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1. Introduction

Let G be a reductive algebraic group over a field k of prime characteristic p. The Weyl modules play a fundamental role in the study of finite dimensional representations of G. One of the important tools in investigating the structure of Weyl modules is their Jantzen filtration. The sum of the characters of the filtration terms obeys a sum formula analogous to the Verma module case, [11]. This formula was first proved by J. C. Jantzen some 30 years ago [10] with some mild restrictions on p. Later the first author [1] gave another proof valid for all p based on the fact that Weyl modules are special cases of cohomology of line bundles on the flag manifold for G and exploring natural homomorphisms between such cohomology modules.

More recently a similar formula [5] turned up in the theory for tilting modules for G. This time one filters the space of homomorphisms from a Weyl module into a tilting module. However, the proof in loc. cit. needs p to be at least the Coxeter number for G.

Both the above mentioned sum formulas are related to Ext-groups involving integral versions of Weyl modules, see [14] and [5]. In this paper we start out by proving an Euler-type formula for such Ext-groups using techniques from [1] and [14]. Then we are able to deduce the two sum formulas from this. In particular, our results work for all p. It also reveals that the two cases are in fact equivalent.

Let U_q denote the quantum group corresponding to G. When q is a root of unity (and U_q is obtained via the Lusztig divided power construction) there are completely analogous sum formulas for U_q. Our proof applies in this case as well and it avoids the restrictions on the order of q in [5].

We have taken the opportunity to recall the arguments from [1], [5] and [14] that we need. In this way our proof of the sum formulas for G is completely self-contained relying only on basic facts on Weyl modules, cohomology on line bundles, and tilting modules (which can all be found in [12]). In the quantum case everything works in the same way and we have only given the statements in that case leaving the analogous proofs to the readers.

Some of the results in this paper date back several years. At the meeting AMS Scand 2000 in Odense, Denmark the second author gave a talk, “Ext groups and Jantzen's sum formula” in which he presented the Weyl module sum formula in terms of Ext-groups. This can be found in [14], and it is also referred to in the preprint [15] where he proves the equivalence with the sum formula for tilting modules. Shortly after the appearance of this preprint we realized how to give the uniform proof presented below.

2. Notation

2.1. **Roots.** Throughout this paper k will denote an algebraically closed field of characteristic $p > 0$ and G will denote a reductive algebraic group over k. We choose a maximal torus T in G and a Borel subgroup B containing T. Then R will be the root system for (G,T). We fix a set of simple roots S in R by requiring that the roots of B are the corresponding negative roots $-R^+$. The number of positive roots is called N. This is also the dimension of the flag variety G/B.

The character group for T (and B) is denoted X. We let X^+ be the set of dominant characters, i.e., $X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}$.

The Weyl group $W = N_G(T)/Z_G(T)$ for G acts naturally on X. If $\alpha \in R$ then the reflection $s_\alpha \in W$ corresponding to α is given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for all $\lambda \in X$. We shall also use the 'dot-action' defined by $w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W, \lambda \in X$. Here ρ is the half sum of the positive roots.

Each element $w \in W$ is a product of simple reflections (reflections for simple roots) and we have the corresponding length function l on W taking w into the minimal number of such simple reflections needed to express w. The unique longest element in W is denoted w_0. It has length $l(w_0) = N$.

2.2. **Weights.** If M is a finite dimensional T-module and $\lambda \in X$ then the weight space M_λ is defined by $M_\lambda = \{m \in M \mid tm = \lambda(t)m \text{ for all } t \in T\}$. We say that λ is a weight of M if $M_\lambda \neq 0$. The character $\operatorname{ch} M$ is $\operatorname{ch} M = \sum_{\lambda \in X} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[X]$.

For each $\lambda \in X^+$ we have a Weyl module $\Delta(\lambda)$ for G with highest weight λ. Its contragredient dual $\Delta(\lambda)^*$ is denoted $\nabla(-w_0\lambda)$. Note that then the dual Weyl module $\nabla(\mu)$ attached to $\mu \in X^+$ has highest weight μ (because $w_0(\lambda)$ is the smallest weight of $\Delta(\lambda)$).

2.3. **Cohomology modules.** Let M be a finite dimensional B-module. Then we will write $H^0(M)$ for the G-module $\operatorname{Ind}_B^G M$ induced by M. This is also the 0-th cohomology (i.e., the set of global sections) for the vector bundle on G/B associated with M. More generally, we denote by $H^i(M)$ the i-th cohomology of this bundle, or alternatively the value of the i-th right derived functor $R^i \operatorname{Ind}_B^G$ on M. It is well known (as G/B is a projective variety) that the cohomology $H^\bullet(M)$ is finite dimensional, and that $H^i(M) = 0$ for $i > N$.

The Euler character of a B-module M is given by

$$\chi(M) = \sum_{\mu \in X} (-1)^i \operatorname{ch}(H^i(M)).$$

Note that χ is additive, i.e., if $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of finite dimensional B-modules then $\chi(M) = \chi(M_1) + \chi(M_2)$.

In the following the cohomology modules $H^i(\lambda), \lambda \in X$ will play a vital role. In particular, we recall that the Weyl modules above are special instances of such modules. Precisely, we have $\Delta(\lambda) \simeq H^N(w_0 \cdot \lambda)$ for all $\lambda \in X^+$. Also $\nabla(\lambda) = H^0(\lambda)$. Moreover, as we shall see (cf. Section 3.4 below) we have $\chi(\lambda) = \operatorname{ch} \Delta(\lambda) = \operatorname{ch} \nabla(\lambda)$.

2.4. **Chevalley groups.** Let $G_\mathbb{Z}$ be a split and connected reductive algebraic group scheme over $\mathbb{Z}$ corresponding to G. In other words $G_\mathbb{Z}$ is the associated Chevalley group. Then G is obtained from $G_\mathbb{Z}$ by extending scalars to k. More generally, we write G_A for the group scheme over an arbitrary commutative ring A obtained via the base change $\mathbb{Z} \to A$. (The case $A = \mathbb{Z}_p$, the ring of p-adic integers, will be needed in Chapter 5.) We use similar notation relative to the subgroups T and B. In particular, $T_\mathbb{Z}$ is a split maximal torus in $G_\mathbb{Z}$ with $T_k = T$. We will identify R with the root system associated to (G_A, T_A).

Note that for a G_A-module V that is free of finite rank as an A-module, $\mathrm{ch}(V)$ makes sense by considering ranks of weight spaces. If our field k is an A-algebra then we have for such a module $\mathrm{ch}(V) = \mathrm{ch}(V \otimes_A k)$.

For any commutative ring A and any B_A-module M we write $H^i_A(M)$ for the G_A-module $R^i \operatorname{Ind}_{B_A}^{G_A} M$. See [12], I.5 for the general properties of these modules. In particular, we recall that if A is noetherian and M is finitely generated over A, then $H^i_A(M)$ is also finitely generated over A, see [12], Proposition I.5.12 c).

Given any commutative ring A, for each $\lambda \in X^+$ we have the following two G_A-modules: the Weyl module $\Delta_A(\lambda)$ and the dual Weyl module $\nabla_A(\lambda)$. These modules are characteristic-free, i.e., as A-modules both are free of rank equal to $\dim \Delta(\lambda)$ and we have G_A-module isomorphisms $\Delta_A(\lambda) \simeq \Delta_{\mathbb{Z}}(\lambda) \otimes A$ and $\nabla_A(\lambda) \simeq \nabla_{\mathbb{Z}}(\lambda) \otimes A$. Just as for G, we have $\nabla_A(\lambda) = H^0_A(\lambda)$ and $\Delta_A(\lambda) \simeq H^N_A(w_0 \cdot \lambda)$, see Chapter 3.

Any $G_{\mathbb{Z}}$-module M which is finitely generated as a $\mathbb{Z}$-module has finite torsion submodule $M_t = \{m \in M \mid nm = 0 \text{ for some } n \in \mathbb{N}\}$. This is a $G_{\mathbb{Z}}$-submodule and we set $M_f = M/M_t$. Then we refer to M_t and M_f as the torsion part, respectively free part of M.

Any M as above allows a surjection $P_0 \to M$ from a $G_{\mathbb{Z}}$-module P_0 which is free of finite rank as a $\mathbb{Z}$-module. Hence M also has a free presentation $0 \to P_1 \to P_0 \to M \to 0$ with P_0 and P_1 free over $\mathbb{Z}$.

2.5. **Divisors.** Let $\mathcal{D}(\mathbb{Z})$ denote the divisor group for $\mathbb{Z}$, i.e., the free $\mathbb{Z}$-module with basis consisting of all prime numbers p. If $n \in \mathbb{Z} \setminus \{0\}$ then we write $\mathrm{div}(n) \in \mathcal{D}(\mathbb{Z})$ for the divisor associated to n. If M is a finite $\mathbb{Z}$-module of order $|M|$ we write $\mathrm{div}(M) = \mathrm{div}(|M|)$. Clearly, div is additive with respect to short exact sequences of finite $\mathbb{Z}$-modules.

Suppose M is a $T_{\mathbb{Z}}$-module. Then M splits into a direct sum of weight submodules, see [12], I.2.11. When M is finite, in analogy with the situation for T-modules in Section 2.2, this leads us to the following definition of $\mathrm{div}_T(M) \in \mathcal{D}(\mathbb{Z})[X]$

$$\mathrm{div}_T(M) = \sum_{\mu \in X} \mathrm{div}(M_\mu) e^\mu.$$

Again it is clear that div_T is additive on exact sequences of finite $T_{\mathbb{Z}}$-modules.

2.6. **Ext groups.** Consider finitely generated $G_{\mathbb{Z}}$-modules M and N. By [12], II.B, the groups $\mathrm{Ext}^i_{G_{\mathbb{Z}}}(M, N)$ are finitely generated and vanish for large enough i. We will also need the following special cases of some vanishing results from loc. cit.

Proposition 2.1. *For* $\lambda, \mu \in X^+$*,*

a) $\mathrm{Ext}^i_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\mu), \nabla_{\mathbb{Z}}(\lambda)) = 0$ *unless* ($\mu = \lambda$ *and* $i = 0$).
$\mathrm{Hom}_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), \nabla_{\mathbb{Z}}(\lambda)) = \mathbb{Z}$.

b) $\mathrm{Ext}^i_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\mu), \Delta_{\mathbb{Z}}(\lambda)) = 0$ *unless* $\mu < \lambda$ *or* ($\mu = \lambda$ *and* $i = 0$). $\mathrm{Hom}_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), \Delta_{\mathbb{Z}}(\lambda)) = \mathbb{Z}$.

The universal coefficient theorem [12], Proposition I.4.18a gives analogous results over G_A for other commutative rings A. In particular the proposition stays valid after replacing each $\mathbb{Z}$ by $\mathbb{Z}_p$.

2.7. **Tilting modules.** A tilting module for G_A is an A-finite G_A-module Q which has both a Weyl filtration (i.e., a filtration with successive quotients isomorphic to Weyl modules) and a dual Weyl filtration (with successive quotients isomorphic to dual Weyl modules). For a tilting module Q (or more generally any module with a Weyl filtration) we write $(Q : \Delta(\lambda))$ for the number of times $\Delta_A(\lambda)$ occurs in a Weyl filtration of Q. This integer is also uniquely defined by the character equation

$$\operatorname{ch} Q = \sum_{\lambda \in X^+} (Q : \Delta(\lambda))\chi(\lambda).$$

Let $A = \mathbb{Z}_p$ in this paragraph. We have the following standard facts, e.g., from [12], II.E. For each $\lambda \in X^+$ there is a unique indecomposable tilting module $T(\lambda)$ for G (respectively, $T_A(\lambda)$ for G_A) with highest weight λ. Every tilting module for G (respectively, of G_A) is uniquely expressible as a direct sum of the various $T(\lambda)$ (respectively, $T_A(\lambda)$). We have $T_A(\lambda) \otimes_A k \simeq T(\lambda)$. In particular, every tilting module $\bar{Q}$ for G lifts uniquely to a tilting module Q for G_A (i.e., $Q \otimes_A k \simeq \bar{Q}$).

3. The Borel-Weil-Bott theorem and its consequences over $\mathbb{Z}$

3.1. **The Borel-Weil-Bott theorem.** The Borel-Weil-Bott theorem holds over any field of characteristic 0. Here we only state it over $\mathbb{Q}$. The general case then follows by an easy base change argument, compare 3.2 below.

Theorem 3.1 ([6], [7])**.** *Let $\lambda \in X$ and choose $w \in W$ such that $w(\lambda + \rho) \in X^+$. Then we have isomorphisms of $G_{\mathbb{Q}}$-modules*

$$H^i_{\mathbb{Q}}(\lambda) \simeq \begin{cases} H^0_{\mathbb{Q}}(w \cdot \lambda) & \text{if } i = l(w), \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Note that if $\lambda \in X$ is singular, i.e., if there exists $\alpha \in R$ with $\langle \lambda + \rho, \alpha^\vee \rangle = 0$, then $H^i_{\mathbb{Q}}(\lambda) = 0$ for all i. Hence the possible non-uniqueness of w in this statement does not cause ambiguity.

3.2. **Universal coefficients theorem.** Let A be an arbitrary commutative ring. Then for any $G_{\mathbb{Z}}$-module M which is free of finite rank over $\mathbb{Z}$ and for any $i \geq 0$ we have the following short exact sequence of A-modules, cf. e.g., [12] , I.4.18.

$$0 \to H^i_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} A \to H^i_A(M \otimes_{\mathbb{Z}} A) \to \operatorname{Tor}_1^{\mathbb{Z}}(H^{i+1}_{\mathbb{Z}}(M), A) \to 0.$$

3.3. **The Borel-Weil-Bott theorem over $\mathbb{Z}$.** When we combine 3.1 and 3.2 we find

Corollary 3.2. *Let $\lambda \in X$ and suppose $w \in W$ satisfies $w \cdot \lambda \in X^+$. Then*

a) $H^i_{\mathbb{Z}}(\lambda)$ is a finite $\mathbb{Z}$-module for all $i \neq l(w)$.

b) $H^{l(w)}_{\mathbb{Z}}(\lambda)_f \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \Delta_{\mathbb{Q}}(w \cdot \lambda)$.

Remark 3.3. If no $w \in W$ exists with $w \cdot \lambda \in X^+$ (i.e., if λ is singular) then $H^i_{\mathbb{Z}}(\lambda)$ is a finite $\mathbb{Z}$-module for all i.

3.4. **Kempf's theorem.** Recall that Kempf's vanishing theorem [13] says that if λ is dominant then all the higher cohomology modules $H^i(\lambda), i > 0$ vanish. This being true for all fields we get (e.g., via the universal coefficient theorem above)

Theorem 3.4. *Let $\lambda \in X^+$. Then $H^i_{\mathbb{Z}}(\lambda) = 0$ for all $i > 0$.*

This means in particular via the universal coefficient theorem above that for dominant λ we have that $H^0_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k \simeq H^0(\lambda)$. Hence $\nabla_{\mathbb{Z}}(\lambda) \simeq H^0_{\mathbb{Z}}(\lambda)$.

Via Serre duality (which is valid over all fields but not over $\mathbb{Z}$), Kempf's theorem gives also

$$H^i(\lambda) = 0 \text{ for all } i < N \text{ and all } \lambda \text{ with } -\lambda - 2\rho \in X^+. \tag{3.1}$$

Hence for each $\lambda \in X^+$ we conclude that $H^N_{\mathbb{Z}}(w_0 \cdot \lambda)$ has no torsion and the dimension of $H^N(w_0 \cdot \lambda)$ is independent of k. In fact, $H^N(w_0 \cdot \lambda) \simeq H^N_{\mathbb{Z}}(w_0 \cdot \lambda) \otimes_{\mathbb{Z}} k$ and hence $H^N_{\mathbb{Z}}(w_0 \cdot \lambda) \simeq \Delta_{\mathbb{Z}}(\lambda)$.

Corollary 3.5. *Let V be any $G_{\mathbb{Z}}$-module (finitely generated over $\mathbb{Z}$ as always). Then for all $\lambda \in X^+$ we have $H^i_{\mathbb{Z}}(V \otimes_{\mathbb{Z}} \lambda) = 0$ for all $i > 0$ and $H^0_{\mathbb{Z}}(V \otimes_{\mathbb{Z}} \lambda) \simeq V \otimes_{\mathbb{Z}} H^0_{\mathbb{Z}}(\lambda)$.*

Proof: If V is free over $\mathbb{Z}$ then we have the tensor identity [12] I.3.6 $H^i_{\mathbb{Z}}(V \otimes_{\mathbb{Z}} \lambda) \simeq V \otimes_{\mathbb{Z}} H^i_{\mathbb{Z}}(\lambda)$. Hence in this case the corollary results directly from Kempf's theorem. In general, we have from 2.4 a presentation $0 \to P_1 \to P_0 \to V \to 0$ with P_1 and P_0 free over $\mathbb{Z}$. The corollary then holds for P_1 and P_0. It is then immediate to deduce it for V.

Another important consequence of Kempf's theorem is that since it clearly gives $\operatorname{ch} H^0(\lambda) = \sum_{i \geq 0} \operatorname{ch} H^i(\lambda) = \chi(\lambda)$ for all dominant weights λ, we get

$$\operatorname{ch} \nabla(\lambda) = \chi(\lambda) = \operatorname{ch} \Delta(\lambda) \text{ for all } \lambda \in X^+. \tag{3.2}$$

Here the last equality follows by combining Kempf's vanishing and Serre duality, see (3.1) above.

Remark. It is well known that $\chi(\lambda)$ is given by Weyl's character formula, see, e.g., [8] (2.2.6).

3.5. **Rank 1.** The (very easy) proof by Demazure [7] of Bott's theorem relies on an analysis of natural isomorphisms $H^{i+1}_{\mathbb{Q}}(s_\alpha \cdot \lambda) \to H^i_{\mathbb{Q}}(\lambda)$ when α is a simple root with $\langle \lambda, \alpha^\vee \rangle \geq 0$. We shall need the underlying homomorphisms over $\mathbb{Z}$ and hence engage in the following considerations.

Let α be a simple root and denote the corresponding minimal parabolic subgroup P_α in G containing B. Then we denote for any B-module M by $H^i_\alpha(M)$ the module $H^i(P_\alpha/B, M)$. Note that P_α/B is the projective line so that these cohomology modules always vanish for $i > 1$. When working over $\mathbb{Z}$ we write $H^i_{\alpha,\mathbb{Z}}(M)$ for the analogously defined modules for the $\mathbb{Z}$-version $P_{\alpha,\mathbb{Z}}$ of P_α.

Lemma 3.6 (cf. [12] II.5.2 and 8.13)**.** *Let $\lambda \in X$.*

- *a) If $\langle \lambda, \alpha^\vee \rangle \geq 0$ then $H^i_{\alpha,\mathbb{Z}}(\lambda) = 0$ for all $i > 0$ and $H^0_{\alpha,\mathbb{Z}}(\lambda)$ is a free $\mathbb{Z}$-module whose weights are $\lambda, \lambda - \alpha, \ldots, s_\alpha(\lambda)$, all occurring with multiplicity 1.*
- *b) If $\langle \lambda, \alpha^\vee \rangle < -1$ then $H^i_{\alpha,\mathbb{Z}}(\lambda) = 0$ for all $i \neq 1$ and $H^1_{\alpha,\mathbb{Z}}(\lambda)$ is a free $\mathbb{Z}$-module whose weights are $\lambda + \alpha, \lambda + 2\alpha, \ldots, s_\alpha \cdot \lambda$, all occurring with multiplicity 1.*
- *c) If $\langle \lambda, \alpha^\vee \rangle = r \geq 0$ then $\operatorname{Hom}_{P_{\alpha,\mathbb{Z}}}(H^1_{\alpha,\mathbb{Z}}(s_\alpha \cdot \lambda), H^0_{\alpha,\mathbb{Z}}(\lambda)) \simeq \mathbb{Z}$. Moreover, $H^0_{\alpha,\mathbb{Z}}(\lambda)$, respectively $H^1_{\alpha,\mathbb{Z}}(s_\alpha \cdot \lambda)$ has a standard $\mathbb{Z}$-basis $\{v_0, v_1, \ldots, v_r\}$, respectively $\{v'_0, v'_1, \ldots, v'_r\}$ with v_j, respectively v'_j, having weight $\lambda - j\alpha$, $j = 0, 1, \ldots, r$. A generator $c_\alpha(\lambda)$ of $\operatorname{Hom}_{P_{\alpha,\mathbb{Z}}}(H^1_{\alpha,\mathbb{Z}}(s_\alpha \cdot \lambda), H^0_{\alpha,\mathbb{Z}}(\lambda))$ is given by*

$$c_\alpha(\lambda)(v'_j) = \binom{r}{j} v_j, \ j = 0, 1, \ldots, r.$$

3.6. **Passing from Rank 1 to the general case.** Keep the notation from 3.5. By transitivity of induction we have with obvious notation $H^0(M) \simeq H^0(G/P_\alpha, H^0_\alpha(M))$. The same is true over $\mathbb{Z}$. Hence using general properties of $H^i_{\mathbb{Z}}$, cf. [12], II.8 we obtain from Lemma 3.6 a) and b) that if $\lambda \in X$ and $\alpha \in S$ satisfy $\langle \lambda, \alpha^\vee \rangle \geq 0$ then

$$H^i_{\mathbb{Z}}(\lambda) \simeq H^i_{\mathbb{Z}}(H^0_{\alpha,\mathbb{Z}}(\lambda)) \tag{3.3}$$

and

$$H^{i+1}_{\mathbb{Z}}(s_\alpha \cdot \lambda) \simeq H^i_{\mathbb{Z}}(H^1_{\alpha,\mathbb{Z}}(s_\alpha \cdot \lambda)) \tag{3.4}$$

Denote by $Q_\alpha(\lambda)$ the cokernel of the generator $c_\alpha(\lambda)$ from Lemma 3.6 c). Then $Q_\alpha(\lambda)$ is a finite $P_{\alpha,\mathbb{Z}}$-module with weights $\lambda - \alpha, \lambda - 2\alpha, \dots, s_\alpha(\lambda) + \alpha$. Each weight space is cyclic and we have

$$\operatorname{div}_T(Q_\alpha(\lambda)) = \sum_{j=1}^{r-1} \operatorname{div} \binom{r}{j} e^{\lambda - j\alpha}. \tag{3.5}$$

The short exact sequence of $P_{\alpha,\mathbb{Z}}$-modules

$$0 \to H^1_{\alpha,\mathbb{Z}}(s_\alpha \cdot \lambda) \to H^0_{\alpha,\mathbb{Z}}(\lambda) \to Q_\alpha(\lambda) \to 0$$

gives via (3.3) and (3.4) rise to the long exact sequence of $G_{\mathbb{Z}}$-modules

$$\cdots \to H^{i+1}_{\mathbb{Z}}(s_\alpha \cdot \lambda) \to H^i_{\mathbb{Z}}(\lambda) \to H^i_{\mathbb{Z}}(Q_\alpha(\lambda)) \to \cdots .$$

Remark. The isomorphisms over $\mathbb{Q}$ analogous to (3.3) and (3.4) give isomorphisms $H^{i+1}_{\mathbb{Q}}(s_\alpha \cdot \lambda) \simeq H^i_{\mathbb{Q}}(\lambda)$ for all i. This is the key to Demazure's proof [7] of Theorem 3.1.

4. Euler type formulas

4.1. **Euler coefficients for G-modules.** Let V and V' be $G_{\mathbb{Z}}$-modules, both finitely generated over $\mathbb{Z}$. Then $\operatorname{Ext}^i_{G_{\mathbb{Z}}}(V, V')$ is finite for all $i > 0$. This follows from Section 2.6 and the universal coefficient theorem [12], Proposition I.4.18a, because $\operatorname{Ext}^i_{G_{\mathbb{C}}}(A, B) = 0$ for all $i > 0$ and for any two rational $G_{\mathbb{C}}$-modules A and B ($G_{\mathbb{C}}$ being reductive). If the $G_{\mathbb{C}}$-modules $V \otimes_{\mathbb{Z}} \mathbb{C}$ and $V' \otimes_{\mathbb{Z}} \mathbb{C}$ do not have an isomorphic simple summand, then $\operatorname{Hom}_{G_{\mathbb{Z}}}(V, V')$ is finite. This happens in particular when V or V' is finite. By Section 2.6 we have in any case $\operatorname{Ext}^i_{G_{\mathbb{Z}}}(V, V') = 0$ when $i \gg 0$. So whenever $\lambda \in X^+$ and V is a $G_{\mathbb{Z}}$-module such that $(V \otimes_{\mathbb{Z}} \mathbb{C} : \Delta_{\mathbb{C}}(\lambda)) = 0$ (e.g., when V is a finite $G_{\mathbb{Z}}$-module), the following expression gives a well defined element in $\operatorname{Div}(\mathbb{Z})$

$$e^G_\lambda(V) = \sum_{i \geq 0} (-1)^i \operatorname{div}(\operatorname{Ext}^i_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), V)).$$

Clearly, e^G_λ is additive on exact sequences of such $G_{\mathbb{Z}}$-modules (in particular finite $G_{\mathbb{Z}}$-modules).

Remark. We may extend the above definition of $e^G_\lambda(V)$ to all (finitely generated) V by using just the torsion part of $\operatorname{Hom}_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), V)$. Clearly when extended in this way e^G_λ will fail to be additive on arbitrary exact sequences in general. The proofs of Theorem 4.1 and Proposition 4.3 below require careful examination of this failure for particular exact sequences.

4.2. **Euler coefficients for B-modules.** Suppose M is a finite $B_{\mathbb{Z}}$-module. Then for each i the $G_{\mathbb{Z}}$-module $H^i_{\mathbb{Z}}(M)$ is also finite (because $G_{\mathbb{Z}}/B_{\mathbb{Z}}$ is a projective scheme) and it is 0 for $i > N = \dim G_{\mathbb{Z}}/B_{\mathbb{Z}}$. We define for $\lambda \in X^+$

$$e^B_\lambda(M) = \sum_{j\geq 0}(-1)^j e^G_\lambda(H^j_{\mathbb{Z}}(M)).$$

Again we see that e^B_λ is additive on exact sequences of finite $B_{\mathbb{Z}}$-modules.

If the $B_{\mathbb{Z}}$-structure on M extends to $G_{\mathbb{Z}}$ then Corollary 3.5 tells us that $H^j_{\mathbb{Z}}(M) = 0$ for $j > 0$ and $H^0_{\mathbb{Z}}(M) \simeq M$. Hence in this case we have for all $\lambda \in X^+$

$$e^G_\lambda(M) = e^B_\lambda(M). \tag{4.1}$$

4.3. **Formulas for B-modules.**

Theorem 4.1. *Let M be a finite $B_{\mathbb{Z}}$-module. Then*

a) $e^B_\lambda(M) = \sum_{w\in W}(-1)^{l(w)}\operatorname{div}(M_{w\cdot\lambda})$ *for all* $\lambda \in X^+$.

b) $\sum_{i\geq 0}(-1)^i \operatorname{div}_T(H^i_{\mathbb{Z}}(M)) = \sum_{\lambda\in X^+} e^B_\lambda(M)\chi(\lambda)$.

Proof: a) The additivity of e^B_λ immediately allows us to reduce to the case where M is given by the following short exact sequence

$$0 \to \mathbb{Z}_\mu \xrightarrow{n} \mathbb{Z}_\mu \to M \to 0 \tag{4.2}$$

with $\mu \in X$ and $n \in \mathbb{N}$. In this case the formula we want to verify is

$$e^B_\lambda(M) = \begin{cases} (-1)^{l(w)}\operatorname{div}(n) & \text{if } \mu = w\cdot\lambda \text{ for some } w \in W, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that if $\mu = w\cdot\lambda$ for some $w \in W$ then μ is non-singular and w is uniquely determined. This is so because $\lambda \in X^+$).

To prove this we consider the long exact cohomology sequence arising from (4.2)

$$\cdots \to H^i_{\mathbb{Z}}(\mu) \xrightarrow{n} H^i_{\mathbb{Z}}(\mu) \to H^i_{\mathbb{Z}}(M) \to \cdots .$$

If μ is singular (i.e., if there is a $\beta \in R$ with $\langle \mu+\rho, \beta^\vee\rangle = 0$) then all modules in this sequence are finite. In this case the additivity of e^B_λ immediately gives $e^B_\lambda(M) = 0$ as desired.

So suppose μ is non-singular. Then there exists a unique i_0 such that $H^{i_0}_{\mathbb{Z}}(\mu)$ is infinite. Define then $C^t(\mu)$, $C(\mu)$, respectively $C^f(\mu)$ such that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^{i_0}_{\mathbb{Z}}(\mu)_t & \xrightarrow{n} & H^{i_0}_{\mathbb{Z}}(\mu)_t & \longrightarrow & C^t(\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^{i_0}_{\mathbb{Z}}(\mu) & \xrightarrow{n} & H^{i_0}_{\mathbb{Z}}(\mu) & \longrightarrow & C(\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{i_0}_{\mathbb{Z}}(\mu)_f & \xrightarrow{n} & H^{i_0}_{\mathbb{Z}}(\mu)_f & \longrightarrow & C^f(\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

has exact rows. By definition the two first columns are exact and hence it follows that so is the last column, i.e.,

$$0 \to C^t(\mu) \to C(\mu) \to C^f(\mu) \to 0$$

is exact. Now the long exact sequence arising from (4.2) also gives exact sequences (recall that $H^j_{\mathbb{Z}}(\mu)$ is a torsion module for $j \neq i_0$)

$$\cdots \to H^{i_0-1}_{\mathbb{Z}}(\mu) \to H^{i_0-1}_{\mathbb{Z}}(\mu) \to H^{i_0-1}_{\mathbb{Z}}(M) \to H^{i_0}_{\mathbb{Z}}(\mu)_t \to H^{i_0}_{\mathbb{Z}}(\mu)_t \to C^t(\mu) \to 0$$

and

$$0 \to C(\mu) \to H^{i_0}_{\mathbb{Z}}(M) \to H^{i_0+1}_{\mathbb{Z}}(\mu) \to H^{i_0+1}_{\mathbb{Z}}(\mu) \to H^{i_0+1}_{\mathbb{Z}}(M) \to \cdots .$$

This implies

$$e^G_\lambda(C^t(\mu)) = (-1)^{i_0-1} \sum_{j=0}^{i_0-1} (-1)^j e^G_\lambda(H^j_{\mathbb{Z}}(M))$$

and

$$e^G_\lambda(C(\mu)) = (-1)^{i_0} \sum_{j \geq i_0} (-1)^j e^G_\lambda(H^j_{\mathbb{Z}}(M)).$$

We conclude that $e^B_\lambda(M) = \sum_{j \geq 0} (-1)^j e^G_\lambda(H^j_{\mathbb{Z}}(M)) = (-1)^{i_0}(e^G_\lambda(C(\mu)) - e^G_\lambda(C^t(\mu)))$ $= (-1)^{i_0} e^G_\lambda(C^f(\mu))$.

So let $w \in W$ be determined by $w(\mu + \rho) \in X^+$. Then $i_0 = l(w)$ and the weights of $H^{i_0}_{\mathbb{Z}}(\mu)_f$ coincide with those of $\Delta(w \cdot \mu)$. In particular, if $\lambda = w \cdot \mu$ then $H^{i_0}_{\mathbb{Z}}(\mu)_f$ has unique highest weight λ. Therefore $\mathrm{Hom}_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), H^{i_0}_{\mathbb{Z}}(\mu)_f) \simeq \mathbb{Z}$ and $\mathrm{Ext}^i_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), H^{i_0}_{\mathbb{Z}}(\mu)_f) = 0$ for $i > 0$ (since $\lambda \geq \nu$ for all weights ν of $H^{i_0}_{\mathbb{Z}}(\mu)_f$). Hence in this case $e^G_\lambda(C^f(\mu)) = \mathrm{div}(n)$. If on the other hand $\lambda \neq w \cdot \mu$ then $\mathrm{Hom}_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\lambda), H^{i_0}_{\mathbb{Z}}(\mu)_f) = 0$ and the long exact Ext-sequence arising from

$$0 \to H^{i_0}_{\mathbb{Z}}(\mu)_f \xrightarrow{n} H^{i_0}_{\mathbb{Z}}(\mu)_f \to C^f(\mu) \to 0$$

consists entirely of finite $\mathbb{Z}$-modules. It follows that in this case $e^G_\lambda(C^f(\mu)) = 0$ as desired.

b) Both sides of the equation in b) are additive in M and hence just as above we may restrict to the case where M is defined by (4.2). Using a) for the right hand side and recalling that $\chi(\mu) = (-1)^{l(w)} \chi(w \cdot \mu)$ for all $w \in W$ we see that the desired equality in this case is

$$\sum_{i \geq 0} (-1)^i \, \mathrm{div}_T(H^i_{\mathbb{Z}}(M)) = \mathrm{div}(n) \chi(\mu).$$

Note that here $\chi(\mu) = 0$ if μ is singular. Arguing as in the proof of a) above we obtain in fact

$$\sum_{i \geq 0} (-1)^i \, \mathrm{div}_T(H^i_{\mathbb{Z}}(M)) = (-1)^{i_0} \, \mathrm{div}_T(C^f(\mu)) = \mathrm{div}(n) \chi(\mu).$$

Here the last equality results from the definition of $C^f(\mu)$ via the fact that $\mathrm{ch}(H^{i_0}_{\mathbb{Z}}(\mu)_f)$ $= (-1)^{i_0} \chi(\mu)$.

4.4. **Formulas for G-modules.** Combining Theorem 4.1 and the identity (4.1) we obtain

Corollary 4.2. *Let V be a finite $G_{\mathbb{Z}}$-module*

a) $e_\lambda^G(V) = \sum_{w\in W}(-1)^{l(w)} \operatorname{div}(V_{w\cdot\lambda})$ *for all* $\lambda \in X^+$.

b) $\operatorname{div}_T(V) = \sum_{\lambda\in X^+} e_\lambda^G(V)\chi(\lambda)$.

Remark. The second identity in this corollary was obtained by the second author in [14]. The argument there is different.

4.5. **Natural homomorphisms.** Fix now $\mu \in X^+$ and a reduced expression $w_0 = s_1 s_2 \cdots s_N$ for w_0 with s_i denoting the reflection corresponding to the simple root α_i. Then we set

$$\mu_0 = \mu,\ \mu_1 = s_1\cdot\mu_0, \dots, \mu_i = s_i\cdot\mu_{i-1}, \dots, \mu_N = s_N\cdot\mu_{N-1} = w_0\cdot\mu.$$

Since μ is the unique highest weight of $\nabla_{\mathbb{Z}}(\mu)$ we have up to sign a unique generator c_μ for $\operatorname{Hom}_{G_{\mathbb{Z}}}(\Delta_{\mathbb{Z}}(\mu), \nabla_{\mathbb{Z}}(\mu)) \simeq \mathbb{Z}$. We set $Q(\mu) = \operatorname{Coker}(c_\mu)$ so that we have a short exact sequence

$$0 \to \Delta_{\mathbb{Z}}(\mu) \to \nabla_{\mathbb{Z}}(\mu) \to Q(\mu) \to 0.$$

Now we claim that c_μ factors through $H^i_{\mathbb{Z}}(\mu_i)$ for all i. In fact, note that $\langle \mu_{i-1} + \rho, \alpha_i^\vee\rangle = \langle \mu+\rho, s_1 s_2\cdots s_{i-1}(\alpha_i)^\vee\rangle > 0$ because $s_1 s_2 \cdots s_{i-1}(\alpha_i) \in R^+$. Using the notation from Lemma 3.6 we therefore have a short exact sequence

$$0 \to H^1_{\alpha_i,\mathbb{Z}}(\mu_i) \xrightarrow{c_i} H^0_{\alpha_i,\mathbb{Z}}(\mu_{i-1}) \to Q_{\alpha_i}(\mu_{i-1}) \to 0. \tag{4.3}$$

where $c_i = c_{\alpha_i}(\mu_{i-1})$ and $Q_{\alpha_i}(\mu_{i-1}) = \operatorname{Coker}(c_i)$. When we apply $H^{i-1}_{\mathbb{Z}}$ to (4.3) we get (see (3.3) and (3.4))

$$\to H^i_{\mathbb{Z}}(\mu_i) \xrightarrow{\tilde{H}^{i-1}_{\mathbb{Z}}(c_i)} H^{i-1}_{\mathbb{Z}}(\mu_{i-1}) \to H^{i-1}_{\mathbb{Z}}(Q_{\alpha_i}(\mu_{i-1})) \to$$

as part of a long exact sequence. Tracing a highest weight vector we see that (up to sign) c_μ may be identified with the composite

$$\Delta_{\mathbb{Z}}(\mu) \simeq H^N_{\mathbb{Z}}(\mu_N) \xrightarrow{\tilde{c}_N} \cdots \xrightarrow{\tilde{c}_{i+1}} H^i_{\mathbb{Z}}(\mu_i)_f \xrightarrow{\tilde{c}_i} H^{i-1}_{\mathbb{Z}}(\mu_{i-1})_f \xrightarrow{\tilde{c}_{i-1}} \cdots \xrightarrow{\tilde{c}_{i-1}} H^0_{\mathbb{Z}}(\mu_0) \simeq \nabla_{\mathbb{Z}}(\mu).$$

Note that we have passed to the free quotient of $H^i_{\mathbb{Z}}(\mu_i)$ and denoted the homomorphism here induced by $H^{i-1}_{\mathbb{Z}}(c_i)$ by $\tilde{c}_i$. For $i = N$ and $i = 0$ the cohomology modules are free, see Section 3.4 and so in these cases we have omitted the subscript f. If $Q^f_i(\mu)$ denotes the cokernel of $\tilde{c}_i$ then we have a short exact sequence

$$0 \to H^i_{\mathbb{Z}}(\mu_i)_f \xrightarrow{\tilde{c}_i} H^{i-1}_{\mathbb{Z}}(\mu_{i-1})_f \to Q^f_i(\mu) \to 0.$$

4.6. **Formulas for Euler coefficients.** Keep the notation from 4.5. Then we have

Proposition 4.3.

$$e^G_\lambda(Q(\mu)) = \sum_{i=1}^N (-1)^{i-1} e^B_\lambda(Q_{\alpha_i}(\mu_{i-1}))\ \textit{for all}\ \lambda \in X^+.$$

Proof: The factorization $c_\mu = \tilde{c}_1 \circ \tilde{c}_2 \circ \cdots \circ \tilde{c}_N$ from 4.5 gives immediately

$$e^G_\lambda(Q(\mu)) = \sum_{i=1}^N e^G_\lambda(Q^f_i(\mu)) \text{ for all } \lambda \in X^+. \tag{4.4}$$

If we now set $Q_i(\mu) = \mathrm{Coker}(H^{i-1}_{\mathbb{Z}}(c_i))$ and let $Q^t_i(\mu)$ denote the cokernel of the induced homomorphism $H^i_{\mathbb{Z}}(\mu_i)_t \to H^{i-1}_{\mathbb{Z}}(\mu_{i-1})_t$ then we get the following commutative diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^i(\mu_i)_t & \longrightarrow & H^{i-1}(\mu_{i-1})_t & \longrightarrow & Q^t_i(\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^i_{\mathbb{Z}}(\mu_i) & \longrightarrow & H^{i-1}_{\mathbb{Z}}(\mu_{i-1}) & \longrightarrow & Q_i(\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^i(\mu_i)_f & \longrightarrow & H^{i-1}(\mu_{i-1})_f & \longrightarrow & Q^f_i(\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here the rows and two first columns are exact. Hence we deduce that the last column is also exact and we get

$$e^G_\lambda(Q^f_i(\mu)) = e^G_\lambda(Q_i(\mu)) - e^G_\lambda(Q^t_i(\mu)).$$

Now exactly as in the proof of Theorem 4.1 the long exact sequences involved in the above diagram give (note also that all the terms in the following expressions have to do with finite $\mathbb{Z}$-modules)

$$
\begin{aligned}
e^G_\lambda(Q^t_i(\mu)) = & \sum_{j\le i}(-1)^{j-i-1}(e^G_\lambda(H^j_{\mathbb{Z}}(\mu_i)_t) - e^G_\lambda(H^{j-1}_{\mathbb{Z}}(\mu_{i-1})_t)) \\
& + \sum_{j<i-1}(-1)^{j-i}e^G_\lambda(H^j_{\mathbb{Z}}(Q_{\alpha_i}(\mu_{i-1})))
\end{aligned}
$$

and

$$
\begin{aligned}
e^G_\lambda(Q_i(\mu)) = & \sum_{j>i}(-1)^{j-i}(e^G_\lambda(H^j_{\mathbb{Z}}(\mu_i)) - e^G_\lambda(H^{j-1}_{\mathbb{Z}}(\mu_{i-1}))) \\
& - \sum_{j\ge i-1}(-1)^{j-i}e^G_\lambda(H^j_{\mathbb{Z}}(Q_{\alpha_i}(\mu_{i-1}))).
\end{aligned}
$$

When we combine these two equations we obtain

$$e^G_\lambda(Q^f_i(\mu)) = (-1)^i(e^t_i - e^t_{i-1} - e^B_\lambda(Q_{\alpha_i}(\mu_{i-1}))), \tag{4.5}$$

where we have set $e^t_r = \sum_{j\ge 0}(-1)^j e^G_\lambda(H^j_{\mathbb{Z}}(\mu_r)_t)$, $r = 0, 1, 2, \ldots, N$. When we now sum over i in (4.5) we obtain the desired equality since the e^t_i cancel each other leaving only e^t_0 and e^t_N. Both these are 0 by Kempf's theorem, see Section 3.4.

4.7. **Two lemmas.** We still use the notation from 4.5. Now we shall combine Proposition 4.3 with Theorem 4.1. Recall from (3.5) that the weights of $Q_{\alpha_i}(\mu_{i-1})$ are $\mu_{i-1} - \alpha_i, \mu_{i-1} - 2\alpha_i, \ldots, \mu_{i-1} - (r_i - 1)\alpha_i$ where $r_i = \langle \mu_{i-1}, \alpha_i^\vee \rangle$. All weight spaces are cyclic and the order of $Q_{\alpha_i}(\mu_i)_{\mu_{i-1}-m\alpha_i}$ is $\binom{r_i}{m}$, $m = 1, 2, \ldots, r_i - 1$.

Lemma 4.4. *Let $\lambda \in X^+$ and $x, y \in W$. Suppose that both $x \cdot \lambda$ and $y \cdot \lambda$ are weights of $Q_{\alpha_i}(\mu_{i-1})$. Then either $x = y$ or $x = s_i y$.*

Proof: Suppose $x \cdot \lambda = \mu_{i-1} - m\alpha_i$ and $y \cdot \lambda = \mu_{i-1} - m'\alpha_i$ with $0 < m, m' < r_i$. Then $x \cdot \lambda = y \cdot \lambda + (m' - m)\alpha_i$. Hence

$$(\lambda + \rho, \lambda + \rho) = (\lambda + \rho, \lambda + \rho) + (\alpha_i, \alpha_i)(m' - m)(\langle y(\lambda + \rho), \alpha_i^\vee \rangle + (m' - m))$$

and we conclude that either $m' = m$ or $m - m' = \langle y(\lambda + \rho), \alpha_i^\vee \rangle$. In the first case $y \cdot \lambda = x \cdot \lambda$ and therefore $y = x$. In the second case we get $s_i y \cdot \lambda = y \cdot \lambda - \langle y(\lambda + \rho), \alpha_i^\vee \rangle \alpha_i = y \cdot \lambda - (m - m')\alpha_i = x \cdot \lambda$, i.e., $y = s_i x$.

Lemma 4.5. *Let $\lambda \in X^+$. Suppose there exist $x \in W$ and $0 < m < r_i$ with $x \cdot \lambda = \mu_{i-1} - m\alpha_i$. Then*

$$\sum_{w \in W} (-1)^{l(w)} \operatorname{div}(Q_{\alpha_i}(\mu_{i-1})_{w \cdot \lambda}) = (-1)^{l(x)}(\operatorname{div}(r_i + 1 - m) - \operatorname{div}(m)).$$

Proof: This follows from the Lemma 4.4 together with the observation that for all $r \geq m \geq 0$ we have

$$\operatorname{div}(\binom{r}{m}) - \operatorname{div}(\binom{r}{r + 1 - m}) = \operatorname{div}(r + 1 - m) - \operatorname{div}(m).$$

Note in particular that the lemma holds also when $m = 1$ (in which case $s_i x \cdot \lambda$ is not a weight of $Q_{\alpha_i}(\mu_i)$).

4.8. **An Euler type formula.** Let $\lambda, \mu \in X^+$. For each $\beta \in R^+$ we set

$$V_\beta(\lambda, \mu) = \{(x, m) \mid x \in W, 0 < m < \langle \mu + \rho, \beta^\vee \rangle \text{ with } x \cdot \lambda = \mu - m\beta\}.$$

With this notation we have

Theorem 4.6. *The cokernel $Q(\mu)$ of the canonical homomorphism $\Delta_{\mathbb{Z}}(\mu) \to \nabla_{\mathbb{Z}}(\mu)$ satisfies*

$$e_\lambda^G(Q(\mu)) = -\sum_{\beta \in R^+} \sum_{(x,m) \in V_\beta(\lambda,\mu)} (-1)^{l(x)} \operatorname{div}(m).$$

Proof: When we combine Theorem 4.1 a) and Proposition 4.3 we get

$$e_\lambda^G(Q(\mu)) = \sum_{i=1}^{N} (-1)^{i-1} \sum_{w \in W} (-1)^{l(w)} \operatorname{div}(Q_{\alpha_i}(\mu_{i-1})_{w \cdot \lambda}).$$

Note that if we set $\beta_i = s_1 s_2 \cdots s_{i-1}(\alpha_i)$ then $\{\beta_1, \beta_2, \ldots, \beta_n\} = R^+$. Moreover, the equality $x \cdot \lambda = \mu_{i-1} - m\alpha_i$ is equivalent to $s_1 s_2 \cdots s_{i-1} x \cdot \lambda = \mu - m\beta_i$. Also $r_i = \langle \mu_{i-1}, \alpha_i^\vee \rangle = \langle (\mu + \rho, \beta_i^\vee \rangle - 1$. Hence the theorem follows by Lemma 4.4 and Lemma 4.5.

Remark 4.7. The arguments in Lemma 4.4 show that the set $V_\beta(\lambda, \mu)$ is either empty or contains exactly two elements (of the form (x, m) and $(s_\beta x, \langle \mu + \rho, \beta^\vee \rangle - m)$).

4.9. **Variations.** We present some variations of Theorem 4.6 for later use.

Corollary 4.8.

$$e_\lambda^G(\Delta_{\mathbb{Z}}(\mu)) = \sum_{\beta\in R^+} \sum_{(x,m)\in V_\beta(\lambda,\mu)} (-1)^{l(x)} \operatorname{div}(m).$$

Proof: Use Proposition 2.1 with the sequence $0 \to \Delta_{\mathbb{Z}}(\mu) \xrightarrow{c_\mu} \nabla_{\mathbb{Z}}(\mu) \to Q(\mu) \to 0$. (Note that the corollary–as understood by the Remark in Section 4.1–and its proof are valid even for $\lambda = \mu$. We have $e_\lambda^G(\Delta_{\mathbb{Z}}(\lambda)) = 0$, see Proposition 7.1 below.)

Let $\lambda, \mu \in X^+$. For each $\gamma \in R^+$ we set

$$U_\gamma(\lambda,\mu) = \{(w,n) \mid w \in W, n < 0 \text{ or } n > \langle \lambda+\rho, \gamma^\vee\rangle, w\cdot\mu = \lambda - n\gamma\}.$$

With this notation, we can deduce from Theorem 4.6 an alternate expression for $e_\lambda^G(\Delta_{\mathbb{Z}}(\mu))$.

Proposition 4.9.

$$e_\lambda^G(\Delta_{\mathbb{Z}}(\mu)) = \sum_{\gamma\in R^+} \sum_{(w,n)\in U_\gamma(\lambda,\mu)} (-1)^{l(w)} \operatorname{div}(n).$$

Proof: Let $V(\lambda,\mu) = \bigcup_{\beta\in R^+}\{(\beta,x,m) \mid (x,m)\in V_\beta(\lambda,\mu)\}$ and

$$U(\lambda,\mu) = \bigcup_{\gamma\in R^+} \{(\gamma,w,n) \mid (w,n)\in U_\gamma(\lambda,\mu)\}.$$

By Corollary 4.8 it is enough to produce a bijection between $U(\lambda,\mu)$ and $V(\lambda,\mu)$ for which $m = \pm n$ and $x = w^{-1}$. This is an easy check as follows.

First let $(\gamma,w,n)\in U(\lambda,\mu)$. Since $\lambda - n\gamma = w\cdot\mu$, we have $w^{-1}\cdot\lambda = \mu + n(w^{-1}\gamma)$.

Case 1a. If $w^{-1}\gamma \in R^+$ then let $\beta = w^{-1}\gamma$, $x = w^{-1}$ and $m = -n$. We have

$$\langle \mu+\rho, w^{-1}\gamma^\vee\rangle = \langle w^{-1}(\lambda+\rho) - n(w^{-1}\gamma), w^{-1}\gamma^\vee\rangle = \langle\lambda+\rho,\gamma^\vee\rangle - 2n. \tag{4.6}$$

Since $\langle\mu+\rho,\beta^\vee\rangle > 0$ and $\langle\lambda+\rho,\gamma^\vee\rangle > 0$, the possibility $n > \langle\lambda+\rho,\gamma^\vee\rangle$ in the definition of $U_\gamma(\lambda,\mu)$ cannot be true. So $n < 0$ and hence $m = -n > 0$. Also $\langle\mu+\rho,\beta^\vee\rangle = \langle\lambda+\rho,\gamma^\vee\rangle - 2n > -2n = 2m$. So $0 < m < \frac{1}{2}\langle\mu+\rho,\beta^\vee\rangle$; in particular $(\beta,x,m)\in V(\lambda,\mu)$.

Case 1b. If $w^{-1}\gamma \in -R^+$ then let $\beta = -w^{-1}\gamma$, $x = w^{-1}$ and $m = n$. By (4.6) we have $\langle\mu+\rho,\beta^\vee\rangle = 2n - \langle\lambda+\rho,\gamma^\vee\rangle$. Since $\langle\mu+\rho,\beta^\vee\rangle > 0$ and $\langle\lambda+\rho,\gamma^\vee\rangle > 0$, the possibility $n < 0$ in the definition of $U_\gamma(\lambda,\mu)$ cannot be true. So $n > \langle\lambda+\rho,\gamma^\vee\rangle$. Thus $m = n > 0$. Also $\langle\mu+\rho,\beta^\vee\rangle = 2n - \langle\lambda+\rho,\gamma^\vee\rangle > n = m$, as desired. (Since $0 < \langle\lambda+\rho,\gamma^\vee\rangle = 2n - \langle\mu+\rho,\beta^\vee\rangle$, we actually have $\frac{1}{2}\langle\mu+\rho,\beta^\vee\rangle < n = m < \langle\mu+\rho,\beta^\vee\rangle$.)

For the inverse map, let $(\beta,x,m)\in V(\lambda,\mu)$. Since $\mu - m\beta = x\cdot\lambda$, we have $x^{-1}\cdot\mu = \lambda + m(x^{-1}\beta)$.

Case 2a. If $x^{-1}\beta\in R^+$ then let $\gamma = x^{-1}\beta$, $w = x^{-1}$ and $n = -m$. We have $n < 0$ since $m > 0$, so $(\gamma,w,n)\in U(\lambda,\mu)$. Clearly this case is inverse to Case 1a. (m must satisfy the bounds in the last sentence of Case 1a by the calculation there.)

Case 2b. If $x^{-1}\beta \in -R^+$ then let $\gamma = -x^{-1}\beta$, $w = x^{-1}$ and $n = m$. Now via a calculation similar to (4.6) we have $\langle\lambda+\rho,\gamma^\vee\rangle = 2m - \langle\mu+\rho,\beta^\vee\rangle < 2m - m = n$. so $(\gamma,w,n)\in U(\lambda,\mu)$. Clearly this case is inverse to Case 1b (and again the bounds obtained there on m must hold in this case).

Remark. Note that the above bijection pairs $V_\beta(\lambda,\mu)$ and $U_\gamma(\lambda,\mu)$ where $\gamma = x^{-1}\beta$ with $x \in W$ chosen such that $(x,m) \in V_\beta(\lambda,\mu)$ and $x^{-1}\beta \in R^+$. (This is always possible by replacing x with $s_\beta x$ if necessary, see Remark 4.7.)

5. Sum Formulas

5.1. **Sum formula for Weyl modules.** Let $\Delta^i_{\mathbb{Z}}(\mu) = c_\mu^{-1}(p^i\nabla_{\mathbb{Z}}(\mu)) \subset \Delta_{\mathbb{Z}}(\mu)$. Jantzen's filtration is a descending filtration of $\Delta(\mu)$ defined by $\Delta^i(\mu)$ = the G-submodule generated by the image of $\Delta^i_{\mathbb{Z}}(\mu)$ under the canonical map $\Delta_{\mathbb{Z}}(\mu) \to \Delta(\mu)$. We now have Jantzen's sum formula cf. [10] and [1].

Corollary 5.1. *Let ν_p denote the p-adic valuation. Then*

$$\sum_{i>0} \operatorname{ch}(\Delta^i(\mu)) = \sum_{\beta\in R^+} \sum_{0<m<\langle\mu+\rho,\beta^\vee\rangle} \nu_p(m)\chi(\mu - m\beta).$$

Proof: It is well-known that the left hand side is the coefficient of $[p]$ in $\operatorname{div}_T(Q(\mu))$, e.g., diagonalize c_μ and calculate each expression. The result follows by Corollary 4.2b and Theorem 4.6.

5.2. **A filtration associated to tilting modules.** Henceforth in this chapter we let $A = \mathbb{Z}_p$, the ring of p-adic integers. Fix Q, a tilting G_A-module. Also fix $\lambda \in X^+$. Following [4] we define two descending filtrations as follows. First, let $F_\lambda(Q) = \operatorname{Hom}_{G_A}(\Delta_A(\lambda), Q)$. Define

$$F_\lambda(Q)^j = \{\varphi \in F_\lambda(Q) \mid \psi\circ\varphi \in p^j Ac_\lambda \text{ for all } \psi \in \operatorname{Hom}_{G_A}(Q, \nabla_A(\lambda))\},$$

where c_λ now denotes a generator of $\operatorname{Hom}_{G_A}(\Delta_A(\lambda), \nabla_A(\lambda))$. Next, let $\bar{Q} = Q \otimes_A k$. Recall that Q is determined uniquely by $\bar{Q}$. Let $\bar{F}_\lambda(\bar{Q}) = \operatorname{Hom}_G(\Delta(\lambda), \bar{Q}) = \operatorname{Hom}_{G_A}(\Delta_A(\lambda), Q) \otimes_A k$ (by, e.g., universal coefficients and Proposition 2.1 a)). Define $\bar{F}_\lambda(\bar{Q})^j$ = the k-vector space spanned by the image of $F_\lambda(Q)^j$ in $F_\lambda(Q) \otimes_A k$. In the remaining sections we will prove a sum formula for the latter filtration.

5.3. **Homological considerations.** Continue with the notation in 5.2. Following [5], Chapter 1, we relate the desired sum formula to certain Ext groups via an equivalent description of $F_\lambda(Q)^j$. For this, fix an enumeration of dominant weights such that $\lambda_i < \lambda_j$ implies $i < j$. Let $(Q : \Delta_A(\lambda_j)) = n_j$. We will freely use Proposition 2.1 in the following analysis without further mention. A first application gives that Q has a finite filtration $Q = Q_0 \supset Q_1 \supset Q_2 \cdots$ with $Q_{i-1}/Q_i = \Delta_A(\lambda_i)^{n_i}$ for some $n_i \geq 0$. Now fix i such that the chosen $\lambda = \lambda_i$. Consider the two short exact sequences

$$0 \to Q_{i-1} \to Q \to Q/Q_{i-1} \to 0 \quad \text{and} \quad 0 \to Q_i \to Q_{i-1} \xrightarrow{\pi} \Delta_A(\lambda)^{n_i} \to 0. \tag{5.1}$$

Apply $\operatorname{Hom}_{G_A}(\Delta_A(\lambda), -)$ to these. In the first long exact sequence, for $t > 0$, $\operatorname{Ext}^t_{G_A}(\Delta_A(\lambda), Q/Q_{i-1}) = 0$ (since $(Q/Q_{i-1} : \Delta_A(\lambda_j)) = 0$ for any $\lambda_j > \lambda$) and $\operatorname{Ext}^t_{G_A}(\Delta_A(\lambda), Q) = 0$ (since Q has a dual Weyl filtration). Also $\operatorname{Hom}_{G_A}(\Delta_A(\lambda), Q/Q_{i-1}) = 0$, since $(Q/Q_{i-1} : \Delta_A(\lambda)) = 0$. Hence $\operatorname{Ext}^t_{G_A}(\Delta_A(\lambda), Q_{i-1}) = 0$ for $t > 0$ and the entire sequence reduces to the isomorphism $\operatorname{Hom}_{G_A}(\Delta_A(\lambda), Q_{i-1}) \simeq F_\lambda(Q)$.

Next we use this information in the second long exact sequence. Since $\operatorname{Ext}^t_{G_A}(\Delta_A(\lambda), \Delta_A(\lambda)) = 0$ for $t > 0$, we get $\operatorname{Ext}^t_{G_A}(\Delta_A(\lambda), Q_i) = 0$ for $t > 1$. Also $\operatorname{Hom}_{G_A}(\Delta_A(\lambda), Q_i) = 0$ since $(Q_i : \Delta_A(\lambda)) = 0$. So the entire sequence reduces to

$$0 \to F_\lambda(Q) \xrightarrow{\Phi} \operatorname{End}_{G_A}(\Delta_A(\lambda))^{\oplus n_i} \to \operatorname{Ext}^1_{G_A}(\Delta_A(\lambda), Q_i) \to 0. \tag{5.2}$$

Still following [5], we take a closer look at certain maps between several Hom-groups. First, note that $\Phi(\varphi) = \pi \circ \varphi$. (This makes sense since, by the previous paragraph, any map $\varphi \in F_\lambda(Q)$ factors through Q_{i-1}.) Note that Φ is an injection between free A-modules, each of rank n_i. Next, apply $\mathrm{Hom}(-, \nabla_A(\lambda))$ to the two short exact sequences (5.1). Each of the resulting long exact sequences reduces to just Hom-terms. Since $(Q_i : \Delta_A(\lambda)) = 0 = (Q/Q_{i-1} : \Delta_A(\lambda))$, we get isomorphisms $\mathrm{Hom}_{G_A}(Q, \nabla_A(\lambda)) \simeq \mathrm{Hom}_{G_A}(Q_{i-1}, \nabla_A(\lambda)) \simeq \mathrm{Hom}_{G_A}(\Delta_A(\lambda), \nabla_A(\lambda))^{\oplus n_i}$. This sequence of bijections pairs $\psi \in \mathrm{Hom}_{G_A}(Q, \nabla_A(\lambda))$ first with its restriction $\psi|_{Q_{i-1}}$ and then to $\bar{\psi} \in \mathrm{Hom}_{G_A}(\Delta_A(\lambda)^{\oplus n_i}, \nabla_A(\lambda))$ such that $\bar{\psi} \circ \pi = \psi|_{Q_{i-1}}$. So $\psi \circ \varphi = \bar{\psi} \circ \pi \circ \varphi = \bar{\psi} \circ \Phi(\varphi)$. This easily gives (see [5], Proposition 1.6):

$$F_\lambda(Q)^j = \{\varphi \in F_\lambda(Q) \mid \Phi(\varphi) \in p^j \,\mathrm{End}_{G_A}(\Delta_A(\lambda))^{\oplus n_i}\}. \tag{5.3}$$

5.4. **A sum formula involving tilting modules.** Keep the notation from 5.2 and 5.3. Additionally, for arbitrary $\xi \in X$, we make the following notation. If there exists $w \in W$ with $\mu = w \cdot \xi$ dominant, define $[Q : \chi(\xi)] = (-1)^{\ell(w)}(Q : \Delta_A(\mu))$. Otherwise let $[Q : \chi(\xi)] = 0$. This makes sense by Theorem 3.1. We now prove the following sum formula, which was discovered (and proved when $p \geq h$) in [5].

Theorem 5.2.

$$\sum_{j>0} \dim \bar{F}_\lambda(\bar{Q})^j = -\sum_{\alpha \in R^+} \sum_{n<0 \text{ or } n>\langle \lambda+\rho, \alpha^\vee\rangle} \nu_p(n)[\bar{Q} : \chi(\lambda - n\alpha)].$$

Proof: From (5.2) and (5.3) it is standard (e.g., by diagonalizing Φ) to see that

$$\sum_{j>0} \dim \bar{F}_\lambda(\bar{Q})^j = \nu_p(\mathrm{Ext}^1_{G_A}(\Delta_A(\lambda), Q_i)).$$

Since $\mathrm{Ext}^t_{G_A}(\Delta_A(\lambda), Q_i) = 0$ for $t \neq 1$, we have

$$\nu_p(\mathrm{Ext}^1_{G_A}(\Delta_A(\lambda), Q_i)) = -\sum_t (-1)^t \nu_p(\mathrm{Ext}^t_{G_A}(\Delta_A(\lambda), Q_i)).$$

Recall that $(Q_i : \Delta(\lambda_j))$ is n_j if $j > i$ and 0 otherwise. So

$$\sum_t (-1)^t \nu_p(\mathrm{Ext}^t_{G_A}(\Delta_A(\lambda), Q_i)) = \sum_{j>i} n_j \sum_t (-1)^t \nu_p(\mathrm{Ext}^t_{G_A}(\Delta_A(\lambda), \Delta_A(\lambda_j))).$$

(Note that all the Hom-terms in the previous equation are zero, so additivity of Euler characteristic holds.) The last alternating sum in the preceding equation may be replaced by the coefficient of $[p]$ in $e^G_\lambda(\Delta_{\mathbb{Z}}(\lambda_j))$. Then we may take the outer sum over all j as $e^G_\lambda(\Delta_{\mathbb{Z}}(\lambda_j)) = 0$ for $j \leq i$. Altogether we have

$$\sum_{j>0} \dim \bar{F}_\lambda(\bar{Q})^j = -\text{ the coefficient of } [p] \text{ in } \sum_j n_j e^G_\lambda(\Delta_{\mathbb{Z}}(\lambda_j)).$$

The result follows by Proposition 4.9.

6. Quantum Groups

6.1. **Passing to the quantum case.** The sum formulas Corollary 5.1 and Theorem 5.2 have direct analogues for quantum groups at roots of 1, see [2], [17] and [5]. We shall show in this section that our approach above carries over to the quantum case. In particular, this allows us to get rid of the condition in loc. cit. that the order of the root of unity must be at least equal to the Coxeter number. In the Weyl module case the reason for this restriction was that the quantized Kempf vanishing theorem had only been proved in that case (see [2] and [3]). This restriction was removed by Ryom-Hansen's general proof [9]. In the tilting module case the reason for the restriction was that the proof in [5] required a regular weight as its starting point.

We carefully set up the quantized version of the approach in Sections 6.2–5. Once this is done the arguments are completely parallel and we shall leave to the reader the task of repeating the proofs leading to the quantized versions of the sum formulas.

6.2. **The quantum parameter.** Throughout this section k will denote an arbitrary field. We set $p = \mathrm{char}(k) \geq 0$. For technical reasons we need $p \neq 2$ and also that $p \neq 3$ if the root system in question contains type G_2. Then we fix a root of unity $q \in k$ of order l or $2l$ with $l \in \mathbb{N}$ odd.

We let v denote an indeterminate and set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, $A = k[v, v^{-1}]$. The natural homomorphism $\mathcal{A} \to A$ mapping $v \in \mathcal{A}$ to $v \in A$ makes A into an $\mathcal{A}$-algebra. We make k into an $\mathcal{A}$-algebra by specializing v to q. Of course so far q could be any non-zero element in k but as we shall see the only interesting case for our present purposes is when q is a root of unity.

6.3. **Roots and weights.** As in Section 2.1 we denote by R a (finite) root system and we choose a set of positive roots R^+. This takes place in some euclidian space $E = \mathbb{R}^n$ and we let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an enumeration of the set of simple roots $S \subset R^+$. Moreover, we denote by $X \subset E$ the set of integral weights, i.e.,

$$X = \{\lambda \in E \mid \langle \lambda, \beta^\vee \rangle \in \mathbb{Z},\ \beta \in R\}.$$

Then $X \simeq \mathbb{Z}^n$. As before we set X^+ equal to the set of dominant weights in X.

The Weyl group W of R acts naturally on E and X. Again we also have the dot-action given by $w \cdot \lambda = w(\lambda + \rho) - \rho,\ w \in W, \lambda \in E$ with $\rho = \frac{1}{2}\sum_{\beta \in R^+} \beta$.

6.4. **Quantum groups over k.** Let U denote the quantum group over $\mathbb{Q}(v)$ associated with R. This is the $\mathbb{Q}(v)$-algebra defined by some generators $E_i, F_i, K_i^{\pm},\ i = 1, 2, \dots, n$ and certain relations, see e.g. [16]. It has a triangular decomposition $U = U^- U^0 U^+$ with U^-, respectively U^0, U^+ denoting the subalgebra generated by all F_i's, respectively $K_i^{\pm}$'s, E_i's.

Inside U we have an $\mathcal{A}$-subalgebra $U_{\mathcal{A}}$, the Lusztig $\mathcal{A}$-form of U. It is defined via the (gaussian) divided powers $E_i^{(m)}$ and $F_i^{(m)}$, $m \in \mathbb{N}$, $i = 1, 2, \dots, n$, see [16]. Then for each $\mathcal{A}$-algebra A' we set $U_{A'} = U_{\mathcal{A}} \otimes_{\mathcal{A}} A'$ and call this the quantum group over A' associated with R. In particular, $U = U_{\mathbb{Q}(v)}$ when $\mathbb{Q}(v)$ is given the natural $\mathcal{A}$-structure obtained by sending v to v. In the case where $A' = k$ with $\mathcal{A}$-structure as above we often write U_q instead of U_k.

The above triangular decomposition of U generalizes to $U_{A'} = U_{A'}^- U_{A'}^0 U_{A'}^+$ for appropriate A'-subalgebras $U_{A'}^-, U_{A'}^0$, and $U_{A'}^+$, see [16]. We set $B_{A'} = U_{A'}^- U_{A'}^0$. Again we write B_q instead of B_k.

6.5. **Integrable modules and induction functors.** Let $\lambda \in X$. Then for any A' as above λ gives rise to a character $\chi_\lambda : U_{A'}^0 \to A'$ which extends uniquely to a character of $B_{A'}$ (taking all $F_i^{(m)}$'s to 0). Then if M is a $U_{A'}^0$-module we set

$$M_\lambda = \{m \in M \mid um = \chi_\lambda(u)m,\ u \in U_{A'}^0\}$$

and call this the λ-weight space in M.

We denote by $\mathcal{C}_{A'}$, respectively $\mathcal{C}_{A'}^-$ the category consisting of all integrable $U_{A'}$-, respectively $B_{A'}$-modules. A module is integrable if it splits into a direct sum of its weight spaces (as $U^0_{A'}$-module) and all high enough divided powers of all relevant generators vanish on any given element in the module, see [2].

We have a natural induction functor $\mathrm{Ind}_{B_{A'}}^{U_{A'}} : \mathcal{C}^-_{A'} \to \mathcal{C}_{A'}$, see [2]. As in loc. cit. we shall denote the right derived functors of this functor by $H^j_{A'}$, $j \geq 0$. These functors share many of the properties of the G/B-sheaf cohomology functors from Section 2.3. In particular, we have (cf. [2] Theorem 5.8)

(1) If $M \in \mathcal{C}^-_{A'}$ is finitely generated as an A'-module then each $H^j_{A'}(M) \in \mathcal{C}_{A'}$ is also finitely generated over A',

(2) $H^j_{A'} = 0$ for all $j > N$

(as before N denotes the number of positive roots).

6.6. **Weyl and dual Weyl modules.** Keep the notation from above and fix now $\mu \in X^+$. Then we set

$$\Delta_{A'}(\mu) = H^N_{A'}(w_0 \cdot \mu) \text{ and } \nabla_{A'}(\mu) = H^0_{A'}(\mu).$$

We call these the Weyl module and the dual Weyl module for $U_{A'}$ with highest weight μ. Because of the quantized Kempf's vanishing theorem (which was proved for special A''s in [2] and in general by Ryom-Hansen in [9])we have in analogy with Section 3.4

$$\nabla_{A'}(\mu) = \nabla_{\mathcal{A}}(\mu) \otimes_{\mathcal{A}} A'. \tag{6.1}$$

Since $H^{N+1}_{\mathcal{A}} = 0$ we also have

$$\Delta_{A'}(\mu) = \Delta_{\mathcal{A}}(\mu) \otimes_{\mathcal{A}} A'. \tag{6.2}$$

Just as in Section 3.4 we get quantized Weyl character formulas

$$\mathrm{ch}\,\Delta_q(\mu) = \mathrm{ch}\,\nabla_q(\mu) = \chi(\lambda). \tag{6.3}$$

6.7. **Simple modules.** We shall now consider the case $A' = k$. Then each $\nabla_q(\mu)$ (again we use index q instead of k here) contains a unique simple submodule which we denote $L_q(\mu)$, cf. [2]. The family $\{L_q(\mu)\}_{\mu \in X^+}$ is then up to isomorphisms the set of simple modules in $\mathcal{C}_q$ (and in fact this family and their sign-twists constitute all finite dimensional simple U_q-modules, see loc. cit.).

Serre duality gives that $L_q(\mu)$ is also the unique simple quotient of $\Delta_q(\mu)$. In fact, we have up to scalars a unique homomorphism $c_q(\mu) : \Delta_q(\mu) \to \nabla_q(\mu)$ and the image is $L_q(\mu)$. This homomorphism may be obtained by specialization from a generator $c_{\mathcal{A}}(\mu)$ of $\mathrm{Hom}_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\mu), \nabla_{\mathcal{A}}(\mu))$. We shall now study the corresponding homomorphism $c_\mu = c_A(\mu) \in \mathrm{Hom}_{\mathcal{C}_A}(\Delta_A(\mu), \nabla_A(\mu))$ (obtained from $c_{\mathcal{A}}(\mu)$ by the base change $\mathcal{A} \to A$) just as we studied the corresponding homomorphism in Section 4.5.

Remark. If q was not a root of unity then we would have $\nabla_q(\mu) = L_q(\mu) = \Delta_q(\mu)$ for all $\mu \in X^+$, and $\mathcal{C}_q$ would be semisimple, see [2].

6.8. **Rank 1.** Fix $i \in \{1, 2, \ldots, n\}$. Then we set $P_A(i)$ equal to the A-subalgebra of U_A generated by B_A and all $E_i^{(n)}$, $n \in \mathbb{N}$, see [2]. We let $H^0_{A,i}$ denote the induction functor from $\mathcal{C}^-_A$ to $\mathcal{C}(P_A(i))$ where this last category consists of all integrable $P_A(i)$-modules. The right derived functors are denoted $H^j_{A,i}$, $j \geq 0$. Then we have the following analogue of Lemma 3.6.

Lemma 6.1 (cf. [2] Section 4)**.** *Let* $\lambda \in X$*.*

a) If $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ *then* $H^j_{A,i}(\lambda) = 0$ *for all* $j > 0$ *and* $H^0_{A,i}(\lambda)$ *is a free* A*-module whose weights are* $\lambda, \lambda - \alpha_i, \dots, s_{\alpha_i}(\lambda)$*, all occurring with multiplicity* 1*.*

b) If $\langle \lambda, \alpha_i^\vee \rangle < -1$ *then* $H^j_{A,i}(\lambda) = 0$ *for all* $j \neq 1$ *and* $H^1_{A,i}(\lambda)$ *is a free* A*-module whose weights are* $\lambda + \alpha_i, \lambda + 2\alpha_i, \dots, s_{\alpha_i} \cdot \lambda$*, all occurring with multiplicity* 1*.*

c) If $\langle \lambda, \alpha_i^\vee \rangle = r \geq 0$ *then* $\mathrm{Hom}_{P_A(i)}(H^1_{A,i}(s_{\alpha_i} \cdot \lambda), H^0_{A,i}(\lambda)) \simeq A$*. Moreover,* $H^0_{A,i}(\lambda)$*, respectively* $H^1_{A,i}(s_{\alpha_i} \cdot \lambda)$ *has a standard* A*-basis* $\{v_0, v_1, \dots, v_r\}$*, respectively* $\{v'_0, v'_1, \dots, v'_r\}$ *with* v_j*, respectively* v'_j*, having weight* $\lambda - j\alpha_i$*,* $j = 0, 1, \dots, r$*. A generator* $c_i(\lambda)$ *of* $\mathrm{Hom}_{P_{A,i}}(H^1_{A,i}(s_{\alpha_i} \cdot \lambda), H^0_{A,i}(\lambda))$ *is given by (with* $(d_1, d_2, \dots, d_n)$ *being a minimal* n*-tuple in* $\mathbb{N}$ *making the Cartan matrix for* R *symmetric)*

$$c_i(\lambda)(v'_j) = \begin{bmatrix} r \\ j \end{bmatrix}_{d_i} v_j, \; j = 0, 1, \dots, r.$$

The gaussian binomial coefficients $\begin{bmatrix} r \\ j \end{bmatrix}_{d_i}$ occurring in c) are defined like the usual binomial numbers with each integer $m \in \mathbb{N}$ replaced by $[m]_{d_i} = \frac{v^{d_i m} - v^{-d_i m}}{v^{d_i} - v^{-d_i}}$. If $d_i = 1$ we omit this subscript.

6.9. **Divisors.** Let $\mathcal{D}(A)$ denote the divisor group for A, i.e., the free $\mathbb{Z}$-module with basis consisting of all irreducible polynomials in A with leading coefficients equal to 1. If $a \in A \setminus \{0\}$ then we write $\mathrm{div}(a) \in \mathcal{D}(A)$ for the divisor associated with a. The coefficient corresponding to $v - q$ in $\mathrm{div}(a)$ we shall denote $\mathrm{div}_q(a)$.

The following formulas are easy exercises, see [17], Lemma 5.2: If $\mathrm{char}\, k = 0$ we have

$$\mathrm{div}_q([m]) = \begin{cases} 1 & \text{if } l \text{ divides } m, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

If $\mathrm{char}\, k = p > 0$ we have

$$\mathrm{div}_q([m]) = \begin{cases} p^{\nu_p(m)} & \text{if } l \text{ divides } m, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Also if M is a finitely generated torsion A-module then $M \simeq \bigoplus_i A/(a_i)$ for some $a_i \in A$ and we write $\mathrm{div}(M) = \sum_i \mathrm{div}(a_i)$. Clearly, div is then additive on short exact sequences of finitely generated torsion A-modules. Again $\mathrm{div}_q(M)$ picks out the coefficient in $\mathrm{div}(M)$ corresponding to $v - q$.

If M is a U^0_A-module which is a direct sum of its weight spaces M_μ, and if M is a finitely generated torsion A-module then we define $\mathrm{div}_{U^0}(M) \in \mathcal{D}(A)[X]$ by

$$\mathrm{div}_{U^0}(M) = \sum_{\mu \in X} \mathrm{div}(M_\mu) e^\mu.$$

6.10. **Euler type formulas.** We have now reached the point where we can just mimic what we did in Chapter 4. In particular, for each $\lambda \in X^+$ and for any $V \in \mathcal{C}_A$ which is a finitely generated torsion A-module we define

$$e^U_\lambda(V) = \sum_{i \geq 0} (-1)^i \,\mathrm{div}(\mathrm{Ext}^i_{\mathcal{C}_A}(\Delta_A(\lambda), V)).$$

Note that just as we had finiteness results for H^j_A in Section 6.5 we also have such results for $\mathrm{Ext}^j_{\mathcal{C}_A}$ so that this definition makes sense.

Likewise if $M \in \mathcal{C}_A^-$ is a finitely generated torsion A-module then

$$e_\lambda^B(M) = \sum_{j \geq 0} (-1)^j e_\lambda^U(H_A^j(M)).$$

Then the direct analogue of Theorem 4.1 holds with the same proof: for a) we first reduce to the case (corresponding to (4.2)) where M is determined by the exact sequence

$$0 \to A_\mu \xrightarrow{a} A_\mu \to M \to 0$$

with $\mu \in X$ and $a \in A \setminus \{0\}$. Then we proceed as in Section 4.3. Also the proof of b) is a direct translation.

The next step is for a fixed $\mu \in X^+$ to compute $e_\lambda^U(Q(\mu))$ where $Q(\mu)$ is the cokernel of the homomorphism c_μ discussed in Section 6.7. Just as in Section 4.5 we factorize $c_\mu = \tilde{c}_1 \circ \tilde{c}_2 \circ \cdots \circ \tilde{c}_N$ (relative to some reduced decomposition of w_0) and then proceed as in Section 4.6.

Lemma 6.1 c) tells us that the cokernel $Q_i(\lambda)$ of $c_i(\lambda)$ has weights $\lambda - \alpha_i, \lambda - \alpha_i, \ldots, s_{\alpha_i} \cdot \lambda$ and that the weight space $Q_i(\lambda)_{\lambda - j\alpha_i}$ equals $A/(\begin{bmatrix} r \\ j \end{bmatrix}_{d_i})$ with $r = \langle \lambda, \alpha_i^\vee \rangle$.

All this leads exactly as in Sections 4.7-8 to the following

Theorem 6.2. *Let $\lambda, \mu \in X^+$. The cokernel $Q(\mu)$ of the canonical homomorphism $\Delta_A(\mu) \to \nabla_A(\mu)$ satisfies*

$$e_\lambda^U(Q(\mu)) = -\sum_{\beta \in R^+} \sum_{(x,m) \in V_\beta(\lambda,\mu)} (-1)^{l(x)} \operatorname{div}([m]_{d_\beta}).$$

6.11. **Sum formulas for quantized Weyl modules.** We now deduce sum formulas by proceeding as in Chapter 5. We first define for $\mu \in X^+$ a filtration of $\Delta_A^i(\mu)$ of $\Delta_A(\mu)$ by setting $\Delta_A^i(\mu) = c_\mu^{-1}((v-q)^i \nabla_A(\mu))$. The quantized Jantzen's filtration is then the descending filtration of $\Delta_q(\mu)$ defined by setting $\Delta_q^i(\mu)$ equal to the image of $\Delta_A^i(\mu)$ under the canonical projection $\Delta_A(\mu) \to \Delta_A(\mu) \otimes_A k \simeq \Delta_q(\mu)$.

Taking into account the identities (6.4-5) we now get

Theorem 6.3. *Let $\mu \in X^+$.*

a) Assume $\operatorname{char} k = 0$. *Then*

$$\sum_{i>0} \operatorname{ch}(\Delta_q^i(\mu)) = \sum_{\beta \in R^+} \sum_{0 < m < \langle \mu + \rho, \beta^\vee \rangle} \chi(\mu - ml\beta).$$

b) Assume $\operatorname{char} k = p > 0$. *Then*

$$\sum_{i>0} \operatorname{ch}(\Delta_q^i(\mu)) = \sum_{\beta \in R^+} \sum_{0 < m < \langle \mu + \rho, \beta^\vee \rangle} p^{\nu_p(m)} \chi(\mu - ml\beta).$$

(As in Chapter 5 ν_p denotes the p-adic valuation.)

6.12. **Sum formulas for quantized tilting modules.** Tilting modules for U_q are defined in direct analogy with the way it was done for G. This means that a finite dimensional U_q-module Q is tilting if it has a filtration where the quotients are Weyl modules $\Delta_q(\lambda)$ as well as a filtration where the quotients are dual Weyl modules $\nabla_q(\lambda)$. Moreover, each such tilting module Q has a unique lift to a tilting module $\tilde{Q}$ for $U_{\tilde{A}}$ where $\tilde{A}$ denotes the localization $A_{(v-q)}$ of A at the maximal ideal generated by $v - q$.

For each $\lambda \in X^+$ we set $\bar{F}_\lambda(Q) = \mathrm{Hom}_{U_q}(\Delta_q(\lambda), Q)$ and $F_\lambda(\tilde{Q}) = \mathrm{Hom}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(\lambda), \tilde{Q})$. Then $F_\lambda(\tilde{Q}) \otimes_{\tilde{A}} k \simeq \bar{F}_\lambda(Q)$.

We define a filtration of $F_\lambda(\tilde{Q})$ consisting of the $\tilde{A}$-submodules

$$F_\lambda(\tilde{Q})^j = \{\phi \in F_\lambda(\tilde{Q}) \mid \psi \circ \phi \in (v-q)^j \tilde{A} c_\lambda \text{ for all } \psi \in \mathrm{Hom}_{U_{\tilde{A}}}(\tilde{Q}, \nabla_{\tilde{A}}(\lambda))\}.$$

The image in $\bar{F}_\lambda(Q)$ of this filtration is then a k-space filtration whose j'th term we denote $\bar{F}_\lambda(Q)^j$.

Using notation analogous to the one in Chapter 5 we now have the sum formulas, cf. [5]

Theorem 6.4. *Let Q be a tilting module for U_q and let $\lambda \in X^+$.*

a) Assume $\mathrm{char}\, k = 0$. *Then*

$$\sum_{j>0} \dim \bar{F}_\lambda(Q)^j = -\sum_{\alpha \in R^+} \sum_{n<0 \text{ or } nl > \langle \lambda+\rho, \alpha^\vee \rangle} [Q : \chi(\lambda - nl\alpha)].$$

b) Assume $\mathrm{char}\, k = p > 0$. *Then*

$$\sum_{j>0} \dim \bar{F}_\lambda(Q)^j = -\sum_{\alpha \in R^+} \sum_{n<0 \text{ or } nl > \langle \lambda+\rho, \alpha^\vee \rangle} p^{\nu_p(n)} [Q : \chi(\lambda - nl\alpha)].$$

7. Root subsets. Examples.

In this section we have collected some remarks and examples concerning the sets $V(\lambda,\mu)$ and $U(\lambda,\mu)$ occurring in Chapter 4. These sets play important roles in our proof of the sum formulas. Even though they are defined in a completely elementary way they are somewhat complicated to describe explicitly. Fixing distinct dominant weights λ and μ we will explore the implications of the following key condition involved in the definition of these sets.

$$\lambda - n\gamma = w \cdot \mu \text{ for some } n \in \mathbb{Z}, \gamma \in R \text{ and } w \in W. \tag{7.1}$$

Note that this condition is symmetric in λ and μ. Further, by switching the signs of n and γ if necessary, we may require $\gamma \in R^+$, but we prefer not to do so here. Instead we define, again for distinct $\lambda, \mu \in X^+$,

$$S(\lambda,\mu) = \{\gamma \in R^+ \mid \lambda - n\gamma = w \cdot \mu \text{ for some } n \in \mathbb{Z} \text{ and } w \in W\}. \tag{7.2}$$

7.1. **Alternative descriptions of $V(\lambda,\mu)$ and $U(\lambda,\mu)$.** We will show that (7.1) forces $\lambda < \mu$ or $\mu < \lambda$, leading to the following result.

Proposition 7.1. *For $\lambda, \mu \in X^+$ the sets $U(\lambda,\mu)$ and $V(\lambda,\mu)$ are empty unless $\lambda < \mu$. If $\lambda < \mu$, then*

$$U(\lambda,\mu) = \{(\gamma, w, n) \mid \gamma \in R^+, w \in W, n \in \mathbb{Z}, \lambda - n\gamma = w \cdot \mu\},$$
$$V(\lambda,\mu) = \{(\beta, x, m) \mid \beta \in R^+, x \in W, m \in \mathbb{Z}, \mu - m\beta = x \cdot \lambda\}.$$

Proof: Throughout the proof suppose that $w \cdot \mu = \lambda - n\gamma$ with $\gamma \in R^+$. So $\langle \mu+\rho, w^{-1}\gamma^\vee \rangle = \langle \lambda+\rho, \gamma^\vee \rangle - 2n$ as in (4.6). Note that $\lambda, \mu \in X^+$ implies $w \cdot \mu \leq \mu$ and $w^{-1} \cdot \lambda \leq \lambda$ with equalities iff w is the identity.

It will be convenient to prove the claims for $U(\lambda,\mu)$ and $V(\mu,\lambda)$ simultaneously. (Note the interchanged roles of λ and μ in the latter set.) It suffices to show that one must have $\lambda < \mu$, $\lambda = \mu$ or $\mu < \lambda$ appropriately depending on the value of n.

Case I. If $n < 0$, then $\lambda < \lambda - n\gamma = w \cdot \mu \leq \mu$.

Case II. If $n = 0$, then $w =$ the identity and $\mu = \lambda$.

Case III. If $0 < n < \frac{1}{2}\langle \lambda + \rho, \gamma^\vee \rangle$, then $\langle \mu + \rho, w^{-1}\gamma^\vee \rangle > 0$. So $w^{-1}\gamma \in R^+$ and hence $\mu < \mu + n(w^{-1}\gamma) = w^{-1} \cdot \lambda \leq \lambda$.

$n = \frac{1}{2}\langle \lambda + \rho, \gamma^\vee \rangle$ is impossible, e.g., because that would mean $\langle \mu + \rho, w^{-1}\gamma^\vee \rangle = 0$.

To deal with the remaining possibilities, we use $s_\gamma w \cdot \mu = \lambda - (\langle \lambda + \rho, \gamma^\vee \rangle - n)\gamma$ (see Remark 4.7). If $\frac{1}{2}\langle \lambda + \rho, \gamma^\vee \rangle < n < \langle \lambda + \rho, \gamma^\vee \rangle$, one reduces to Case III and concludes that $\mu < \lambda$. Similarly if $n = \langle \lambda + \rho, \gamma^\vee \rangle$, then $\mu = \lambda$ via Case II and if $n > \langle \lambda + \rho, \gamma^\vee \rangle$, then $\lambda < \mu$ via Case I.

7.2. **Explicit determination of the sets $U(\lambda, \mu)$ and $V(\lambda, \mu)$.** We start by making some easy reductions towards computing these sets. First, we remark that it is enough to determine the sets $S(\lambda, \mu)$ defined in (7.2). By the proof of Proposition 7.1, $S(\lambda, \mu)$ is nonempty precisely when $U(\lambda, \mu) \cup V(\mu, \lambda)$ is nonempty. If this happens, exactly one of the sets in the union is nonempty (depending on whether $\lambda < \mu$ or $\mu < \lambda$). Then, by Remark 4.7, the size of this set is $2|S(\lambda, \mu)|$. Further, we will see that in all our examples, for each $\gamma \in S(\lambda, \mu)$, the two associated values of n (and the corresponding $w \in W$) are easy to determine.

Next, we reduce to the case of irreducible root systems. Note that $S(\lambda, \mu)$ is described directly in terms of the root system R. Clearly, for (7.1) to hold, λ and μ must differ only in the component of R to which γ belongs. In particular the largest possible cardinality of $S(\lambda, \mu)$ for R is the maximum of this cardinality for the irreducible components of R. In the remaining sections we will describe all the different possibilities that can occur for (simply connected almost simple) groups of classical types A, B, C and D. We start by summarizing part of the findings.

Proposition 7.2. *When nonempty, the sets $V(\lambda, \mu)$ and $U(\lambda, \mu)$ have cardinality* 2 *for type A_m, cardinality* 2 *or* 4 *for types D_m ($m > 3$) and B_2, and cardinality* 2, 4 *or* 6 *for types B_m and C_m ($m > 2$).*

Looking at (7.1), it makes sense that the sets in question are smaller for sparser root systems. For type G_2 one can check that the cardinality of these sets is again 0, 2, 4 or 6. We did not work out the types F_4, E_6, E_7 and E_8.

7.3. **Notation.** Let us fix some notation that will be in force throughout the remaining sections. We will realize the classical root systems in standard ways (recalled below) in $\mathbb{R}^m$. We will use a fixed orthonormal basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ for $\mathbb{R}^m$. Then the set of weights X is a subset of $\mathbb{R}^m$. For any $\lambda \in X$ we set

$$\lambda + \rho = \sum_{i=1}^{m} \lambda_i \epsilon_i \quad \text{and} \quad I_\lambda = \{\lambda_1, \ldots, \lambda_m\}.$$

Moreover, if also $\mu \in X$ we define the difference sets

$$D_{\lambda\mu} = I_\lambda \setminus I_\mu \quad \text{and} \quad D_{\mu\lambda} = I_\mu \setminus I_\lambda.$$

In the following we fix two dominant weights λ and μ and for the classical types we describe the set $S(\lambda, \mu)$. Since I_λ and I_μ both have cardinality m, we get $|D_{\lambda\mu}| = |D_{\mu\lambda}|$. Nonemptiness of $S(\lambda, \mu)$ will be characterized by these set differences having cardinality exactly 1 or 2 along with some easy numerical conditions.

7.4. **Type A.** For $m > 1$, we realize R of type A_{m-1} as the subset $R = \{\epsilon_i - \epsilon_j \mid i \neq j, 1 \leq i, j \leq m\}$ of $\mathbb{R}^m$ with positive roots defined by the condition $i < j$. A vector $\sum_{i=1}^m q_i \epsilon_i$ is a weight precisely when each $q_i - q_j \in \mathbb{Z}$ and $\sum_{i=1}^m q_i = 0$ (i.e., when, for some $t \in \mathbb{Z}$, every $q_i \in \frac{t}{m} + \mathbb{Z}$.) The Weyl group acts on $\mathbb{R}^m$ by permuting the ϵ_i. So using the notation from Section 7.3 we see that the W-orbit (under the 'dot' action) of any weight η consists of those $\eta' \in X$ for which $I_\eta = I_{\eta'}$. Note that $\eta \in X^+$ is equivalent to the condition $\eta_1 > \eta_2 > \cdots > \eta_m$.

Now assume that (7.1) holds for (our fixed and distinct $\lambda, \mu \in X^+$ and) $\gamma = \epsilon_a - \epsilon_b$. This is equivalent to having

$$I_\mu = (I_\lambda \setminus \{\lambda_a, \lambda_b\}) \cup \{\lambda_a - n, \lambda_b + n\},$$

which implies that $D_{\lambda\mu} = \{\lambda_a, \lambda_b\}$. Thus a positive γ is uniquely determined by λ and μ. Morover, if $D_{\mu\lambda} = \{\mu_c, \mu_d\}$, then one must have $n = \lambda_a - \mu_c = \mu_d - \lambda_b$ or $n = \lambda_a - \mu_d = \mu_c - \lambda_b$.

Conversely, we always have $|S(\lambda, \mu)| = 0$ or 1. The latter occurs precisely when $|D_{\lambda\mu}| = |D_{\mu\lambda}| = 2$. In that case, letting $D_{\lambda\mu} = \{\lambda_a, \lambda_b\}$ and $D_{\mu\lambda} = \{\mu_c, \mu_d\}$, we automatically have $\lambda_a + \lambda_b = \mu_c + \mu_d$ (since $\sum_{i=1}^m \lambda_i = \sum_{i=1}^m \mu_i = 0$). Then the corresponding values of n can be read off as above and the associated permutations w are also easy to describe explicitly.

Suppose $G = GL_m$ instead of SL_m. Take λ and μ to be partitions with at most m parts. (This just gives a different language to address the question at hand without altering it–the equivalence is given via translation by a (possibly fractional) multiple of the W-invariant vector $\sum_{i=1}^m \epsilon_i$.) Then it turns out that $|S(\lambda, \mu)| = 1$ precisely when the Young diagrams of λ and μ "differ by connected skew hooks", see [14].

7.5. **Type B.** For $m > 1$, we realize R of type B_m as the subset $R = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j \mid i \neq j, 1 \leq i, j \leq m\}$ of $\mathbb{R}^m$. Positive roots are those of the form either $\epsilon_i \pm \epsilon_j$ with $i < j$ or ϵ_i. A vector $\sum_{i=1}^m q_i\epsilon_i$ is a weight precisely when each $q_i \in \mathbb{Z}$ or each $q_i \in \frac{1}{2} + \mathbb{Z}$. The Weyl group acts on $\mathbb{R}^m$ by permuting the ϵ_i and by changing the signs of ϵ_i. Hence two weights η and η' belong to the same W-orbit (under the 'dot' action) if and only if the two sets I_η and $I_{\eta'}$ coincide up to signs. Using the notation in Section 7.3, a weight η is dominant if and only if the condition $\eta_1 > \eta_2 > \cdots > \eta_m > 0$ is satisfied.

Assume that (7.1) holds for λ and μ. To analyze the implications of this, we separate into three cases depending on the form of the root γ in question.

Case 1. $\gamma = \epsilon_a$. Then (7.1) is equivalent to having

$$I_\mu = (I_\lambda \setminus \{\lambda_a\}) \cup \{|\lambda_a - n|\},$$

which implies that $D_{\lambda\mu} = \{\lambda_a\}$. Moreover, if $D_{\mu\lambda} = \{\mu_c\}$, then we have $|\lambda_a - n| = \mu_c$, i.e., $n = (\lambda_a \pm \mu_c)$. On the other hand, Case 1 clearly arises whenever $|D_{\lambda\mu}| = |D_{\mu\lambda}| = 1$ since, for $m > 1$, this guarantees that $n = (\lambda_a \pm \mu_c) \in \mathbb{Z}$. The associated w are easily deduced.

Case 2. $\gamma = \epsilon_a - \epsilon_b$ with $a \neq b$. (It is convenient not to assume γ to be positive.) Then (7.1) is equivalent to having

$$I_\mu = (I_\lambda \setminus \{\lambda_a, \lambda_b\}) \cup \{|\lambda_a - n|, |\lambda_b + n|\}.$$

There are now two possibilities.

Subcase 2.1. $|D_{\lambda\mu}| = 1$. By reversing the signs of γ and n, we may assume without loss of generality that $D_{\lambda\mu} = \{\lambda_a\}$. Suppose $D_{\mu\lambda} = \{\mu_c\}$. Then one of the following two options must be true.

(i) $\lambda_b + n = -\lambda_b$ and $\lambda_a - n = \pm\mu_c$, or
(ii) $\lambda_b + n = \pm\mu_c$ and $\lambda_a - n = \pm\lambda_b$.

By adding, in both cases we have $\lambda_b+\lambda_a=\pm\lambda_b\pm\mu_c$. By positivity constraints and since $\mu_c\neq\lambda_a$ by assumption, we must have $\lambda_b+\lambda_a=-\lambda_b+\mu_c$, i.e., $\lambda_a-\mu_c=-2\lambda_b$.

Conversely, clearly Subcase 2.1 arises exactly when both the following conditions hold: $|D_{\lambda\mu}|=|D_{\mu\lambda}|=1$ and (letting $D_{\lambda\mu}=\{\lambda_a\}$ and $D_{\mu\lambda}=\{\mu_c\}$) there exists a necessarily unique b such that $\lambda_a-\mu_c=-2\lambda_b$. The corresponding two values of n are easily found to be $n=-2\lambda_b=\lambda_a-\mu_c$ (leading to (i)) and $n=\mu_c-\lambda_b=\lambda_a+\lambda_b$ (leading to (ii)). The associated w are easily deduced.

Subcase 2.2. $|D_{\lambda\mu}|=2$. Clearly $D_{\lambda\mu}=\{\lambda_a,\lambda_b\}$. Let $D_{\mu\lambda}=\{\mu_c,\mu_d\}$ with $c<d$. Then one of the following two options must be true. Either $\lambda_b+n=\pm\mu_d$ and $\lambda_a-n=\pm\mu_c$, or $\lambda_b+n=\pm\mu_c$ and $\lambda_a-n=\pm\mu_d$. Adding and using $\mu_c>\mu_d$, in both cases we have $\lambda_a+\lambda_b=\mu_c\pm\mu_d$.

Conversely, Subcase 2.2 arises exactly when the following conditions hold: $|D_{\lambda\mu}|=|D_{\mu\lambda}|=2$ and (setting $D_{\mu,\lambda}=\{\mu_c,\mu_d\}$ with $c<d$) one of the equalities $\lambda_a+\lambda_b=\mu_c\pm\mu_d$ holds. Then the corresponding two values of n and the associated w are easily deduced.

Case 3. $\gamma=\epsilon_a+\epsilon_b$ with $a\neq b$. The analysis is very similar to Case 2, so we skip some details.

Subcase 3.1. $|D_{\lambda\mu}|=1$. Suppose $D_{\lambda\mu}=\{\lambda_a\}$ and $D_{\mu\lambda}=\{\mu_c\}$. As in Subcase 2.1, we deduce $\lambda_b-\lambda_a=-\lambda_b\pm\mu_c$, i.e., $\lambda_a\pm\mu_c=2\lambda_b$.

Conversely, clearly Subcase 3.1 arises exactly when both the following conditions hold. First, $|D_{\lambda\mu}|=|D_{\mu\lambda}|=1$. Next, there must exist a necessarily unique b such that $\lambda_a-\mu_c=2\lambda_b$ and/or a necessarily unique b' such that $\lambda_a+\mu_c=2\lambda_{b'}$. If $\lambda_a-\mu_c=2\lambda_b$, then $n=2\lambda_b=\lambda_a-\mu_c$ or $n=\lambda_a-\lambda_b=\lambda_b+\mu_c$. If $\lambda_a+\mu_c=2\lambda_{b'}$, then $n=2\lambda_{b'}=\lambda_a+\mu_c$ or $n=\lambda_a-\lambda_{b'}=\lambda_{b'}-\mu_c$.

Subcase 3.2. $|D_{\lambda\mu}|=2$. Clearly $D_{\lambda\mu}=\{\lambda_a,\lambda_b\}$. Suppose $a<b$. Let $D_{\mu\lambda}=\{\mu_c,\mu_d\}$ with $c<d$. As before, we deduce $\lambda_a-\lambda_b=\mu_c\pm\mu_d$.

Conversely, Subcase 3.2 arises exactly when the following conditions hold. $|D_{\lambda\mu}|=|D_{\mu\lambda}|=2$ and one of the equalities $\lambda_a-\lambda_b=\mu_c\pm\mu_d$ holds. Again the corresponding two values of n and the associated w are easily deduced.

Using the above cases, we sketch a procedure to calculate the sets $S(\lambda,\mu)$ (and hence the sets $U(\lambda,\mu)$ and $V(\mu,\lambda)$). Clearly $S(\lambda,\mu)$ is empty unless $|D_{\lambda\mu}|=|D_{\mu\lambda}|=1$ or 2.

Suppose $D_{\lambda\mu}=\{\lambda_a,\lambda_b\}$ with $a<b$ and $D_{\mu\lambda}=\{\mu_c,\mu_d\}$ with $c<d$. Then at most one of cases 2.2 and 3.2 can occur, since $\lambda_a+\lambda_b=\mu_c\pm\mu_d$ and $\lambda_a-\lambda_b=\mu_c\pm\mu_d$ cannot be true simultaneously. So $|S(\lambda,\mu)|=0$ or 1.

Suppose $D_{\lambda\mu}=\{\lambda_a\}$ and $D_{\mu\lambda}=\{\mu_c\}$. Then Case 1 will always occur and cases 2.1 and 3.1 will occur depending on the existence of b satisfying one of the three conditions $\lambda_a-\mu_c=-2\lambda_b$, $\lambda_a-\mu_c=2\lambda_b$ and $\lambda_a+\mu_c=2\lambda_b$. Clearly at most two of these can be satisfied (since at most one of the first two can be true), each by a unique b. Note also that if the rank $m=2$, then at most one of the three conditions can hold. So $|S(\lambda,\mu)|=3$ (provided $m>2$) or 2 or 1.

An example with $|S(\lambda,\mu)|=3$ for type B_3 is given by $\lambda+\rho=5\epsilon_1+3\epsilon_2+2\epsilon_3$ and $\mu+\rho=3\epsilon_1+2\epsilon_2+\epsilon_3$.

7.6. **Type C.** The analysis can be lifted almost verbatim from that for type B, so we indicate only the changes that need to be made there. In R we replace $\pm\epsilon_i$ by $\pm2\epsilon_i$, with $2\epsilon_i\in R^+$. For weights we require each $q_i\in\mathbb{Z}$. Again we make three cases. Only Case 1 needs any change. Here we take $\gamma=2\epsilon_a$. Then (7.1) leads to $n=\frac{1}{2}(\lambda_a\pm\mu_c)$. Conversely, this case arises exactly when $|D_{\lambda\mu}|=|D_{\mu\lambda}|=1$ and

$(\lambda_a \pm \mu_c)$ is even. Except for the inclusion of the evenness condition, the procedure to calculate $S(\lambda, \mu)$ stays unchanged. (In particular $|D_{\lambda\mu}| = |D_{\mu\lambda}| = 1$ no longer guarantees $|S(\lambda, \mu)| \geq 1$.)

7.7. **Type D.** The analysis is again similar to that for type B, so we indicate only the changes. Here they are more significant. Now $R = \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j, 1 \leq i, j \leq m\}$ with positive roots those of the form $\epsilon_i \pm \epsilon_j$ with $i < j$. The weights stay the same, i.e., $\sum_{i=1}^m q_i \epsilon_i$ with each $q_i \in \mathbb{Z}$ or each $q_i \in \frac{1}{2} + \mathbb{Z}$. The Weyl group acts by permuting the ϵ_i and by changing the signs of an even number of ϵ_i. This means that two weights η and η' belong to the same W-orbit (under the 'dot' action) if and only if the two sets I_η and $I_{\eta'}$ coincide up to an even number of signs. In this case we shall therefore find it convenient to work with the set $I'_\eta = \{|\eta_1|, |\eta_2|, \dots, |\eta_m|\}$ instead of I_η, and we replace D by D' for the corresponding difference sets. Using otherwise the notation in Section 7.3, $\eta \in X^+$ is equivalent to the condition $\eta_1 > \eta_2 > \cdots > \eta_{m-1} > |\eta_m|$. Note that η_m may be 0 or negative. If $\eta \in X^+$ with $\eta_m \geq 0$ then $I'_\eta = I_\eta$.

Assuming (7.1), for $\gamma = \epsilon_a \pm \epsilon_b$, we get

$$I'_\mu = (I'_\lambda \setminus \{|\lambda_a|, |\lambda_b|\}) \cup \{|\lambda_a - n|, |\lambda_b \mp n|\}.$$

Now just as for type B, we get the following consequences.

If $\gamma = \epsilon_a + \epsilon_b$ (respectively, $\epsilon_a - \epsilon_b$) and $D'_{\lambda\mu} = \{|\lambda_a|, |\lambda_b|\}$, then letting $a < b$ and $D'_{\mu\lambda} = \{\mu_c, |\mu_d|\}$ with $c < d$, we get $\lambda_a - \lambda_b = \mu_c \pm \mu_d$ (respectively, $\lambda_a + \lambda_b = \mu_c \pm \mu_d$).

If $\gamma = \epsilon_a + \epsilon_b$ (respectively, $\epsilon_a - \epsilon_b$) and $D'_{\lambda\mu} = \{|\lambda_a|\}$, then letting $D'_{\mu\lambda} = \{|\mu_c|\}$, we get $\lambda_a - \mu_c = 2\lambda_b$ (respectively, $\lambda_a \pm \mu_c = -2\lambda_b$; unlike for type B, here it requires some work to rule out $\lambda_a + \mu_c = 2\lambda_b$. One sees that the latter equality only arises when $\mu_c = 0$, in which case one may as well use $\lambda_a - \mu_c = -2\lambda_b$).

Conversely, we now describe exactly when (7.1) holds for a given γ. We make the convention that $\operatorname{sign}(0) = 0$. For a weight η, define $\operatorname{sign}(\eta) = \Pi_{i=1}^m \operatorname{sign}(\eta_i)$. Note that for all η in a W-orbit (under the 'dot' action), $\operatorname{sign}(\eta)$ remains the same. Clearly, making cases as for Type B, the validity of (7.1) in each case is characterized by the respective numerical constraints along with the requirement $\operatorname{sign}(\mu) = \operatorname{sign}(\lambda - n\gamma)$. We make this explicit below. Since $\mu \in X^+$, $\operatorname{sign}(\mu) = \operatorname{sign}(\mu_m)$ (and so the sign condition is vacuous if $\mu_m = 0$). To calculate $\operatorname{sign}(\lambda - n\gamma)$, we have used the values of n obtained in each case.

Suppose $|D'_{\lambda\mu}| = |D'_{\mu\lambda}| = 2$. Let $D'_{\lambda\mu} = \{\lambda_a, |\lambda_b|\}$ with $a < b$ and $D'_{\mu\lambda} = \{\mu_c, |\mu_d|\}$ with $c < d$. Then (7.1) holds for $\gamma = \epsilon_a + \epsilon_b$ iff $\lambda_a - \lambda_b = \mu_c \pm \mu_d$ and

$$\operatorname{sign}(\mu_m) = \begin{cases} \operatorname{sign}(-\lambda_a + \lambda_b + \mu_c)\operatorname{sign}(\lambda_m) & \text{if } b < m; \\ \operatorname{sign}(-\lambda_a + \lambda_m + \mu_c) & \text{if } b = m. \end{cases}$$

Similarly, (7.1) holds for $\gamma = \epsilon_a - \epsilon_b$ iff $\lambda_a + \lambda_b = \mu_c \pm \mu_d$ and

$$\operatorname{sign}(\mu_m) = \begin{cases} \operatorname{sign}(\lambda_a + \lambda_b - \mu_c)\operatorname{sign}(\lambda_m) & \text{if } b < m; \\ \operatorname{sign}(\lambda_a + \lambda_m - \mu_c) & \text{if } b = m. \end{cases}$$

Note that in the above cases, respectively, $-\lambda_a + \lambda_b + \mu_c = \mp\mu_d$ and $\lambda_a + \lambda_b - \mu_c = \pm\mu_d$.

Suppose $|D'_{\lambda\mu}| = |D'_{\mu\lambda}| = 1$. Let $D'_{\lambda\mu} = \{|\lambda_a|\}$ and $D'_{\mu\lambda} = \{|\mu_c|\}$. Then (7.1) holds for $\gamma = \epsilon_a + \epsilon_b$ iff $\lambda_a \pm \mu_c = 2\lambda_b$ and

$$\operatorname{sign}(\mu_m) = \begin{cases} -\operatorname{sign}(\lambda_a - 2\lambda_b)\operatorname{sign}(\lambda_m) & \text{if } a < m; \\ 1 & \text{if } a = m. \end{cases}$$

Likewise (7.1) holds for $\gamma = \epsilon_a - \epsilon_b$ iff $\lambda_a - \mu_c = -2\lambda_b$ and

$$\operatorname{sign}(\mu_m) = \begin{cases} -\operatorname{sign}(\lambda_a + 2\lambda_b)\operatorname{sign}(\lambda_m) & \text{if } a < m; \\ -1 & \text{if } a = m. \end{cases}$$

Again note that in these cases we have, respectively, $\lambda_a - 2\lambda_b = \mp\mu_c$ and $\lambda_a + 2\lambda_b = \mu_c$.

By easy extensions of the arguments for type B, one easily deduces the following. If $|D'_{\lambda\mu}| = |D'_{\mu\lambda}| = 2$, then $|S(\lambda, \mu)| \leq 1$. If $|D'_{\lambda\mu}| = |D'_{\mu\lambda}| = 1$, then $|S(\lambda, \mu)| \leq 2$. In fact $|S(\lambda, \mu)| = 2$ can occur only when $\mu_m = \lambda_m = 0$. This can be seen using the sign constraints above. In particular, for types D_2 and D_3, $|S(\lambda, \mu)| \leq 1$ (as we already know from the result for type A). For type D_4, the example given for type B_3 provides an instance where $|S(\lambda, \mu)| = 2$.

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