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by Victor Beresnevich, Maurice Dodson,
Simon Kristensen and Jason Levesley

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Ny Munkegade, Bldg. 1530
DK-8000 Aarhus C, Denmark
http://www.imf.au.dk
institut@imf.au.dk

AN INHOMOGENEOUS WAVE EQUATION AND NON-LINEAR DIOPHANTINE APPROXIMATION

VICTOR BERESNEVICH, MAURICE DODSON, SIMON KRISTENSEN,
AND JASON LEVESLEY

Abstract. A nonlinear Diophantine condition involving perfect squares and arising from an inhomogeneous wave equation on the torus guarantees the existence of a smooth solution. The exceptional set associated with the failure of the Diophantine condition and hence of the existence of a smooth solution is studied. Both the Lebesgue and Hausdorff measures of this set are obtained.

1. Introduction

Diophantine phenomena occur frequently in the theory of partial differential equations through the 'notorious problem of small denominators' [1, 20] which can compromise the convergence of solutions to the PDEs. The Diophantine phenomena in question often take the form of a Diophantine inequality connected with near resonances and it can be shown that the exceptional set of points where the inequality fails to hold is small, typically of Lebesgue measure 0. Thus if this exceptional set is ignored, the differential equations under consideration are guaranteed to have solutions. This interplay between the solvability of certain class of PDEs and related Diophantine conditions was exploited by Petronilho [18, 19] to establish a correspondence between the global Gevrey solvability and global Gevrey hypoellipticity for a class of sublaplacians on a torus with coefficients in the Gevrey class $G^s(\mathbb{T}^N)$ and a certain Diophantine inequality.

The 'size' of of these exceptional sets is therefore a question of real interest since it is desirable to know that any obstacles to solvability are confined to as small a set as possible. As stated above, these exceptional sets are typically of Lebesgue measure zero and it is natural to use Hausdorff dimension and measure to obtain more precise information about their structure. These metric type results have been studied extensively for a variety of different PDEs. For example, Hausdorff dimension results were obtained for KAM theory by Dodson, Pröschel, Rynne and Vickers in [7] and by Dodson and Vickers in [8], by Kristensen for the Schrödinger equation [14] and by Dickinson, Gramchev and Yoshino for a class of hypoelliptic operators [11].

In this paper, the existence of a smooth solution of the inhomogeneous wave equation in n spatial dimensions and one temporal dimension (time t) on a torus is shown to depend on a Diophantine condition, which is nonlinear for $n > 1$. Moreover, the finer metrical structure of the associated exceptional set, where the solubility is problematic, is obtained. The Diophantine condition is essentially linear when $n = 1$

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and Novák [17] showed that the inhomogeneous wave equation was soluble except on a set of measure zero, an account is in [2, §7.3]. The Hausdorff dimension of the exceptional set associated with the Schrödinger equation for a particle moving on a two-dimensional torus has been studied in [14]. The underlying Diophantine inequality is only partly nonlinear, in the sense that the Diophantine approximation is in terms of distance from the integers. This allows the Hausdorff dimension to be determined using results from the metrical theory of linear Diophantine approximation [21]. By contrast the corresponding problem for the inhomogeneous wave equation is 'fully' nonlinear in the sense that the Diophantine inequality is in terms of the distance from a nonlinear subset of $\mathbb{Z}$, namely the perfect squares.

There has been considerable progress in the one dimensional metrical theory of 'fully' non-linear Diophantine approximation, where the numerator and denominator of the rational approximants are restricted to non-linear subsets of the integers, such as primes or sums of squares, which are of number theoretic interest, see [6, 5, 13] for the Lebesgue measure theory and [3] for the complete metric theory. In higher dimensions the theory is less developed. There are some partial results, see [22]. The results obtained below are, as far as we are aware, the first complete treatment with respect to Lebesgue measure of an exceptional set arising from a fully nonlinear Diophantine approximation problem. Furthermore, we obtain analogous results for the more delicate notion of Hausdorff s-measure. The problem is described fully in §2 and the associated Diophantine problem derived. The two main results of the paper are stated in §3, together with some consequences. The results are proved for $n = 2$ in §§4–5 and an outline for the proof of the n-dimensional case is given in §6.

2. The solvability of the wave equation and a related Diophantine problem

Let $n \in \mathbb{N}$, $\alpha_i > 0$ for $i = 1, \ldots, n$, $\beta > 0$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be periodic in all variables with period α_i in the i'th variable and period β in the $n+1$'st. We denote the n first variables by $x_1, \ldots, x_n$ and the $n+1$'st by t. Suppose furthermore that f is a smooth function of any of the variables x_i, t, *i.e.*, f has continuous partial derivatives of all orders. The inhomogeneous partial differential equation given by

$$\frac{\partial^2 u(\boldsymbol{x}, t)}{\partial t^2} - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, t \in \mathbb{R}, \tag{1}$$

where the solution u is smooth and periodic with the same periods as f, describes an n-dimensional wave. Here Δ denotes the usual Laplacian, *i.e.*,

$$\Delta u(\boldsymbol{x}, t) = \sum_{i=1}^{n} \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2}.$$

The periodicity and smoothness conditions on f are well-known to be equivalent to the condition that f has an expansion into a Fourier series

$$f(\boldsymbol{x}, t) = \sum_{(\boldsymbol{a}, b) \in \mathbb{Z}^{n+1}} f_{\boldsymbol{a}, b} \exp\left(2\pi i \left[\sum_{i=1}^{n} \frac{a_i}{\alpha_i} x_i + \frac{b}{\beta} t\right]\right),$$

where $\boldsymbol{a} = (a_1, \ldots, a_n)$, such that the coefficients $f_{\boldsymbol{a}, b}$ decay faster than the reciprocal of any polynomial in $a_1, \ldots, a_n, b$ as $\max\{|a_1|, \ldots, |a_n|, |b|\}$ tends to infinity.

Suppose for the moment that (1) has a solution u satisfying the periodicity and smoothness conditions. Clearly, u must also have the following Fourier expansion

$$u(\boldsymbol{x},t)=\sum_{(\boldsymbol{a},b)\in\mathbb{Z}^{n+1}}u_{\boldsymbol{a},b}\exp\left(2\pi i\left[\sum_{i=1}^{n}\frac{a_i}{\alpha_i}x_i+\frac{b}{\beta}t\right]\right).$$

Substituting this into (1) and comparing coefficients, we obtain

$$u_{\boldsymbol{a},b}=\frac{\beta^2}{4\pi^2}\frac{f_{\boldsymbol{a},b}}{\sum_{i=1}^{n}a_i^2\frac{\beta^2}{\alpha_i^2}-b^2}. \tag{2}$$

Now, since $\alpha_1,\dots,\alpha_n,\beta$ are fixed, and since $f_{\boldsymbol{a},b}$ decays faster than the reciprocal of any polynomial, for u to be smooth it suffices to verify that

$$\left|\sum_{i=1}^{n}a_i^2\frac{\beta^2}{\alpha_i^2}-b^2\right|\geq C\max\{|a_1|,\dots,|a_n|\}^{-w},$$

for some $C>0$, $w>1$ for all $(\boldsymbol{a},b)\in\mathbb{Z}^{n+1}$ with $\boldsymbol{a}\neq\boldsymbol{0}$. It is easy to see that this condition can only fail if for any $w>1$ the inequality

$$\left|\sum_{i=1}^{n}a_i^2\frac{\beta^2}{\alpha_i^2}-b^2\right|<\max\{|a_1|,\dots,|a_n|\}^{-w} \tag{3}$$

holds for infinitely many $(\boldsymbol{a},b)\in\mathbb{Z}^{n+1}$ with $\boldsymbol{a}\neq\boldsymbol{0}$.

Note that the condition given in (3) is sufficient for the solvability of (1), but not necessary. The Diophantine problem considered in this paper is a natural generalisation inequality (3).

3. Statement of results

Throughout $\mathbb{Z}_{\geq 0}$ will denote the set of non-negative integer numbers and $|A|$ the Lebesgue measure of a set $A\subset\mathbb{R}^n$. Given an n-tuple $\boldsymbol{a}\in\mathbb{Z}^n_{\geq 0}$, define the *height* $h_{\boldsymbol{a}}$ of $\boldsymbol{a}$ by setting $h_{\boldsymbol{a}}:=\max(\,|\,a_1\,|\,\dots,|\,a_n\,|\,)$, that is $h_{\boldsymbol{a}}$ is the highest coefficient of $\boldsymbol{a}$ in absolute value.

Let $\psi:(0,\infty)\to(0,\infty)$ be a function such that $\psi(h)\to 0$ as $h\to\infty$ and define the set $W_n(\psi)$ to be

$$W_n(\psi):=\{\boldsymbol{x}\in[0,1]^n:|\,\boldsymbol{a^2}\cdot\boldsymbol{x}-b^2|<\psi(h_{\boldsymbol{a}}),\ \text{holds for infinitely many } (\boldsymbol{a},b)\in\mathbb{Z}^{n+1}_{\geq 0}\ \},$$

where $\boldsymbol{a^2}:=(a_1^2,\dots,a_n^2)$.

The following statements constitute the main results of this paper.

Theorem 3.1. *Let $\psi:(0,\infty)\to(0,\infty)$ be monotonic. Then*

$$|W_n(\psi)|=\begin{cases}0, & \sum_{h=1}^{\infty}h^{n-2}\psi(h)<\infty\,,\\ 1, & \sum_{h=1}^{\infty}h^{n-2}\psi(h)=\infty\,.\end{cases}$$

Theorem 3.2. *Let $\psi:(0,\infty)\to(0,\infty)$ be a monotonic. Given any positive $s<n$, the s-dimensional Hausdorff measure $\mathcal{H}^s(W_n(\psi))$ of $W_n(\psi)$ is given by*

$$\mathcal{H}^s(W_n(\psi))=\begin{cases}0, & \sum_{h=1}^{\infty}\psi(h)^{s-(n-1)}h^{3n-2-2s}<\infty\,,\\ \infty, & \sum_{h=1}^{\infty}\psi(h)^{s-(n-1)}h^{3n-2-2s}=\infty\,.\end{cases}$$

Corollary 3.3. *Let $\psi : (0,\infty) \to (0,\infty)$ be a monotonic function with $\lim_{h\to\infty} \psi(h) = 0$. Define $\lambda_\psi \in [0,\infty]$, the lower order of $1/\psi(2^r)$ at infinity, by setting*

$$\lambda_\psi = \liminf_{r\to\infty} \frac{-\log \psi(2^r)}{r \log 2} \ .$$

If $n-1 \leq \lambda_\psi < \infty$ then

$$\dim W_n(\psi) = (n-1) + \frac{n+1}{2+\lambda_\psi} \, .$$

In particular, if $\psi(r) = r^{-v}$ for some $v > n-1$ then

$$\dim W_n(r \mapsto r^{-v}) = (n-1) + \frac{n+1}{2+v} \, .$$

In terms of the wave equation, we can immediately derive the following corollary:

Corollary 3.4. *Let $\alpha_1, \ldots, \alpha_n, \beta > 0$ and consider the partial differential equation (1). Let $\delta_i = \beta^2/\alpha_i^2$ for $i = 1, \ldots, n$. If f is smooth and periodic in $x_1, \ldots, x_n, t$ with periods $\alpha_1, \ldots, \alpha_n, \beta$ respectively, then (1) is solvable with u smooth and periodic with the same periods whenever $(\delta_1, \ldots, \delta_n)$ does not belong to*

$$\bigcap_{v>1} W_n(r \mapsto r^{-v}),$$

a null set of Hausdorff dimension $n-1$.

4. Proof of Theorem 3.1

We first prove the result for the case $n = 2$ as the argument is easiest to follow in this dimension.

4.1. **The case of convergence.** For every triple $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$ define the sets

$$\sigma_{a,b}(c) := \{(x,y) \in [0,1]^2 : |\, a^2 x + b^2 y - c^2| < \psi(h_{a,b})\},$$
$$\sigma_{a,b} := \bigcup_{c\in\mathbb{Z}_{\geq 0}} \sigma_{a,b}(c).$$

Without loss of generality we can assume that $a + b > 0$. It is easy to verify that

$$|\sigma_{a,b}(c)| \ll \frac{\psi(h_{a,b})}{h^2_{a,b}} \, .$$

Given a pair $(a,b) \in \mathbb{Z}^2_{\geq 0} \setminus \{\mathbf{0}\}$, $\sigma_{a,b}(c) \neq \emptyset$ implies that $c \ll h_{a,b}$. It follows that

$$|\sigma_{a,b}| \ll \sum_{c\in\mathbb{Z}_{\geq 0} \,:\, \sigma_{a,b}(c)\neq\emptyset} \frac{\psi(h_{a,b})}{h^2_{a,b}} \ll \frac{\psi(h_{a,b})}{h_{a,b}} \, .$$

Now assume that $\sum_{h=1}^\infty \psi(h) < \infty$. Then,

$$\sum_{h=1}^\infty \sum_{\substack{(a,b)\in\mathbb{Z}^2_{\geq 0}\setminus\{\mathbf{0}\};\\ h_{a,b}=h}} |\sigma_{a,b}| \ll \sum_{h=1}^\infty \sum_{\substack{(a,b)\in\mathbb{Z}^2_{\geq 0}\setminus\{\mathbf{0}\};\\ h_{a,b}=h}} \frac{\psi(h)}{h} \ll \sum_{h=1}^\infty \psi(h) < \infty. \tag{4}$$

As the set $W_2(\psi)$ is exactly the set of points (x,y) in the unit square that fall into infinitely many sets $\sigma_{a,b}$, we can apply the Borel-Cantelli Lemma to (4) to conclude that the set $W_2(\psi)$ has zero Lebesgue measure.

4.2. **The case of divergence: auxiliary lemmas.** It should be noted that the main difficulty in proving Theorem 3.1 is in the case of divergence, to be considered in sections 4.3 and 4.4. The line of investigation of this case will rely on the following standard auxiliary measure theoretic statements.

Lemma 4.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let $|A|$ be the Lebesgue measure of A. Let E be a Borel subset of $\mathbb{R}^n$. Assume that there are constants $r_0, c > 0$ such that for any ball B of radius $r(B) < r_0$ in Ω we have*

$$|E \cap B| \geq c\,|B|\,.$$

Then E has full measure in Ω, i.e. $|\Omega \setminus E| = 0$.

Lemma 4.2. *Let (Ω, A, μ) be a probability space and E_n be a sequence of μ-measurable sets such that $\sum_{n=1}^{\infty}\mu(E_n) = \infty$. Then*

$$\mu(\limsup_{n\to\infty} E_n) \;\geq\; \limsup_{Q\to\infty} \frac{\left(\sum_{s=1}^{Q}\mu(E_s)\right)^2}{\sum_{s,t=1}^{Q}\mu(E_s\cap E_t)} \qquad .$$

Lemma 4.1 follows easily from Lebesgue's density theorem. A proof of Lemma 4.2 can be found in [13, Lemma 2.3].

In our particular problem we will take E_n to be a subsequence of the sequence of sets $\sigma_{a,b}$. More precisely, we will estimate pairwise intersections of $\sigma_{a,b}$ restricted to a fixed ball B on average. The corresponding limsup set will be contained in $W_2(\psi)\cap B$. On applying Lemma 4.2, we will arrive at a lower bound of the form $|W_2(\psi)\cap B| \geq c|B|$ for some positive absolute constant. Lemma 4.1 will complete the proof.

Further, to avoid painful and unnecessary calculation we will restrict B to be a ball lying inside $\Omega = [\varepsilon, 1]^2$ for some arbitrarily small $\varepsilon > 0$. The corresponding probability measure μ will be taken to be the normalized Lebesgue measure in Ω.

4.3. **Estimates for the measure of $\sigma_{a,b}\cap B$ and their pairwise intersections.** Fix an arbitrary positive number $\varepsilon < 1$ and set $\Omega = [\varepsilon, 1]^2$. Take any ball B in $\mathbb{R}^2$ lying in Ω.

4.3.1. *Restrictions on c.* Assume that $\sigma_{a,b}(c)\cap B \neq \emptyset$. Then there is a point $(x, y) \in B \subset [\varepsilon, 1]^2$ satisfying $|a^2x + b^2y - c^2| < \psi(h_{a,b})$. If $h_{a,b}$ is sufficiently large then $\psi(h_{a,b}) < \varepsilon$. Therefore, $c^2 < \varepsilon + a^2x + b^2y \leq 1 + 2h_{a,b}^2$. Hence,

$$|c| < 2h_{a,b}\,.$$

On the other hand,

$$c^2 > a^2x + b^2y - \psi(h) > \varepsilon(a^2+b^2) - \varepsilon \geq \varepsilon(h^2-1).$$

Therefore,

$$|c| > \varepsilon h_{a,b}/2$$

if $h_{a,b}$ is sufficiently large. Therefore, for all $(a, b) \in \mathbb{Z}^2_{\geq 0}$ with sufficiently large $h_{a,b}$ and all positive c with $\sigma_{a,b}(c)\cap B \neq \emptyset$ we have

$$\frac{\varepsilon}{2}\,h_{a,b} < |c| < 2\,h_{a,b}\;. \tag{5}$$

4.3.2. *The number of different* c. Define the line $R_{a,b,c} := \{(x,y) \in \mathbb{R}^2 : a^2x + b^2y - c^2 = 0\}$. It is readily verified that $\sigma_{a,b}(c) \cap B \neq \emptyset$ is equivalent to $R_{a,b,c} \cap B \neq \emptyset$, except possibly for 2 'extremal' cases when $\sigma_{a,b}(c) \cap B \neq \emptyset$ but the corresponding lines do not hit the ball B but lie sufficiently close to B.

To evaluate the number of different c such that $\sigma_{a,b}(c) \neq \emptyset$ we will estimate the number of lines $R_{a,b,c}$ that hit the ball B and then add 2 to the upper estimate.

Let (x_0, y_0) be the center of B and r be the radius of B. Any point (x,y) in B can be written as

$$x = x_0 + \theta r \cos\phi, \quad y = y_0 + \theta r \sin\phi, \qquad 0 \leq \theta < 1,\ 0 \leq \phi < 2\pi\,. \tag{6}$$

Clearly, $R_{a,b,c} \cap B \neq \emptyset$ if and only if there is a choice of (x,y) subject to (6) if and only if c is in the interval

$$\left[\ \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4+b^4}},\ \sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4+b^4}}\ \right]. \tag{7}$$

The length of interval (7) is

$$\begin{aligned}\xi_{a,b,B} &= \sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4+b^4}} - \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4+b^4}} \\ &= \frac{2r\sqrt{a^4+b^4}}{\sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4+b^4}} + \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4+b^4}}}.\end{aligned}$$

Taking into account that $\varepsilon \leq x_0, y_0 \leq 1$ and $r < 1$, it follows that

$$\frac{1}{2}\, r\, h_{a,b} \leq \xi_{a,b,B} \leq \frac{8}{\varepsilon}\, r\, h_{a,b}\,.$$

Now, the number of possible values for c lies between $\xi_{a,b,B}$ and $\xi_{a,b,B} + 3$ and is therefore $\asymp r\, h_{a,b}$.

4.3.3. *The measure of* $\sigma_{a,b} \cap B$. Given a c, it is easily verified that $|\sigma_{a,b}(c) \cap B| \leq 4r\psi(h_{a,b})/\sqrt{a^4+b^4} \leq 4r\psi(h_{a,b})/h_{a,b}^2$, where r is the radius of B.

The number of possible values of c such that $\sigma_{a,b}(c) \cap B \neq \emptyset$ is bounded above by $\xi_{a,b,B} + 3 \leq \frac{10}{\varepsilon}\, r\, h_{a,b}$ if $h_{a,b}$ is sufficiently large. Therefore,

$$|\sigma_{a,b} \cap B| \leq 4r\psi(h_{a,b})/h_{a,b}^2 \times \frac{10}{\varepsilon}\, r\, h_{a,b} = c_2|B|\, \frac{\psi(h_{a,b})}{h_{a,b}},$$

where $c_2 = \frac{40}{\varepsilon\pi}$ and $h_{a,b}$ is sufficiently large.

Let $\frac{1}{2}B$ be the ball centred at the same point as B of radius $r/2$. Then it is an elementary geometric task to compute that $|\sigma_{a,b}(c) \cap B| \geq r\psi(h_{a,b})/h_{a,b}^2$ whenever $\sigma_{a,b}(c) \cap \frac{1}{2}B \neq \emptyset$ and $h_{a,b}$ is sufficiently large.

The number of possible values of c such that $\sigma_{a,b}(c) \cap \frac{1}{2}B \neq \emptyset$ is bounded below by $\xi_{a,b,\frac{1}{2}B} \geq \frac{1}{4}\, r\, h_{a,b}$. Therefore,

$$|\sigma_{a,b} \cap B| \geq r\psi(h_{a,b})/h_{a,b}^2 \times \frac{1}{4}\, r\, h_{a,b} = c_1|B|\frac{\psi(h_{a,b})}{h_{a,b}},$$

where $c_1 = \frac{1}{4\pi}$.

The upshot of the above is that

$$c_1|B|\frac{\psi(h_{a,b})}{h_{a,b}} \leq |\sigma_{a,b} \cap B| \leq c_2|B|\frac{\psi(h_{a,b})}{h_{a,b}} \tag{8}$$

for all sufficiently large $h_{a,b}$, where c_1, c_2 are absolute positive constants.

4.3.4. *Additional conditions on* (a,b). Throughout the remainder of the proof of Theorem 3.1 we will assume that the following conditions on (a,b) hold:

$$\gcd(a,b)=1, \tag{9}$$

where gcd means the greatest common divisor, and

$$1/2 \le a/b \le 2. \tag{10}$$

The above conditions sift out elements of the sequence of sets $\sigma_{a,b}$ which prevent us from having sufficiently good estimates for the measures of pairwise intersections of these sets. On the other hand, the remaining 'thinned out' part of the sequence $\sigma_{a,b}$ is still rich enough to ensure that the sum

$$\sum |\sigma_{a,b}| \tag{11}$$

diverges over this restricted sequence. Such a condition as that of Equation (11) is necessary to apply Lemma 4.2. Indeed, to verify that (11) diverges over $(a,b)\in\mathbb{Z}^2_{\ge 0}$ satisfying (9) and (10) define N_k to be the number of (a,b) satisfying (9) and (10) with $2^k \le h_{a,b} < 2^{k+1}$. Then in view of symmetry of the set of such (a,b) we get

$$N_k = 2\sum_{2^k\le a<2^{k+1}}\ \sum_{\substack{b<a\\ \gcd(a,b)=1\\ 1/2\le a/b\le 2}} 1 = 2\sum_{2^k\le a<2^{k+1}} \Big(\varphi(a)-\varphi([a/2])\Big),$$

where φ is the Euler function. It is well known that

$$\sum_{1\le q\le Q}\varphi(q) = \frac{3}{\pi^2}Q^2 + O(Q\log Q).$$

Then

$$2\sum_{2^k\le a<2^{k+1}}\varphi(a) = \frac{6}{\pi^2}((2^{k+1})^2-(2^k)^2) + O(k2^k) = \frac{18}{\pi^2}\,2^{2k} + O(k2^k)$$

and

$$\begin{aligned} 2\sum_{2^k\le a<2^{k+1}}\varphi([a/2]) &= 4\sum_{2^{k-1}\le x<2^k}\varphi(x) \\ &= \frac{12}{\pi^2}((2^k)^2-(2^{k-1})^2) + O(k2^k) = \frac{9}{\pi^2}\,2^{2k} + O(k2^k). \end{aligned}$$

It follows that

$$N_k = \frac{9}{\pi^2}\,2^{2k} + O(k2^k).$$

Now the estimated sum is

$$\begin{aligned} \sum_{\substack{(a,b)\in\mathbb{Z}^2_{\ge 0}\\ (9)\text{ and }(10)\text{ are satisfied}}} |\sigma_{a,b}\cap B| &= \sum_{k=0}^{\infty}\ \sum_{2^k\le h<2^{k+1}}\ \sum_{\substack{(a,b)\in\mathbb{Z}^2_{\ge 0}:\,h_{a,b}=h\\ (9)\text{ and }(10)\text{ are satisfied}}} |\sigma_{a,b}| \\ &\gg |B|\sum_{k=0}^{\infty}\ \sum_{2^k\le h<2^{k+1}}\ \sum_{\substack{(a,b)\in\mathbb{Z}^2_{\ge 0}:\,h_{a,b}=h\\ (9)\text{ and }(10)\text{ are satisfied}}} \frac{\psi(2^{k+1})}{2^k} \\ &\asymp |B|\sum_{k=0}^{\infty}2^k\psi(2^k) \asymp |B|\sum_{h=1}^{\infty}\psi(h) = \infty. \end{aligned}$$

Finally, note that the limsup set for the 'thinned out' sequence $\sigma_{a,b}$ is contained in the limsup set for the complete sequence $\sigma_{a,b}$, which is $W_2(\psi)$. Therefore, it will be sufficient to prove that the thinned out limsup set is of full Lebesgue measure in order to ensure that $W_2(\psi)$ is also of full measure.

An immediate consequence of condition (9) is that for any two pairs (a,b) and (a',b') satisfying (9) the assumption $(a,b)\neq(a',b')$ implies that (a,b) and (a',b') are not collinear. Moreover, (a^2,b^2) and (a'^2,b'^2) are not collinear. Therefore we can assume that the (smaller) angle between (a^2,b^2) and (a'^2,b'^2), which will be denoted by $\alpha=\alpha(a,b,a',b')$, is not zero. The analysis of the measures of intersections $\sigma_{a,b}\cap\sigma_{a',b'}\cap B$ will rely on the behaviour of this angle and is given in the following sections.

4.3.5. *The measure of intersections in the case of a large angle.* We will assume that $(a,b)\neq(a',b')$. Within this subsection we set $h=h_{a,b}$ and $h'=h_{a',b'}$. For simplicity we will assume that $h\geq h'$. Now

$$\sigma_{a,b}\cap\sigma_{a',b'}\cap B=\bigcup_{c'\in\mathbb{Z}_{\geq 0}}\sigma_{a,b}\cap\sigma_{a',b'}(c')\cap B. \tag{12}$$

For a fixed c' the set $\sigma_{a',b'}(c')\cap B$ is covered with a strip of length $2r$ (recall that r is the radius of B) and width $\psi(h')/h'^2$. This strip is a piece of the $\psi(h')/h'^2$-neighbourhood of the line

$$a'^2x+b'^2y-c'^2=0. \tag{13}$$

To estimate the measure in (12) we first estimate the measure of the intersection of $\sigma_{a,b}$ with such a strip.

The angle $\alpha=\alpha(a,b,a',b')$ introduced in the previous section is the (smaller) angle between the line defined in (13) and the family of parallel lines

$$a^2x+b^2y-c^2=0,\ \text{where } c\in\mathbb{Z}_{\geq 0}. \tag{14}$$

Using (5) it is readily verified that the distance between two adjacent lines in the family (14) is $\asymp h^{-1}$.

Let A and B be the intersection points of the line (13) and two neighbouring lines in (14), say $\mathcal{L}_1$ and $\mathcal{L}_2$. The distance between A and B is the distance between $\mathcal{L}_1$ and $\mathcal{L}_2$ divided by $\sin\alpha$. In other words, the distance is $\asymp\frac{1}{h\sin\alpha}$. Since the piece of the line (13) of interest is of length at most $2r$, there are at most

$$\ll rh\sin\alpha+1$$

non-empty intersections $\sigma_{a,b}(c)\cap\sigma_{a',b'}(c')\cap B$ when c runs over all integers.

As the set $\sigma_{a,b}(c)\cap\sigma_{a',b'}(c')$ is a parallelepiped with area $\ll\frac{\psi(h)}{h^2}\frac{\psi(h')}{h'^2}\frac{1}{\sin\alpha}$, the upshot of the above is that

$$|\sigma_{a,b}\cap\sigma_{a',b'}(c')\cap B|\ll\frac{\psi(h)}{h^2}\frac{\psi(h')}{h'^2}\frac{1}{\sin\alpha}\times(rh\sin\alpha+1).$$

Further, since there are $\ll rh'$ values of c' that need to be considered, we have that

$$\begin{aligned}|\sigma_{a,b}\cap\sigma_{a',b'}\cap B|\ll&\ \frac{\psi(h)}{h^2}\frac{\psi(h')}{h'^2}\frac{1}{\sin\alpha}\times(rh\sin\alpha+1)rh'\\ &\asymp|B|\frac{\psi(h)}{h}\frac{\psi(h')}{h'}\left(1+\frac{1}{rh\sin\alpha}\right).\end{aligned} \tag{15}$$

Assuming that $\frac{1}{rh\sin\alpha}\leq 1$, or equivalently that

$$\sin\alpha\geq\frac{1}{rh}, \tag{16}$$

gives

$$|\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \, \frac{\psi(h)}{h} \, \frac{\psi(h')}{h'} \, . \tag{17}$$

Finally, since there are $\asymp h$ integer vectors (a,b) with $h_{a,b} = h$ and $\asymp h'$ integer vectors (a',b') with $h_{a',b'} = h'$, summing the measures of intersections $|\sigma_{a,b} \cap \sigma_{a',b'} \cap B|$ in the case under consideration results in

$$\sum_{\substack{h_{a,b} \le H, \; h_{a',b'} \le H \\ (a,b) \ne (a',b') \text{ and (16) holds}}} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \left(\sum_{h=1}^{H} \psi(h) \right)^2 .$$

4.3.6. *The measure of intersections in the case of a small angle.* In this section we will deal with the case of

$$\sin\alpha < \frac{1}{rh} \, . \tag{18}$$

Again we will assume that $(a,b) \ne (a',b')$ and given a matrix A, $|A|$ will denote its determinant and $\|A\|$ the absolute value of its determinant.

Since α is the angle between the vectors (a^2,b^2) and (a'^2,b'^2) it follows that

$$h^2 h'^2 \sin\alpha \asymp \sqrt{a^4+b^4}\sqrt{a'^4+b'^4} \;\, \sin\alpha = \left\| \begin{array}{cc} a^2 & b^2 \\ a'^2 & b'^2 \end{array} \right\| = \left\| \begin{array}{cc} a & b \\ a' & b' \end{array} \right\| \times \left\| \begin{array}{cc} a & -b \\ a' & b' \end{array} \right\| . \tag{19}$$

If β denotes the (smaller) angle between (a,b) and $(a,-b)$ then by (10),

$$\sin\beta = \frac{1}{a^2+b^2} \left\| \begin{array}{cc} a & b \\ a & -b \end{array} \right\| = \frac{2|ab|}{a^2+b^2} \ \ge \ \frac{1}{2} \, .$$

Hence, $\beta \ge \pi/6$ and the angle between (a',b') and at least one of the vectors (a,b) and $(a,-b)$ is at least $\pi/12$. Without loss of generality we can assume that such an angle is between $(a,-b)$ and (a',b'). Then

$$\left\| \begin{array}{cc} a & -b \\ a' & b' \end{array} \right\| \ge \sqrt{a^2+b^2}\sqrt{a'^2+b'^2}\,\sin\pi/12 \gg h\,h' \, .$$

It now follows from (18) and (19) that

$$1 \le \left\| \begin{array}{cc} a & b \\ a' & b' \end{array} \right\| \ll h\,h' \,\sin\alpha \le \frac{h'}{r} \, . \tag{20}$$

This means that for every fixed a',b',a there are at most $\ll \frac{1}{r}$ possible values for b. Indeed, $|ab'-a'b| \ll h'r^{-1}$, that is $|b - ab'/a'| \ll h'r^{-1}/a' \ll r^{-1}$. Moreover, (20) implies that

$$\sin\alpha \gg \frac{1}{h\,h'} \, . \tag{21}$$

To complete the analysis for this case we consider two specific subcases.

Subcase (i) – moderately small angle.

Assume for the moment that

$$\sin\alpha \ge \frac{1}{r^2\,h\,h'} \, . \tag{22}$$

Using (15), (18) and (22) it follows that

$$|\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \, \frac{\psi(h)}{h} \, \frac{\psi(h')}{h'} \, \frac{1}{rh\sin\alpha} \ll |B| \, \frac{\psi(h)}{h} \, \psi(h') \; r \; .$$

Now the sum of intersections for this subcase can be estimated as follows,

$$\sum_{h_{a,b}\le H,\ h_{a',b'}\le H} |\sigma_{a,b}\cap\sigma_{a',b'}\cap B| \ll \sum_{h=1}^{H}\sum_{h'=1}^{h-1}\sum_{h_{a',b'}=h'}\sum_{h_{a,b}=h} |\sigma_{a,b}\cap\sigma_{a',b'}\cap B|$$
$$\ll \sum_{h=1}^{H}\sum_{h'=1}^{h-1}\sum_{h_{a',b'}=h'}\sum_{h_{a,b}=h} |B|\,\frac{\psi(h)}{h}\,\psi(h')\,r.$$

On using the fact that for every fixed a', b', a there are at most $\ll \frac{1}{r}$ possible values for b, this is

$$\ll \sum_{h=1}^{H}\sum_{h'=1}^{h-1} h'|B|\,\frac{\psi(h)}{h}\,\psi(h') \ll |B|\sum_{h=1}^{H}\sum_{h'=1}^{h-1}\psi(h)\,\psi(h') \ll |B|\left(\sum_{h=1}^{H}\psi(h)\right)^2. \qquad (23)$$

Subcase (ii) – very small angle.
To complete the analysis of all possible values of α it remains to consider the case when

$$\sin\alpha < \frac{1}{r^2\,h\,h'}\,.$$

Then

$$\left\|\begin{array}{cc} a & b\\ a' & b'\end{array}\right\| \ll h\,h'\,\sin\alpha < \frac{1}{r^2}\,. \qquad (24)$$

and

$$|\sigma_{a,b}\cap\sigma_{a',b'}\cap B| \ll |B|\,\frac{\psi(h)}{h}\,\frac{\psi(h')}{h'}\,\frac{1}{rh\sin\alpha} \ll |B|\,\frac{\psi(h)}{h}\,\psi(h')\frac{1}{r}. \qquad (25)$$

Now we estimate the number of quadruples (a, b, a', b') satisfying (9), (10), (24), $2^k \le h_{a,b} < 2^{k+1}$ and $2^l \le h_{a',b'} < 2^{l+1}$. Given fixed a and b', (24) means that a', b can only be chosen to satisfy $|ab'-a'b| \ll r^{-2}$. This means that there are $\ll r^{-2}$ possible values for $t = a'b$. In turn, for a fixed t there are at most $d(t)$ possible values for a' and b, where $d(t)$ is the number of divisors of t. It is well known, [16, Theorem 7.2], that for any $\delta > 0$ there is a constant $c_\delta > 0$ such that $d(t) \le c_\delta t^\delta$ for all t. Taking $\delta = 1/4$ we get that the number of possible quadruples a, b, a', b' is $\ll (2^k\,2^l)^{5/4}r^{-2}$.

Without loss of generality we assume that $\psi(h) \le h^{-1}$. Then the sum of intersections for this subcase is estimated as follows

$$\sum_{h_{a,b}\le H,\ h_{a',b'}\le H} |\sigma_{a,b}\cap\sigma_{a',b'}\cap B| = \sum_{k=1}^{[\log H]+1}\ \sum_{l=1}^{[\log H]+1}\ \sum_{\substack{2^k\le h_{a,b}<2^{k+1}\\ 2^l\le h_{a',b'}<2^{l+1}}} |\sigma_{a,b}\cap\sigma_{a',b'}\cap B|$$
$$\ll \sum_{k=1}^{[\log H]+1}\ \sum_{l=1}^{k}|B|\,\frac{\psi(2^k)}{2^k}\,\psi(2^l)\frac{1}{r}\times(2^k\ 2^l)^{5/4}r^{-2}$$
$$\ll \frac{1}{r}\sum_{k=1}^{[\log H]+1}\ \sum_{l=1}^{k}2^{k/4}\psi(2^k)2^{5l/4}\psi(2^l)$$
$$\ll \frac{1}{r}\sum_{k=1}^{[\log H]+1}\ \sum_{l=1}^{k}2^{3k/4}\psi(2^k)2^{3l/4}\psi(2^l)$$
$$\ll \frac{1}{r}\sum_{k=1}^{\infty}\ \sum_{l=1}^{\infty}2^{-k/4}2^{-l/4} < \infty.$$

We are now in a position to complete the proof of Theorem 3.1 for the divergence case.

4.4. **Completion of the proof of Theorem 3.1.** The upshot of the above computations is the following estimates:

$$S_1(H) = \sum_{(a,b)\in\mathcal{Z}_H} |\sigma_{a,b}\cap B| \gg |B|\left(\sum_{h=1}^{H}\psi(h)\right)$$

$$S_2(H) = \sum_{(a,b)\in\mathcal{Z}_H}\ \sum_{(a',b')\in\mathcal{Z}_H} |\sigma_{a,b}\cap\sigma_{a',b'}\cap B| \ll |B|\left(\sum_{h=1}^{H}\psi(h)\right)^2$$

where $\mathcal{Z}_H = \{(a,b)\in\mathbb{Z}^2_{\geq 0},\ (9) \text{ and } (10) \text{ hold and } \ h_{a,b}\leq H\}$. Therefore,

$$\frac{S_1(H)^2}{S_2(H)} \gg |B|$$

for all sufficiently large H. Since $\limsup_{h_{a,b}\to\infty}\sigma_{a,b}\cap B\subset W_2(\psi)\cap B$, by Lemma 4.2

$$|W_2(\psi)\cap B| \geq |\limsup_{h_{a,b}\to\infty}\sigma_{a,b}\cap B| \gg |B|.$$

This holds for any ball B in Ω with the implied constant independent of B. Therefore, by Lemma 4.1, $W_2(\psi)$ has full measure in $\Omega=[\varepsilon,1]^2$. Since $\varepsilon>0$ is arbitrary, $W_2(\psi)$ has full measure in $[0,1]^2$. This completes the proof of Theorem 3.1.

5. Proof of Theorem 3.2

5.1. **Hausdorff measures and dimension.** In this section we give a very brief introduction to the theory of Hausdorff measures and dimension. For further details consult [15].

Let s be a positive real number. The Hausdorff s-measure will be denoted throughout by $\mathcal{H}^s$ and is defined as follows. Suppose F is a non-empty subset of $\mathbb{R}^k$. Suppose that $\rho>0$. A ρ-cover of F is a countable collection $\{B_i\}$ of balls in $\mathbb{R}^k$ with radii $r_i\leq\rho$ for each i such that

$$F\subset\bigcup_i B_i.$$

Define the function $\mathcal{H}^s_\rho$ by

$$\mathcal{H}^s_\rho(F) := \inf\left\{\sum_i r_i^s\right\},$$

where the infimum is taken over all possible ρ-covers of F. Then $\mathcal{H}^s(F)$ of the set F is defined by

$$\mathcal{H}^s(F) := \lim_{\rho\to 0}\mathcal{H}^s_\rho(F) \ = \ \sup_{\rho>0}\mathcal{H}^s_\rho(F)\ .$$

Let F be an infinite set. The Hausdorff dimension of F is the (unique) number

$$\dim F = \inf\{s>0 : \mathcal{H}^s(F)=0\} = \sup\{s>0 : \mathcal{H}^s(F)=+\infty\}.$$

Note that $\mathcal{H}^k$ is a multiple of the k-dimensional Lebesgue measure in $\mathbb{R}^k$ when $k\in\mathbb{N}$.

5.2. **Proof of Theorem 3.2. The case of convergence.** The proof of convergence is straightforward. Recall from above that $W_2(\psi)$ can be expressed as a limsup set of the form

$$W_2(\psi) = \bigcap_{h=1}^{\infty} \bigcup_{\substack{(a,b)\in\mathbb{Z}^2,\\ h_{a,b}=h}}^{\infty} \bigcup_{c\in\mathbb{Z}} \sigma_{a,b}(c).$$

Each $\sigma_{a,b}(c)$ can be covered by a family $C^c_{a,b}$ of balls each of radius $\psi(h_{a,b})/h^2_{a,b}$ where

$$\#C^c_{a,b} \ll \frac{h^2_{a,b}}{\psi(h_{a,b})}.$$

By assumption $\psi(h) \to 0$ as $h \to \infty$. Therefore, given any $N \in \mathbb{N}$, $\psi(h)/h^2 \leq 1/N$ for sufficiently large h. It follows that

$$\begin{aligned}\mathcal{H}^s_{1/N}(W_2(\psi)) &\ll \sum_{\substack{(a,b)\in\mathbb{Z}^2,\\ h_{a,b}\geq N}} \left(\frac{\psi(h_{a,b})}{h^2_{a,b}}\right)^s \frac{h^2_{a,b}}{\psi(h_{a,b})} h_{a,b} \ll \sum_{h\geq N} \left(\frac{\psi(h)}{h^2}\right)^s \psi(h)^{-1} h^2 h h \\ &= \sum_{h\geq N} \psi(h)^{s-1} h^{4-2s} \to 0 \text{ as } N \to \infty.\end{aligned}$$

Therefore $\mathcal{H}^s(W_2(\psi)) = 0$, as required.

5.3. **Proof of Theorem 3.2. The case of divergence.** To prove the divergence case of Theorem 3.2 we appeal to a recent result of Beresnevich and Velani [4] in which a mass transference principle for linear forms based on a 'slicing' technique is established. The result allows one to transfer statements about the Lebesgue measure of general limsup sets occurring in Diophantine approximation to ones involving Hausdorff measure. The ideas outlined below are specialised to suit the particular Diophantine approximation problems posed in this paper and are therefore simplified versions of those given in [4]. The general framework of [4] is far richer and allows one to address Diophantine problems involving systems of linear forms, inhomogeneous approximation and general measure functions in one consuming package.

Let $\mathcal{R} = (R_\alpha)_{\alpha\in J}$ be a family of lines in $\mathbb{R}^2$ indexed by an infinite countable set J. For every $\alpha \in J$ and $\delta \geq 0$ define the δ–neighborhood $\Delta(R_\alpha, \delta)$ of R_α by

$$\Delta(R_\alpha, \delta) := \{\boldsymbol{x} \in \mathbb{R}^2 : \operatorname{dist}(\boldsymbol{x}, R_\alpha) < \delta\} .$$

Next, let

$$\Upsilon : J \to \mathbb{R}^+ : \alpha \mapsto \Upsilon(\alpha) := \Upsilon_\alpha$$

be a non-negative, real valued function on J. Further, assume that for every $\epsilon > 0$ the set $\{\alpha \in J : \Upsilon_\alpha > \epsilon\}$ is finite. This condition implies that $\Upsilon_\alpha \to 0$ as α runs through J. Now define the following 'lim sup' set,

$$\Lambda(\Upsilon) = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} \in \Delta(R_\alpha, \Upsilon_\alpha) \text{ for infinitely many } \alpha \in J\} .$$

Theorem 5.1. *Let $\mathcal{R}$ and Υ as above be given. Let V be a line in $\mathbb{R}^2$ and*
(i) $V \cap R_\alpha \neq \emptyset$ *for all* $\alpha \in J$,
(ii) $\sup_{\alpha\in J} \operatorname{diam}(V \cap \Delta(R_\alpha, 1)) < \infty$.
Let f and $g : r \to g(r) := r^{-1} f(r)$ be dimension functions such that $r^{-2} f(r)$ is monotonic and let Ω be a ball in $\mathbb{R}^2$. Suppose for any ball B in Ω

$$\mathcal{H}^2\big(B \cap \Lambda(g(\Upsilon))\big) = \mathcal{H}^2(B) .$$

Then

$$\mathcal{H}^f\big(B\cap\Lambda(\Upsilon)\big) \;=\; \mathcal{H}^f(B)\;.$$

Now, let $f : r \to r^s$. As $1 < s < 2$ it follows that $r^{-2}f(r)$ is monotonic and f and g, defined as above, are both dimension functions. Further, let Ω be the unit square $[0,1)^2$, $J := \{(a,b,c) \in \mathbb{Z}^3_{\geq 0} : h_{a,b} = |a|\}$,

$$R_{(a,b,c)} = \{(x,y) \in \mathbb{R}^2 : a^2x + b^2y = c^2\}$$

and $\Upsilon_{(a,b,c)} := \psi(h_{a,b})/h^2_{a,b}$. Define sets $S_2(\psi)$ and $S_2'(\psi)$ by

$$S_2(\psi) := \Lambda(\Upsilon) \cap [0,1)^2 \quad \text{and} \quad S_2'(\psi) := \Lambda(g(\Upsilon)) \cap [0,1)^2.$$

Note that $S_2(\psi) \subset W_2(\psi)$. Note also, that under the divergence assumptions of Theorem 3.2, the same argument used for the divergence case of Theorem 3.1 can easily be adapted to show that $|S_2'(\psi)| = 1$. To complete the proof of Theorem 3.2, it is sufficient to prove the divergence case for $S_2(\psi)$. With this in mind, let $V := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\}$. It is straightforward to verify that conditions (i) and (ii) of Theorem 5.1 hold in this case. From the divergence case of Theorem 3.1, it follows that $\mathcal{H}^2(S_2'(\psi)) = 1 = \mathcal{H}^2([0,1)^2)$. Therefore, $\mathcal{H}^s(S_2(\psi)) = \mathcal{H}^s([0,1)^2) = \infty$ and Theorem 3.2 is proved.

5.4. **Proof of Corollary 3.3.** By the definition of the lower order for any $\delta > 0$ the inequality $\lambda_\psi + \delta \geq \frac{\log\frac{1}{\psi(2^r)}}{\log 2^r}$ for infinitely many r. It follows that

$$\psi(2^r) \geq (2^r)^{-\lambda_\psi - \delta} \quad \text{for infinitely many } r\,. \tag{26}$$

Take $s = 1 + \frac{3}{2+\lambda_\psi+\delta} - \delta$. Then

$$\psi(2^r)^{s-1}(2^r)^{5-2s} \geq (2^r)^{-(\lambda_\psi+\delta)(s-1)+5-2s} = (2^r)^{-(\lambda_\psi+2+\delta)(s-1)+3} = (2^r)^{\delta(\lambda_\psi+2+\delta)} > 1$$

for infinitely many r. Therefore,

$$\sum_{r=1}^{\infty} \psi(2^r)^{s-1}(2^r)^{5-2s} = \infty\,.$$

Since ψ is monotonic, using a simple 'condensation' argument it is easy to verify that

$$\sum_{h=1}^{\infty} \psi(h)^{s-1}h^{4-2s} = \infty\,.$$

Hence, by Theorem 3.2,

$$\mathcal{H}^s(W_2(\psi)) = \infty \qquad \text{and} \qquad \dim W_2(\psi) \geq s = 1 + \frac{3}{2+\lambda_\psi+\delta} - \delta.$$

Since $\delta > 0$ is arbitrary, we have $\dim W_2(\psi) \geq 1 + \frac{3}{2+\lambda_\psi}$.

Again, by the definition of the lower order, for any $\delta > 0$ the inequality $\lambda_\psi - \delta \leq \frac{\log\frac{1}{\psi(2^r)}}{\log 2^r}$ holds for all sufficiently large r. It follows that

$$\psi(2^r) \leq (2^r)^{-\lambda_\psi+\delta} \quad \text{for all sufficiently large } r\,. \tag{27}$$

Take $s = 1 + \frac{3}{2+\lambda_\psi-\delta} + \delta$. Then

$$\psi(2^r)^{s-1}(2^r)^{5-2s} \leq (2^r)^{-(\lambda_\psi-\delta)(s-1)+5-2s} = (2^r)^{-\delta(\lambda_\psi+2-\delta)}$$

for infinitely many r. Therefore,

$$\sum_{r=1}^{\infty} \psi(2^r)^{s-1}(2^r)^{5-2s} < \sum_{r=1}^{\infty}(2^r)^{-\delta(\lambda_\psi+2-\delta)} < \infty\,.$$

Since ψ is monotonic, using the 'condensation' argument it is easy to verify that

$$\sum_{h=1}^{\infty} \psi(h)^{s-1}h^{4-2s} < \infty\,.$$

Hence

$$\mathcal{H}^s(W_2(\psi)) < \infty \qquad \text{and} \qquad \dim W_2(\psi) \le s = 1 + \frac{3}{2+\lambda_\psi - \delta} + \delta.$$

Since $\delta > 0$ is arbitrary, we have $\dim W_2(\psi) \le 1 + \frac{3}{2+\lambda_\psi}$. Therefore, we have the equality $\dim W_2(\psi) = 1 + \frac{3}{2+\lambda_\psi}$.

5.5. **Proof of Corollary 3.4.** As $W_2(r \mapsto r^{-\tau'}) \subset W_2(r \mapsto r^{-\tau})$ for $\tau' > \tau$. It follows by continuity of $\dim(\cdot)$ that

$$\dim\left(\bigcap_{v>1} W_2(r \mapsto r^{-v})\right) = \lim_{v\to\infty} \dim\left(W_2(r\mapsto r^{-v})\right) = \lim_{v\to\infty}\left(1+\frac{3}{2+v}\right) = 1.$$

This establishes Corollary 3.4.

6. Outline of the General case $n \ge 3$

The convergence case of Theorem 3.1 for $n \ge 3$ is almost immediate. For every $(n+1)$-tuple $(\boldsymbol{a}, b) \in \mathbb{Z}_{\ge 0}^{n+1}$, let

$$\sigma_{\boldsymbol{a}}(b) := \{\boldsymbol{x} \in [0,1]^n : |\boldsymbol{a^2}\cdot \boldsymbol{x} - b^2| < \psi(h_{\boldsymbol{a}})\}$$

and

$$\sigma_{\boldsymbol{a}} := \bigcup_{b\in\mathbb{Z}} \sigma_{\boldsymbol{a}}(b)$$

where $\boldsymbol{a^2}$ is the vector $(a_1^2, a_2^2, \ldots, a_n^2)$. It is easy to see that each set $\sigma_{\boldsymbol{a}}(b)$ is a neighbourhood of an $(n-1)$-dimensional hyperplane of measure $|\sigma_{\boldsymbol{a}}(b)| \ll \psi(h_{\boldsymbol{a}})/h_{\boldsymbol{a}}^2$. Fix an $\boldsymbol{a} \in \mathbb{Z}_{\ge 0}^n$, $\sigma_{\boldsymbol{a}} \ne \emptyset$ implies that $b \ll h_{\boldsymbol{a}}$. Note that the number of vectors $\boldsymbol{a}$ for which $h_{\boldsymbol{a}} = h$ is $\ll h^{n-1}$. Now

$$\sum_{h=1}^{\infty} \sum_{\substack{\boldsymbol{a}\in\mathbb{Z}_{\ge 0}^n\setminus\{\boldsymbol{0}\};\\ h_{\boldsymbol{a}}=h}} \sum_{\substack{b\in\mathbb{Z}:\\ \sigma_{\boldsymbol{a}}(b)\ne\emptyset}} |\sigma_{\boldsymbol{a}}(b)| \ll \sum_{h=1}^{\infty} h^{n-2}\psi(h) < \infty$$

by assumption. It follows that $|W_n(\psi)| = 0$ and we are done.

Assuming for a moment the validity of the divergence part of Theorem 3.1 when $n \ge 3$. Establishing Theorem 3.2 is relatively straightforward.

In the convergence case we note that

$$W_n(\psi) = \bigcap_{h=1}^{\infty} \bigcup_{\substack{\boldsymbol{a}\in\mathbb{Z}^n,\\ h_{\boldsymbol{a}}=h}}^{\infty} \bigcup_{b\in\mathbb{Z}} (\sigma_{\boldsymbol{a}}(b) \cap [0,1)^n)$$

and each $\sigma_{\boldsymbol{a}}(b)$ can be covered by a family $C_{\boldsymbol{a}}^b$ of balls each of radius $\psi(h_{\boldsymbol{a}})/h_{\boldsymbol{a}}^2$ such that

$$\#C_{\boldsymbol{a}}^b \ll \left(h_{\boldsymbol{a}}^2/\psi(h_{\boldsymbol{a}})\right)^{n-1}.$$

It is then a simple matter to amend the proof in the case when $n = 2$ for $n \geq 3$ and deduce that $\mathcal{H}^s(W_n(\psi)) = 0$.

The divergence case of Theorem 3.2 can be proved with only minor modifications of the proof for the case when $n = 2$. The main changes to be made to the general framework of Theorem 5.1 are that $\mathcal{R}$ is now a countable family of $(n-1)$-dimensional hyperplanes, $\boldsymbol{x} \in \mathbb{R}^n$, V is a linear subspace of $\mathbb{R}^n$, f is a dimension function such that $r^{-n}f(r)$ is monotonic and $g : r \to r^{-(n-1)}f(r)$ is a dimension function.

Now, let $f : r \to r^s$, Ω be the unit hypercube $[0,1)^n$, $J := \{(\boldsymbol{a}, b) \in \mathbb{Z}^{n+1}_{\geq 0} : h_{\boldsymbol{a}} = |a_1|\}$,

$$R_{(\boldsymbol{a},b)} := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a^2} \cdot \boldsymbol{x} = b^2\}$$

and $\Upsilon_{(\boldsymbol{a},b)} := \psi(h_{\boldsymbol{a}})/h_{\boldsymbol{a}}^2$. The rest of the argument is essentially the same as that given above with 2 replaced by n and $V := \{\boldsymbol{x} \in \mathbb{R}^n : x_n = 0\}$.

It remains to establish the divergence part of Theorem 3.1 for the cases when $n \geq 3$. As noted above, the family of lines that we considered in § 4 have now been replaced by $(n-1)$-dimensional hyperplanes, but the analysis again hinges on the angle between the members of two non-parallel families of hyperplanes. It is relatively easy to see that the restrictions that applied to c in § 4.3.1 must also apply to b in the above argument and further, that the number of such b must also be $\asymp rh_{\boldsymbol{a}}$. This follows from the fact that the geometry in the n-dimensional case can be reduced to the same problem as that of the 2-dimensional case by projecting the ball B and the $(n-1)$-dimensional hyperplanes onto a 2-dimensional plane perpendicular to the family of hyperplanes defined by the equations

$$\boldsymbol{a^2} \cdot \boldsymbol{x} - b^2 = 0$$

where $b \in \mathbb{Z}$. A simple geometric argument implies that

$$|\sigma_{\boldsymbol{a}}(b) \cap B| \ll r^{n-1}\frac{\psi(h_{\boldsymbol{a}})}{h_{\boldsymbol{a}}^2}$$

where r is the radius of B. As the number of possible b such that $\sigma_{\boldsymbol{a}}(b) \cap B \neq \emptyset$ is $\ll rh_{\boldsymbol{a}}$ it follows that

$$|\sigma_{\boldsymbol{a}} \cap B| \ll r^n \frac{\psi(h_{\boldsymbol{a}})}{h_{\boldsymbol{a}}^2} h_{\boldsymbol{a}} \ll |B|\frac{\psi(h_{\boldsymbol{a}})}{h_{\boldsymbol{a}}},$$

and by an analogous argument to that in § 4.3.3 it can be shown that

$$|\sigma_{\boldsymbol{a}} \cap B| \gg |B|\frac{\psi(h_{\boldsymbol{a}})}{h_{\boldsymbol{a}}}$$

where the constants implied by the $\ll$ and $\gg$ are absolute. Recall that conditions (9) and (10) were imposed on a and b in the 2-dimensional cases. For the higher dimensional cases the corresponding conditions become

$$\gcd(a_1, a_2, \ldots, a_n) = 1 \tag{28}$$

and

$$1/2 \leq a_1/a_2 \leq 2, \tag{29}$$

with the same consequences as in § 4.3.4, namely a sufficient quantity of vectors to maintain divergence of our sum and non-collinearity of any two vectors satisfying (28).

As in the 2-dimensional case considered above, take any two vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ with $\boldsymbol{a} \neq \boldsymbol{a}'$, which must be linearly independent by (28). The upshot of linear independence is that the angle between the normals to the two hyperplanes, and therefore the hyperplanes themselves, is non-zero. Strictly speaking there are two angles, but we shall take the smaller of the two and call this α. The result of § 4.3.5 also holds in this case. It is a simple geometric argument to show that the volume of the parallelepiped obtained by intersecting any two members of the two families is now

$$\ll r^{n-2}\frac{\psi(h_{\boldsymbol{a}})}{h_{\boldsymbol{a}}^2}\frac{\psi(h_{\boldsymbol{a}'})}{h_{\boldsymbol{a}'}^2}.$$

An analogous argument to that presented in § 4.3.5 with the restriction that $\sin\alpha \geq \frac{1}{rh}$ yields the desired estimate for the sum of the measures of the intersections subject to the above restriction on α.

To complete the proof requires taking care of the cases when the angle α is such that $\sin\alpha \leq 1/rh$. Recall that in the 2-dimensional case, § 4.3.6, this naturally split into two cases; that of a moderately small angle and a very small angle. It was shown in the former case that the same estimate as that of the big angle case could be deduced and in the latter, that the sum of the intersections over the class of vectors with very small angle was in fact convergent and could therefore be disregarded. It is precisely these conclusions that can be shown to hold in the general case and the divergence part of Theorem 3.1 will follow in exactly the same manner as in the 2-dimensional case.

The analysis in § 4.3.6 relied on a key observation that the angle α couldn't get too small. More precisely that $\sin\alpha \gg 1/h_{\boldsymbol{a}}h_{\boldsymbol{a}'}$. This was a consequence of the assumption that $1/2 \leq a_1/a_2 \leq 2$. To establish this fact we used the standard result from elementary geometry that $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}||\sin\beta|$ where β is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. In higher dimensions the cross product $\times$ is replaced by the wedge product $\wedge$ where

$$\boldsymbol{a}\wedge\boldsymbol{b} = \left\{ \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \ : \ 1 \leq i < j \leq n \right\}.$$

Note without any loss of generality we can assume that the first two coordinates give the biggest determinant by reordering if necessary and it is this observation, coupled with the assumption that $1/2 \leq a_1/a_2 \leq 2$ that allows us to conclude that $\sin\alpha \gg 1/h_{\boldsymbol{a}}h_{\boldsymbol{a}'}$. The argument for the case when the angle is moderately small is exactly the same as for the 2-dimensional case. This leaves only the case when

$$\sin\alpha < \frac{1}{r^2 h_{\boldsymbol{a}}h_{\boldsymbol{a}'}}. \tag{30}$$

As there is a free choice in all but the first two components of either of the vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ the number of pairs of vectors that we need to consider is $h_{\boldsymbol{a}}^{n-2}h_{\boldsymbol{a}'}^{n-2}\#\{(a_1,a_2,a_1',a_2')\}$. Using the estimate we deduced in § 4.3.6 it follows that the sum we are estimating is convergent and can therefore be disregarded.

The final steps in proving the divergence part of Theorem 3.1 follow in exactly the same manner as that of the 2-dimensional case.

There are only minor modifications needed to the proofs of Corollaries 3.3 and 3.4 to establish them in the general case and the details are left to the reader.

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Victor Beresnevich, Department of Mathematics, University of York, Heslington, York, YO10 5DD, United Kingdom
E-mail address: vb8@york.ac.uk

Maurice Dodson, Department of Mathematics, University of York, Heslington, York, YO10 5DD, United Kingdom
E-mail address: mmd1@york.ac.uk

Simon Kristensen, School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, King's Buildings, Mayfield Road, Edinburgh, EH9 3JZ, United Kingdom
Current address: Department of Mathematical Sciences, Faculty of Science, University of Aarhus, Ny Munkegade, Building 530, DK-8000 Aarhus C, Denmark
E-mail address: sik@imf.au.dk

Jason Levesley, Department of Mathematics, University of York, Heslington, York, YO10 5DD, United Kingdom
E-mail address: jl107@york.ac.uk