begin{aligned}
&H_0(\xi; A_\Lambda)\\
&\quad \geq 1\!\!1(N \leq |\xi|/(2\Lambda))\kappa(\xi)\omega(\xi - d\Gamma(k)) + 1\!\!1(N > |\xi|/(2\Lambda))\frac{\omega_0}{2\Lambda}|\xi|\\
&\quad \geq 1\!\!1(N \leq |\xi|/(2\Lambda))\kappa(\xi)C_\omega^{-1} f(|\xi|/2) + 1\!\!1(N > |\xi|/(2\Lambda))\frac{\omega_0}{2\Lambda} g(|\xi|)f(|\xi|)\\
&\quad \geq C_1 \min\{\kappa(\xi), \Lambda^{-1} g(|\xi|)\}\omega(\xi).
\end{aligned}$$

Here $\kappa(\xi)$ is defined in (5.4) and $C_1 = C_\omega^{-1}\min\{\rho C_\omega^{-1}, \omega_0/2\}$.

As for (5.2) we also have $\Sigma_{0,\Lambda}(0) \leq 0$. If Λ and ξ are such that

$$\xi \in \mathcal{U},\ C_1\kappa(\xi) \geq 1 \text{ and } C_1 g(|\xi|) \geq \Lambda \tag{5.12}$$

then we get as in the previous proof that $\Sigma_{0,\Lambda}(\xi) = \Sigma_{\mathrm{ess},\Lambda}(\xi)$ (cf. (5.6) and (5.7)).

From Lemma 3.6, Corollary 2.5 i) and (5.3) we find that

$$\begin{aligned}
0 &\leq \Sigma_{0,\Lambda}(\xi) - \Sigma_0(\xi)\\
&\leq C\|1\!\!1(|k| > \Lambda)v\|\big(\Sigma_0(\xi)^{\frac{1}{2}} + \|v\|^{\frac{1}{2}}\big)\\
&\leq C_2\|1\!\!1(|k| > \Lambda)v\|\omega(\xi)^{\frac{1}{2}}.
\end{aligned}$$

Now put $\Lambda = C_1^{-1} g(|\xi|)$, cf. (5.12). We can assume without loss that $C_1 \leq 1$. Then, using Corollary 2.5 i), we have the large $\xi \in \mathcal{U}$ asymptotic bound

$$\begin{aligned}\Sigma_{\mathrm{ess}}(\xi) - \Sigma_0(\xi) &= \Sigma_{\mathrm{ess}}(\xi) - \Sigma_{\mathrm{ess},\Lambda}(\xi) + \Sigma_{0,\Lambda}(\xi) - \Sigma_0(\xi)\\ &\leq \Sigma_{0,\Lambda}(\xi) - \Sigma_0(\xi)\\ &\leq C_2 \|\mathbb{1}(|k| > C_1^{-1} g(|\xi|)) v\| \omega(\xi)^{\frac{1}{2}}\\ &\leq C_2 \|\mathbb{1}(|k| > g(|\xi|)) v\| \omega(\xi)^{\frac{1}{2}}\\ &\leq o\big([g(|\xi|)/g^{-1}(g(|\xi|))]^{\frac{1}{2}}\big) f(|\xi|)^{\frac{1}{2}} = o(1).\end{aligned}$$

This establishes the theorem. □

Proof of Theorem 2.1 iii): If $\omega(k) \to \infty$, then iii) follows from ii). We can hence assume that (2.3) is satisfied.

Let $\xi \in \mathbb{R}^\nu$ and $E \geq \Sigma_{\mathrm{ess}}(\xi)$. Let n be the smallest integer such that $E \geq \Sigma_0^{(n)}(\xi)$ and $E < \Sigma_0^{(\tilde{n})}(\xi)$, for $\tilde{n} > n$. Let $\underline{k} = (k, \ldots, k)$, and consider a sequence with $|k| \to \infty$. Then, by Theorem 5.2, $\Sigma_0(\xi - nk) - \Sigma_{\mathrm{ess}}(\xi - nk) \to 0$, and we can thus for any $\epsilon > 0$ pick k such that $\Sigma_0(\xi - nk) \geq \Sigma_{\mathrm{ess}}(\xi - nk) - \epsilon$. For such a k we get

$$\begin{aligned}\Sigma_0^{(n)}(\xi, \underline{k}) &\geq \Sigma_{\mathrm{ess}}(\xi - nk) + n\omega(k) - \epsilon\\ &\geq \Sigma_0^{(\tilde{n})}(\xi - nk; \underline{\tilde{k}}) + n\omega(k) - 2\epsilon\\ &= \Sigma_0^{(n+\tilde{n})}(\xi; (\underline{k}, \underline{\tilde{k}})) - 2\epsilon\\ &\geq \Sigma_0^{(n+\tilde{n})}(\xi) - 2\epsilon,\end{aligned}$$

where $\tilde{n}$ and $\underline{\tilde{k}} \in \mathbb{R}^{\tilde{n}\nu}$ are chosen such that $\Sigma_{\mathrm{ess}}(\xi - nk) \geq \Sigma_0^{(\tilde{n})}(\xi - nk; \underline{\tilde{k}}) - \epsilon$. By continuity there exists k such that $E = \Sigma_0^{(n)}(\xi, \underline{k})$. The result now follows from ii).

Lemma 5.4. *Let $v \in L^2(\mathbb{R}^\nu)$ and assume Conditions 2.1, 2.2 and 2.4. Let $\xi \in \mathbb{R}^\nu$ and suppose $n \geq 1$ is such that $\Sigma_0^{(n)}(\xi) = \Sigma_{\mathrm{ess}}(\xi) < \min_{n'>n} \Sigma_0^{(n')}(\xi)$. Then there exists $R = R(\xi) > 0$ such that*

$$\inf_{\underline{k} \in \mathbb{R}^{n\nu}: |k^{(n)}| \geq R} \Sigma_0^{(n)}(\xi; \underline{k}) > \Sigma_0^{(n)}(\xi).$$

Proof. This is clearly true under the hypothesis (2.12). We can hence assume that (2.13) holds true.

Assume the lemma is false for some ξ. Fix $\mathcal{U} := \{\eta | \Sigma_0(\eta) \leq \Sigma_0^{(n)}(\xi)\}$, which is not an empty set since $\xi \in \mathcal{U}$. There exists a sequence $\{\underline{k}_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^{n\nu}$ such that $\xi - k_j^{(n)} \in \mathcal{U}$, $|k_j^{(n)}| \to \infty$ for $j \to \infty$, and $\Sigma_0^{(n)}(\xi) = \lim_{j\to\infty} \Sigma_0^{(n)}(\xi; \underline{k}_j)$. Let $\epsilon > 0$. There exists by Theorem 5.2 a j_0 such that for $j \geq j_0$

$$\begin{aligned}\Sigma_0^{(n)}(\xi) &\geq \Sigma_0^{(n)}(\xi; \underline{k}_j) - \epsilon\\ &= \Sigma_0(\xi - k_j^{(n)}) + \sum_{i=1}^{n} \omega(k_{j;i}) - \epsilon\\ &\geq \Sigma_{\mathrm{ess}}(\xi - k_j^{(n)}) + \sum_{i=1}^{n} \omega(k_{j;i}) - 2\epsilon.\end{aligned}$$

Fix a $j \geq j_0$ and pick $n' \geq 1$ and $\underline{k} \in \mathbb{R}^{n'\nu}$ such that

$$\Sigma_0^{(n)}(\xi) \geq \Sigma_0^{(n')}(\xi - k_j^{(n)}; \underline{k}) + \sum_{i=1}^{n} \omega(k_{j;i}) - 3\epsilon = \Sigma_0^{(n+n')}(\xi; (\underline{k}_j, \underline{k})) - 3\epsilon.$$

This estimate contradicts the choice of n. □

Proof of Corollary 2.5 ii): Suppose first $\Sigma_{0,2}(\xi)$ is an isolated eigenvalue. Under the extra assumption $v_1 \neq 0$ a.e. and $v_1 \neq v_2$, we get from (2.10) and (4.5) that $\Sigma_{0,1}(\xi) < \Sigma_{0,2}(\xi)$. Now suppose $\Sigma_{0,2}(\xi) = \Sigma_{\mathrm{ess},2}(\xi)$. By a compactness argument we conclude from Lemmas 3.7 and 5.4 as well as (4.6) that $\Sigma_{\mathrm{ess},1}(\xi) < \Sigma_{\mathrm{ess},2}(\xi)$. This completes the proof.

Recall that subadditivity implies that $n \to \Sigma_0^{(n)}(\xi)$ are non-decreasing functions. In the following lemma we show that under Condition 2.5 the functions are strictly increasing.

Lemma 5.5. *Assume Conditions 2.1, 2.2 and 2.5. Then for any $\xi \in \mathbb{R}^\nu$ and $n \geq 1$ we have $\Sigma_0^{(n)}(\xi) < \Sigma_0^{(n+1)}(\xi)$.*

Proof. If $\lim_{|k|\to\infty} \omega(k)^{-1} = 0$, the result follows from strict subadditivity (2.15). Hence we only have to deal with ω's satisfying (2.16).

Suppose the conclusion of the lemma is false. That is, there exist ξ and $n \geq 2$ such that $\Sigma_0^{(n-1)}(\xi) = \Sigma_0^{(n)}(\xi)$. We can without loss of generality suppose that $\min_{n'>n} \Sigma_0^{(n')}(\xi) > \Sigma_0^{(n)}(\xi)$. Let $\{\underline{k}_\ell\} \subset \mathbb{R}^{n\nu}$ be a sequence with $\lim_{\ell\to\infty} \Sigma_0^{(n)}(\xi; \underline{k}_\ell) = \Sigma_0^{(n)}(\xi)$. By Lemma 5.4 there exists $R > 0$ such that $|k_\ell^{(n)}| \leq R$ (for ℓ large). Suppose first that there exists R' such that for all $1 \leq j \leq n$ and $\ell \geq 1$ we have $|k_{\ell;j}| \leq R'$. Then by strict subadditivity (2.15), we obtain a contradiction. If all the $k_{\ell;j}$'s cannot be uniformly bounded, there exist $1 \leq i < j \leq n$ such that $|k_{\ell;i}|$ and $|k_{\ell;j}|$ diverge. This together with (2.16) also yields a contradiction. □

Remark 5.6. The assumption (2.16) is not just technical. As a counterexample consider the borderline case $\omega(k) = m(1+\exp(-k^2/2))$, which is strictly subadditive but just fails to satisfy (2.16) We take $v \equiv 0$ and for Ω we take $\eta^2/(2M)$ with m and M chosen such that $mM < 1$. This choice ensures that the function $k \to \Omega(-k) + \omega(k)$ has a global minimum at $k = 0$. We prove that for this example we have $\Sigma_0^{(2)}(0) = \Sigma_0^{(1)}(0)$: Assume this equalty is false, that is $\Sigma_0^{(1)}(0) < \Sigma_0^{(2)}(0)$. Then the assumption of Lemma 3.7 is satisfied with $\xi = 0$ and $n = 1$, which implies that

$$\Sigma_0^{(1)}(0) = \Sigma_{\mathrm{ess}}(0) = \min_k(\Omega(-k) + \omega(k)) = 2m.$$

On the other hand we obtain using $\Sigma_0(0) = \Omega(0) = 0$ for the uncoupled model

$$\Sigma_0^{(2)}(0) = \inf_{k_1,k_2}(\Sigma_0(-k_1 - k_2) + \omega(k_1) + \omega(k_2)) \leq \inf_k(\Sigma_0(0) + 2\omega(k)) = 2m.$$

This is a contradiction.

Proof of Theorem 2.6: We begin with i).

As demonstrated by Spohn, cf. [40, (5.14)], it suffices to show that $k \to \Sigma_0^{(1)}(\xi;k)$ has a minimizer for all ξ, with $\xi - k \in \mathcal{I}_0$. Here we refer to the simpler proof given in [31] which uses the assumption $\omega(k) \to \infty$ to obtain a minimizer.

We remark that the proof given in [31] only relies on: 1) The HVZ Theorem, 2) That ground state eigenfunctions are unique and can be chosen strictly positive with respect to the cone

$$\mathbf{C} := \{\psi \in \mathcal{F} | \psi^{(0)} \geq 0 \text{ and } \forall n \geq 1: \ (-1)^n \bar{v} \otimes \cdots \otimes \bar{v}\, \psi^{(n)} \geq 0 \text{ a.e.}\}.$$

There are n-copies of $\bar{v}$ in the tensor product. 3) For $\xi \notin \mathcal{I}_0$, the function $k \to \Sigma_0^{(1)}(\xi;k)$ attains its infimum at a momentum $k \in \mathcal{I}_0^{(1)}(\xi)$, cf. (3.26).

The proof given in [31], which is formulated for real-valued v, goes through for complex-valued v provided 1)-3) above is satisfied. Alternatively employ the transformation (4.2).

We remark that $\psi \in \mathbf{C}$ is said to be strictly positive if $\langle \psi, \varphi \rangle > 0$ for all $\varphi \in \mathbf{C} \backslash \{0\}$. The property 2) follows from Theorem 2.3 under the assumption (2.17).

To verify 3), we recall that subadditivity of ω implies $\Sigma_{\mathrm{ess}}(\xi) = \Sigma_0^{(1)}(\xi)$. For $\xi \notin \mathcal{I}_0$ we have $\Sigma_0(\xi) = \Sigma_{\mathrm{ess}}(\xi) = \Sigma_0^{(1)}(\xi)$. The existence of a minimizer k with $\xi - k \in \mathcal{I}_0$ now follows from Lemmas 5.4 and 5.5, applied with $n = 1$.

As for ii), the proof given in [31] relies on 1) and 2) above together with: 3') For $\xi \notin \mathcal{I}_0$, the set of global minima of $k \to \Sigma_0^{(1)}(\xi;k)$ is a bounded subset of $\mathcal{I}_0^{(1)}(\xi)$.

As above, 3') follows from Lemmas 5.4 and 5.5 and as above the proof is formulated for real-valued v, but goes through also for complex-valued v.

Let $E_0(\xi) := \inf \sigma(H_0(\xi))$ be the bottom of the spectrum for the uncoupled system. Then, by Corollary 2.5 i) and Lemma 3.6 (applied with $v_2 = 0$), there exist $C > 0$ such that

$$E_0(\xi) - C E_0(\xi)^{\frac{1}{2}} - C \leq \Sigma_0(\xi) \leq E_0(\xi). \tag{5.13}$$

The following proposition is concerned with the rate of growth at large total momentum of the bottom of the spectrum. It turns out that at linear growth a transition in behavior occurs

Proposition 5.7. *Assume Conditions 2.1 and 2.2.*

i) Suppose there exist $c > 0$ and $0 \leq s \leq 1$ such that

$$\omega(k) \geq c\langle k \rangle^s \text{ and } \ \Omega(\eta) \geq c\langle \eta \rangle^s - c. \tag{5.14}$$

Then there exists $C > 0$ such that

$$C^{-1}\langle \xi \rangle^s - C \leq \Sigma_0(\xi) \leq \omega(\xi) \tag{5.15}$$

ii) Suppose we have at least linear growth

$$\sup_{k \in \mathbb{R}^\nu} \frac{|k|}{\omega(k)} < \infty \quad \text{and} \quad \sup_{\eta \in \mathbb{R}^\nu} \frac{|\eta|}{\Omega(\eta) + 1} < \infty. \tag{5.16}$$

Then there exists $C \geq 1$ such that

$$C^{-1}|\xi| - C \leq \Sigma_0(\xi) \leq C|\xi| + C. \tag{5.17}$$

Proof. By (5.13) it suffices to establish the result with $v = 0$.

The upper bound in (5.15) follows from (5.3).

As for the upper bound in (5.17) let $n_\xi = [|\xi|] + 1$, where $[|\xi|]$ is the smallest integer less than or equal to $|\xi|$. Then

$$\Sigma_0(\xi) \le \Sigma_{\mathrm{ess}}(\xi) \le \Sigma_0(0) + n_\xi \omega(\xi/n_\xi) \le (|\xi| + 1)\omega(\xi/n_\xi).$$

For the second inequality in (5.17), we can thus take $C = \sup_{|k|\le 1} \omega(k)$.

Let $0 < \rho < 1$. For the lower bound we estimate

$$H_0(\xi) \ge \mathbb{1}(|\xi - d\Gamma(k)| \le \rho|\xi|) d\Gamma(\omega) + \mathbb{1}(|\xi - d\Gamma(k)| \ge \rho|\xi|)\Omega(\xi - d\Gamma(k)). \quad (5.18)$$

Let $C_1 := \sup_{k\in\mathbb{R}^\nu} \frac{|k|}{\omega(k)}$. Then for the first term we can estimate, using the constraint $|\xi - d\Gamma(k)| \le \rho|\xi|$ and the subadditivity of $t \to \langle t\rangle^s$ for $0 \le s \le 1$,

$$\begin{array}{ll} \text{Case i)} & d\Gamma(\omega) \ge c d\Gamma(\langle k\rangle^s) \ge c\langle d\Gamma(|k|)\rangle^s \ge c(1-\rho)^s|\xi|^s \\ \text{Case ii)} & d\Gamma(\omega) \ge C_1^{-1} d\Gamma(|k|) \ge C_1^{-1}|d\Gamma(k)| \ge (1-\rho)C_1^{-1}|\xi|. \end{array} \quad (5.19)$$

Write $C_2 = \sup_{\eta\in\mathbb{R}^\nu} \frac{|\eta|}{\Omega(\eta)+1}$. Then, under the constraint $|\xi - d\Gamma(k)| \ge \rho|\xi|$ we have

$$\begin{array}{ll} \text{Case i)} & \Omega(\xi - d\Gamma(k)) \ge c\langle \xi - d\Gamma(k)\rangle^s - c \ge c\rho^s|\xi|^s - c. \\ \text{Case ii)} & \Omega(\xi - d\Gamma(k)) \ge C_2^{-1}|\xi - d\Gamma(k)| - 1 \ge \rho C_2^{-1}|\xi| - 1. \end{array} \quad (5.20)$$

Combining (5.18)–(5.20) we obtain the lower bounds in (5.15) and (5.17). □

A Geometric partition of unity

We recall briefly here the construction of a partition of unity introduced by Dereziński and Gérard in [7, Section 2.13]. See also [31, Section 2.3].

We assume the reader is familiar with the second quantization functor Γ (cf. [4] or the two references mentioned in the previous paragraph).

First let $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ be Hilbert spaces. Then I is the canonical isomorphism $I : \Gamma(\mathfrak{h}_0 \oplus \mathfrak{h}_\infty) \to \Gamma(\mathfrak{h}_0) \otimes \Gamma(\mathfrak{h}_\infty)$ given by

$$I a^{\#}((f,g)) = (a^{\#}(f) \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\infty)} + \mathbb{1}_{\Gamma(\mathfrak{h}_0)} \otimes a^{\#}(g)) I \quad \text{and} \quad I\Omega = \Omega \otimes \Omega.$$

Here $a^{\#}(h)$ denotes either $a(h)$ or $a^*(h)$.

Let $j = (j_0, j_\infty) : \mathfrak{h} \to \mathfrak{h}_0 \oplus \mathfrak{h}_\infty$ satisfy $j_0^* j_0 + j_\infty^* j_\infty = \mathbb{1}_{\mathfrak{h}}$. Then the geometric partition of unity $\check\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}_0) \otimes \Gamma(\mathfrak{h}_\infty)$ is given by

$$\check\Gamma(j) := I\Gamma(j) \quad (A.1)$$

and is an isometry

$$\check\Gamma(j)^* \check\Gamma(j) = \mathbb{1}_{\Gamma(\mathfrak{h})}. \quad (A.2)$$

If furthermore $j_0 j_0^* = \mathbb{1}_{\mathfrak{h}_0}$ and $j_\infty j_\infty^* = \mathbb{1}_{\mathfrak{h}_\infty}$ then $\check\Gamma(j)$ is unitary.

B Weak ultraviolet singularities

In this appendix we treat the following type of couplings

$$V_{\mathrm{I}} := \{v \in L^2_{\mathrm{loc}}(\mathbb{R}^\nu) | \omega^{-\frac{1}{2}} v \in L^2(\mathbb{R}^\nu)\}. \tag{B.1}$$

We remark that if ω is bounded $V_{\mathrm{I}} = L^2(\mathbb{R}^\nu)$, and there are no ultraviolet singular couplings in this class. We nevertheless include this appendix for two reasons. For the Nelson model this is of relevance and secondly it may serve as a warmup for the more involved renormalization procedures discussed at the end of Section 2.

If $v \in V_{\mathrm{I}}$, the Hamiltonian H can be constructed using the KLMN theorem [35], via the following lemma

Lemma B.1. *Let $v \in V_{\mathrm{I}}$. For $\psi \in L^2(\mathbb{R}^\nu) \otimes \mathcal{C}_0^\infty$ (algebraic tensor product) and $\varphi \in \mathcal{C}_0^\infty$:*

$$|\langle \psi, \Phi(e^{-\mathrm{i}k\cdot \mathrm{x}} v)\psi\rangle| \leq 2\|\omega^{-\frac{1}{2}} v\| \|\psi\| \langle \psi, 1\!\!1_{L^2(\mathbb{R}^\nu)} \otimes d\Gamma(\omega)\psi\rangle^{\frac{1}{2}} \tag{B.2}$$

$$|\langle \varphi, \Phi(v)\varphi\rangle| \leq 2\|\omega^{-\frac{1}{2}} v\| \|\varphi\| \langle \varphi, d\Gamma(\omega)\varphi\rangle^{\frac{1}{2}}. \tag{B.3}$$

Proof. The bound (B.3) was proved in (3.10). The bound (B.2) follows from (B.3) by applying the Lee-Low-Pines transformation (1.5) and using the second bound fiber by fiber. □

We can now define H as the self-adjoint operator associated with the closure of the semi-bounded form

$$\mathcal{D}(\Delta) \otimes \mathcal{C}_0^\infty \ni \psi, \varphi \to \langle \psi, [H_0 + \Phi(e^{-\mathrm{i}k\cdot \mathrm{x}} v)]\varphi\rangle$$

and the form domain of H equals $\mathcal{D}(H_0^{\frac{1}{2}})$. Note that this is the same Hamiltonian one obtains by applying [5, Theorem 2.2] with $\mathcal{K} = \mathbb{C}$ and $K = 0$. We can also construct $H(\xi)$ as the self-adjoint operator associated with the closure of the form

$$C_0^\infty \ni \psi, \phi \to \langle \psi, [H_0(\xi) + \Phi(v)]\psi\rangle$$

and the form domain of $H(\xi)$ is independent of ξ and equals $\mathcal{D}^{\frac{1}{2}} := \mathcal{D}(H_0(\xi)^{\frac{1}{2}})$. Note that for the (UV-regular) Hamiltonian H_Λ with coupling $v_\Lambda = 1\!\!1(|k| \leq \Lambda)v$ or $v_\Lambda = \exp(-|k|/\Lambda)v$, we have: $H_\Lambda \to H$ in norm-resolvent sense. Similarly, the fiber Hamiltonians $H_\Lambda(\xi)$ converge to $H(\xi)$ in norm-resolvent sense locally uniformly in ξ. From this observation it follows easily that H is translation invariant and the Lee-Low-Pines operator (1.5) transforms H into $\oint_{\mathbb{R}^\nu} H(\xi) d\xi$ as usual. See [5, 17] for a more refined analysis of confined linearly coupled models defined as forms.

We proceed to discuss how to establish our main results for this larger class of interactions.

The HVZ Theorem: We begin with i). The idea is the same as when we passed from compactly supported to square integrable v's in Section 3. We just need to replace Lemma 3.6 by

Lemma B.2. *Let $v_1, v_2 \in V_{\mathrm{I}}$. Suppose Conditions 2.1 and 2.2. Then for any $\sigma > 0$*

$$\Sigma_{0,2}(\xi) - \Sigma_{0,1}(\xi) \leq \|\omega^{-\frac{1}{2}}(v_1 - v_2)\| \big(4|\Sigma_{0,1}(\xi)|^{\frac{1}{2}} + 6\|\omega^{-\frac{1}{2}} v_1\|\big).$$

Proof. In order to verify this lemma one should first observe that the estimates (3.7) and (3.8) can, using Lemma B.1, be replaced by: For $\psi \in \mathcal{D}^{\frac{1}{2}}$ we have

$$|\langle\psi, \Phi(v)\psi\rangle| \leq 2\|\omega^{-\frac{1}{2}}v\|\|\psi\|\big\{|\langle\psi, [H_0(\xi) + \Phi(\tilde{v})]\psi\rangle|^{\frac{1}{2}} + |\langle\psi, \Phi(\tilde{v})\psi\rangle|^{\frac{1}{2}}\big\} \tag{B.4}$$

$$|\langle\psi, \Phi(v)\psi\rangle| \leq 4\|\omega^{-\frac{1}{2}}v\|\|\psi\||\langle\psi, [H_0(\xi) + \Phi(v)]\psi\rangle|^{\frac{1}{2}} + 4\|\omega^{-\frac{1}{2}}v\|^2\|\psi\|^2. \tag{B.5}$$

Following the same strategy as in the proof of Lemma 3.6 we arrive at the result. □

As for Theorem 2.1 ii) and iii) they follow from norm resolvent convergence of $H_\Lambda(\xi)$ to $H(\xi)$. (As usual $H_\Lambda(\xi)$ is defined with v replaced by $v_\Lambda = \mathbb{1}(|k| \leq \Lambda)v$.)

Uniqueness of groundstates: Assume $v \neq 0$ almost everywhere. That the resolvent $(H(\xi) - \mu)^{-1}$ is positivity improving follows by approximating v by $v_\Lambda = \exp(-|k|/\Lambda)v \in L^2(\mathbb{R}^\nu)$, and noting that for $\psi, \varphi \in \mathbf{C}\backslash\{0\}$ the expectation value $\langle\psi, (H_\Lambda(\xi)-\mu)^{-1}\varphi\rangle$ is non-zero and strictly increasing (expand in a Neumann series). The same argument works if $v \neq 0$ a.e. in $A \subset \mathbb{R}^\nu$ (measurable) and we restrict the operators to $\Gamma(L^2(A))$.

The argument that isolated ground states are unique, also when v vanishes on a set of non-zero measure, goes through unaltered.

Monotonicity: Let $v_1, v_2 \in V_{\mathrm{I}}$. Suppose (2.14) is satisfied. By norm resolvent convergence $\Sigma_{0,1}(\xi) \leq \Sigma_{0,2}(\xi)$ (approximate v_i by $\mathbb{1}(|k| \leq \Lambda)v_i$). As for $\Sigma_{\mathrm{ess},i}(\xi)$ we observe that Corollary 3.8 remains valid for $v \in V_{\mathrm{I}}$ (again by an approximation argument), and hence; the proof given in Section 4 goes through.

Large total momentum: Theorem 5.2 remains true and in the proof one should use Lemma B.2 instead of Lemma 3.6. Hence Theorem 2.4 remains true under the assumption $v \in V_{\mathrm{I}}$.

Before continuing we remark that Lemmas 3.7, 5.4 and 5.5 also remain valid.

Strict monotonicity: From the KLMN theorem we found that the form domains of $H_1(\xi)$ and $H_2(\xi)$ coincide and equals $\mathcal{D}^{\frac{1}{2}}$. By Lemma B.1 we have that $\mathcal{D}^{\frac{1}{2}}$ is contained in the form domains of the $\Phi(v_i)$'s. Hence the computation (4.5) remains valid and the proof given in Section 5 goes through unchanged.

Existence of ground states: First we need to address the proof given in [31, Section 3.3]. The key was the following inequality ([31, Lemma 3.7]) which holds for $v \in L^2(\mathbb{R}^\nu)$ and $z < \Sigma_0(\xi)$ and goes back to [40].

$$\begin{aligned}\langle\Omega, (H(\xi) - z)^{-1}\Omega\rangle^{-1} \\ &\leq \Omega(\xi) - z - \int_{\mathbb{R}^\nu} |v(k)|^2\big\langle\Omega, (H(\xi - k) + \omega(k) - z)^{-1}\Omega\big\rangle\, dk.\end{aligned} \tag{B.6}$$

Note that in [31] (B.6) was formulated as the right-hand side being strictly positive, but the above bound is what was actually proved. (It was also assumed that $v \geq 0$, which is superfluous, cf. the discussion in Section 4.)

For $v \in V_{\mathrm{I}}$ apply (B.6) to $H_\Lambda(\xi)$ (defined by replacing v by $\mathbb{1}(|k| \leq \Lambda)v$) and take the limit $\Lambda \to \infty$. The left-hand side and the integrand on the right-hand side in (B.6) converge. Hence, the inequality in (B.6) remains valid in the limit by Fatou's lemma. (Alternatively one could apply both the dominated and the monotone convergence theorem to the right-hand side.)

The rest of the proof goes through unchanged (with the additional comments in Section 5 to handle bounded ω's.)

Non-existence of embedded ground states:

Here we need to impose an extra assumption, namely: There exists $0 < \rho \leq 1$ and $C'_\Omega > 0$ such that

$$|\nabla\Omega(\eta)| \leq C'_\Omega \Omega(\eta)^{1-\rho} + C'_\Omega. \tag{B.7}$$

The argument relies on a pull-through formula which we first need to estasblish for the renormalized Hamiltonian.

We equip the form domain $\mathcal{D}^{\frac{1}{2}}$ of $H_0(\xi)$ with the norm $\|\psi\|_{\mathcal{D}^{\frac{1}{2}}} = \|(H_0(0)+1)^{\frac{1}{2}}\psi\|$. Consider the operator $N^{\frac{1}{2}}$ as a densely defined operator on $\mathcal{D}^{\frac{1}{2}}$, with domain

$$\mathcal{D}^{\frac{1}{2}}_N = (N+1)^{-\frac{1}{2}}\mathcal{D}^{\frac{1}{2}} = (H_0(0)+1)^{-\frac{1}{2}}(N+1)^{-\frac{1}{2}}\mathcal{F}.$$

We equip $\mathcal{D}^{\frac{1}{2}}_N$ with the norm $\|\psi\|_{\mathcal{D}^{\frac{1}{2}}_N} = \|(N+1)^{\frac{1}{2}}\psi\|_{\mathcal{D}^{\frac{1}{2}}}$. By duality this gives the sequence of spaces

$$\mathcal{D}^{\frac{1}{2}}_N \subset \mathcal{D}^{\frac{1}{2}} \subset \mathcal{F} \subset \mathcal{D}^{\frac{1}{2}*} \subset \mathcal{D}^{\frac{1}{2}*}_N,$$

where $\mathcal{D}^{\frac{1}{2}*}$ is the completion of $\mathcal{F}$ in the norm $\|\psi\|_{\mathcal{D}^{\frac{1}{2}*}} = \|(H_0(0)+1)^{-\frac{1}{2}}\psi\|$ and $\mathcal{D}^{\frac{1}{2}*}_N$ is the completion of $\mathcal{F}$ with respect to the norm $\|\psi\|_{\mathcal{D}^{\frac{1}{2}*}_N} = \|(N+1)^{-\frac{1}{2}}\psi\|_{\mathcal{D}^{\frac{1}{2}*}}$. We wish to extend the annihilation operator $a(\cdot)$, viewed as a map from $\mathcal{D}(N^{\frac{1}{2}}) \to L^2(\mathbb{R}^\nu;\mathcal{F})$ to $\mathcal{D}^{\frac{1}{2}*}$.

Below we will use two representation formulas for square roots. We have for $t > 0$:

$$t^{\frac{1}{2}} = -\frac{1}{\pi}\int_0^\infty \big((t+x)^{-1} - x^{-1}\big)x^{\frac{1}{2}}dx \tag{B.8}$$

$$t^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty (t+y)^{-1}y^{-\frac{1}{2}}dy. \tag{B.9}$$

Either one can be verified by direct computation, and either one follow from the other. We learned the first of these formulas from [38, Lemma B.3].

We have the following

Lemma B.3. *For any compact set $K \subset \mathbb{R}^\nu$, the map $\mathcal{D}(N^{\frac{1}{2}}) \ni \psi \to a(\cdot)\psi \in L^2(K;\mathcal{F})$ extends by continuity to a bounded operator from $\mathcal{D}^{\frac{1}{2}*}$ to $L^2(K;\mathcal{D}^{\frac{1}{2}*}_N)$.*

Proof. Let $\varphi \in L^2(K;\mathcal{D}^{\frac{1}{2}}_N)$ (the space dual to $L^2(K;\mathcal{D}^{\frac{1}{2}*}_N)$). Write $\varphi = (N+1)^{-\frac{1}{2}}(H_0(0)+1)^{-\frac{1}{2}}\tilde{\varphi}$ with $\tilde{\varphi} \in L^2(K;\mathcal{F})$. Then for $\psi \in \mathcal{C}_0^\infty$

$$\langle\varphi, a(\cdot)\psi\rangle = \int_K \langle\tilde{\varphi}(k), (H_0(0)+1)^{-\frac{1}{2}}a(k)(N+2)^{-\frac{1}{2}}\psi\rangle\, dk.$$

In order to commute $(H_0(0)+1)^{-\frac{1}{2}}$ with $a(k)$ we compute using the formula (B.9) and the pull-through formula (see e.g. [31, Proposition 2.2]) with $v \equiv 0$

$$\begin{aligned}[(H_0(0)+1)^{-\frac{1}{2}}, a(k)] = -\frac{1}{\pi}\int_0^\infty &(H_0(0)+1+y)^{-1} \\ &\times \big\{H_0(0) - H_0(-k) - \omega(k)\big\} a(k)(H_0(0)+1+y)^{-1} y^{-\frac{1}{2}} dy.\end{aligned}$$

This identity is in the sense of operators on $\mathcal{C}_0^\infty$. We compute the following difference locally uniformly in k

$$H_0(0) - H_0(-k) - \omega(k) = \nabla\Omega(-d\Gamma(k)) \cdot k + O(1).$$

The extra assumption (B.7) on Ω yields the estimate

$$\|[(H_0(0)+1)^{-\frac{1}{2}}, a(\cdot)]\psi\|_{L^2(K;\mathcal{F})} \leq C\|(H_0(0)+1)^{-\frac{1}{2}}\psi\|,$$

for some $C > 0$. We thus get

$$\begin{aligned}|\langle\varphi, a(\cdot)\psi\rangle| &\leq \|\tilde{\varphi}\|_{L^2(K;\mathcal{F})}(\|a(\cdot)(N+2)^{-\frac{1}{2}}(H_0(0)+1)^{-\frac{1}{2}}\psi\|_{L^2(K;\mathcal{F})} \\ &\quad + C\|(H_0(0)+1)^{-\frac{1}{2}}\psi\|_{\mathcal{F}}) \\ &\leq (1+C)\|\varphi\|_{L^2(K;\mathcal{D}_N^{\frac{1}{2}})}\|\psi\|_{\mathcal{D}^{\frac{1}{2}*}}.\end{aligned}$$

This concludes the proof. □

Lemma B.4. *Let $z \in \mathbb{C}$, $\mathrm{Re}(z) \neq 0$. For any compact set $K \subset \mathbb{R}^\nu$, there exists $C > 0$ such that for all $0 < \Lambda < \infty$ and $\psi \in \mathcal{F}$ we have $\|(H_\Lambda(\xi) - z)^{-1}\psi\|_{\mathcal{F}} \leq C\|\psi\|_{\mathcal{D}_N^{\frac{1}{2}*}}$ uniformly in $\xi \in K$.*

Proof. Let $\mu < \inf_{\Lambda,\xi\in K} \Sigma_{0,\Lambda}(\xi) - 1$. The lemma reduces to the following commutator bound:

$$T_\Lambda^\xi(\psi) := \|[(H_\Lambda(\xi)-\mu)^{-\frac{1}{2}}, (N+1)^{\frac{1}{2}}]\psi\|_{\mathcal{F}} \leq C\|\psi\|_{\mathcal{D}^{\frac{1}{2}*}}, \tag{B.10}$$

for $\psi \in \mathcal{C}_0^\infty$. Here C should be independen of Λ and $\xi \in K$. In order to obtain this estimate we use the formulas (B.8) and (B.9)

$$T_\Lambda^\xi(\psi) \leq \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \|[(H_\Lambda(\xi)-\mu+y)^{-1}, (N+1+x)^{-1}]\psi\|_{\mathcal{F}} y^{-\frac{1}{2}} x^{\frac{1}{2}} dx\, dy. \tag{B.11}$$

We have

$$\begin{aligned}&\|[(H_\Lambda(\xi)-\mu+y)^{-1}, (N+1+x)^{-1}]\psi\|_{\mathcal{F}} \\ &\quad = \|(N+1+x)^{-1}(H_\Lambda(\xi)-\mu+y)^{-1} \\ &\qquad \times \Phi(\mathrm{i}v_\Lambda)(H_\Lambda(\xi)-\mu+y)^{-1}(N+1+x)^{-1}\psi\|_{\mathcal{F}} \\ &\quad \leq 2(1+x)^{-2}(1+y)^{-1}\|\mathbf{a}(\mathrm{i}v_\Lambda)(H_0(0)+1)^{-\frac{1}{2}}\| \\ &\qquad \times \|(H_\Lambda(\xi)-\mu+y)^{-\frac{1}{2}}(H_0(0)+1)^{\frac{1}{2}}\|^2\|\psi\|_{\mathcal{D}^{\frac{1}{2}*}}\end{aligned} \tag{B.12}$$

The first norm on the right-hand side is bounded uniformly in Λ by (3.10). The second norm on the right-hand side is finite for each Λ and ξ. In order to get uniformity we argue as follows. For $\lambda > 0$ we estimate

$$\begin{aligned}
&\|(H_\Lambda(\xi) - \mu + y)^{-\frac{1}{2}}(H_0(\xi) + \lambda)^{\frac{1}{2}}\|^2 \\
&\quad = \|(H_\Lambda(\xi) - \mu + y)^{-\frac{1}{2}}(H_0(0) + \lambda)(H_\Lambda(\xi) - \mu + y)^{-\frac{1}{2}}\| \\
&\quad \leq 1 + \lambda + \|(d\Gamma(\omega) + \lambda)^{-\frac{1}{2}}\Phi(v_\Lambda)(d\Gamma(\omega) + \lambda)^{-\frac{1}{2}}\| \\
&\qquad \times \|(H_\Lambda(\xi) - \mu + y)^{-\frac{1}{2}}(H_0(\xi) + \lambda)^{\frac{1}{2}}\|^2
\end{aligned}$$

By (B.3) we find that $\|(d\Gamma(\omega) + \lambda)^{-\frac{1}{2}}\Phi(v_\Lambda)(d\Gamma(\omega) + \lambda)^{-\frac{1}{2}}\| \leq C\lambda^{-\frac{1}{2}}$, and hence for $\lambda \geq (C/2)^2$ we get uniformly in Λ and ξ:

$$\|(H_\Lambda(\xi) - \mu + y)^{-\frac{1}{2}}(H_0(\xi) + \lambda)^{\frac{1}{2}}\|^2 \leq 2(1 + \lambda).$$

Together with (B.11), (B.12) and the fact that $\|(H_0(0) + 1)^{\frac{1}{2}}(H_0(\xi) + \lambda)^{-\frac{1}{2}}\|$ is bounded locally uniformly in ξ, we get the required bound (B.10). $\square$

As a consequence of this lemma we get: Each resolvent $(H_\Lambda(\xi) - z)^{-1}$, $0 < \Lambda \leq \infty$, extend by continuity to $\mathcal{B}(\mathcal{D}_N^{\frac{1}{2}*}; \mathcal{F})$ with norm bounded uniformly in Λ and locally uniformly in ξ. Furthermore, $(H_\Lambda(\xi) - z)^{-1}$ converge strongly to $(H(\xi) - z)^{-1}$ in $\mathcal{B}(\mathcal{D}_N^{\frac{1}{2}*}; \mathcal{F})$, locally uniformly in ξ.

We finally get the following extension of the pull-through formula as presented in [18, Proposition 3.4] (and [31, Proposition 2.3]). Our formula is closely related to the one presented in [5] except for the precence of the dispersive term $\Omega(\xi - d\Gamma(k))$.

Proposition B.5. (Pull-through) *Let $z \in \mathbb{C}$, $\mathrm{Re}(z) < \Sigma_0(\xi)$. For any $\psi \in \mathcal{D}^{\frac{1}{2}}$ we have as an $L^2(\mathbb{R}^\nu; \mathcal{F})$ identity*

$$a(k)\psi = (H(\xi - k) + \omega(k) - z)^{-1}a(k)(H(\xi) - z)\psi + v(k)(H(\xi - k) + \omega(k) - z)^{-1}\psi.$$

Remark B.6. The first expression on the right-hand should be understood in the sense of composition of operators between weighted spaces. That the result is in $L^2(\mathbb{R}^\nu; \mathcal{F})$ (and not just $L^2_{\mathrm{loc}}(\mathbb{R}^\nu; \mathcal{F})$ as we get from the lemma's above), is a consequence of the remaining two terms in the equation being square integrable.

Proof. We have by the usual pull-through formula [18] the result with v replaced by $v_\Lambda = \mathbb{1}(|k| \leq \Lambda)v$ and $\psi \in \mathcal{D}$. That the formula extends to $\psi \in \mathcal{D}^{\frac{1}{2}}$ follows from Lemma's B.3 and B.4. Finally we appeal again to Lemma's B.3 and B.4 to remove the ultraviolet cutoff Λ. $\square$

Now Theorem 2.6 ii), with the extra assumption (B.7), follows from the pull-through formula just as in [31, Section 3.3], together with the considerations made in Section 5.

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