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SPONTANEOUS GENERATION OF EIGENVALUES

by Thomas Branson and Bent Ørsted

Spontaneous generation of eigenvalues

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Abstract: We show that the action of conformal vector fields on functions on the sphere determines the spectrum of the Laplacian (or the conformal Laplacian), without further input of information. The spectra of intertwining operators (both differential and non-local) with principal part a power of the Laplacian follows as a corollary. An application of the method is the sharp form of Gross' entropy inequality on the sphere. The same method gives the spectrum of the Dirac operator on the sphere, as well as of a continuous family of nonlocal intertwinors, and an infinite family of odd-order differential intertwinors.

Introduction

Among compact Riemannian manifolds, the sphere is remarkable in several ways. For example, it admits a large group of conformal transformations, and the corresponding conformal vector fields are known to contain important information about the geometry of the sphere. What is perhaps not so well-known is the fact that the action of these vector fields also determines a large part of global analysis on the sphere, notably the exact eigenvalues of most natural differential and pseudodifferential operators. Already the example of the Laplace operator on functions is perhaps a little surprising.

In [3], a method was developed for finding the spectrum of intertwining operators for certain representations of semisimple groups. In this calculation scheme, one first inputs the spectral data of a differential *spectrum generating operator*. The idea is that this spectral data is readily accessible, obtained just from quadratic Casimir data for finite-dimensional representations of compact groups. Using this information, one may generate the (much less accessible) spectral data on the intertwining operators, which occur in series parameterized by a complex number r. Typically, a subseries of *differential* operators occurs at values of r which are positive integral, in a sense appropriate to the particular series.

A special situation that may occur in such a calculation is when the spectrum generating operator is closely related to one of the differential intertwinors. The simplest interpretations of "closely related" here are (1) when the operators are identical, or differ by a constant additive shift; (2) when the spectrum generating operator is a polynomial in one of the differential intertwinors. In some of these cases, there may be some many relations among the spectral data that one can generate the spectra of the operators in question with no input at all; that is, the spectra are *spontaneously generated*.

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In this paper, we execute this process of spontaneous generation in some examples, and derive some related results. Among the central relations are a quadratic equation (9) relating "adjacent" eigenvalues of the conformal Laplacian on scalar fields, and a cubic equation (38) which serves a similar purpose for the Dirac operator on spinor fields. Once the spectra of these fundamental operators are in place, one may go on to find parameterized families of operators of various orders which are *intertwining* for conformal transformations and vector fields in the same sense as are the conformal Laplacian and Dirac operator.

An interesting feature of this construction is that it proves completeness of the eigenvalue list it obtains – any "wrong" eigenvalue will generate lower and lower ones, until some basic estimate is violated. And even though no information on spherical harmonics is input, the fact that eigenfunctions are spherical harmonics is a consequence of the construction. (See, for example, Corollary 6.)

Our method, combined with Beckner's sharp Hardy-Littlewood- Sobolev inequalities on the sphere, is perfectly suited for deriving the sharp form of Gross' entropy inequality on the sphere, estimating the integral of $f^2 \log(f)$ for a positive smooth function. This argument culminates in Theorem 8 below.

Of course, some of the spectral resolutions we get here have been known for some time; for example, the spectra of the Laplacian (on functions), and of the Dirac operator on the sphere. The spectra of the other operators is less well known, but nevertheless can be obtained by specializing formulas in the literature (for example [3]). However, the philosophy is avoid all heavy machinery, and derive the spectra just from a couple of elementary operator commutation relations. Among the possible virtues of this sort of "primitivist" derivation is the prospect of deriving such spectra at an early point of a course in differential geometry, quantum mechanics, or relativity.

The Laplacian and conformal Laplacian on the sphere

Let $n \geq 2$, and let ρ_0 be the *azimuthal angle* on the sphere S^n with homogeneous coordinate functions $x_0, x_1, \ldots, x_n$. That is,

$$x_0 = \cos \rho_0,$$

so that $\rho(p)$ is the angle between an indeterminate point p and the point $(1, 0, \ldots, 0)$. More invariantly, we can define an azimuthal angle $\rho(p, q)$ with any desired point q in place of $(1, 0, \ldots, 0)$. In particular, the azimuthal angle from the point where $x_i = 1$ will be denoted ρ_i (for each i).

The proper conformal vector fields on S^n are generated by the $T_i := (\sin \rho_i)(\partial/\partial \rho_i)$ for $j = 0, \ldots, n$. The meaning of $\partial/\partial \rho_i$ is: complete ρ_i to a coordinate system by taking a coordinate system on the latitude $\{\rho_i = \text{const}\}$ (a copy of the sphere S^{n-1}), and compute with these other coordinates held constant. The T_i are conformal vector fields:

$$L(T_i)g = 2x_i g,$$

where g is the round metric on S^n and L denotes the Lie derivative.

In the more general setting of pseudo-Riemannian manifolds (M, g), let d be the exterior derivative on functions, and let δ be its formal adjoint. The operator $\Delta = \delta d$ is the *Laplacian*, and the *conformal Laplacian* is the operator

$$D := \Delta + \frac{n-2}{4(n-1)}K,$$

where K is the scalar curvature of g. This satisfies the *conformal covariance* relation

$$\Omega^{(n+2)/2}D_{\Omega^2 g}f = D_g(\Omega^{(n-2)/2}f) \tag{1}$$

for $\Omega > 0$ and f smooth functions on M. If (M, g) admits a conformal vector field T with $L_T g = 2\omega g$, then (1) immediately leads, via the local flow, to

$$D\left(T + \frac{n-2}{2}m(\omega)\right) = \left(T + \frac{n+2}{2}m(\omega)\right)D, \tag{2}$$

where $m(\omega)$ is multiplication by ω. On the other hand,

$$[D, m(\varphi)] = [\Delta, m(\varphi)] = m(\Delta f) - 2\iota(d\varphi)d, \tag{3}$$

where ι is interior multiplication.

Specializing to the round sphere S^n, let

$$U_j := T_j + \frac{n}{2}x_j.$$

Then (2) reads
$$D(U_i - m(x_i)) = (U_i + m(x_i))D. \tag{4}$$

In particular, applying this relation to the function 1 and using the fact that the scalar curvature of round S^n is $n(n-1)$, so that

$$D = \Delta + \frac{n(n-2)}{4} \quad \text{on } S^n, \tag{5}$$

we get
$$\left(\Delta + \frac{n(n-2)}{4}\right)\frac{n-2}{2}x_i = \frac{n+2}{2} \cdot x_i \cdot \frac{n(n-2)}{4},$$

that is,
$$\Delta x_i = nx_i. \tag{6}$$

Specializing (3), we have

$$\begin{aligned}
[D, m(x_i)] &= m(\Delta x_i) - 2\iota(dx_i)d \\
&= nm(x_i) + 2T_i \\
&= 2U_i,
\end{aligned} \tag{7}$$

where we have used (6) together with the fact that $-dx_i$ corresponds to T_i under the metric identification.

Let $E(\lambda)$ denote the λ-eigenspace of D, and suppose for some λ we have $0 \neq \varphi \in E(\lambda)$. Given another number μ and an operator T, denote the compression of T acting $E(\lambda) \to E(\mu)$ by $_\mu| T |_\lambda$.

Given μ, if we compress (7) and (4) to operators from $E(\lambda)$ to $E(\mu)$, we have

$$_\mu| U_i |_\lambda = \frac{\mu - \lambda}{2} {}_\mu| m(x_i) |_\lambda,$$
$$\mu \cdot {}_\mu| (U_i - m(x_i)) |_\lambda = \lambda \cdot {}_\mu| (U_i + m(x_i)) |_\lambda,$$

(8)

from which

$$\mu \left(\frac{\mu - \lambda}{2} - 1 \right) {}_\mu| m(x_i) |_\lambda = \lambda \left(\frac{\mu - \lambda}{2} + 1 \right) {}_\mu| m(x_i) |_\lambda.$$

For fixed λ, this is a quadratic equation in μ:

$$\mu^2 - 2\lambda\mu - 2\mu + \lambda^2 - 2\lambda = 0.$$

(9)

The solutions

$$\lambda^\pm := \lambda + 1 \pm \sqrt{4\lambda + 1};$$

(10)

are candidates for new eigenvalues of D. Since Δ is a nonnegative operator, (5) gives

$$\lambda \geq \frac{n(n-2)}{4}.$$

(11)

Thus

$$\lambda^- < \lambda < \lambda^+$$

unless $n = 2$ and $\lambda = 0$, in which case $\lambda^- = \lambda$.

Moreover, since (9) is symmetric in μ and λ, we have that μ is a root of the equation obtained starting with λ iff λ is a root of the equation obtained starting with μ. That is,

$$(\lambda^+)^- = \lambda = (\lambda^-)^+.$$

The obvious main questions are:

- If $E(\lambda) \neq 0$, are the spaces $E(\lambda^\pm)$ necessarily nonzero?

- Can we generate all μ for which $E(\mu) \neq 0$ by starting with a single $E(\lambda) \neq 0$ and iterating the process (10)?

The answer to each question is yes, with the exception that $E(\lambda^-)$ vanishes for what should be the bottom eigenvalue λ, namely $n(n-2)/4$. In fact, both questions are answered by the same calculation: any "wrong" eigenvalue keeps generating lower and lower "wrong" eigenvalues, until we get one that violates the estimate (11).

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A weakness of the calculation is that it cannot immediately tell the dimension of the eigenspaces $E(\lambda)$. Of course these dimensions are easily obtainable with an injection of a small amount of Lie theory, but that would violate the spirit of the present calculation. Note however, that in Corollary 6 we prove (from our point of view) the correspondence with spherical harmonics.

Stipulating the statements just above (which will be proved below), we may start either with the fact that

$$1 \in E(0, \Delta) = E(n(n-2)/4, D)$$

or

$$x_i \in E(n, \Delta) = E((n+2)n/4, D) \quad \text{(from (6))}$$

and iterate (10) to compute that the eigenvalues of D are the

$$\lambda_j := \left(\frac{n-2}{2} + j \right) \left(\frac{n}{2} + j \right) \quad \text{for} \quad j = 0, 1, 2, \ldots, \tag{12}$$

or equivalently, that the eigenvalues of Δ are the $j(n-1+j)$.

To begin to answer the bullet point questions above, let us look more closely at the eigenfunctions (as opposed to just eigenvalues) that our process is generating. If $\varphi \in E(\lambda, D)$, then

$$D(x_i\varphi) = \lambda x_i\varphi + 2U_i\varphi \tag{13}$$

by (7). By (4) and (13),

$$D(U_i\varphi) = D(x_i\varphi) + \lambda(U_i + x_i)\varphi = (\lambda + 2)U_i\varphi + 2\lambda x_i\varphi. \tag{14}$$

If we would like $(U_i + cx_i)\varphi$ to be in $E(\mu, D)$, then

$$\mu(U_i + cx_i)\varphi = (\lambda + 2 + 2c)U_i\varphi + (2 + c)\lambda x_i\varphi.$$

If the U_i and x_i terms on the two sides are to agree, we need

$$c(\lambda + 2 + 2c) = \lambda(2 + c).$$

This is a quadratic equation on c, with the roots

$$c^{\pm} = (\lambda - \lambda^{\mp})/2.$$

The eigenvalue μ corresponding to c^+ (resp. c^-) is easily computed to be λ^+ (resp. λ^-). Thus we have:

Proposition 1. *If $\varphi \in E(\lambda, D)$, then*

$$P_i\varphi := \left(U_i + \tfrac{1}{2}(\lambda - \lambda^-)x_i\right) \quad \varphi \in E(\lambda^+, D),$$
$$M_i\varphi := \left(U_i + \tfrac{1}{2}(\lambda - \lambda^+)x_i\right) \quad \varphi \in E(\lambda^-, D).$$

Of course there is *a priori* the danger that $P_i\varphi$ vanishes for $i = 0, \ldots, n$, or that each $M_i\varphi$ does, even though $\varphi \neq 0$. We now proceed to rule out this danger, except in the case where it is expected (applying M_i to the eigenspace where 1 lives). That is, we shall compute

$$\sum_i M_i P_i \varphi \quad \text{and} \quad \sum_i P_i M_i \varphi.$$

If the first of these is nonzero, it must be that $P_i\varphi \neq 0$ for some i; similarly, if the second is nonzero, some $M_i\varphi$ does not vanish identically. We begin with

$$\sum_i M_i P_i \varphi = \sum_i \left(U_i + \frac{-1 - \sqrt{1 + 4\lambda^+}}{2} x_i \right) \left(U_i + \frac{-1 + \sqrt{1 + 4\lambda}}{2} x_i \right) \varphi.$$

Since

$$\sqrt{1 + 4\lambda^\pm} = \pm 2 + \sqrt{1 + 4\lambda}, \tag{15}$$

the above is

$$\sum_i \left(U_i + \frac{-3 - \sqrt{1 + 4\lambda}}{2} x_i \right) \left(U_i + \frac{-1 + \sqrt{1 + 4\lambda}}{2} x_i \right) \varphi$$

$$= \sum_i \left((U_i - m(x_i)) + c^- m(x_i) \right) \left((U_i - m(x_i)) - c^- m(x_i) \right) \varphi.$$

This simplifies to

$$\left\{ \sum_i (U_i - m(x_i))^2 - (c^-)^2 m \left(\sum_i x_i^2 \right) \right.$$
$$\left. - c^- \left(\sum_i \{ (U_i - m(x_i)) m(x_i) - m(x_i)(U_i - m(x_i)) \} \right) \right\} \varphi. \tag{16}$$

Now

$$\sum_i x_i^2 = 1, \tag{17}$$

and commuting this relation with D,

$$0 = [D, m(1)]$$
$$= \sum_i \{ m(x_i)[D, m(x_i)] + [D, m(x_i)] m(x_i) \}$$
$$= 2 \sum_i \{ m(x_i) U_i + U_i m(x_i) \} \tag{18}$$

by (7). This shows in turn that

$$\sum_i (U_i + a m(x_i))^2 = a^2 + \sum_i U_i^2 \tag{19}$$

for any number a. On the other hand,

$$\sum_i [U_i, m(x_i)] = \sum_i T_i \cos \rho_i = -\sum_i \sin^2 \rho_i$$

$$= -\sum_i (1 - \cos^2 \rho_i) = 1 - (n+1) = -n. \tag{20}$$

It is clear that we shall also need a simplification of $\sum_i U_i^2$. This is provided by:

Lemma 2. $\sum_i T_i^2 = -\Delta$.

Proof: Consider one term of the sum on the left, and suppress the subscript i for now, so that $T_i = T$, $x_i = x$. In abstract index notation, one term from the left side of the identity is

$$x^a \nabla_a (x^b \nabla_b) = (\nabla^a x)\{(\nabla^b x)\nabla_a + \nabla_a \nabla^b x\}\nabla_b. \tag{21}$$

Since $\nabla^a x = (dx)^a$, the first term is $(dx \otimes dx)^{ab} \nabla_a \nabla_b$. The second term involves

$$\nabla_a \nabla_b x = (\text{Hess } x)_{ba}.$$

But the conformal Killing equation $L_T g = 2xg$ reads, in abstract index notation,

$$\nabla_a T_b + \nabla_b T_a = 2x g_{ab}.$$

Since $T_a = -(dx)_a$, this says that $2\text{Hess } x = -2xg$, so that the quantity (21) becomes

$$(dx \otimes dx)^{ab}\nabla_a \nabla_b - (\nabla^a x)x \delta_a{}^b \nabla_b = (dx \otimes dx)^{ab}\nabla_a \nabla_b - (\nabla^a x)x\nabla_a.$$

Re-inserting the subscript i and summing over it, the above quantity becomes

$$g^{ab}\nabla_a \nabla_b - \tfrac{1}{2}(\nabla^a 1)\nabla_a = g^{ab}\nabla_a \nabla_b = -\Delta,$$

since $\sum_i dx_i \otimes dx_i$ is the pullback of the ambient flat $\mathbb{R}^{n+1}$ metric, i.e. the round metric, and

$$\sum_i x_i \nabla^a x_i = \frac{1}{2}\nabla^a \underbrace{\left(\sum_i x_i^2\right)}_{1}. \qquad \square$$

By the lemma and (19),

$$-\left(D - \frac{n(n-2)}{4}\right) = \sum_i U_i^2 + \frac{n^2}{4},$$

so that

$$\sum_i U_i^2 = -D - n/2.$$

Using all the identities just derived to evaluate (16), we get

$$\sum_i M_i P_i \varphi = \left(-\lambda - \frac{n}{2} + 1 - (c^-)^2 + nc^-\right)\varphi.$$

It is convenient to write this in terms of

$$\nu := \sqrt{1 + 4\lambda},$$

so that

$$\lambda = \frac{\nu^2 - 1}{4}, \qquad c^- = -\frac{1 + \nu}{2},$$

and

$$\sum_i M_i P_i \varphi = -\frac{1}{2}(\nu + n - 1)(\nu + 2)\varphi. \tag{22}$$

This shows that some $P_i\varphi$ is nonzero unless $\nu = 1 - n$ or $\nu = -2$; in particular it is nonzero for all positive values of ν. But by (11), $\nu \geq n - 1$.

The corresponding calculation with M_i and P_i in the other order begins with

$$\sum_i P_i M_i \varphi = \sum_i \left(U_i + \frac{-1 + \sqrt{1 + 4\lambda^-}}{2}x_i\right)\left(U_i + \frac{-1 - \sqrt{1 + 4\lambda}}{2}x_i\right)\varphi,$$

and (by virtue of (15)) produces a version of (16) with c^+ in place of c^-, yielding

$$\sum_i P_i M_i \varphi = \left(-\lambda - \frac{n}{2} + 1 - (c^+)^2 + nc^+\right)\varphi.$$

Since $c^+ = (\nu - 1)/2$, we get

$$\sum_i P_i M_i \varphi = -\frac{1}{2}(\nu - n + 1)(\nu - 2)\varphi. \tag{23}$$

Thus some $M_i\varphi$ is nonzero unless $\nu = n - 1$ or $\nu = 2$. In the first case, $\lambda = n(n-2)/4$; this was expected, since (11) and $1 \in E(n(n-2)/4, D)$ show that this is the bottom eigenvalue of D. Indeed, a look back at the formula for $M_i\varphi$ in Proposition 1 shows that $M_i 1 = 0$. In the second case, $\nu = 2$, we have $\lambda = 3/4$. By (11), this implies that $n = 3$ and equality holds in (11); this is a special case of the situation just discussed. We have proved:

Proposition 3. *For $\lambda \geq n(n-2)/4$, if $0 \neq \varphi \in E(\lambda, D)$, then $P_i\varphi$ is a nonzero element of $E(\lambda^+, D)$ for some i. If $\lambda > n(n-2)/4$, then $M_k\varphi$ is a nonzero element of $E(\lambda^-, D)$ for some k.*

The mechanism by which we have generated the eigenvalues (12) also rules out any other numbers occurring as eigenvalues. Indeed, suppose λ is an eigenvalue of D not on the list, so that (by (11)) there is some natural number j with $\lambda_j < \lambda < \lambda_{j+1}$. Since the map $\lambda \to \lambda^-$ is strictly monotonic, we have $\lambda_{j-1} < \lambda^- < \lambda_j$, $\lambda_{j-2} < \lambda^{--} < \lambda_{j-1}$ and so on, until we reach eigenvalues $\mu \in (\lambda_0, \lambda_1)$, $\mu^- < \lambda_0$, contradicting (11). The key ingredient, of course, is the assurance from Proposition 3 that each lower eigenspace in the induction is truly nonzero. We have:

Proposition 4. *The λ_j of (12) give the complete list of eigenvalues of D on S^n. As a result, the $j(n-1+j)$ for $j = 0, 1, 2, \ldots$ give the complete list of eigenvalues of Δ on S^n.*

We can also harvest the following corollary. Let $E_j := E(\lambda_j, D)$.

Corollary 5. *The span of the $m(x_i)E_j$ and $Dm(x_i)E_j$ is $E_{j-1} \oplus E_{j+1}$ (where $E_{-1} = 0$).*

Proof: The inclusion $\subset$ is immediate from the first line of (8) together with Proposition 1. For the inclusion $\supset$, (22,23) show that $E_{j-1} \oplus E_{j+1}$ is contained in the sum of the $m(x_i)E_j$ and the

$$U_i E_j \subset Dm(x_i)E_j + m(x_i)DE_j = Dm(x_i)E_j + m(x_i)E_j. \qquad \square$$

Note that we have not injected any information on spherical harmonics into our procedure for generating the eigenvalues. We may, however, get the interpretation of the eigenspaces as spaces of spherical harmonics as a consequence of what we have done:

Corollary 6. *E_j is exactly the set of restrictions from $\mathbb{R}^{n+1}$ to S^n of j-homogeneous harmonic polynomials in the x_i.*

Proof: First, the elements of E_j are restrictions of j-homogeneous polynomials, since this is true of E_0, so follows from the previous corollary by induction on j. Second, if we compute the Laplacian of $\mathbb{R}^{n+1}$ in spherical coordinates, we get

$$\Delta_{\mathbb{R}^{n+1}} = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^n},$$

so that if $\varphi \in E_j$, and we extend to $\mathbb{R}^{n+1}$ by extending the j-homogeneous polynomial formula,

$$\Delta_{\mathbb{R}^{n+1}}\varphi = [-j^2 - (n-1)j + j(n-1+j)]\varphi = 0.$$

This calculation also shows that each j-homogeneous harmonic polynomial in $\mathbb{R}^{n+1}$ gives rise to an element of E_j. $\qquad \square$

As a bonus result, we can give operators A_{2r} which are functions of D (or of Δ), which satisfy generalizations of the conformal covariance relation (4), namely

$$D(U_i - r \cdot m(x_i)) = (U_i + r \cdot m(x_i))D, \tag{24}$$

for each $r \in \mathbb{C}$, each of which takes an eigenvalue on $E_j := E(\lambda_j, D)$. That is, we can find functions f_r on the spectrum of D for which

$$A_{2r}|_{E_j} = f_r(\lambda_j)\mathrm{Id}_{E_j}.$$

In the usual notation of functional calculus, we write $A_{2r} = f_r(D)$.

Remark: Though we have shown that we have all eigenvalues for the conformal Laplacian, our derivation does not contain a proof that the span of the corresponding

eigenfunctions is dense in $L^2(S^n)$. Of course, we have this by general elliptic theory. This sort of completeness is implicitly used later, when we describe other covariant operators, or *intertwinors*, as functions of a basic one (for example D, or the operator A_1 of (27), or the Dirac operator P as it is used in Theorem 12.

First note that $\lambda_j^\pm = \lambda_{j\pm1}$. Compressing (24), and letting μ_j be the putative eigenvalue for A_{2r} on E_j, we have

$$\mu_{j\pm1}\left(\frac{\lambda_{j\pm1}-\lambda_j}{2}-r\right)E_{j\pm1}|\,m(x_i)|_{E_j} = \mu_j\left(\frac{\lambda_{j\pm1}-\lambda_j}{2}+r\right)E_{j\pm1}|\,m(x_i)|_{E_j}.$$

Since the λ_j are known, we may generate the various μ_j inductively by demanding

$$\mu_{j\pm1}\left(\lambda_{j\pm1}-\lambda_j-2r\right) = \mu_j\left(\lambda_{j\pm1}-\lambda_j+2r\right). \tag{25}$$

Since *a priori* this gives two relations between adjacent μ_j, we must check for consistency. Since $\lambda_{j+1}-\lambda_j = n+2j$, we have this, provided we handle occurrences of vanishing $\lambda_{j\pm1}-\lambda_j \pm' 2r$ correctly. We emerge with a choice of $\mu_j^{(2r)}$ that is unique up to a constant (independent of j) factor:

Proposition 7. *For fixed $r \notin \{-n/2, -n/2-1, \dots\}$,*

$$Z(r,j) := \frac{\Gamma(n/2+j+r)}{\Gamma(n/2+j-r)} \tag{26}$$

is, up to a constant nonzero factor, the unique function of $j \in \{0,1,2,\dots\}$ that satisfies (25) and does not vanish identically. For $r = -n/2-j_0$ with j_0 a nonnegative integer, the residue of the above expression (viewed as a meromorphic function of r) at $-n/2-j$ is, up to a constant nonzero factor, the unique function of $j \in \{0,1,2,\dots\}$ that satisfies (25) and does not vanish identically.

If $r \in \frac{1}{2}\mathbb{Z}^+$, then

$$Z(r,j) = (n/2+j+r-1)\cdots(n/2+j-r).$$

If $-r \in \frac{1}{2}\mathbb{Z}^+$, then

$$Z(r,j) = \left(\prod_{1\le p\le -2r}^{\bullet}(n/2+j-r-p)\right)^{-1},$$

where $\prod^{\bullet}$ is the product over nonzero factors.

The operator A_1 has eigenvalue $Z(\frac{1}{2},j) = (n-1)/2+j$ on E_j; thus

$$A_1 := \sqrt{\Delta + \left(\frac{n-1}{2}\right)^2}. \tag{27}$$

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This implies via (26) that

$$A_{2r} = \frac{\Gamma(A_1 + \frac{1}{2} + r)}{\Gamma(A_1 + \frac{1}{2} - r)}, \quad r \notin \{-n/2, -n/2 - 1, \dots\}.$$

Similarly, A_{2r} (for r outside the exceptional set given above) may be written as a function of A_{2q} for any nonzero q outside the exceptional set. In particular, for $r \in \mathbb{Z}^+$ we get the sequence of differential operators

$$\prod_{p=1}^{r} \left\{ \Delta + \left(\frac{n}{2} + p - 1 \right) \left(\frac{n}{2} - p \right) \right\},$$

also written down in [2], Remark 2.23.

The entropy inequality

Equation (25), together with the sharp Hardy-Littlewood-Sobolev inequalities of Beckner [1] on the sphere, give an argument for an optimal form of Gross' entropy inequality on the sphere.

Let r be the parameter of (25), and denote the r-derivative at $r = 0$ by a prime. Differentiating (25) and normalizing so that the intertwinor A_0 is the identity, we have

$$\mu'_{j\pm1}(\lambda_{j\pm1} - \lambda_j) - 2 = \mu'_j(\lambda_{j\pm1} - \lambda_j) + 2.$$

Thus

$$(\mu_{j\pm1} - \mu_j)' = \frac{4}{\lambda_{j\pm1} - \lambda_j}. \tag{28}$$

The $j+1$ and $j-1$ relation lists are consistent, so the information in (28) is equivalent to

$$(\mu_{j+1} - \mu_j)' = \frac{4}{\lambda_{j+1} - \lambda_j}. \tag{29}$$

Since

$$\lambda_{j+1} - \lambda_j = n + 2j,$$

this gives (with $m := n/2$)

$$\mu'_j = \mu'_0 + \frac{2}{m} + \frac{2}{m+1} + \cdots + \frac{2}{m+j-1}.$$

This is in fact a formula for the eigenvalues of A'_{2r}.

It is convenient to pick a normalization of the series A_{2r} with the property that $\mu'_0 = 0$. This is obtained by multiplying the spectral function (26) by $\Gamma(m - r)/\Gamma(m + r)$ – this factor is independent of j, so for fixed r, gives a constant multiple of the A_{2r} described by (26). With this normalization,

$$\mu'_j = \frac{2}{m} + \frac{2}{m+1} + \cdots + \frac{2}{m+j-1}.$$

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Let us denote the intertwinors normalized in this way by B_{2r}. These operators also appear in Beckner's sharp Hardy-Littlewood-Sobolev inequalities for $r \in [0, n/2)$: if F is a positive smooth function on the sphere,

$$\int F^{(n-2r)/2} B_{2r} F^{(n-2r)/2} \geq \left(\int F^n \right)^{(n-2r)/n},$$

where all integrals are with respect to normalized measure. One has equality for every F when $r = 0$ (each side is $\int F^n$). But for $r > 0$, equality holds exactly when F is a constant multiple of a conformal (diffeomorphism) factor.

Now write the inequality as

$$0 \leq - \left(\int F^n \right)^{(n-2r)/n} + \int F^{(n-2r)/2} B_{2r} F^{(n-2r)/2},$$

and take $(d/dr)|_{r=0}$ of each side. (Of course this differentiation of the inequality depends on the fact that equality holds for every F at $r = 0$.) This gives

$$0 \leq \frac{2}{n} \left(\int F^n \right) \log \int F^n - 2 \int F^n \log F + \int F^{n/2} B'_{2r} F^{n/2},$$

or in a slightly better form,

$$2 \int F^n \log F \leq \frac{2}{n} \left(\int F^n \right) \log \int F^n + \int F^{n/2} B'_{2r} F^{n/2}.$$

What we know immediately about the case of equality is that it includes *at least* constant multiples of the conformal factors F. But since the quantities in play are analytic in r near $r = 0$, we get *precisely* these functions.

With $f := F^{n/2}$, we can rewrite as

$$\frac{4}{n} \int f^2 \log f \leq \frac{2}{n} \left(\int f^2 \right) \log \int f^2 + \int f A'_{2r} f,$$

for

$$\mathrm{eig}(A'_{2r}, E_j) = \frac{2}{m} + \frac{2}{m+1} + \cdots + \frac{2}{m+j-1}.$$

It is easily verified that these eigenvalues are $\leq$ (but very close to) those of $2 \log(2A_1/(n-1))$, where A_1 is as in (27). (The harmonic sum is a certain Riemann sum for the integral defining the log.) Thus we may write

$$\frac{2}{n} \int f^2 \log f \leq \frac{1}{n} \left(\int f^2 \right) \log \int f^2 + \int f \left(\log \frac{2A_1}{n-1} \right) f, \tag{30}$$

giving away a little sharpness. To summarize:

Theorem 8. *For smooth positive f on the sphere S^n, in normalized measure,*

$$\frac{4}{n} \int f^2 \log f \le \frac{2}{n} \left(\int f^2 \right) \log \int f^2 + \int f A'_{2r} f, \tag{31}$$

where A'_{2r} takes the eigenvalue

$$\frac{2}{m} + \frac{2}{m+1} + \cdots + \frac{2}{m+j-1}$$

on $j^{\underline{\text{th}}}$ order spherical harmonics. In particular the weaker statement (30) holds. Equality holds in (31) if and only if $f^{2/n}$ is a positive constant multiple of a conformal (diffeomorphism) factor.

Spinor operators

Let P be the Dirac operator on the spinor bundle Σ. By the Lichnerowicz formula,

$$P^2 = \nabla^* \nabla + \tfrac{1}{4} K,$$

so that

$$P^2 = \nabla^* \nabla + \tfrac{1}{4} n(n-1) \quad \text{on } S^n.$$

As a result, if λ is an eigenvalue of P on the sphere, then λ is real with

$$\lambda^2 \ge n(n-1)/4. \tag{32}$$

The analogue of (7) is

$$[P^2, m(\omega)] = [\nabla^* \nabla, m(\omega)] = 2\nabla_T + nm(\omega). \tag{33}$$

The conformal covariance relation satisfied by P is

$$P \left(L_T + \frac{n-1}{2} \omega \right) = \left(L_T + \frac{n+1}{2} \omega \right) P, \tag{34}$$

on general pseudo-Riemannian spin manifolds, where $L_T g = 2\omega g$. If X is an arbitrary smooth vector field, [4] shows that the Lie and covariant derivatives on spinors are related by

$$L_X - \nabla_X = -\tfrac{1}{8} (dX)_{ab} \gamma^a \gamma^b,$$

where dX is the exterior derivative of the 1-form corresponding to X under the metric, and γ is the fundamental tensor-spinor (a section of $TM \otimes \text{End}(\Sigma)$). This assumes that the internal conformal weight 0 has been assigned to the spinor bundle; assigning internal weight $\pm \tfrac{1}{2}$ as in [5] results in an extra term involving $\text{div}\, X$. Our proper conformal vector fields T_i on S^n have the $-dx_i$ as their metric correspondents, so we may specialize (33) to

$$[P^2, m(x_i)] = 2L(T_i) + nm(x_i) =: 2U_i. \tag{35}$$

The conformal covariance relation (34) specializes to

$$P(U_i - \tfrac{1}{2}m(x_i)) = (U_i + \tfrac{1}{2}m(x_i))P. \tag{36}$$

Compressing the operators in (35) to act between two eigenspaces for P, $E(\lambda, P) \to E(\mu, P)$, we have

$$_\mu|\,U_i\,|_\lambda = \frac{\mu^2 - \lambda^2}{2}{}_\mu|\,m(x_i)\,|_\lambda, \tag{37}$$

after which (36) implies that

$$\mu(\mu^2 - \lambda^2 - 1)_\mu|\,m(x_i)\,|_\lambda = \lambda(\mu^2 - \lambda^2 + 1)_\mu|\,m(x_i)\,|_\lambda.$$

This is implied by

$$\mu(\mu^2 - \lambda^2 - 1) = \lambda(\mu^2 - \lambda^2 + 1),$$

which may be rewritten

$$(\mu + \lambda)(\mu - \lambda + 1)(\mu - \lambda - 1) = 0. \tag{38}$$

(Note that all these equations are symmetric in μ and λ.)

The cubic equation (38) suggests the possible nonvanishing of three "adjacent" eigenspaces $E(-\lambda, P)$, $E(\lambda + 1, P)$, and $E(\lambda - 1, P)$, given $E(\lambda, P) \neq 0$. Let $0 \neq \psi \in E(\lambda, P)$, and write the Dirac operator as $\gamma^a \nabla_a$ (using abstract index notation, in which repetition of an index, once up and once down, denotes a contraction). To avoid excessive super- and subscripting, denote $m(x_i)$ and U_i by x and U for now. In analogy with (13,14), we have:

$$P(x\psi) = \gamma^a \nabla_a(x\psi) = \lambda x \psi + \gamma^a(\nabla_a x)\psi,$$
$$P(U\psi) = P(\tfrac{1}{2}x\psi) + (U + \tfrac{1}{2}x)\lambda\psi = \lambda x \psi + \lambda U \psi + \tfrac{1}{2}\gamma^a(\nabla_a x)\psi.$$

In contrast to (13,14), however, this system doesn't close. We need in addition:

$$P(\gamma^a(\nabla_a x)\psi) = \gamma^b \nabla_b(\gamma^a(\nabla_a x)\psi) = \gamma^b \gamma^a((\nabla_a x)\nabla_b \psi + (\nabla_b \nabla_a x)\psi),$$

where we have used the relation $\nabla \gamma = 0$ between the spin connection and the fundamental tensor-spinor. By the Clifford relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = -2g^{ab},$$

we may rewrite the above as

$$P(\gamma^a(\nabla_a x)\psi) = -\gamma^a(\nabla_a x)\lambda\psi - 2(\nabla^a x)\nabla_a \psi - (\nabla^a \nabla_a x)\psi = 2U\psi - \lambda\gamma^a(\nabla_a x)\psi,$$

where we have used (6) to simplify $-\nabla^a \nabla_a x = \Delta x$.

Abbreviating

$$[P, m(x)] = \gamma^a(\nabla_a x) =: y, \tag{39}$$

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we have

$$A\psi := (U + \lambda x + \tfrac{1}{2}y)\psi \in E(\lambda + 1, P),$$
$$S\psi := (U - \lambda x - \tfrac{1}{2}y)\psi \in E(\lambda - 1, P), \tag{40}$$
$$N\psi := (U - \tfrac{1}{2}x - \lambda y)\psi \in E(-\lambda, P).$$

Recall that we have suppressed the subscript $i \in \{0, \ldots, n\}$, so that we really have operators A_i, S_i, and N_i. To find whether we really get something nonzero by the above processes, we compute

$$\sum_i S_i A_i \psi = \sum_i (U_i - (\lambda + 1)x_i - \tfrac{1}{2}y_i)(U_i + \lambda x_i + \tfrac{1}{2}y_i)\psi,$$

$$\sum_i A_i S_i \psi = \sum_i (U_i + (\lambda - 1)x_i + \tfrac{1}{2}y_i)(U_i - \lambda x_i - \tfrac{1}{2}y_i)\psi, \tag{41}$$

$$\sum_i N_i N_i \psi = \sum_i (U_i - \tfrac{1}{2}x_i + \lambda y_i)(U_i - \tfrac{1}{2}x_i - \lambda y_i)\psi.$$

In simplifying this, note that

$$yy = \gamma^a \gamma^b (\nabla_a x) \nabla_b x = -(\nabla_a x)\nabla^a x = -|dx|^2.$$

Thus

$$\sum_i y_i^2 = -\sum_i |dx_i|^2 = -\sum_i \sin^2 \rho_i = -\sum_i (1 - \cos^2 \rho_i) = 1 - (n+1) = -n.$$

We also have

$$yx = xy = x\gamma^a(\nabla_a x) = \tfrac{1}{2}\gamma^a \nabla_a(xx),$$

so that

$$\sum_i y_i x_i = \sum_i x_i y_i = \frac{1}{2}\gamma^a \nabla_a \left(\sum_i x_i^2\right) = \frac{1}{2}\gamma^a \nabla_a 1 = 0.$$

(18), (19), and (20) are still valid (with the new meaning of U_i, and P^2 in place of D in the intermediate steps of (18)).

In addition,

$$Uy = U[P, x] = \tfrac{1}{2}[P^2, x][P, x] = \tfrac{1}{2}(P[P, x] + [P, x]P)[P, x] = Py^2 + yPy,$$
$$yU = [P, x]U = \tfrac{1}{2}[P, x][P^2, x] = \tfrac{1}{2}[P, x](P[P, x] + [P, x]P) = yPy + y^2P,$$

so that

$$[U, y] = [P, y^2],$$

and

$$\sum_i [U_i, y_i] = \left[P, \sum_i y_i^2\right] = [P, -n] = 0. \tag{42}$$

The argument of Lemma 2 goes through formally as written (with ∇ now involving the spin connection), and gives

$$\sum_i \nabla^2_{T_i} = -\nabla^*\nabla = -P^2 + \frac{n(n-1)}{4}.$$

With the discussion preceding (35), this gives

$$\sum_i \left(U_i - \tfrac{n}{2}x_i\right)^2 = -P^2 + \frac{n(n-1)}{4}.$$

By the analogues of (18,19,20),

$$\sum_i (U_i + ax_i)^2 = a^2 + \sum_i U_i^2$$
$$= a^2 - P^2 + \frac{(n(n-1))}{4} - \frac{n^2}{4}$$
$$= a^2 - P^2 - \frac{n}{4}.$$

In particular,

$$\sum_i U_i^2 = -P^2 - \frac{n}{4}.$$

Using all these identities, for arbitrary a and b,

$$\sum_i (U_i - (a+1)x_i - by_i)(U_i + ax_i + by_i) = -P^2 - \frac{3n}{4} - a^2 - a(n+1) + b^2 n.$$

In the first line of (41), we take $a = \lambda$ and $b = \frac{1}{2}$, and apply to a λ-eigenspinor ψ to get

$$\sum_i S_i A_i \psi = \left(-2\lambda^2 - \frac{n}{2} - (n+1)\lambda\right)\psi = -2\left(\lambda + \frac{n}{2}\right)\left(\lambda + \frac{1}{2}\right)\psi.$$

In the second line, take $a = -\lambda$ and $b = -\frac{1}{2}$ to get

$$\sum_i A_i S_i \psi = \left(-2\lambda^2 - \frac{n}{2} + (n+1)\lambda\right)\psi = -2\left(\lambda - \frac{n}{2}\right)\left(\lambda - \frac{1}{2}\right)\psi.$$

In the third line, take $a = -\frac{1}{2}$ and $b = -\lambda$ to get

$$\sum_i N_i N_i \psi = (n-1)\left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right)\psi.$$

This establishes:

Proposition 9. *If P has a nonzero eigenspace on S^n, then the nonzero eigenspaces of P on round S^n are exactly the*

$$F_j := E(n/2 + j, P), \qquad G_j := E(-(n/2 + j), P)$$

for $j = 0, 1, 2, \ldots$.

Proof: The last three displayed identities and (32) show that if $E(\lambda, P) \neq 0$, then

$$
\begin{aligned}
E(-\lambda, P) &\neq 0, \\
E(\lambda - 1, P) &\neq 0 &&\text{unless } \lambda = n/2, \\
E(\lambda + 1, P) &\neq 0 &&\text{unless } \lambda = -n/2.
\end{aligned}
$$

Thus if there is an eigenvalue outside the set of $\pm(n/2 + j)$, we may generate an eigenvalue contradicting (32). Given any nonzero eigenspace $E(\lambda, P)$ with λ of the form $\pm(n/2 + j)$ however, we may generate nonzero eigenspaces corresponding to *all* such λ. $\qquad\square$

The beginning of the statement of the last proposition is a bit awkward; we need to assume there *is* some eigenspace in order to get a foothold analogous to that provided by the function 1 in the scalar case. We can get this foothold by taking a nonzero parallel spinor in $\mathbb{R}^n$ and stereographically injecting it (with the proper conformal weight) to S^n. Alternatively, we could do just enough elementary elliptic theory to conclude that the minimizer for the Rayleigh quotient based on P^2 provides us with an eigensection.

The analogue of Corollary 5 is

Corollary 10. *The span of the $m(x_i)E(\lambda, P)$, $Pm(x_i)E(\lambda, P)$, and $P^2m(x_i)E(\lambda, P)$ is $E(\lambda + 1, P) \oplus E(\lambda - 1, P) \oplus E(-\lambda, P)$.*

Proof: The inclusion $\subset$ follows from (40), (37), and the fact (from (39), in the notation of that display) that $_\mu|y|_\lambda = (\mu - \lambda)_\mu|m(x)|_\lambda$. For the inclusion $\supset$, note first that (41) puts $E(\lambda+1, P)\oplus E(\lambda-1, P)\oplus E(-\lambda, P)$ in the span of the $A_iE(\lambda, P)$, $N_iE(\lambda, P)$, and $S_iE(\lambda, P)$. By (41), this is in the span of the $x_iE(\lambda, P)$, $U_iE(\lambda, P)$, and $y_iE(\lambda, P)$. By (35) and (39), this is in the span of the $m(x_i)E(\lambda, P)$, $Pm(x_i)E(\lambda, P)$, and $P^2m(x_i)E(\lambda, P)$. $\qquad\square$

In analogy with Proposition 7, we may seek intertwinors $\mathcal{A}_{2k+1}$ for complex-valued k, satisfying the intertwining relation

$$\mathcal{A}_{2k+1}(U - (k + \tfrac{1}{2})x) = (U + (k + \tfrac{1}{2})x)\mathcal{A}_{2k+1} \tag{43}$$

(which extends (36)). In fact, such intertwinors exist and commute with P. The subfamily of these for which k is a nonnegative integer yields a family of odd-order (in fact, order $2k + 1$) differential operators which are polynomial in P:

Theorem 11. *For $k = 0, 1, 2, \ldots,$ and*

$$\mathcal{P}_{2k+1} = (P-k)(P-k+1)\cdots P \cdots (P+k-1)(P+k) = P(P^2-1)(P^2-4)\cdots(P^2-k^2),$$

we have

$$\mathcal{P}_{2k+1}(U - (k + \tfrac{1}{2})x) = (U + (k + \tfrac{1}{2})x)\mathcal{P}_{2k+1},$$

where U is any U_i, and x is any x_i.

Proof: Fix k, and consider

$$\alpha(\lambda) := (\lambda - k)(\lambda - k + 1)\cdots\lambda\cdots(\lambda + k - 1)(\lambda + k),$$

the eigenvalue taken by $\mathcal{P}_{2k+1}$ on $E(\lambda, P)$. In view of (37), it is enough to show that

$$
\begin{aligned}
&l(\mu - k)(\mu - k + 1)\cdots\lambda\cdots(\mu + k - 1)(\mu + k)\left(\frac{\mu^2 - \lambda^2}{2} - (k + \tfrac{1}{2})\right) \\
&\stackrel{?}{=} (\lambda - k)(\lambda - k + 1)\cdots\lambda\cdots(\lambda + k - 1)(\lambda + k)\left(\frac{\mu^2 - \lambda^2}{2} + (k + \tfrac{1}{2})\right)
\end{aligned}
\tag{44}
$$

for $\mu = \lambda \pm 1$ and for $\mu = -\lambda$. This is easily verified in each of the three cases. $\square$

For arbitrary order $k + 1/2$, we might expect a nonlocal intertwining operator. This is provided by:

Theorem 12. *If $k + n/2 \notin \mathbb{Z}$, the operators*

$$\mathcal{A}_{2k+1} := \operatorname{sgn}(P)^{n+1} \frac{\Gamma(P + k + 1)}{\Gamma(P - k)} \tag{45}$$

satisfy the intertwining relations (43), where for each function f, the operator $f(P)$ takes the eigenvalue $f(\lambda)$ on $E(\lambda, P)$.

Proof: Fix k and let $\alpha[\lambda]$ be the eigenvalue on $E(\lambda, P)$ of a putative intertwinor $\mathcal{A}_{2k+1}$. It is immediate from (37) that $\alpha[-\lambda] = -\alpha[\lambda]$, and that

$$\alpha[\lambda + 1](\lambda - k) = (\lambda + 1 + k)\alpha[\lambda]. \tag{46}$$

Moreover, these equations for all P-eigenvalues λ are sufficient for the intertwining property (the condition relating $\alpha[\lambda - 1]$ to $\alpha[\lambda]$ is the same as (46), with λ shifted to $\lambda - 1$.) Implementing the recursion this gives, we get the intertwinor

$$\operatorname{sgn}(P) \frac{\Gamma(|P| + k + 1)}{\Gamma(|P| - k)}, \tag{47}$$

and one may check directly that this is intertwining. Using the identity

$$\Gamma(z)\Gamma(-z)z\sin(\pi z) = -\pi,$$

we get

$$\frac{\Gamma(-P+k+1)}{\Gamma(-P-k)} = \frac{\Gamma(P+k+1)}{\Gamma(P-k)} \cdot \frac{\sin \pi(P+k)}{\sin \pi(P-k-1)}.$$

The trigonometric factor in this is

$$\frac{\sin(\pi P)\cos(\pi k) + \cos(\pi P)\sin(\pi k)}{\sin(\pi(P-1))\cos(\pi k) - \cos(\pi(P-1))\sin(\pi k)}.$$

If n is even, $k \notin \mathbb{Z}$ and P takes integral eigenvalues, so this becomes 1. If n is odd, P takes properly half-integral eigenvalues and k is not a proper half-integer, so the above becomes -1. This shows that

$$\frac{\Gamma(|P|+k+1)}{\Gamma(|P|-k)} = \text{sgn}(P)^n \frac{\Gamma(P+k+1)}{\Gamma(P-k)},$$

so that the intertwinor (47) is the $\mathcal{A}_{2k+1}$ of (45). $\qquad \square$

Remark: When n is even, we may set $k = 1/2$ in the last theorem to obtain the intertwinor

$$\text{sgn}(P)\frac{\Gamma(P+\frac{3}{2})}{\Gamma(P-\frac{1}{2})} = \text{sgn}(P)(P+\tfrac{1}{2})(P-\tfrac{1}{2}) = P|P| - \tfrac{1}{4}P/|P|. \tag{48}$$

This is also a formula for an intertwinor when n is odd, though this is not covered by Theorem 12. To see this, we just feed the formula for the operator (48) into a calculation similar to (44). Similar considerations hold for the other values of k excluded by the hypotheses of Theorem 12. Of course it is known that even and odd dimensions act very differently in many ways with respect to the spinor bundle. For example, in even dimensions the bundle Σ itself admits a chirality decomposition into two subbundles. In contrast, Σ is irreducible in odd dimensions, but each eigenspace of P^2 admits a chiral decomposition. In particular, the relation between $E(\lambda, P)$ and $E(-\lambda, P)$ has very different interpretations in the two dimension parities. The technique we use here seems remarkably attuned to deriving Theorem 12 using only things that are common to the two dimension parities, the power on $\text{sgn}(P)$ being the only hint of the differences.

Remark: Our technique can also be used to obtain the eigenvalue specta of D, P and other intertwinors on the non-Riemannian hyperboloids [6]; i.e. on other real forms of the sphere, again manifolds with large Lie algebras of conformal vector fields. This provides some explanation, in terms of conformal geometry, of the remarkable agreement between most of the eigenvalues of the Laplace operator on the various real forms of the sphere.

References

[1] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality.* Annals of Math. **138** (1993) 213–242.

[2] T. Branson, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987) 199–291.

[3] T. Branson, G. Ólafsson, and B. Ørsted, *Spectrum generating operators, and intertwining operators for representations induced from a maximal parabolic subgroup.* J. Funct. Anal. **135** (1996) 163–205.

[4] Y. Kosmann, *Dérivées de Lie des spineurs*, Ann. Mat. Pura Appl. (4) **XCI** (1972) 317–395.

[5] R. Penrose and W. Rindler, "Spinors and space-time" vols. 1,2, Cambridge University Press, Cambridge, 1984 and 1986.

[6] R. Strichartz, *Harmonic analysis on hyperboloids*, J. Funct. Anal. **12** (1973) 341–383.

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