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Abstract

We consider a class of classical lattice spin systems, with $\mathbb{R}^n$-valued spins and two-body interactions. Our main result states that the associated Gibbs measure localizes in certain cylindrical neighbourhoods of the global minima of the unperturbed Hamiltonian. As an application we establish existence of a first order phase transition at low temperature, for a reflection positive mexican hat model on $\mathbb{Z}^d$, $d \geq 3$, with a non-ferromagnetic interaction.

I Assumptions and Main Result

Let $n \in \mathbb{N}$ and Λ be a finite set. Elements of Λ are denoted by x, y and z. We pick and fix one element $o \in \Lambda$ which plays the distinguished role of an origin. We write $\Omega = (\mathbb{R}^n)^\Lambda$ for the vector space of spin configurations $\varphi = \{\varphi_x\}_{x\in\Lambda}$ over Λ, where $\varphi_x \in \mathbb{R}^n$. We use the symbols ψ and φ for elements of Ω. The letters u and v are used for vectors in $\mathbb{R}^n$, and $|u|$ denotes the euclidean norm of u. It is assumed that Λ comes equipped with a metric ρ which satisfies

$$\max_{x\in\Lambda} \sum_{y\in\Lambda} e^{-\rho(x,y)} \leq C_\rho < \infty. \tag{I.1}$$

We study a Hamiltonian function $H_\Lambda \in C^1(\Omega;\mathbb{R})$ of the form

$$H_\Lambda(\varphi) = \sum_{x\in\Lambda} f_x(\varphi_x) + J \sum_{x,y\in\Lambda, x\neq y} w_{xy}(\varphi_x, \varphi_y).$$

The self-energies $\{f_x\}_{x\in\Lambda}$ and the interactions $\{w_{xy}\}_{x,y\in\Lambda,x\neq y}$ should satisfy assumptions specified in the following Conditions I.1 and I.2, respectively.

We introduce some notation. We write $\partial_{|u|} = \frac{u}{|u|} \cdot \nabla_u$ for the radial derivative with respect to the $\mathbb{R}^n$-valued variable u, and $B_r(u) := \{v \in \mathbb{R}^n | |u-v| \leq r\}$, for the closed ball of radius r and centered at u.

Condition I.1. *There are positive constants $R, c_f, C_f > 0$ such that the family $\{f_x\}_{x\in\Lambda}$ of functions $f_x \in C^1(\mathbb{R}^n;\mathbb{R})$ satisfy (i)–(iv) as follows:*

(i) $f_x \geq 0$ *and* $\min_{u\in\mathbb{R}^n} f_x(u) = 0$.

(ii) The set $\mathcal{G}_0 := \{u \in \mathbb{R}^n | f_o(u) = 0\}$ *of global minima satisfies* $\mathcal{G}_0 \subset B_R(0)$.

(iii) For all $x \in \Lambda$ and $u \in \mathbb{R}^n$, with $|u| \geq R$, we have $\partial_{|u|} f_x(u) \geq c_f$.

(iv) For all $x, y \in \Lambda$ and $u, v \in \mathbb{R}^n$, with $|v| \geq |u| \geq R$, we have

$$\partial_{|u|} f_x(u) \leq C_f \partial_{|v|} f_y(v).$$

For $j \in \{1, 2\}$, we write $\nabla_j w_{xy}$ for the gradient of w_{xy} with respect to the j'th variable. The w_{xy}'s are required to be dominated by the f_x's, as specified by the next condition

Condition I.2. *There exist constants $C_a^\rho > 0$ and $\{a_{xy}\}_{x,y\in\Lambda}$, with $a_{xy} = a_{yx} \geq 0$, $a_{xx} = 0$, and*

$$\max_{x\in\Lambda} \Big\{ \sum_{y\in\Lambda} a_{xy} e^{\rho(x,y)} \Big\} \leq C_a^\rho, \tag{I.2}$$

such that the family $\{w_{xy}\}_{x,y\in\Lambda}$ of functions $w_{xy} \in C^1(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R})$, with $w_{xx} \equiv 0$, obeys the following bounds:

$$\begin{aligned} &\max\{|\nabla_1 w_{xy}(u,v)|, |\nabla_2 w_{xy}(u,v)|\} \\ &\quad \leq a_{xy}\Big(1 + \mathbb{1}_{[|u|\geq 4R]} \partial_{|u|} f_x(u) + \mathbb{1}_{[|v|\geq 4R]} \partial_{|v|} f_y(v)\Big) \end{aligned} \tag{I.3}$$

Eq. (I.2) expresses exponential decay of the interaction with respect to the metric ρ. Let $C_a^0 \le C_a^\rho$ be such that

$$\max_{x\in\Lambda}\Big\{\sum_{y\in\Lambda} a_{xy}\Big\} \le C_a^0. \tag{I.4}$$

For polynomially bounded, measurable functions $u : \Omega \to \mathbb{C}$, we use the following notation for expectations with respect to the Gibbs state in finite volume Λ and inverse temperature β:

$$\mathbb{E}_\Lambda[u] := \mathcal{Z}_\Lambda^{-1} \int_\Omega u(\varphi) e^{-\beta H_\Lambda(\varphi)}\, d^\Lambda\varphi. \tag{I.5}$$

Here $\mathcal{Z}_\Lambda = \int_\Omega e^{-\beta H_\Lambda(\varphi)}\, d^\Lambda\varphi$ is the partition function. For the integral in (I.5) to exist under Conditions I.1 and I.2, we require $|J| < \Gamma_0^{-1}$, where

$$\Gamma_0 := 2C_a^0\big(1 + C_f + c_f^{-1}\big). \tag{I.6}$$

See Lemma A.1. We note that for many examples, including the example in Section II, the f_x's grow at a faster rate than the w_{xy}'s such that no assumption on $|J|$ is needed to make polynomially bounded observables integrable. For the probability of a (measurable) event $\mathcal{A} \subset \Omega$, we write

$$\mathbb{P}_\Lambda[\mathcal{A}] := \mathbb{E}_\Lambda[\mathbb{1}_\mathcal{A}].$$

For $\zeta > 0$, we introduce level sets for f_o

$$\mathcal{G}_\zeta := \big\{u \in \mathbb{R}^n | f_o(u) \le \zeta\big\}.$$

We are now ready to formulate the main result of the paper.

Theorem I.1. *Assume (I.1). Let $\{f_x\}_{x\in\Lambda}$ satisfy Condition I.1 and $\{w_{xy}\}_{x,y\in\Lambda}$ satisfy Condition I.2. Let*

$$\Gamma_1 := 4RC_a^0(\tfrac{1}{3}C_\rho + 3), \qquad \Gamma_2 = \Gamma_0 + \tfrac{102}{5}C_a^\rho, \qquad \Gamma_3 := \tfrac{2}{3}c_f R, \tag{I.7}$$

and

$$J_0 := \Gamma_3 \min\Big\{\frac{3}{4\Gamma_1}, \frac{1}{\Gamma_1 + \Gamma_2\Gamma_3}\Big\}. \tag{I.8}$$

For $|J| < J_0$, $\beta > 0$ and $\delta > 2|J|\Gamma_1$, we have

$$\mathbb{P}_\Lambda[\{\varphi \in \Omega | \varphi_o \notin \mathcal{G}_\delta\}] \le Ce^{-\beta\sigma}, \tag{I.9}$$

where

$$C = 2\max\Big\{\Big[\frac{4R}{\min\{\delta, \frac{3}{2}\Gamma_3\}} \sup_{|u|\le 2R} |\nabla f_o(u)|\Big]^n, (\tfrac{3}{5})^n\Big\} e^{\frac{1}{2}nC_\rho}, \tag{I.10}$$

and σ is a strictly positive constant given by

$$\sigma = \min\big\{\tfrac{1}{2}\delta, \tfrac{3}{4}\Gamma_3, \Gamma_3(1 - |J|\Gamma_2)\big\} - |J|\Gamma_1. \tag{I.11}$$

Remark I.2.
(1) We stress that the constants J_0, $\{\Gamma_j\}_{j=1,2,3}$, C, and σ, only depend on ρ, $\{f_x\}_{x\in\Lambda}$ and $\{w_{xy}\}_{x,y\in\Lambda}$ through the constants C_ρ, c_f, C_f, R, C_a^0 and C_a^ρ. In particular, they are independent of Λ and the choice of origin o.

(2) The set of φ's with $\varphi_o \in \mathcal{G}_\delta$ is a cylinder set containing the global minima of the unperturbed (that is, $J = 0$) Hamiltonian. The condition $\delta > 2|J|\Gamma_1$, ensures that the global minima of the perturbed Hamiltonian remain contained in this cylinder set.

(3) The proof goes through without modifications if $\mathbb{R}^n$ is replaced by a convex subset thereof containing 0.

(4) A choice was made here to present the method for a class of Hamiltonians without any special symmetry. For models with $O(n)$ symmetry, like the example discussed in Section II, one can tweak the proof to get better constants.

(5) The restriction to two-body interactions is made for simplicity. The method extends to models with many-body interactions.

The derivation of the bound (I.9) follows a scheme used in [1, Section 3], to derive low temperature localization bounds for models with a unique global minimum at 0. The method developed in [1] was in turn inspired by work going back to Sjöstrand [11], see also [7, 9]. The common idea in the papers cited in this paragraph is to systematically shift points in the set of φ's with $\varphi_o \notin \mathcal{G}_\delta$, towards the global mimima and measure the resulting decrease in energy. In this paper and in [1] the shift is implemented by a single transformation T, with the property that $\inf_{\varphi,\varphi_o\notin\mathcal{G}_\delta}[H(\varphi) - H(T(\varphi))] \geq \sigma > 0$. It is this σ which contributes to the exponential localization in (I.9). In the papers [7, 9, 11] the idea is slightly different: The configuration space is cut up into pieces, each of which is translated into a neighbourhood of a unique global minimum, and the contributions are then summed up. We remark that in [9], the interaction does not shift the global minimum away from 0, which makes it possible to localize arbitrarily close to 0 while keeping the coupling constant J fixed (see Remark I.2 (2)).

II Motivating Example

Let $\Lambda =]-L, L]^d \cap \mathbb{Z}^d$, be the d-dimensional hypercubic lattice of sidelength $2L$, for some $d \geq 3$ and $L \in \mathbb{N}$. We view Λ as the torus $\mathbb{Z}^d/2L\mathbb{Z}^d$, equipped with the metric $\rho(x, y) = \min_{z\in\mathbb{Z}^d} |x-y-2Lz|_1$, where $|z|_p$, is the p-norm of $z \in \mathbb{R}^d$. As self-energies we take

$$f_x(u) = |u|^4 - 2|u|^2 + 1, \tag{II.1}$$

and as interaction we take

$$w_{xy}(u, v) = -\mathcal{J}_\Lambda(x - y)u \cdot v, \tag{II.2}$$

where $\mathcal{J}_\Lambda$ is periodic and defined from an underlying interaction $\mathcal{J} \in \ell^1(\mathbb{Z}^d; \mathbb{R})$ by

$$\mathcal{J}_\Lambda(x) := \sum_{y\in\mathbb{Z}^d} \mathcal{J}(x + 2Ly). \tag{II.3}$$

In general, one should take a reflection positive interaction, with respect to a suitable reflection, in order to get an infrared bound. Here we specialize to the following example

$$\mathcal{J}(x) := \begin{cases} 1, & |x|_2 = 1, \\ -b, & |x|_2 = \sqrt{2}. \end{cases} \tag{II.4}$$

In order to obtain a reflection positive interaction, we impose the restriction

$$J > 0, \qquad 0 < b < \frac{1}{2(d-1)}.$$

Note that the ferromagnetic case $-b \geq 0$, can be treated by methods already established. We introduce the correlation function $F_\Lambda : \Lambda \to \mathbb{R}$ by

$$F_\Lambda(x) := \mathbb{E}_\Lambda[\varphi_0 \cdot \varphi_x].$$

Let $\Lambda^* =]-\pi, \pi]^d \cap (\frac{\pi}{L}\mathbb{Z}^d)$ be the dual lattice and $\hat{F}_\Lambda : \Lambda^* \to \mathbb{R}$ the Fourier transform of F_Λ. That is, $\hat{F}_\Lambda(\xi) = \sum_{x\in\Lambda} \exp(-\mathrm{i}x \cdot \xi) F_\Lambda(x)$. Similarly, for $\xi \in \Lambda^*$,

$$\hat{\mathcal{J}}_\Lambda(\xi) = \hat{\mathcal{J}}(\xi) = 2a \sum_{j=1}^{d} \cos(\xi_j) - 4b \sum_{1\leq i<j\leq d} \cos(\xi_i)\cos(\xi_j)$$

is the Fourier transform of $\mathcal{J}_\Lambda$. Here $\hat{\mathcal{J}}(\xi) = \sum_{x\in\mathbb{Z}^d} e^{-\mathrm{i}x\cdot\xi} \mathcal{J}(x)$. The model defined by (II.1) and (II.4) is translation invariant and reflection positive, cf. [3, 6], and hence it satisfies an infrared bound of the form

$$0 \leq \hat{F}_\Lambda(\xi) \leq \frac{n}{J\beta}\big(\hat{\mathcal{J}}(0) - \hat{\mathcal{J}}(\xi)\big)^{-1}, \tag{II.5}$$

for $\xi \in \Lambda^*\backslash\{0\}$. Here n is the dimension of the single spin space. For a proof of this bound see [6, Proposition 20.12]. See [4] for a discussion of the critical case, where $2(d-1)b = 1$.

As usual, (II.5) implies that

$$\begin{aligned} \mathbb{E}_\Lambda[\varphi_0 \cdot \varphi_x] &= \mathbb{E}_\Lambda[|\varphi_0|^2] + F_\Lambda(x) - F_\Lambda(0) \\ &= \mathbb{E}_\Lambda[|\varphi_0|^2] + \frac{1}{|\Lambda|}\sum_{\xi\in\Lambda^*}(e^{\mathrm{i}x\cdot\xi} - 1)\hat{F}_\Lambda(\xi) \\ &\geq \mathbb{E}_\Lambda[|\varphi_0|^2] - \frac{2n}{J\beta|\Lambda|}\sum_{\xi\in\Lambda^*\backslash\{0\}} \big(\hat{\mathcal{J}}(0) - \hat{\mathcal{J}}(\xi)\big)^{-1}. \end{aligned} \tag{II.6}$$

Note that $\hat{\mathcal{J}}(0) = \hat{\mathcal{J}}(\xi)$ if, and only if, $\xi = 0$, and

$$\hat{\mathcal{J}}(0) - \hat{\mathcal{J}}(\xi) \sim (1 - 2(d-1)b)\xi^2,$$

near $\xi = 0$. In dimension $d \geq 3$ this implies the existence of a first order phase transition at low temperature (large β), provided one can verify the following moment inequality

$$\mathbb{E}_\Lambda[|\varphi_0|^2] \geq c > 0. \tag{II.7}$$

Here c should be independent of L. The estimate (II.6) then says that a (necessarily translation invariant) limit state $\mathbb{E}_\infty = \mathrm{w} - \lim_{\Lambda \to \mathbb{Z}^d} \mathbb{E}_\Lambda$ is not ergodic, hence not a pure phase. See [3, 5, 10] and in particular [6, Theorem 20.15], for a reference dealing with possibly unbounded spins. For models constrained to the unit sphere (or in general to closed subsets of $\mathbb{R}^n$ not containing 0) the bound (II.7) is trivial.

If $\mathcal{J}(x) \geq 0$, for all x, i.e. the model is ferromagnetic, there are two general methods one can use to verify (II.7) for models with interaction of the type (II.2). If $n = 1$ one can use the FKG inequalities [2], which imply monotonicity of the second moment in $\mathcal{J}(x)$ (for any fixed x). This can be used to reduce the moment inequality to a one-dimensional problem which can be analyzed explicitly. See [8] for a discussion of this idea. Another argument applies under the additional assumption of reflection positivity of the interaction (II.2). Then the so-called chessboard estimate [6, Chapter 17.1] applies (a key ingredient in the proof of (II.5) and the reason for the choice of a reflection positive model as our example). The chessboard estimate together with ferromagnetism, i.e. positivity of $\mathcal{J}$, also leads to a moment inequality; no restrictions on n are needed. See [6, Lemma 20.8].

If the interaction is not ferromagnetic there seems to be no method available in the literature to deal with the innocuous looking moment inequality (II.7). This is where our main result comes in. Clearly, Theorem I.1 gives explicit J_0 and α_0 such that, for $0 < J < J_0$ and $J\beta > \alpha_0$, we have $\mathbb{E}_\Lambda[|\varphi_0|^2] \geq c > 0$, for an equally explicit constant c. Here one should take $0 < \delta < 1$, such that $\mathcal{G}_\delta$ is an annulus. Moreover, as opposed to the methods of the preceding paragraph, Theorem I.1 is robust and does not rely on correlation inequalities or indeed on any non-trivial properties of the underlying Gibbs measure. We have thus extended the applicability of Georgii's result [6, Theorem 20.15] to interactions $\mathcal{J}$, which need not be ferromagnetic. (Recall that Georgii in this case requires the single spin space to be bounded away from 0, cf. [6, Comments 20.18 (3)]).

For the above concrete model, we estimated the constant J_0, fixed a $J < J_0$ and estimated σ and C, for which the bound (I.9) is valid. As for J_0 we got $J_0 \sim 10^{-4}$, which seems small, but is in fact only a factor of 10 smaller than Γ_0^{-1}. Recall that Γ_0^{-1} was the upper limit for coupling strengths such that all models satisfying the conditions, with the same constants, are well-defined. This also serves to illustrate Remark I.2 (4). We then took $J = \frac{1}{2}10^{-4}$ and found $\sigma \sim \frac{1}{20}$ and $C \sim e^{60}$. To get a probability less than 1 in (I.9), one has to take $\beta > \beta_0$ with $\beta_0 \sim 1200$. We note that we did not try to optimize carefully over possible choices of metric and the constant R. (For R we chose $R = 1.03$. For the metric we chose $\kappa\rho$, with $\kappa = \ln(2)$. Here ρ is the metric given at the beginning of this section.)

III Transformation T^ζ

The purpose of this section is to construct a transformation of the space Ω, and estimate its Jacobian.

We begin by analyzing the size of the level sets $\mathcal{G}_\zeta$. Let $\zeta \geq 0$ and $u \in \mathbb{R}^n$ be such that $|u| > 2\zeta c_f^{-1} + R$, where the constants c_f and R are taken from Condition I.1.

Let $u' := Ru/|u|$, and for $0 \le t \le 1$,

$$u_t := tu + (1-t)u' = \left(\frac{R}{|u|} + t\Big(1 - \frac{R}{|u|}\Big)\right) u.$$

Then, using that $\dot u_t/|\dot u_t| = u_t/|u_t|$, we estimate

$$\begin{aligned} f_o(u) \ge f_o(u) - f_o(u') &= \int_0^1 \dot u_t \cdot \nabla f_o(u_t) dt \\ &= (|u| - R) \int_0^1 (\partial_{|u|} f_o)(u_t) dt \\ &\ge 2\zeta c_f^{-1} \int_0^1 c_f dt = 2\zeta. \end{aligned}$$

This implies that

$$\mathcal{G}_{2\zeta} \subset B_{R+2\zeta c_f^{-1}}(0). \tag{III.1}$$

We shall henceforth assume that $0 < \zeta \le \frac{1}{2} c_f R$, which is equivalent to

$$R < R_\zeta := R + \frac{2\zeta}{c_f} \le 2R. \tag{III.2}$$

We introduce the size r_ζ, of the largest ball contained in $\mathcal{G}_\zeta$:

$$r_\zeta := \sup\big\{ r \ge 0 | \exists u \in \mathcal{G}_\zeta :\ B_r(u) \subset \mathcal{G}_\zeta \big\}. \tag{III.3}$$

Fix an $\eta_\zeta \in \mathcal{G}_\zeta$, for which $B_{r_\zeta}(\eta_\zeta) \subset \mathcal{G}_\zeta$. Such an η_ζ exists by the choice (III.3) of r_ζ. By (III.1) we get a bound from below on r_ζ

$$r_\zeta \ge \operatorname{dist}(\mathcal{G}_\zeta^c, \mathcal{G}_0) = \operatorname{dist}(B_{R_\zeta}(0) \cap \mathcal{G}_\zeta^c, \mathcal{G}_0).$$

Let $u \in \mathcal{G}_0$ and $v \in B_{R_\zeta}(0) \cap \mathcal{G}_\zeta^c$. Then

$$\zeta \le f_o(v) - f_o(u) \le |v - u| \sup_{u \in B_{R_\zeta}(0)} |\nabla f_o(u)|.$$

This implies

$$r_\zeta \ge \frac{\zeta}{\sup_{u \in B_{R_\zeta}(0)} |\nabla f_o(u)|}. \tag{III.4}$$

We pick a function $\theta \in C^\infty(\mathbb{R}; [0,1])$ with $\theta \equiv 0$ on $(-\infty, \frac{1}{2}]$, $\theta \equiv 1$ on $[1, \infty)$, and $\theta' \ge 0$. Note that $\operatorname{supp}(\theta') \subset (\frac{1}{2}, 1)$.

The final input is a family of scaling factors $\{\epsilon_x\}_{x \in \Lambda}$. We choose them to be of the form

$$\epsilon_x := \epsilon_o e^{-\rho(x,o)}. \tag{III.5}$$

Here ϵ_o is chosen such that

$$0 < \epsilon_o < \tfrac{1}{2}(1 + \|\theta'\|_\infty)^{-1}, \tag{III.6}$$

where $\|\theta'\|_\infty = \max_{t \in [\frac{1}{2}, 1]} |\theta'(t)| > 2$. (We will optimize over ϵ_o and θ in Section VI.)

We define a transformation $T^\zeta : \Omega \to \Omega$ as follows: $(T^\zeta(\varphi))_x := T^\zeta_x(\varphi_x)$ and

$$\forall x \neq o: \quad T^\zeta_x(u) := \Big(1 - \epsilon_x \theta\Big(\frac{|u|}{4R}\Big)\Big)\, u, \tag{III.7}$$

$$T^\zeta_o(u) := \begin{cases} (1-\epsilon_o)u, & |u| \geq 4R, \\ \Big(\frac{r_\zeta}{4R}\Big)\, u + \eta_\zeta, & |u| < 4R. \end{cases} \tag{III.8}$$

The transformation T^ζ is not a global diffeomorphism, but we work below in the sectors $\{\varphi : \; |\varphi_o| < 4R\}$ and $\{\varphi : \; |\varphi_o| > 4R\}$ separately, and T^ζ restricted to these sectors is a smooth transformation.

We end this section with an estimate on the determinant of $\mathrm{Jac}(T^\zeta)$, the Jacobian of T^ζ.

Lemma III.1. *Let T^ζ be the transformation defined in (III.7) and (III.8). We have the bound*

$$\big|\det \mathrm{Jac}(T^\zeta)(\varphi)\big| \geq \min\Big\{\frac{\zeta}{4R \sup_{|u| \leq R_\zeta} |\nabla f_o(u)|}, 1 - \epsilon_o\Big\}^n \times \exp\big[-n\epsilon_o(1 + \|\theta'\|_\infty) C_\rho\big], \tag{III.9}$$

for all $\varphi \in \Omega$, with $|\varphi_o| \neq 4R$.

Proof: The Jacobian of T^ζ (away from $|\varphi_o| = 4R$) is a block diagonal matrix with $n \times n$-blocks given by

$$\forall x \neq o: \quad \mathrm{Jac}(T^\zeta)_{xx}(\varphi) = \Big(1 - \epsilon_x \theta\Big(\frac{|\varphi_x|}{4R}\Big)\Big)\, I_n - \frac{\epsilon_x |\varphi_x|}{4R} \theta'\Big(\frac{|\varphi_x|}{4R}\Big) P_{\varphi_x},$$

$$\mathrm{Jac}(T^\zeta)_{oo}(\varphi) = \begin{cases} (1-\epsilon_o) I_n, & |\varphi_o| > 4R, \\ \Big(\frac{r_\zeta}{4R}\Big)\, I_n, & |\varphi_o| < 4R. \end{cases}$$

Here I_n is the identity matrix in $\mathbb{R}^n$, and $P_u := |u|^{-2} |u\rangle\langle u|$, is the orthogonal projection onto $\mathrm{span}\{u\}$, for $u \in \mathbb{R}^n \backslash \{0\}$.

Note that, for $x \neq o$,

$$\frac{\epsilon_x |\varphi_x|}{4R}\, \theta'\Big(\frac{|\varphi_x|}{4R}\Big) P_{\varphi_x} \leq \epsilon_x \|\theta'\|_\infty I_n.$$

Using this observation, we estimate the determinant of the Jacobian as follows

$$\text{for } |\varphi_o| < 4R: \quad \big|\det \mathrm{Jac}(T^\zeta)(\varphi)\big| \geq \Big[\frac{r_\zeta}{4R} \prod_{x (\neq o)} \big\{1 - \epsilon_x (1 + \|\theta'\|_\infty)\big\}\Big]^n,$$

$$\text{for } |\varphi_o| > 4R: \quad \big|\det \mathrm{Jac}(T^\zeta)(\varphi)\big| \geq \Big[(1 - \epsilon_o) \prod_{x (\neq o)} \big\{1 - \epsilon_x (1 + \|\theta'\|_\infty)\big\}\Big]^n.$$

Using the bound $\ln(1-t) \geq -2t$, for $0 \leq t \leq \frac{1}{2}$, together with (I.1), (III.5) and (III.6), we arrive at (III.9). $\square$

IV Estimating the Interaction

In this section we estimate the effect of the transformation T^ζ on the interaction $W(\varphi) = \sum_{x\neq y} w_{xy}(\varphi_x,\varphi_y)$. We prove the following lemma which is the central technical step in the proof of Theorem I.1. The constant Γ_0 below is defined in (I.6).

Lemma IV.1. *For $0 < \zeta \le \frac{1}{2}c_f R$, we have, for all $\varphi \in \Omega$, the bound*

$$\big|W(\varphi) - W(T^\zeta(\varphi))\big| \le C_W^1 + C_W^2 \sum_{x\in\Lambda} \mathbb{1}_{[|\varphi_x|\ge 4R]}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big), \tag{IV.1}$$

where

$$C_W^1 := 4RC_a^0(2\epsilon_o C_\rho + 3), \qquad C_W^2 := \Gamma_0 + C_a^\rho\big(3\epsilon_o^{-1} + 2(1-\epsilon_o)^{-1}\big). \tag{IV.2}$$

Proof: Let $\varphi \in \Omega$, $x, y \in \Lambda$ with $x \neq y$. For $z \in \{x, y\}$ we abbreviate $u_z(t) = t\varphi_z + (1-t)T_z^\zeta(\varphi_z)$.

Using the fundamental theorem of calculus, together with Condition I.2, we estimate

$$\begin{aligned}
&|w_{xy}(\varphi_x,\varphi_y) - w_{xy}((T^\zeta(\varphi))_x,(T^\zeta(\varphi))_y)| \\
&\quad = \Big|\int_0^1 \big\{(\varphi_x - T_x^\zeta(\varphi_x))\cdot\nabla_1 w_{xy}(u_x(t),u_y(t)) \\
&\qquad\qquad + (\varphi_y - T_y^\zeta(\varphi_y))\cdot\nabla_2 w_{xy}(u_x(t),u_y(t))\big\}dt\Big| \\
&\quad \le a_{xy}\big(|\varphi_x - T_x^\zeta(\varphi_x)| + |\varphi_y - T_y^\zeta(\varphi_y)|\big) \qquad\qquad\qquad\qquad \text{(IV.3)}\\
&\qquad \times\Big[1 + \int_0^1 \Big\{\mathbb{1}_{[|u_x(t)|\ge 4R]}(\partial_{|u|}f_x)(u_x(t)) + \mathbb{1}_{[|u_y(t)|\ge 4R]}(\partial_{|u|}f_y)(u_y(t))\Big\}dt\Big] \\
&\quad = a_{xy}\big(S_1^x(\varphi) + S_1^y(\varphi) + S_2^{xy}(\varphi) + S_2^{yx}(\varphi)\big),
\end{aligned}$$

where

$$\begin{aligned}
S_1^z(\varphi) &= |\varphi_z - T_z^\zeta(\varphi_z)|\Big[1 + \int_0^1 \mathbb{1}_{[|u_z(t)|\ge 4R]}(\partial_{|u|}f_z)(u_z(t))dt\Big], \\
S_2^{zz'}(\varphi) &= |\varphi_z - T_z^\zeta(\varphi_z)|\int_0^1 \mathbb{1}_{[|u_{z'}(t)|\ge 4R]}(\partial_{|u|}f_{z'})(u_{z'}(t))dt.
\end{aligned}$$

We proceed to estimating $S_1^z(\varphi)$ and $S_2^{zz'}(\varphi)$, for all $\varphi \in \Omega$.

To estimate S_1^z we recall (III.7), and observe the bound

$$|\varphi_z - T_z^\zeta(\varphi_z)| \le \epsilon_z|\varphi_z|, \tag{IV.4}$$

which holds true if $z \neq o$, or $z = o$ and $|\varphi_o| \ge 4R$. To deal with the complementary case, where $z = o$ and $|\varphi_o| < 4R$, we note that in this case (III.8) implies $|T_o^\zeta(\varphi_0)| \le R_\zeta \le 2R$, and hence

$$|u_o(t)| < 4R \quad\text{ and }\quad |\varphi_o - T_o^\zeta(\varphi_o)| \le 6R. \tag{IV.5}$$

Combining (IV.4) and (IV.5) yields for all z and $\varphi \in \Omega$,

$$S_1^z(\varphi) \le 6R\delta_{zo} + \epsilon_z|\varphi_z|\Big[1 + \int_0^1 \mathbb{1}_{[|u_z(t)|\ge 4R]}(\partial_{|u|}f_z)(u_z(t))dt\Big]. \tag{IV.6}$$

Here we used that $|\varphi_o| \ge 4R$ on the support of $\mathbb{1}_{[|u_o(t)|\ge 4R]}$.

As for $S_2^{zz'}$ we split Ω into two regions:

$$\Omega_{\mathrm{I}}^{zz'} := \big\{\varphi \in \Omega \big| |\varphi_z| \ge |\varphi_{z'}|/(1-\epsilon_z)\big\} \text{ and } \Omega_{\mathrm{II}}^{zz'} = \Omega \backslash \Omega_{\mathrm{I}}^{zz'}.$$

From Condition I.1 (iv) we get the bound

$$\mathbb{1}_{[\varphi\in\Omega_{\mathrm{I}}^{zz'}]}\mathbb{1}_{[|u_{z'}(t)|\ge 4R]}(\partial_{|u|}f_{z'})(u_{z'}(t)) \le C_f \mathbb{1}_{[|u_z(t)|\ge 4R]}(\partial_{|u|}f_z)(u_z(t)), \tag{IV.7}$$

because on the support of the indicator functions we have

$$|u_z(t)| \ge (1-\epsilon_z)|\varphi_z| \ge |\varphi_{z'}| \ge |u_{z'}(t)| \ge 4R.$$

Complementing (IV.7) we now consider the region $\Omega_{\mathrm{II}}^{zz'}$. We obtain, for $z \ne o$, or $z = o$ and $|\varphi_o| \ge 4R$,

$$|\varphi_z - T_z^\zeta(\varphi_z)|\mathbb{1}_{[\varphi\in\Omega_{\mathrm{II}}^{zz'}]}\mathbb{1}_{[|u_{z'}(t)|\ge 4R]} \le \frac{\epsilon_z}{1-\epsilon_z}|\varphi_{z'}|\mathbb{1}_{[|u_{z'}(t)|\ge 4R]}. \tag{IV.8}$$

Here (IV.4) was used. We note that, for $|\varphi_o| \le 4R$, we have from (IV.5)

$$|\varphi_o - T_o^\zeta(\varphi_o)|\mathbb{1}_{[|u_{z'}(t)|\ge 4R]} \le 6R\mathbb{1}_{[|u_{z'}(t)|\ge 4R]} \le \tfrac{3}{2}|\varphi_{z'}|\mathbb{1}_{[|u_{z'}(t)|\ge 4R]}. \tag{IV.9}$$

Combining (IV.7)–(IV.9) yields for $z \ne z'$ and $\varphi \in \Omega$:

$$\begin{aligned} S_2^{zz'}(\varphi) \le{}& \epsilon_z|\varphi_z|C_f\int_0^1 \mathbb{1}_{[|u_z(t)|\ge 4R]}(\partial_{|u|}f_z)(u_z(t))dt \\ &+ \epsilon_{z'}|\varphi_{z'}|\Big\{\frac{\epsilon_z}{(1-\epsilon_z)\epsilon_{z'}} + \frac{3}{2\epsilon_{z'}}\delta_{oz}\Big\}\int_0^1 \mathbb{1}_{[|u_{z'}(t)|\ge 4R]}(\partial_{|u|}f_{z'})(u_{z'}(t))dt. \end{aligned} \tag{IV.10}$$

Inserting the bounds (IV.6) and (IV.10) into (IV.3), we get the following estimate for all $x \ne y$ and $\varphi \in \Omega$

$$\begin{aligned} &|w_{xy}(\varphi_x,\varphi_y) - w_{xy}((T^\zeta(\varphi))_x,(T^\zeta(\varphi))_y)| \\ &\quad\le a_{xy}\epsilon_x|\varphi_x|\Big(1 + \Big\{1 + C_f + \frac{3}{2\epsilon_x}\delta_{yo} + \frac{\epsilon_y}{(1-\epsilon_y)\epsilon_x}\Big\} \\ &\qquad\times\int_0^1 \mathbb{1}_{[|u_x(t)|\ge 4R]}(\partial_{|u|}f_x)(u_x(t))dt\Big) \\ &\qquad+ a_{xy}\epsilon_y|\varphi_y|\Big(1 + \Big\{1 + C_f + \frac{3}{2\epsilon_y}\delta_{xo} + \frac{\epsilon_x}{(1-\epsilon_x)\epsilon_y}\Big\} \\ &\qquad\times\int_0^1 \mathbb{1}_{[|u_y(t)|\ge 4R]}(\partial_{|u|}f_y)(u_y(t))dt\Big) \\ &\qquad+ a_{xy}6R(\delta_{xo}+\delta_{yo}). \end{aligned} \tag{IV.11}$$

Observe that

$$|\varphi_x| \geq |u_x(t)| \geq 4R \quad \text{implies} \quad T_x^\zeta(\varphi_x) = (1-\epsilon_x)\varphi_x, \tag{IV.12}$$

and hence

$$\begin{aligned}
&\epsilon_x|\varphi_x| \int_0^1 \mathbb{1}_{[|u_x(t)|\geq 4R]}(\partial_{|u|}f_x)(u_x(t))dt \\
&\quad \leq \mathbb{1}_{[|\varphi_x|\geq 4R]} \int_0^1 \epsilon_x|\varphi_x|(\partial_{|u|}f_x)(u_x(t))dt \\
&\quad = \mathbb{1}_{[|\varphi_x|\geq 4R]} \int_0^1 \big(\frac{d}{dt}[f_x(u_x(t))]\big)dt \\
&\quad = \mathbb{1}_{[|\varphi_x|\geq 4R]}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big).
\end{aligned} \tag{IV.13}$$

As an application we get the bound, cf. Condition I.1 (iii) and (IV.12),

$$\begin{aligned}
\epsilon_x|\varphi_x| &= \epsilon_x|\varphi_x|\mathbb{1}_{[|\varphi_z|<4R]} + \epsilon_x|\varphi_x|\mathbb{1}_{[|\varphi_x|\geq 4R]} \\
&\leq \epsilon_x 4R\mathbb{1}_{[|\varphi_x|<4R]} + c_f^{-1}\mathbb{1}_{[|\varphi_x|\geq 4R]}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big).
\end{aligned} \tag{IV.14}$$

Inserting (IV.13) and (IV.14) into (IV.11) we get, for $\varphi \in \Omega$,

$$\begin{aligned}
&\big|w_{xy}(\varphi_x,\varphi_y) - w_{xy}((T^\zeta(\varphi))_x,(T^\zeta(\varphi))_y)\big| \\
&\quad \leq a_{xy}\Big[\epsilon_x 4R + 6R\delta_{xo} + \Big\{1 + c_f^{-1} + C_f + \frac{3}{2\epsilon_x}\delta_{yo} + \frac{\epsilon_y}{(1-\epsilon_y)\epsilon_x}\Big\} \\
&\qquad\quad \times \mathbb{1}_{[|\varphi_x|\geq 4R]}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big)\Big] \\
&\qquad + a_{xy}\Big[\epsilon_y 4R + 6R\delta_{yo} + \Big\{1 + c_f^{-1} + C_f + \frac{3}{2\epsilon_y}\delta_{xo} + \frac{\epsilon_x}{(1-\epsilon_x)\epsilon_y}\Big\} \\
&\qquad\quad \times \mathbb{1}_{[|\varphi_y|\geq 4R]}\big(f_y(\varphi_y) - f_y(T_y^\zeta(\varphi_y))\big)\Big].
\end{aligned}$$

We now recall (I.1), (I.2), (I.4) and (III.5), before we sum up and obtain, for $\varphi \in \Omega$,

$$\begin{aligned}
&\big|W(\varphi) - W(T^\zeta(\varphi))\big| \leq 8\epsilon_o R C_a^0 C_\rho + 12RC_a \\
&\quad + 2\sum_{x\in\Lambda}\Big\{C_a^0\big(1 + c_f^{-1} + C_f\big) + \tfrac{3}{2}C_a^\rho\epsilon_o^{-1} + \sum_{y\in\Lambda}\frac{a_{xy}\epsilon_y}{(1-\epsilon_o)\epsilon_x}\Big\} \\
&\quad \times \mathbb{1}_{[|\varphi_x|\geq 4R]}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big).
\end{aligned} \tag{IV.15}$$

The following bound is a consequence of (I.2) and the triangle inequality for ρ

$$\sum_{y\in\Lambda}\frac{a_{xy}\epsilon_y}{(1-\epsilon_o)\epsilon_x} \leq \frac{C_a^\rho}{1-\epsilon_o}. \tag{IV.16}$$

See also the proof of [1, Lemma 3.2].

From (IV.15) and (IV.16) we conclude the lemma with the constants given in (IV.2). □

V Estimating the Hamiltonian

Lemma V.1. *Let $0 < \zeta \le c_f R/2$, $|J| < \widetilde{J}_0(\zeta)$, and*

$$\sigma_T^\zeta := \min\big\{\zeta, 4\epsilon_o c_f R(1-|J|C_W^2)\big\} - |J|C_W^1, \tag{V.1}$$

where

$$\widetilde{J}_0(\zeta) := \min\Big\{\frac{\zeta}{C_W^1}, \frac{4\epsilon_o c_f R}{C_W^1 + 4\epsilon_o c_f R C_W^2}\Big\}. \tag{V.2}$$

Then, for all $\varphi \in \Omega$, with $\varphi_o \notin \mathcal{G}_{2\zeta}$, we have the bound

$$H_\Lambda(\varphi) - H_\Lambda(T^\zeta(\varphi)) \ge \sigma_T^\zeta. \tag{V.3}$$

Proof: We begin by analyzing the self-energy difference between φ and $T^\zeta(\varphi)$.

For $x \ne o$ we get by definition of T_x^ζ, cf. (III.7), and Condition I.1 (iii) that $f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x)) \ge 0$. In particular we get

$$f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x)) \ge \mathbb{1}_{[|\varphi_x|\ge 4R]}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big) \ge 0. \tag{V.4}$$

For $x = o$, we distinguish two cases. First consider $|\varphi_o| \ge 4R$. Here $|T_o^\zeta(\varphi_o)| = (1-\epsilon_o)|\varphi_o| \ge R$ and hence, by Condition I.1 (iii),

$$f_o(\varphi_o) - f_o(T_o^\zeta(\varphi_o)) \ge \epsilon_o c_f |\varphi_o| \ge 4\epsilon_o c_f R. \tag{V.5}$$

Secondly consider the case $|\varphi_o| < 4R$ and $\varphi_o \notin \mathcal{G}_{2\zeta}$. Then $T_o^\zeta(\varphi_o) \in \mathcal{G}_\zeta$ and thus

$$f_o(\varphi_o) - f_o(T_o^\zeta(\varphi_o)) \ge \zeta. \tag{V.6}$$

Putting (IV.1) and (V.4)-(V.6) together, we obtain the desired lower bound on $H(\varphi) - H_\Lambda(T^\zeta(\varphi))$,

$$\begin{aligned} &H_\Lambda(\varphi) - H_\Lambda(T^\zeta(\varphi)) \\ &\quad\ge \Big[\sum_{x\ne o} \mathbb{1}_{[|\varphi_x|\ge 4R]}\big\{1 - |J|C_W^2\big\}\big(f_x(\varphi_x) - f_x(T_x^\zeta(\varphi_x))\big)\Big] - |J|C_W^1 \\ &\qquad + \big\{\mathbb{1}_{[|\varphi_o|<4R]} + \mathbb{1}_{[|\varphi_o|\ge 4R]}(1-|J|C_W^2)\big\}\big(f_o(\varphi_o) - f_o(T_o^\zeta(\varphi_o))\big) \\ &\quad\ge \min\big\{\zeta, 4\epsilon_o c_f R(1-|J|C_W^2)\big\} - |J|C_W^1, \end{aligned} \tag{V.7}$$

where we use $\varphi_o \notin \mathcal{G}_{2\zeta}$ and also $|J|C_W^2 < \widetilde{J}_0(\zeta)C_W^2 \le 1$. □

VI Localization

In this section we prove the main result, Theorem I.1.

We begin separating into two regions

$$\begin{aligned} \mathbb{P}_\Lambda\big[\varphi_o \notin \mathcal{G}_{2\zeta}\big] &= \mathbb{P}_\Lambda\big[|\varphi_o| \ge 4R\big] + \mathbb{P}_\Lambda\big[\varphi_o \in B_{4R}(0)\backslash\mathcal{G}_{2\zeta}\big] \\ &= \mathcal{Z}_\Lambda^{-1}\Big(\int_{\{|\varphi_o|\ge 4R\}} e^{-\beta H_\Lambda(\varphi)} d^\Lambda\varphi + \int_{\{\varphi_o\in B_{4R}(0)\backslash\mathcal{G}_{2\zeta}\}} e^{-\beta H_\Lambda(\varphi)} d^\Lambda\varphi\Big). \end{aligned} \tag{VI.1}$$

Let $\mathcal{A}_1 = \{|\varphi_o| \geq 4R\}$ and $\mathcal{A}_2 = \{\varphi_o \in B_{4R}(0) \backslash \mathcal{G}_{2\zeta}\}$. We estimate using Lemma V.1, for $j = 1, 2$,

$$\begin{aligned}
\int_{\mathcal{A}_j} e^{-\beta H_\Lambda(\varphi)} d^\Lambda \varphi &\leq \sup_{\varphi \in \mathcal{A}_j} \left\{ e^{-\beta [H_\Lambda(\varphi) - H_\Lambda(T^\zeta(\varphi))]} \right\} \int_{\mathcal{A}_j} e^{-\beta H_\Lambda(T^\zeta(\varphi))} d^\Lambda \varphi \\
&= e^{-\beta \inf_{\varphi \in \mathcal{A}_j} [H_\Lambda(\varphi) - H_\Lambda(T^\zeta(\varphi))]} \int_{T^\zeta(\mathcal{A}_j)} e^{-\beta H_\Lambda(\psi)} \frac{d^\Lambda \psi}{|(\det \mathrm{Jac} T^\zeta)(T^{\zeta^{-1}}(\psi))|} \\
&\leq \frac{e^{-\beta \sigma_T^\zeta}}{\inf_{\varphi \in \mathcal{A}_j} |(\det \mathrm{Jac} T^\zeta)(T^{\zeta^{-1}}(\varphi))|} \mathcal{Z}_\Lambda, \qquad \text{(VI.2)}
\end{aligned}$$

provided $|J| < \widetilde{J}_0(\zeta)$ and $0 < \zeta \leq c_f R/2$. Inserting (VI.2) into (VI.1), together with the estimate (III.9) on the determinant of the Jacobian of T^ζ, we get

$$\begin{aligned}
\mathbb{P}_\Lambda\big[\varphi_o \notin \mathcal{G}_{2\zeta}\big] \leq \max &\Big\{ \big[4R\zeta^{-1} \sup_{|u| \leq R_\zeta} |\nabla f_o(u)|\big]^n, (1 - \epsilon_o)^{-n} \Big\} \\
&\times \exp\big[n \epsilon_o (1 + \|\theta'\|_\infty) C_\rho\big] e^{-\beta \sigma_T^\zeta}. \qquad \text{(VI.3)}
\end{aligned}$$

Taking infimum over admissible θ's and ϵ_o's, yields the estimate with $\|\theta'\|_\infty$ replaced by 2 and ϵ_o replaced by $\frac{1}{6}$ (see (III.6)), We have thus obtained the bound

$$\mathbb{P}_\Lambda\big[\varphi_o \notin \mathcal{G}_{2\zeta}\big] \leq 2 \max \Big\{ \big[4R\zeta^{-1} \sup_{|u| \leq R_\zeta} |\nabla f_o(u)|\big]^n, (\tfrac{6}{5})^n \Big\} e^{\frac{1}{2} n C_\rho} e^{-\beta \sigma_T^\zeta}. \qquad \text{(VI.4)}$$

We recapitulate: The constants in (IV.2), (V.1) and (V.2), with $\|\theta'\|_\infty = 2$ and $\epsilon_o = \frac{1}{6}$, become

$$C_W^1 = 4RC_a^0(\tfrac{1}{3}C_\rho + 3), \qquad C_W^2 = \Gamma_0 + \tfrac{102}{5} C_a^\rho, \qquad \text{(VI.5)}$$

$$\widetilde{J}_0(\zeta) = \min\Big\{ \frac{\zeta}{C_W^1}, \frac{\frac{2}{3} c_f R}{C_W^1 + \frac{2}{3} c_f R C_W^2} \Big\}, \qquad \text{(VI.6)}$$

$$\sigma_T^\zeta = \min\big\{\zeta, \tfrac{2}{3} c_f R (1 - |J| C_W^2)\big\} - |J| C_W^1. \qquad \text{(VI.7)}$$

With these constants and for $|J| < \widetilde{J}_0(\zeta)$, $0 < \frac{1}{2}\zeta \leq c_f R$, and $\beta > 0$ the localization bound (VI.3) holds true.

We end by explaining how to derive the assertion of Theorem I.1 from here. Note that $\Gamma_1 = C_W^1$ and $\Gamma_2 = C_W^2$. Comparing (VI.6) to (I.8), we further notice that $\widetilde{J}_0(\frac{1}{2} c_f R) = J_0$ and that $\widetilde{J}_0(\zeta) \leq J_0$, whenever $\zeta \leq \frac{1}{2} c_f R$. By assumption, we have $|J| \leq \min\{J_0, \frac{1}{2}\delta\Gamma_1^{-1}\}$. We distinguish the cases $\delta \geq c_f R$ and $\delta < c_f R$.

If $\delta \geq c_f R$ then we choose $\zeta := \frac{1}{2} c_f R$ and observe that $|J| \leq J_0 = \widetilde{J}_0(\zeta)$. The claim now follows from (VI.3), the trivial bound $\mathbb{P}_\Lambda[\varphi_o \notin \mathcal{G}_\delta] \leq \mathbb{P}_\Lambda[\varphi_o \notin \mathcal{G}_{c_f R}]$, and the fact that $\min\{\delta, \frac{3}{2}\Gamma_3\} = \min\{\delta, c_f R\} = c_f R = 2\zeta$.

Conversely, if $\delta < c_f R$ then we choose $\zeta := \frac{1}{2}\delta < \frac{1}{2} c_f R$. Since $\delta > 2|J|\Gamma_1$, also this choice insures that $|J| \leq \widetilde{J}_0(\zeta)$, namely, $|J| < \frac{1}{2}\zeta\Gamma_1^{-1} \leq \widetilde{J}_0(\zeta)$. Now the claim follows directly from (VI.3) and $\min\{\delta, \frac{3}{2}\Gamma_3\} = \min\{\delta, c_f R\} = \delta = 2\zeta$.

This completes the proof of Theorem I.1. □

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A Controlling the interaction

In this appendix we prove a basic bound on the interaction, which shows that it can be dominated by the self-energy. This is only used to ensure that polynomially bounded observables are integrable, and in particular that the partition function is finite.

Lemma A.1. *Suppose Conditions I.1 and I.2. There exists a constant A, which may depend on Λ, such that for all $\varphi \in \Omega$*

$$\Big| \sum_{x\neq y, x,y\in\Lambda} \omega_{xy}(\varphi_x,\varphi_y)\Big| \leq A + 2C_a^0\big(1+C_f+c_f^{-1}\big)\sum_{x\in\Lambda} f_x(\varphi_x).$$

Proof. Let $u, v \neq 0$. In the following A_j, $j\in\{1,2,3\}$, denote non-negative constants, which contribute to the A in the lemma. We estimate using Conditions I.1 and I.2

$$\begin{aligned}
|w_{xy}(u,v)| &\leq A_1 + |w_{xy}(u,v) - w_{xy}(0,0)| \\
&\leq A_1 + \int_0^1 \big|u\cdot(\nabla_1 w_{xy})(tu,tv) + v\cdot(\nabla_2 w_{xy})(tu,tv)\big|dt \\
&\leq A_1 + a_{xy}\int_0^1 (|u|+|v|)\big[1 + 1\!\!1_{[t|u|\geq 4R]}(\partial_{|u|}f_x)(tu) + 1\!\!1_{[t|v|\geq 4R]}(\partial_{|v|}f_y)(tv)\big]dt \\
&\leq A_1 + a_{xy}\Big\{|u|+|v| \\
&\qquad + (1+C_f)\int_0^1 \big[1\!\!1_{[t|u|\geq 4R]}|u|(\partial_{|u|}f_x)(tu) + 1\!\!1_{[t|v|\geq 4R]}|v|(\partial_{|v|}f_y)(tv)\big]dt\Big\} \\
&= A_1 + a_{xy}\Big\{|u|+|v| \\
&\qquad + (1+C_f)\int_0^1 \Big[1\!\!1_{[t|u|\geq 4R]}\frac{d}{dt}f_x(tu) + 1\!\!1_{[t|v|\geq 4R]}\frac{d}{dt}f_y(tv)\Big]dt\Big\} \\
&= A_1 + a_{xy}\Big\{|u|+|v| \\
&\qquad + (1+C_f)\Big[1\!\!1_{[|u|\geq 4R]}\int_{\frac{4R}{|u|}}^1 \frac{d}{dt}f_x(tu)dt + 1\!\!1_{[|v|\geq 4R]}\int_{\frac{4R}{|v|}}^1 \frac{d}{dt}f_y(tv)dt\Big]\Big\} \\
&\leq A_2 + a_{xy}\Big\{|u|+|v| + (1+C_f)\big[f_x(u)+f_y(v)\big]\Big\}.
\end{aligned}$$

To conclude the proof we observe the following bound

$$|u| \leq A_3 + 1\!\!1_{[|u|\geq R]}|u| \leq A_3 + c_f^{-1}1\!\!1_{[|u|\geq R]}f_x(u) \leq A_3 + c_f^{-1}f_x(u),$$

and sum up, using (I.4). □

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