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REPRESENTATIONS, CONVOLUTION SQUARE ROOTS AND SPHERICAL VECTORS

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Representations, convolution square roots and spherical vectors

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Abstract

We extend results on square integrable representations of a locally compact group to subrepresentations of a representations induced from a unitary character of a closed subgroup. We do so working in the framework of quotient representations of $*$-algebras.

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Contents

1 Introduction

This paper grew out of an attempt to understand three classical results about the left regular representation L of a locally compact group G and the relations between them:

(I) That any square-integrable representation is contained in the left regular representation of G on $L^2(G)$.

(II) Schur's orthogonality relations for matrix-elements of square-integrable representations.

(III) Godement's theorem saying that any continuous and positive definite function $f \in L^2(G)$ on G has a positive convolution square root, i.e. may be written in the form $f = f_0 * f_0$, where $f_0 \in L^2(G)$ is of positive type.

The left regular representation on $L^2(G)$ can be characterized as the one induced from the trivial subgroup. We extend (I) and (II) to the representation U^χ induced from a unitary character χ on a closed subgroup H of a unimodular group G, instead of just the special case of H being the trivial subgroup. We get an extension of Godement's theorem, that applies not just to square integrable functions on G, but to functions in the representation space of such a representation.

Our results are valid for locally compact groups, and we do not need any structure theory for semi-simple Lie groups and the like. We do not discuss results for particular groups like $G = \mathbb{R}^n$ where special results are available on, say, convolution roots (see [7]). To accomplish our program we apply functional analytic methods to matrix elements of the representations in question. These representations are often just cyclic, not necessarily irreducible.

Rieffel [14] and Phillips [12], [13] generalized to Hilbert algebras the theory of square-integrable representations of locally compact groups. A Hilbert algebra sits as a dense subspace inside its Hilbert space completion. The present paper treats a new direction by detaching the algebra from the Hilbert space: Our set up (see Section 3) is that of a quotient representation of a $*$-algebra $\mathfrak{A}$ the elements of which need not belong to the Hilbert space. A typical example of $\mathfrak{A}$ is the convolution algebra $C_c(G)$ of a locally compact group G. In the papers mentioned above the self-adjoint idempotents of the Hilbert algebra play an important role. They play no role in our treatment. Indeed, they cannot, because our framework is so general that our algebra $\mathfrak{A}$ need not contain any non-zero self-adjoint idempotents.

Working in the framework of quotient representations we get a transparent treatment of various aspects of sub-representations of a representation induced from a character χ on a subgroup H of the given group G. A novel feature seems to be the observation of the role of the spherical vectors for subrepresentations of the induced representation U^χ and for the orthogonality relations. Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$. We say that a vector $x_0' \in \mathcal{H}'$ is *spherical*, if $\sigma(h)x_0' = \chi(h)x_0'$ for all $h \in H$. All vectors are spherical if $H = \{e\}$.

Assuming that G/H has a G-invariant measure $d\mu$, we prove that any unitary representation σ of G with a cyclic and spherical vector x_0' such that $|\langle x_0', \sigma(\cdot)x_0'\rangle| \in L^2(G/H, d\mu)$ is a subrepresentation of U^χ (Theorem 7.10). The converse is also true if G is unimodular and $H = \langle K, A\rangle$ is generated by a compact subgroup K of G and a closed central subgroup A of G: More precisely, any irreducible unitary subrepresentation σ of U^χ with a spherical vector x_0' satisfies that $|\langle x_0', \sigma(\cdot)x_0'\rangle| \in L^2(G/H)$ (Theorem 9.4).

For $H = \langle K, A\rangle$ of the form above we prove a version of Schur's orthogonality relations for matrix-elements $\langle x', \sigma(\cdot)x_0'\rangle$ of an irreducible, unitary representation σ, where x' is arbitrary while x_0' ranges over the spherical vectors (Theorem 9.6).

The irreducible subrepresentations of the left regular representation L of G, i.e. the case of $H = \{e\}$, are known to be the same as the square integrable representations, also called the discrete series of G. This is an active field with a long history (see for example Dixmier [4], Carey [2], Duflo and Moore [5] and Rieffel [15]). The investigations are continued in the present paper in the case of $H \neq \{e\}$, mainly for G unimodular, although some of our results (for example Theorems 5.1 and 7.10) also hold for non-unimodular groups.

There are already some investigations of what happens when $H \neq \{e\}$:

(i) Representations that are square integrable modulo the center $Z(G)$ of G, were studied by A. Borel [1, §5.13]. They fit perfectly into our framework (see Subsection 9.4): Such a representation σ is a subrepresentation of U^χ, where χ is the central character of σ, and $H = Z(G)$.

(ii) Cassinelli and De Vito described in [3] the sub-representations σ of an induced representation U^τ, where τ is a unitary representation of a subgroup H of G, in terms of admissible maps modulo (H, τ) (defined in [3, Definition 3.1]). We relate this to our paper by producing such admissible maps modulo (H, χ), when σ is a unitary spherical representation, which is irreducible or just cyclic and has a square integrable matrix coefficient (see Remark 7.12 for details).

Any square-integrable representation σ of G is a subrepresentation of the left-regular representation of G on $L^2(G)$. Carey [2] and Dixmier [4] used Godement's classical theorem to construct the orthogonal projection onto a subspace of $L^2(G)$ on which σ is realized. We obtain such a projection directly from Theorem 4.4 (see Corollary 4.6).

Theorem 4.4 also provides us with an extension of Godement's theorem, that applies not just to square integrable functions on G, but even to functions in the representation space of a representation induced from a unitary character of a subgroup of G (Section 5). Godement's theorem has been extended to Hilbert algebras by Rieffel [14] and Phillips [12], [13]. However, our extension of Godement's theorem is quite different from theirs.

In [10] Kunze considered a unitary representation of a unimodular group. Assumimg that all its matrix-coefficients are square integrable he concluded [10, Corollary] that it is discretely decomposable, i.e. it can be written as an orthogonal sum of irreducible subrepresentations. In Section 6 we extend Kunze's result to our setting of a quotient representation.

The paper is organized in the following way:

The notation (which is standard) and the general set up are fixed in Sections 2 and 3.

The technical key to our work, Theorem 4.4, is the main content of Section 4.

Section 5 presents a necessary and sufficient condition for an element to have a convolution square root. Godement's classical theorem is derived as a corollary.

Section 6 extends the results of [10] about decomposition theory. The remaining sections do not depend on it.

In Section 7 we compute in details what the operators in the general set up of Section 3 amount to for a representation which is induced from a unitary character of a subgroup of G. We state a sufficient condition for certain spherical, cyclic representations of G to be contained in the induced representation.

Section 8 discusses in a general framework (i) the necessity of the condition from Section 7 and (ii) Schur's orthogonality relations.

Section 9 applies the results of Section 8 to a representation induced from a unitary character of a subgroup of the form $\langle K, A\rangle$, where K is a compact subgroup of G, and A is a closed central subgroup of G.

2 Notation and terminology

All vector spaces are taken over the field $\mathbb{C}$ of complex numbers. For Hilbert spaces and operators between them we use the standard notation as found in [6]. The commutator algebra of $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}')$ will be denoted $\mathcal{M}'$. For the polar decomposition of a closed operator between two Hilbert spaces we refer to [11, Section XII.7].

In the present paper an algebra need not possess a unit. By a $*$-representation of a $*$-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ we mean a $*$-homomorphism $\sigma : \mathfrak{A} \to \mathcal{B}(\mathcal{H}')$. A vector $x' \in \mathcal{H}'$ is said to be *cyclic* for the representation σ, if $\overline{\operatorname{span}}\{\sigma(a)x' \mid a \in \mathfrak{A}\} = \mathcal{H}'$. The representation σ is said to be *irreducible*, if each non-zero vector in $\mathcal{H}'$ is cyclic for σ.

We let G denote a locally compact topological group, equipped with a choice of left Haar measure ds on G. We include the Hausdorff property in the definition of locally compact. The modular function on G will be denoted Δ_G. If F is a complex-valued function on G then we let $\widetilde{F}$ denote the function $\widetilde{F}(s) = \overline{F(s^{-1})}$, $s \in G$. We let L be the left regular representation of G on the complex vector space of all complex valued functions on G. The Hilbert space $L^2(G) = L^2(G, ds)$ is equipped with its usual inner product $\langle f, g\rangle = \int_G f(s)\,\overline{g(s)}\,ds$, $f, g \in L^2(G)$.

The convolution $*$-algebra of compactly supported continuous functions on G is denoted $C_c(G)$. The group G acts by $*$-automorphisms as a transformation group on $C_c(G)$ according to the prescription

$$(s \cdot f)(t) = \Delta_G(s) f(s^{-1}ts), \qquad f \in C_c(G),\ s, t \in G. \tag{2.1}$$

If H is a subgroup of G we equip G/H with the quotient topology and identify $C(G/H)$ with the functions in $C(G)$ that are constant on the left cosets sH, $s \in G$.

We incorporate continuity in the definition of a representation σ of a topological group G on a topological vector space V: We require that the map $(s, v) \mapsto \sigma(s)v$ of $G \times V$ to V is continuous. If σ is a unitary representation of G we introduce the corresponding $*$-representation of $C_c(G)$, again denoted σ, by $\sigma(f) = \int_G f(s)\sigma(s)\,ds$

for $f \in C_c(G)$. Like for representations we incorporate continuity in the definition of a unitary group character.

A function $f \in L^1_{\text{loc}}(G)$ is said to be of *positive type*, if

$$\int_G (\phi^* * \phi)(s)\overline{f(s)}\, ds \geq 0 \quad \text{for all } \phi \in C_c(G).$$

We shall use that $f = \widetilde{f}$ locally almost everywhere if f is of positive type [4, 13.7.7], and that a continuous function on G is of positive type if and only if it is positive definite [4, Proposition 13.4.4].

3 Set up

Throughout this paper $\mathfrak{A}$ denotes a $*$-algebra and $\mathcal{H}$ a Hilbert space. They are related by a linear map $i : \mathfrak{A} \to \mathcal{H}$ and a $*$-representation π of $\mathfrak{A}$ on $\mathcal{H}$, such that

(A) $i(\mathfrak{A})$ is dense in $\mathcal{H}$, and

(B) $\pi(a)i(b) = i(ab)$ for all $a, b \in \mathfrak{A}$.

Remark. We do **not** assume that:

(a) $i : \mathfrak{A} \to \mathcal{H}$ is injective. See Remark 7.1 for an example where it is not injective. It is injective for Hilbert algebras (Example 3.5 below).

(b) $i : \mathfrak{A} \to \mathcal{H}$ relates the $*$-operation in $\mathfrak{A}$ to the norm in $\mathcal{H}$. In particular we do not in general assume that $\|i(a^*)\| = \|i(a)\|$ for all $a \in \mathfrak{A}$. In our studies of spherical vectors for induced representations we get by with the weaker condition $\|i(a_0^*)\| = \|i(a_0)\|$, $a_0 \in \mathfrak{A}_0$, where $\mathfrak{A}_0$ is a certain sub-$*$-algebra of $\mathfrak{A}$ (Section 8),

(c) $\mathfrak{A}$ comes equipped with a topology.

Let σ be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and let $x_0', x_1' \in \mathcal{H}'$ be given. We shall so often impose the following condition on σ and x_0', x_1', that it almost becomes part of the set up: There exists a constant $C = C(x_0', x_1')$ such that

$$|\langle \sigma(a)x_0', x_1' \rangle| \leq C\, \|i(a)\| \quad \text{for all } a \in \mathfrak{A}. \tag{3.1}$$

Definition 3.1. An element $x \in \mathcal{H}$ is said to be *positive*, if $\langle i(a^*a), x \rangle \geq 0$ for all $a \in \mathfrak{A}$.

Definition 3.2. An element $y \in \mathcal{H}$ is said to be a *convolution square root* of $x \in \mathcal{H}$, if $\langle \pi(a)y, y \rangle = \langle i(a), x \rangle$ for all $a \in \mathfrak{A}$.

It is immediate that $x \in \mathcal{H}$ is positive if it has a convolution square root. We do not require that convolution square roots are positive. However, as we shall see later, if $x \in \mathcal{H}$ has a convolution square root then it also has a positive one (Proposition 5.1(b)).

The terminology in Definitions 3.1 and 3.2 is justified and explained in the following motivating Example 3.3.

Example 3.3. If G is a locally compact group, then the set up described in the beginning of this Section 3 holds with $\mathfrak{A} = C_c(G)$, $\mathcal{H} = L^2(G)$, $i : C_c(G) \to L^2(G)$ the inclusion map, and $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ given by $\pi(a)f = a * f$ for $a \in C_c(G)$ and $f \in L^2(G)$.

That $f \in L^2(G)$ is positive means here that $\int_G (a^* * a)(s)\overline{f(s)}\, ds \geq 0$ for all $a \in C_c(G)$, i.e. that f is of positive type.

Let $f_0 \in L^2(G)$. From the formula $\langle \pi(a)f_0, f_0\rangle = \int_G a(s)\overline{\langle f_0, L(s)f_0\rangle}\, ds$ we see that f_0 is a convolution square root of $f \in L^2(G)$, if and only if $f(s) = \langle f_0, L(s)f_0\rangle$ for almost all $s \in G$ or equivalently $f = f_0 * \widetilde{f_0}$ almost everywhere. If f_0 is a positive convolution square root of f, then $f = f_0 * \widetilde{f_0} = f_0 * f_0$, which gives a justification for the terminology square root.

The condition (3.1) is satisfied for a unitary representation σ of G on a Hilbert space $\mathcal{H}'$ if and only if the matrix-element $s \mapsto \langle x_1', \sigma(\cdot)x_0'\rangle$ is square integrable over G (A detailed argument for this can be found for the more general case of induced representations in the proof of Lemma 7.7).

Example 3.4. The representation π in the first part of Example 3.3 is the left regular representation of G which is a very special case of a representation induced from a character of a subgroup of G. The more general situation of representations induced from characters also fits into our set up and will be discussed in Section 7.1.

Example 3.5. Let $\mathfrak{A}$ be a Hilbert algebra and $\mathcal{H}$ the Hilbert space completion of $\mathfrak{A}$. Our set up holds with $i : \mathfrak{A} \to \mathcal{H}$ as the inclusion map and $\pi(a) = \overline{U}_a$ for $a \in \mathfrak{A}$, where we use the notation of [14].

Remark 3.6. We have right after Definition 3.2 noted that a necessary condition for an element in $\mathcal{H}$ to possess a convolution square root is that the element is positive. However, it is in general not true that positive elements have convolution square roots, so the condition is not sufficient:

Take in Example 3.3 the group $G = \mathbb{R}$. The Fourier transform $\widehat{f}$ of the function

$$f(t) = \frac{1}{1+|t|}, \quad t \in \mathbb{R},$$

is a square integrable function of positive type, because f is square integrable and $f(t) \geq 0$ for all $t \in \mathbb{R}$. We obtain a contradiction, if we assume that $\widehat{f}$ possesses a convolution square root $f_0 \in L^2(\mathbb{R})$. Indeed, applying the inverse Fourier transform to $\widehat{f} = f_0 * f_0^*$ we get that $f = |\check{f}_0|^2$. Thus

$$|\check{f}_0| = \frac{1}{\sqrt{1+|t|}}, \quad t \in \mathbb{R}.$$

But then $\check{f}_0 \notin L^2(\mathbb{R})$, and so we obtain the contradiction $f_0 \notin L^2(\mathbb{R})$.

4 Two key operators

Let σ be a $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and let $x_0' \in \mathcal{H}'$. In this section we associate to σ and x_0' two unbounded linear operators $R_{x_0'}$ and $T_{x_0'}$ and list a number of their pertinent properties (Theorem 4.4).

First, however, the following lemma that will be used without mentioning.

Lemma 4.1. *Let σ be a $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and let $x'_0 \in \mathcal{H}'$. Let $P_{\mathcal{K}}$ denote the orthogonal projection of $\mathcal{H}'$ onto the closure $\mathcal{K}$ of $\sigma(\mathfrak{A})x'_0$ in $\mathcal{H}'$. Then*

(a) $\mathcal{K}$ is a σ-invariant closed subspace of $\mathcal{H}'$.

(b) $P_{\mathcal{K}}\sigma(a) = \sigma(a)P_{\mathcal{K}}$ for all $a \in \mathfrak{A}$.

(c) Let $x'_1 \in \mathcal{H}'$. Then $P_{\mathcal{K}}x'_1$ is a cyclic vector for $\sigma|\mathcal{K}$, if x'_1 is cyclic, or if $x'_1 = x'_0$.

Proof. (a): This is trivial, since σ is a representation of $\mathfrak{A}$ on $\mathcal{H}'$.
(b): Since $\sigma(a)P_{\mathcal{K}}x' \in \mathcal{K}$ for any $x' \in \mathcal{H}'$ we see that $P_{\mathcal{K}}\sigma(a)P_{\mathcal{K}} = \sigma(a)P_{\mathcal{K}}$. Taking adjoints and using that $\sigma(a)^* = \sigma(a^*)$ for all $a \in \mathfrak{A}$ we get that $\sigma(a)P_{\mathcal{K}} = P_{\mathcal{K}}\sigma(a)$.
(c): The first statement is trivial. For the second we note that $\sigma(a)P_{\mathcal{K}}x'_0 = P_{\mathcal{K}}\sigma(a)x'_0 = \sigma(a)x'_0$ so $\overline{\operatorname{span}}\{\sigma(a)P_{\mathcal{K}}x'_0 \mid a \in \mathfrak{A}\} = \overline{\operatorname{span}}\{\sigma(a)x'_0 \mid a \in \mathfrak{A}\} = \mathcal{K}$. □

Definition 4.2. Let σ be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$. For any $x' \in \mathcal{H}'$ such that

$$a \in \mathfrak{A},\ i(a) = 0 \quad \Rightarrow \quad \sigma(a)x' = 0 \tag{4.1}$$

we unambiguously define a linear operator $R_{x'} : i(\mathfrak{A}) \to \mathcal{H}'$ by $R_{x'}(i(a)) = \sigma(a)x'$ for $a \in \mathfrak{A}$.

The condition (4.1) on x' is satisfied by any $x' \in \mathcal{H}'$, if $i : \mathfrak{A} \to \mathcal{H}$ is injective. But injectivity of i is not part of our set-up.

If we take $\sigma = \pi$ in Example 3.3 then the operator $R_{x'}$ becomes convolution by the function $x' \in L^2(G)$ from the right (that is our reason for choosing the notation $R_{x'}$), i.e. $R_{x'}a = a * x'$ for $a \in C_c(G)$. In the Hilbert algebra case (Example 3.5) we have in the notation of [14] that $R_{x'} = V_{x'}$.

Let σ be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$. We fix for a moment $x'_0, x'_1 \in \mathcal{H}'$ and assume that the inequality (3.1) holds. We observe that the map $i(a) \mapsto \langle \sigma(a)x'_0, x'_1 \rangle$ in that case defines a continuous linear functional on $\mathcal{H}$, so that there exists exactly one element $x_1 \in \mathcal{H}$ such that

$$\langle \sigma(a)x'_0, x'_1 \rangle = \langle i(a), x_1 \rangle \quad \text{for all } a \in \mathfrak{A}.$$

The observation is the basis for the definition of the operator $T_{x'_0}$:

Definition 4.3. Let σ be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and fix $x'_0 \in \mathcal{H}'$. Let $\mathcal{D}(T_{x'_0})$ denote the subspace of $\mathcal{H}'$ consisting of those $x' \in \mathcal{H}'$ for which there exists a constant $C(x')$ such that $|\langle \sigma(a)x'_0, x' \rangle| \le C(x')\|i(a)\|$ for all $a \in \mathfrak{A}$. For $x' \in \mathcal{D}(T_{x'_0})$ define $T_{x'_0}x' \in \mathcal{H}$ by

$$\langle \sigma(a)x'_0, x' \rangle = \langle i(a), T_{x'_0}x' \rangle \quad \text{for all } a \in \mathfrak{A}. \tag{4.2}$$

The following technical Theorem 4.4 lists a number of properties of the operators $R_{x'_0}$ and $T_{x'_0}$ and of their relations to the polar decomposition of $T_{x'_0}$. The theorem is the foundation for our further work.

By (c) of the theorem $(R_{x'_0})^* = T_{x'_0}$, so it is a matter of convemience whether to work with $T_{x'_0}$ or $R_{x'_0}$. We have chosen to mainly work with $T_{x'_0}$.

Theorem 4.4 may be seen as a rather far reaching generalization of classical results on square-integrable representations: In Example 3.3 the elements $T_{x'_0}x' \in \mathcal{H}$, $x' \in \mathcal{H}'$, are matrix-elements, some of which are in $L^2(G)$, when σ is square integrable. The characterization of $T_{x'_0}x'$ as a matrix-element extends to representations induced from a unitary character (See Proposition 7.7).

Theorem 4.4. *Let σ be a $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and fix $x_0' \in \mathcal{H}'$. Let P denote the orthogonal projection of $\mathcal{H}'$ onto $\overline{\sigma(\mathfrak{A})x_0'}$, and $T_{x_0'} = W|T|$ the polar decomposition of $T_{x_0'}$.*

We assume there exists a vector $x_1' \in \mathcal{D}(T_{x_0'})$, meaning that

$$|\langle \sigma(a)x_0', x_1'\rangle| \leq C \,\|i(a)\| \quad \text{for all } a \in \mathfrak{A}, \tag{4.3}$$

for some constant C not depending on $a \in \mathfrak{A}$, such that Px_1' is cyclic for the restriction of σ to $\overline{\sigma(\mathfrak{A})x_0'}$. Then

(a) $R_{x_0'}$ is a well defined, closable, densely defined, linear operator from $\mathcal{H}$ to $\mathcal{H}'$.

(b) $T_{x_0'}$ is a closed, densely defined, linear operator from $\mathcal{H}'$ to $\mathcal{H}$. Furthermore $\pi(a)T_{x_0'} \subseteq T_{x_0'}\sigma(a)$ and $\sigma(a)|T| \subseteq |T|\,\sigma(a)$ for all $a \in \mathfrak{A}$.

(c) $(R_{x_0'})^* = T_{x_0'}$.

(d) $\pi(a)W = W\sigma(a)$ for all $a \in \mathfrak{A}$. The initial domain of W is $\overline{\sigma(\mathfrak{A})x_0'}$, and its range is $\overline{\pi(\mathfrak{A})y_0}$ where $y_0 := Wx_0' \in \mathcal{H}$.

(e) R_{y_0} is a well defined operator from $\mathcal{H}$ to $\mathcal{H}$. It is positive and essentially self-adjoint: Indeed, $W|T|W^*$ is self-adjoint, and $\overline{R_{y_0}} = W|T|W^*$. Furthermore $\pi(a)\overline{R_{y_0}} \subseteq \overline{R_{y_0}}\pi(a)$ for all $a \in \mathfrak{A}$. Finally, y_0 is positive.

(f) If $x_0' \in \mathcal{D}(T_{x_0'})$, then $T_{x_0'}x_0' \in \mathcal{H}$ has a positive convolution square root, viz. y_0.

(g) If x_0' is cyclic, then $T_{x_0'}$ is injective, and W is an isometry of $\mathcal{H}'$ into $\mathcal{H}$. In this case σ is a subrepresentation of π, with W as an intertwining isometry from σ to π.

Proof. A couple of times during the proof we use without explicit mentioning that $(BA)^* = A^*B^*$, when A is a densely defined linear operator and B a bounded, everywhere defined linear operator.

We will in the proof abbreviate $T_{x_0'}$ by T.

(a) We start by showing that $R_{x_0'}$ is well defined, i.e. that $i(a) = 0$ implies $\sigma(a)x_0' = 0$. For any $b \in \mathfrak{A}$ we find that

$$\begin{aligned}|\langle \sigma(a)x_0', \sigma(b)Px_1'\rangle| &= |\langle \sigma(b^*a)x_0', Px_1'\rangle| = |\langle \sigma(b^*a)x_0', x_1'\rangle| \\ &\leq C\|i(b^*a)\| = C\|\pi(b^*)i(a)\| = C\|\pi(b^*)0\| = 0,\end{aligned}$$

so $\langle \sigma(a)x_0', \sigma(b)Px_1'\rangle = 0$. But Px_1' is by assumption a cyclic vector for $\overline{\text{span}}\{\sigma(b)x_0' \mid b \in \mathfrak{A}\}$, and $b \in \mathfrak{A}$ is arbitrary, so $\sigma(a)x_0' = 0$.

We next establish that $R_{x_0'}$ is closable. Let $a_1, a_2, \ldots$ be a sequence from $\mathfrak{A}$ and let $x' \in \mathcal{H}'$. We shall show that if $i(a_n) \to 0$ and $R_{x_0'}(i(a_n)) = \sigma(a_n)x_0' \to x'$ for $n \to \infty$, then $x' = 0$. For any $a \in \mathfrak{A}$ we get on the one hand that $\langle \sigma(a_n)x_0', \sigma(a)x_1'\rangle \to \langle x', \sigma(a)x_1'\rangle$, and on the other hand that

$$\begin{aligned}|\langle \sigma(a_n)x_0', \sigma(a)x_1'\rangle| &= |\langle \sigma(a^*a_n)x_0', x_1'\rangle| \\ &\leq C\|i(a^*a_n)\| = C\|\pi(a^*)i(a_n)\| \to 0,\end{aligned}$$

so $\langle x', \sigma(a)x_1'\rangle = 0$. Since $x' = \lim_n \sigma(a_n)x_0' \in \overline{\text{span}}\{\sigma(a)x_0' \mid a \in \mathfrak{A}\}$ we have $Px' = x'$. Now $0 = \langle x', \sigma(a)x_1'\rangle = \langle Px', \sigma(a)x_1'\rangle = \langle x', \sigma(a)Px_1'\rangle$. But Px_1' is cyclic for the restriction of σ to $\overline{\text{span}}\{\sigma(a)x_0' \mid a \in \mathfrak{A}\}$, so $x' = 0$.

(c) Let $x' \in \mathcal{D}(T)$. Then $\langle i(a), Tx'\rangle = \langle \sigma(a)x'_0, x'\rangle = \langle R_{x'_0} i(a), x'\rangle$ for all $a \in \mathfrak{A}$, so $x' \in \mathcal{D}((R_{x'_0})^*)$ and $(R_{x'_0})^* x' = Tx'$. This means that $T \subseteq (R_{x'_0})^*$. It is thus left to show that $\mathcal{D}((R_{x'_0})^*) \subseteq \mathcal{D}(T)$.

If $x' \in \mathcal{D}((R_{x'_0})^*)$ we get for all $a \in \mathfrak{A}$ that $\langle i(a), (R_{x'_0})^* x'\rangle = \langle R_{x'_0} i(a), x'\rangle = \langle \sigma(a)x'_0, x'\rangle$, which shows that $x' \in \mathcal{D}(T)$.

(b) The first statement follows from general functional analysis ([6, Lemma XII.7]), T being the adjoint of the closed, densely defined linear operator $\overline{R_{x'_0}}$ as just shown. The first commutation property is an easy consequence of Definition 4.3, and the second one follows from the first one by standard arguments (see [11, Ch. IV, § 21]).

(d) We find for any $x' \in \mathcal{D}(|T|) = \mathcal{D}(T)$, that

$$\pi(a)W|T|x' = \pi(a)Tx' = T\sigma(a)x' = W|T|\sigma(a)x' = W\sigma(a)|T|x'.$$

Thus the two bounded operators $\pi(a)W$ and $W\sigma(a)$ coincide on the range $R(|T|)$ of $|T|$, and hence also on its closure $\overline{R(|T|)} = (\ker|T|)^\perp = (\ker T)^\perp$. It is left to show that they also coincide on $\ker T$. The initial domain of W is $(\ker T)^\perp$, so by its very definition $W = 0$ on its orthogonal complement, i.e. on $\ker T$. Thus $\pi(a)W = 0$ on $\ker T$. If $x' \in \ker T$, then we get by (b) that $T\sigma(a)x' = \sigma(a)Tx' = 0$, so $\sigma(a)(\ker T) \subseteq \ker T$. Finally, $W\sigma(a)(\ker T) \subseteq W(\ker T) = \{0\}$.

The initial domain of W is $(\ker T)^\perp$, so we shall prove that $(\ker T)^\perp = \overline{\sigma(\mathfrak{A})x'_0}$ or equivalently that $\ker T = (\sigma(\mathfrak{A})x'_0)^\perp$. If $x' \in \ker T$ then $x' \in \mathcal{D}(T)$ and $\langle \sigma(a)x'_0, x'\rangle = \langle i(a)x'_0, Tx'\rangle = \langle i(a)x'_0, 0\rangle = 0$, so that $\ker T \subseteq (\sigma(\mathfrak{A})x'_0)^\perp$. If conversely $x' \in (\sigma(\mathfrak{A})x'_0)^\perp$, then $\langle \sigma(a)x'_0, x'\rangle = 0$ for all $a \in \mathfrak{A}$). But then $x' \in \mathcal{D}(T)$ and $Tx' = 0$. Thus $(\sigma(\mathfrak{A})x'_0)^\perp \subseteq \ker T$.

Using that W is a partial isometry we find that its range is $W\overline{\sigma(\mathfrak{A})x'_0} = \overline{W\sigma(\mathfrak{A})x'_0} = \overline{\pi(\mathfrak{A})Wx'_0} = \overline{\pi(\mathfrak{A})y_0}$.

(e) The operator $W|T|W^* = WT^*$ is densely defined since so is T and hence also T^* ([6, Lemma XII.7]). Now, $(W|T|W^*)^* = (WT^*)^* = (T^*)^*W^* = TW^* = W|T|W^*$ shows that $W|T|W^*$ is self-adjoint. It is clearly a positive operator. The computation

$$\begin{aligned}\pi(a)W|T|W^* &= \pi(a)TW^* \subseteq T\sigma(a)W^* = T(W\sigma(a^*))^* \\ &= T(\pi(a^*)W)^* = TW^*\pi(a) = W|T|W^*\pi(a)\end{aligned}$$

shows that $\pi(a)$ and $W|T|W^*$ commute.

If $i(a) = 0$, then $0 = R_{x'_0} i(a) = \sigma(a)x'_0$, because $R_{x'_0}$ is well defined by (a). Applying W to this we get $0 = W\sigma(a)x'_0 = \pi(a)Wx'_0 = \pi(a)y_0$, which proves that R_{y_0} is well defined.

We next prove that R_{y_0} is closable, i.e. that if $i(a_n) \to 0$ and $R_{y_0} i(a_n) = \pi(a_n)y_0 \to x_0$ then $x_0 = 0$. Combining $\pi(a_n)y_0 = W\sigma(a_n)x'_0$ with the fact that W is an isometry on its initial domain $\overline{\sigma(\mathfrak{A})x'_0}$ we see that $\{\sigma(a_n)x'_0\}$ is a Cauchy sequence in $\mathcal{H}'$. Hence there exists an $x' \in \mathcal{H}'$ such that $\sigma(a_n)x'_0 \to x'$. This means that $R_{x'_0} i(a_n) \to x'$, and so $x' = 0$, $R_{x'_0}$ being closable. Thus $\sigma(a_n)x'_0 \to 0$. Applying W to this we get that $\pi(a_n)y_0 \to 0$. But $\pi(a_n)y_0 \to x_0$, so $x_0 = 0$.

The computation $R_{y_0} i(a) = \pi(a)y_0 = \pi(a)Wx'_0 = W\sigma(a)x'_0 = WR_{x'_0} i(a)$ which is valid for any $a \in \mathfrak{A}$, shows that $R_{y_0} = WR_{x'_0}$, from which it follows that $\overline{R_{y_0}} = W\overline{R_{x'_0}}$.

We get from the formula $(R_{x'_0})^* = T$ that

$$\overline{R_{y_0}} = W\overline{R_{x'_0}} = W(R_{x'_0})^{**} = WT^* = W(W|T|)^* = W|T|W^*.$$

We now establish that y_0 is positive from the fact that $R_{y_0} \subseteq W|T|W^*$ so that R_{y_0} is a positive operator: For any $a \in \mathfrak{A}$ we find that

$$\langle i(a^*a), y_0\rangle = \langle \pi(a)^* i(a), y_0\rangle = \langle i(a), \pi(a)y_0\rangle = \langle i(a), R_{y_0} i(a)\rangle \geq 0.$$

(f) Note that P is the orthogonal projection of $\mathcal{H}'$ onto the initial domain for W. Using that W is an isometry on its initial domain and is 0 on its orthogonal complement so that $WP = W$ we find that

$$\begin{aligned}\langle \pi(a)y_0, y_0\rangle &= \langle \pi(a)Wx_0', Wx_0'\rangle = \langle W\sigma(a)x_0', WPx_0'\rangle \\ &= \langle \sigma(a)x_0', Px_0'\rangle = \langle P\sigma(a)x_0', x_0'\rangle = \langle \sigma(a)x_0', x_0'\rangle.\end{aligned}$$

From this formula we get for any $a \in \mathfrak{A}$ that

$$\langle i(a), Tx_0'\rangle = \langle \sigma(a)x_0', x_0'\rangle = \langle \pi(a)y_0, y_0\rangle,$$

which shows that y_0 is a convolution square root of Tx_0'. Its positivity was established under (e).

(g) During the proof of (d) we saw that $\ker T = (\sigma(\mathfrak{A})x_0')^\perp$, which proves the statement about injectivity. We also saw that the initial domain of W is $(\ker T)^\perp$, so here we get that it is $(\ker T)^\perp = \overline{\sigma(\mathfrak{A})x_0'} = \mathcal{H}'$. □

By [14, Theorem 4.6]) a unitary representation σ of G with a cyclic vector x_0' such that $\langle x_0', \sigma(\cdot)x_0'\rangle \in L^2(G)$ is a subrepresentation of the left regular representation. Corollary 4.5 contains that fact. It even allows two cyclic vectors that need not coincide. The corollary also entails a more general result about representations induced from a character (Theorem 7.10).

Corollary 4.5. *Let σ be a $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$. Assume that there exist cyclic vectors $x_0', x_1' \in \mathcal{H}'$ and a constant C such that the inequality* (4.3) *holds.*

Then σ is a subrepresentation of π.

In Corollary 4.6 we examine what happens if σ is irreducible and not just cyclic. Theorem 7.11 is an application of it to representations induced from a character. A partial inverse of Corollary 4.6 will be discussed in Theorem 8.4.

Corollary 4.6. *Let σ be an irreducible $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$. Assume that there exist non-zero vectors $x_0', x_1' \in \mathcal{H}'$ and a constant C such that the inequality* (4.3) *holds. Then*

(a) There exists an isometry $W : \mathcal{H}' \to \mathcal{H}$, that intertwines σ and π, so that σ is a subrepresentation of π.

(b) There exists a constant $c > 0$ such that

$$\langle \sigma(a)x_0', x'\rangle = c\,\langle i(a), Wx'\rangle, \quad a \in \mathfrak{A},\ x' \in \mathcal{H}'.$$

(c) $y_0 := Wx_0' \in \mathcal{H}$ is a positive convolution square root of cy_0.

(d) R_{y_0} is a well defined, closable operator, and $c^{-1}\overline{R_{y_0}}$ is the orthogonal projection of $\mathcal{H}$ onto the subspace $W(\mathcal{H}')$ on which σ by (a) *is realized as a subrepresentation of π.*

(e) If $z_0 \in \mathcal{H}$ is any positive convolution square root of cy_0, then R_{z_0} is well defined and $R_{z_0} = R_{y_0}$.

Proof. (a) This is Theorem 4.4(g).

(b) The self-adjoint operator $|T_{x'_0}|$ commutes with the irreducible representation σ according to Theorem 4.4(d), so by Schur's lemma $|T_{x'_0}| = cI$, where $c \in \mathbb{C}$ is a constant. In particular $|T_{x'_0}| \in \mathcal{B}(\mathcal{H}', \mathcal{H})$. Furthermore $c \geq 0$, because $|T_{x'_0}|$ is a positive operator. The polar decomposition $T_{x'_0} = W|T|$ of $T_{x'_0}$ reduces to $T_{x'_0} = cW$. We see from this that c is not zero, because $T_{x'_0}$ is injective by Theorem 4.4(g), so $c > 0$. W is an isometry by Theorem 4.4(g).

(c) We may apply Theorem 4.4(f), because $T_{x'_0}$ here is everywhere defined.

(d) By Theorem 4.4(e) we get that R_{y_0} is well defined and closable, and also, because $|T| = cI$, that $\overline{R_{y_0}} = W|T|W^* = cWW^*$.

(e) Here under (e) we assume for simplicity $c = 1$. When $z_0 \in \mathcal{H}$ is a convolution square root of $T_{x'_0}x'_0$, then

$$\langle \pi(a)z_0, z_0\rangle = \langle i(a), T_{x'_0}x'_0\rangle = \langle \sigma(a)x'_0, x'_0\rangle \text{ for all } a \in \mathfrak{A},$$

which implies that $U\sigma(a)x'_0 := \pi(a)z_0$, $a \in \mathfrak{A}$, defines a linear isometry $U : \sigma(\mathfrak{A})x'_0 \to \mathcal{H}$. This extends uniquely to an isometry, again denoted U, of $\overline{\sigma(\mathfrak{A})x'_0} = \mathcal{H}'$ into $\mathcal{H}$.

If $i(a) = 0$, then $\pi(a)z_0 = U\sigma(a)x'_0 = UR_{x'_0}i(a) = 0$, which means that R_{z_0} is well defined (Definition 4.2). The definition of U says that $R_{z_0} = UR_{x'_0}$, which implies that R_{z_0} is closable and that $\overline{R_{z_0}} = U\overline{R_{x'_0}}$. Now,

$$(\overline{R_{z_0}})^*\overline{R_{z_0}} = (U\overline{R_{x'_0}})^*U\overline{R_{x'_0}} = (\overline{R_{x'_0}})^*U^*U\overline{R_{x'_0}} = (\overline{R_{x'_0}})^*\overline{R_{x'_0}},$$

and similarly $(\overline{R_{y_0}})^*\overline{R_{y_0}} = (\overline{R_{x'_0}})^*\overline{R_{x'_0}}$. Since $\overline{R_{y_0}}$ is self-adjoint (Theorem 4.4(e)), we see that

$$(\overline{R_{z_0}})^*\overline{R_{z_0}} = (\overline{R_{y_0}})^2. \tag{4.4}$$

By (b) the operator $T_{x'_0}$ is bounded. Hence so is $\overline{R_{x'_0}} = T^*_{x'_0}$, and then also $R_{z_0} = UR_{x'_0}$. Furthermore, $z_0 \in \mathcal{H}$ being positive implies that R_{z_0} is positive, so $\overline{R_{z_0}}$ is a positive self-adjoint operator. Due to the uniqueness of the positive square root of an operator we see from (4.4) that $\overline{R_{z_0}} = \overline{R_{y_0}}$. Restricting this to $i(\mathfrak{A})$ we get that $R_{z_0} = R_{y_0}$. □

Remark 4.7. Let σ be a square integrable representation of G. By help of Godement's classical theorem Carey [2, p. 5] following Dixmier [4, p. 268] found a positive convolution square root f_0 of the matrix element $s \mapsto \langle x'_0, \sigma(s)x'_0\rangle$. Both of them then worked with the operator R_{f_0}. We have included point (e) of Corollary 4.6 to point out that R_{f_0} is independent of the choice of positive convolution square root f_0, even in our more general setting.

In the papers mentioned Dixmier and Carey continued by choosing the Friedrichs extension of R_{f_0} to get a self-adjoint extension of it. Actually R_{f_0} is essentially self-adjoint according to Theorem 4.4(e), so $W|T|W^*$ is the only self-adjoint extension of R_{f_0}.

5 Godement's theorem on convolution square roots

Theorem 5.1(a) is our generalization of Godement's theorem in the set up of Section 3. Its proof is based on Theorem 4.4 and hence on considerations from functional analysis.

Theorem 5.1. *Let $x_0 \in \mathcal{H}$.*

(a) x_0 has a positive convolution square root if and only if there exist a $*$-representation σ of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ and a vector $x_0' \in \mathcal{H}'$, such that $\langle \sigma(a)x_0', x_0' \rangle = \langle i(a), x_0 \rangle$ for all $a \in \mathfrak{A}$.

(b) If x_0 has a convolution square root, then it has a positive convolution square root.

Proof. (a): The condition is satisfied by definition if x_0 has a convolution square root. To prove the other direction we refer to Theorem 4.4(f). The theorem applies, because $x_1' = x_0'$ here.
(b): Let $y \in \mathcal{H}$ be a convolution square root of x_0. This means that $\langle \pi(a)y, y \rangle = \langle i(a), x_0 \rangle$ for all $a \in \mathfrak{A}$, so (b) is an immediate consequence of (a). □

Remark 5.2. If $x \in \mathcal{H}$ has a convolution square root then x is integrable (see [14, Definition 3.5] and [12, Definition 1.4]) in the sense that

$$\sup\{\langle i(e), x \rangle \mid e = e^* = e^2 \in \mathfrak{A}\} < \infty.$$

Indeed, let $y \in \mathcal{H}$ be a convolution square root of x and let $e = e^* = e^2 \in \mathfrak{A}$. Noting that $\pi(e)$ is an orthogonal projection we find that $\langle i(e), x \rangle = \langle \pi(e)y, y \rangle = \|\pi(e)y\|^2 \leq \|y\|^2$.

Phillips proved in [12, Theorem 1.10] that the converse is true if $\mathfrak{A}$ is an achieved left Hilbert algebra, generalizing the corresponding result by Rieffel [14, Theorem 3.14] on Hilbert algebras. In the present generality there is no hope for such a converse result, since $\mathfrak{A}$ need not contain any non-zero self-adjoint idempotents.

In the remainder of this section we assume we are given a Hilbert space $\mathcal{H}$ which is a subspace of $L^1_{\text{loc}}(G)$ (in which functions agreeing locally almost everywhere are identified). We assume given a linear map $i : C_c(G) \to \mathcal{H}$ such that $\langle i(a), f \rangle = \int_G a(s)\overline{f(s)}\, ds$ for all $a \in C_c(G)$ and $f \in \mathcal{H}$, and a unitary representation π of G on $\mathcal{H}$ such that $\pi(s)i(a) = i(L(s)a)$ for all $s \in G$ and $a \in C_c(G)$. The set up of Section 3 then holds.

That $f \in \mathcal{H}$ is positive means by Definition 3.1 that $\langle i(a^* * a), f \rangle \geq 0$ for all $a \in C_c(G)$, which by our assumption here becomes

$$\int_G (a^* * a)(s)\overline{f(s)}\, ds \geq 0 \quad \text{for all } a \in C_c(G).$$

In other words, $f \in \mathcal{H}$ positive means f is of positive type.

Similarly we find that $f_0 \in \mathcal{H}$ is a convolution square root of $f \in \mathcal{H}$ if and only if $f(s) = \langle f_0, \pi(s)f_0 \rangle_{\mathcal{H}}$ for locally almost all $s \in G$. In particular a necessary condition for f to possess a convolution square root is that f is continuous and positive definite. The condition is also sufficient, as shown by the following slight extension of Godement's classical theorem.

Theorem 5.3. *Let the assumptions be as above. Let $f \in C(G)$ be a positive definite function on G. If $f \in \mathcal{H}$, then there exists a function $f_0 \in \mathcal{H}$ of positive type such that $f(s) = \langle f_0, \pi(s)f_0 \rangle$ for all $s \in G$.*

Proof. Any continuous, positive definite function f on G can be written in the form $f(s) = \langle x'_0, \sigma(s)x'_0\rangle$, $s \in G$, where σ is a unitary representation of G on a Hilbert space $\mathcal{H}'$ and $x'_0 \in \mathcal{H}'$ ([4, Théorème 13.4.5]). For any $a \in C_c(G)$ we find that

$$\langle i(a), f\rangle = \int_G a(s)\overline{f(s)}\,ds = \int_G a(s)\overline{\langle x'_0, \sigma(s)x'_0\rangle}\,ds$$
$$= \int_G a(s)\langle \sigma(s)x'_0, x'_0\rangle\,ds = \langle \sigma(a)x'_0, x'_0\rangle,$$

so we conclude from Proposition 5.1 that f has a positive convolution square root $f_0 \in \mathcal{H}$. i.e. that $\langle i(a), f\rangle = \langle \pi(a)f_0, f_0\rangle$ for all $a \in C_c(G)$. Thus

$$\int_G a(s)\overline{f(s)}\,ds = \int_G a(s)\langle \pi(s)f_0, f_0\rangle\,ds \text{ for all } a \in C_c(G),$$

which implies that $f(s) = \langle f_0, \pi(s)f_0\rangle$ for locally almost all $s \in G$. Since both sides are continuous, they agree everywhere. □

We get Godement's theorem by taking $\mathcal{H} = L^2(G)$ in Theorem 5.3. We have thus derived Godement's theorem by other methods than those of [8, Théorème 17], [12] and [14]. Theorem 5.3 applies more generally to representations induced from a character χ of a closed subgroup H of G; here $\mathcal{H} = \mathcal{H}^\chi$ and $\pi = L$ (see Section 7.1). These representations include the classical case of $\mathcal{H} = L^2(G)$ as $H = \{e\}$.

6 Decomposition theory

In [10] Kunze considered a unitary representation of a unimodular group. Assuming that all its matrix-coefficients are square integrable he concluded [10, Corollary] that it is discretely decomposable. In this section we derive Theorem 6.1 which has the result corresponding to Kunze's, but in our set-up, as Corollary 6.2. It shows that the assumption in [10] of unimodularity of the group is superfluous.

Theorem 6.1. *Let σ be a cyclic $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$ with cyclic vector $x'_0 \in \mathcal{H}'$, and let $\mathcal{H}'_0 = \overline{\sigma(\mathfrak{A})'x'_0}$. Assume that there to each $x' \in \mathcal{H}'_0$ and each $y' \in \mathcal{H}'$ is a constant $M(x', y') \geq 0$ such that*

$$|\langle \sigma(a)x', y'\rangle| \leq M(x', y')\|i(a)\| \text{ for all } a \in \mathfrak{A}. \tag{6.1}$$

Then σ is discretely decomposable.

Theorem 6.1 applies not just to square integrable representations, but even to the more general case of representations induced from a character (Corollary 7.13). We pause for a moment to present two other corollaries of Theorem 6.1 before embarking on the proof of it.

Corollary 6.2. *Let σ be a $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$. Assume that there to each $x', y' \in \mathcal{H}$ is a constant $M(x', y') \geq 0$ such that*

$$|\langle \sigma(a)x', y'\rangle| \leq M(x', y')\|i(a)\| \text{ for all } a \in \mathfrak{A}. \tag{6.2}$$

Then σ is discretely decomposable.

Proof of Corollary 6.2. By help of Zorn's lemma we write σ as a direct orthogonal sum of cyclic subrepresentations, and then we apply Theorem 6.1 to each of these subrepresentations. $\square$

A $*$-algebra $\mathfrak{A}$ is called a *left H^*-algebra* if $\mathfrak{A}$ is a Hilbert space such that

$$\|a \cdot b\| \leq \text{const}\, \|a\|\|b\| \text{ for all } a, b \in \mathfrak{A}, \text{ and}$$
$$\langle a \cdot b, c\rangle = \langle b, a^* \cdot c\rangle \text{ for all } a, b, c \in \mathfrak{A}.$$

Corollary 6.3. *Any left H^*-algebra is a direct orthogonal sum of irreducible (minimal) closed left ideals.*

Proof. Let $\mathfrak{A}$ be an arbitrary left H^*-algebra. We apply Corollary 6.2 with $\mathcal{H}' = \mathfrak{A}$ and σ the left regular representation of $\mathfrak{A}$, i.e. $\sigma(a)b = a \cdot b$ for $a \in \mathfrak{A}$ and $b \in \mathcal{H}'$. $\square$

Our proof of Theorem 6.1 is composed of the following four lemmas, in each of which we impose the assumptions of Theorem 6.1.

Lemma 6.4. *There exists a constant $M \geq 0$ such that*

$$|\langle\sigma(a)x', y'\rangle| \leq M\|i(a)\|\|x'\|\|y'\| \text{ for all } a \in \mathfrak{A},\ x' \in \mathcal{H}_0' \text{ and } y' \in \mathcal{H}'. \tag{6.3}$$

Proof. The uniform boundedness principle combined with (6.1). $\square$

We see from the inequality (6.3) that the operator $T_{x'}$ from Definition 4.3 for each $x' \in \mathcal{H}_0'$ is everywhere defined and bounded, i.e. $T_{x'} \in \mathcal{B}(\mathcal{H})$, and that $\|T_{x'}\| \leq M\|x'\|$. This holds in particular for $x' = x_0'$, because $x_0' \in \mathcal{H}_0'$. As in Theorem 4.4 we shall work with the polar decomposition $T_{x_0'} = W_{x_0'}|T_{x_0'}|$ of $T_{x_0'} \in \mathcal{B}(\mathcal{H})$. However, here we do not encounter problems with domains of definitions because the operators all are everywhere defined. From Theorem 4.4 we know that $S := |T_{x_0'}|$ is injective and that $S \in \sigma(\mathfrak{A})'$. Observe also for use below that $T_{x'}^* W_{x_0'} \in \sigma(\mathfrak{A})'$ for all $x' \in \mathcal{H}_0'$.

The next three lemmas have the same assumptions of Lemma 6.4. We find it convenient to introduce the following product

$$x' \cdot y' := T_{x'}^* W_{x_0'} y', \quad x' \in \mathcal{H}_0',\ y' \in \mathcal{H}',$$

and we start by noting some of its properties.

Lemma 6.5. *(a) The bilinear map $(x', y') \mapsto x' \cdot y'$ of $\mathcal{H}_0' \times \mathcal{H}'$ into $\mathcal{H}'$ is continuous, since $\|x' \cdot y'\| \leq M\, \|x'\|\, \|y'\|$ for all $x' \in \mathcal{H}_0',\ y' \in \mathcal{H}'$.*

(b) $(Ax') \cdot y' = A(x' \cdot y')$ for all $A \in \sigma(\mathfrak{A})'$, $x' \in \mathcal{H}_0'$ and $y' \in \mathcal{H}'$. In particular $(Ax_0') \cdot y' = ASy'$ for all $A \in \sigma(\mathfrak{A})'$ and $y' \in \mathcal{H}'$, because $x_0' \cdot y' = Sy'$ for all $y' \in \mathcal{H}'$.

(c) $x_1', x_2' \in \mathcal{H}_0' \Rightarrow x_1' \cdot x_2' \in \mathcal{H}_0'$.

(d) The product is associative in the sense that

$$(x_1' \cdot x_2') \cdot y' = x_1' \cdot (x_2' \cdot y') \text{ for all } x_1', x_2' \in \mathcal{H}_0' \text{ and } y' \in \mathcal{H}'.$$

We put $(x')^n := x' \cdot x' \cdots x'$ (n factors) for $x' \in \mathcal{H}_0'$.

(e) Let $P \in \mathcal{B}(\mathcal{H}')$ be an orthogonal projection such that $P \in \sigma(\mathfrak{A})'$ and $PS = SP$. Let $\mathcal{K} = P(\mathcal{H}')$. Then $Px'_0 \in \mathcal{H}'_0$, and we have for each $n = 1, 2, \ldots$ that

(i) $S^n z = (Px'_0)^n \cdot z$ for all $z \in \mathcal{K}$.

(ii) $\|(S|_{\mathcal{K}})^n\| \leq M\,\|(Px'_0)^n\|$.

Proof. (a) The bilinearity of the product is immediate from its very definition. For any $x' \in \mathcal{H}'_0$ and $y' \in \mathcal{H}'$ we get

$$\|x' \cdot y'\| = \|T^*_{x'} W_{x'_0} y'\| \leq \|T^*_{x'}\| \|W_{x'_0}\| \|y'\| = \|T_{x'}\| \|y'\| \leq M \|x'\| \|y'\|.$$

(b) For any $x' \in \mathcal{H}'_0$, $y' \in \mathcal{H}'$ and $A \in \sigma(\mathfrak{A})'$ we get that

$$\begin{aligned} \langle i(a), T_{Ax'} y' \rangle &= \langle \sigma(a) A x', y' \rangle = \langle A \sigma(a) x', y' \rangle \\ &= \langle \sigma(a) x', A^* y' \rangle = \langle i(a), T_{x'} A^* y' \rangle, \end{aligned}$$

which shows that $T_{Ax'} = T_{x'} A^*$, and hence that $T^*_{Ax'} = A T_{x'}$. Finally, $(Ax') \cdot y' = T^*_{Ax'} W_{x'_0} y' = A T^*_{x'} W_{x'_0} y' = A(x' \cdot y')$.

(c) If x'_1 has the special form $x'_1 = Ax'_0$ where $A \in \sigma(\mathfrak{A})'$, and $x'_2 \in \mathcal{H}'_0$ then we get from (b) that

$$x'_1 \cdot x'_2 = (Ax'_0) \cdot x'_2 = A(x'_0 \cdot x'_2) = ASx'_2 \in \mathcal{H}'_0,$$

because $AS \in \sigma(\mathfrak{A})'$. The case of a general $x'_1 \in \mathcal{H}'_0$ now follows from the continuity of the product.

(d) If x'_1 has the special form $x'_1 = Ax'_0$ where $A \in \sigma(\mathfrak{A})'$, then we find from (a) that

$$\begin{aligned} &(x'_1 \cdot x'_2) \cdot y' = (Ax'_0 \cdot x'_2) \cdot y' = (ASx'_2) \cdot y' = AS(x'_2 \cdot y'), \text{ and} \\ &x'_1 \cdot (x'_2 \cdot y') = (Ax'_0) \cdot (x'_2 \cdot y') = AS(x'_2 \cdot y'), \end{aligned}$$

so the associativity holds in this case. The general case follows from the continuity of the product.

(e) $Px'_0 \in \mathcal{H}'_0$ because $P \in \sigma(\mathfrak{A})'$. Using (b) we find for any $z \in \mathcal{K}$ that $Sz = SPz = PSz = P(x'_0 \cdot z) = (Px'_0) \cdot z$, which shows that $S|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ is multiplication from the left by $Px'_0 \in \mathcal{H}'_0$. Thus $S^n z = (Px'_0)^n \cdot z$, so that $\|S^n z\| = \|(Px'_0)^n \cdot z\| \leq M \|(Px'_0)^n\| \|z\|$. ☐

Lemma 6.6. *If $P \in \mathcal{B}(\mathcal{H}')$ is a non-zero orthogonal projection such that $P \in \sigma(\mathfrak{A})'$ and $PS = SP$, then the subspace $P(\mathcal{H}')$ of $\mathcal{H}'$ contains an eigenvector for S.*

Proof. The operator $S_P := S|_{P(\mathcal{H}')}$ is a positive self-adjoint operator on $P(\mathcal{H}')$. It is non-zero because S is injective, so $\Lambda := \|S_P\| > 0$. Let $S_P = \int_{[0,\Lambda]} \lambda E(d\lambda)$ be the spectral decomposition of S_P. From Lemma 6.5(e) we get the estimate $\|S_P^n\| \leq M \|(Px'_0)^n\|$, and that $(Px'_0)^{n+1} = S_P^n P x'_0$, which we use in the following computation:

$$\begin{aligned} 1 = \|S_P\|^{n+1}/\Lambda^{n+1} &= \|S_P^{n+1}\|/\Lambda^{n+1} \leq M \|(Px'_0)^{n+1}\|/\Lambda^{n+1} \\ &= M \|S_P^n P x'_0\|/\Lambda^{n+1} = M \| \int_{[0,\Lambda]} \lambda^n E(d\lambda) P x'_0 \|/\Lambda^{n+1} \\ &= \frac{M}{\Lambda} \| \int_{[0,\Lambda]} (\frac{\lambda}{\Lambda})^n E(d\lambda) P x'_0 \| \to \frac{M}{\Lambda} \|E(\{\Lambda\}) P x'_0\| \text{ as } n \to \infty. \end{aligned}$$

It follows that $E(\{\Lambda\}) P x'_0 \neq 0$. It is an eigenvector for S_P, and hence for S, in $P(\mathcal{H}')$. ☐

Let $\{\mathcal{K}_\lambda\}$ be the family of all eigenspaces of S. Their direct sum $\bigoplus \mathcal{K}_\lambda$ is invariant under S and $\sigma(\mathfrak{A})$, so this is also true for its orthogonal complement. It follows from Zorn's lemma and Lemma 6.6 that $\mathcal{H}' = \bigoplus \mathcal{K}_\lambda$.

Lemma 6.7. *If $\mathcal{K}$ is an eigenspace for S, then $\sigma|\mathcal{K}$ is a finite orthogonal sum of irreducible subrepresentations.*

Proof. Let $\mathcal{K} = \{x' \in \mathcal{H}' \mid Sx' = \lambda x'\}$. Here $\lambda > 0$, because S is positive and injective. If $\sigma|\mathcal{K}$ is irreducible we are done. If not, then we can write $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are closed, σ-invariant, non-zero subspaces of $\mathcal{K}$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are irreducible we are done. If not we continue decomposing $\mathcal{K}$ into subrepresentations, so in the nth step we write $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \cdots \oplus \mathcal{K}_n$, where $\mathcal{K}_1, \mathcal{K}_2, \ldots \mathcal{K}_n$ are closed, σ-invariant, non-zero subspaces of $\mathcal{K}$. However, this process stops and so we are done, because there is a bound on n, viz. $n \leq (M/\lambda)^2 \|x_0'\|^2$. This bound is established by a classical trick:

The orthogonal projections $P_1, P_2, \ldots P_n$ of $\mathcal{H}'$ onto $\mathcal{K}_1, \mathcal{K}_2, \ldots \mathcal{K}_n$ respectively, belong to $\sigma(\mathfrak{A})'$. They commute with S, because $SP_j = \lambda P_j$. Each $e_j := P_j x_0' \neq 0$, because $\mathcal{K}_j \neq \{0\}$ and x_0' is cyclic. From Lemma 6.5(a) we find for any $j = 1, 2, \ldots, n$ that $e_j \cdot e_j = (P_j x_0') \cdot e_j = P_j S e_j = \lambda P_j e_j = \lambda e_j$, and so the estimate $\lambda \|e_j\| = \|e_j \cdot e_j\| \leq M \|e_j\|^2$, implying that $\|e_j\| \geq \lambda/M$. Now, $Px_0' = e_1 + e_2 + \cdots + e_n$ where P denotes the orthogonal projection of $\mathcal{H}'$ onto $\mathcal{K}$, so

$$\|x_0'\|^2 \geq \|Px_0'\|^2 = \sum_{j=1}^{n} \|e_j\|^2 \geq \sum_{j=1}^{n} (\lambda/M)^2 = n(\lambda/M)^2,$$

from which the asserted bound on n follows. □

Proof of Theorem 6.1. The remaining part of the proof of Theorem 6.1 consists in combining the fact that $\mathcal{H}' = \bigoplus \mathcal{K}_\lambda$ (derived just prior to Lemma 6.7) with Lemma 6.7. □

7 On induced representations

7.1 Induced representations and our set up

We assume in this Section 7 and in the later Section 9 that G is second countable, so that we can use the results of [3].

We let H be a closed subgroup of G. We fix a left Haar measure dh on H, and let $\rho(h) = \Delta_H(h)/\Delta_G(h)$, $h \in H$.

We shall construct the unitary representation U^χ of G induced from a unitary group character $\chi : H \to \mathbb{T}$. The character χ will be kept fixed in the sequel and will be implicitly understood whenever we work in the context of induced representations.

For any $\phi \in C_c(G)$ we define the function $\phi^\chi \in C(G)$ by

$$\phi^\chi(s) := \int_H \phi(sh)\rho(h)^{-1/2}\chi(h)\, dh, \quad s \in G.$$

It is well known that the prescription

$$\langle \phi^\chi, \psi^\chi \rangle := \int_G \phi(s)\overline{\psi^\chi(s)}\, ds = \int_G \phi^\chi(s)\overline{\psi(s)}\, ds, \quad \phi, \psi \in C_c(G), \tag{7.1}$$

defines an inner product on the subspace $\{\phi^\chi \mid \phi \in C_c(G)\}$ of $C(G)$. We let $\mathcal{H}^\chi$ denote the Hilbert space completion of $\{\phi^\chi \mid \phi \in C_c(G)\}$ with respect to the inner product (7.1).

The above fits into our general set up from Section 3, i.e. the conditions (A) and (B) hold, when we take

(i) $\mathcal{H} = \mathcal{H}^\chi$.

(ii) $\mathfrak{A}$ as the convolution $*$-algebra $C_c(G)$.

(iii) $i(\phi) = \phi^\chi$ for $\phi \in C_c(G)$, so that $i : \mathfrak{A} = C_c(G) \to \mathcal{H}^\chi$.

(iv) π as the $*$-representation of $C_c(G)$ on $\mathcal{H}^\chi$ given by $\pi(a)\phi^\chi = a * \phi^\chi$ for $a \in \mathfrak{A}$ and $\phi \in C_c(G)$.

Remark 7.1. The linear map $i : \phi \mapsto \phi^\chi$ is in general not injective. To take a simple example let $G = \{z \in \mathbb{C} \mid |z| = 1\}$, $H = \{\pm 1\}$ and $\chi = 1$, so that

$$\phi^\chi(z) = \tfrac{1}{2}(\phi(z) + \phi(-z)), \quad z \in G.$$

The function ϕ defined by $\phi(e^{i\theta}) = e^{i\theta} - e^{-i\theta}$ has $\phi(-e^{i\theta}) = -\phi(e^{i\theta})$, so that $\phi^\chi = 0$, even though $\phi \neq 0$.

To proceed further we need to realize the completion $\mathcal{H}^\chi$ as a space of functions on G. We recall the construction in [3] of $\mathcal{H}^\chi$ and the unitary representation U^χ of G on $\mathcal{H}^\chi$ induced from $\chi : H \to \mathbb{T}$ (Lemma 7.2 and Theorem 7.3 below. Proofs can be found in [3]). In the vector space of complex-valued functions on G we shall as usual identify functions that are equal almost everywhere, and we identify a continuous function with its equivalence class.

Lemma 7.2. *There is a continuous function $\theta : G \to [0, +\infty[$ such that, for all $s \in G$, $\int_H \theta(sh)dh = 1$, and, for any compact subset C of G, $CH \cap \operatorname{supp}\theta$ is compact.*

$\mathcal{H}^\chi$ is the subspace of the vector space of complex-valued functions f on G such that

(I) f is Borel measurable on G;

(II) given $h \in H$, for almost all $s \in G$, $f(sh) = \rho(h)^{1/2}\overline{\chi(h)}f(s)$;

(III) $\int_G |f(s)|^2\theta(s)\, ds < \infty$.

Theorem 7.3. *$\mathcal{H}^\chi$ is a closed subspace of $L^2(G, \theta ds)$ and a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^\chi}$ inherited from $L^2(G, \theta ds)$. Neither $\mathcal{H}^\chi$ nor its inherited inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^\chi}$ depend on the choice of θ. Each $f \in \mathcal{H}^\chi$ is locally integrable over G.*

If $\phi \in C_c(G)$ then $\phi^\chi(s) := \int_H \phi(sh)\rho(h)^{-1/2}\chi(h)dh$, $s \in G$, is a continuous function in $\mathcal{H}^\chi$, and the set $\{\phi^\chi \mid \phi \in C_c(G)\}$ is a dense subspace of $\mathcal{H}^\chi$. Furthermore, for all $\phi \in C_c(G)$ and $f \in \mathcal{H}^\chi$:

$$\langle i(\phi), f\rangle_{\mathcal{H}^\chi} = \langle \phi^\chi, f\rangle_{\mathcal{H}^\chi} = \int_G \phi(s)\overline{f(s)}\, ds. \tag{7.2}$$

The left regular representation of G restricts to a unitary representation U^χ of G on $\mathcal{H}^\chi$.

Example 3.3 is the special case of $H = \{e\}$.

By the remarks just prior to Theorem 5.3 we see that $f \in \mathcal{H}^\chi$ is positive if and only if f is of positive type.

Definition 7.4. Let $\rho = 1$. A *spherical function* or *spherical vector* in $\mathcal{H}^\chi$ is a function $f \in \mathcal{H}^\chi$ for which $U^\chi(h)f = \chi(h)f$ for all $h \in H$. Equivalently $\forall h \in H : f(hsh^{-1}) = f(s)$ for almost all $s \in G$.

Lemma 7.5. *If there is a positive element $f \in \mathcal{H}^\chi \setminus \{0\}$, then $\rho = 1$, and any positive element in $\mathcal{H}^\chi$ is spherical.*

Proof. Since f is positive, it is of positive type, and so $f = \widetilde{f}$ almost everywhere. Fix $h_0 \in H$. Computing modulo null-sets we find that

$$\begin{aligned} \{U^\chi(h_0)f\}(s) &= f(h_0^{-1}s) = \widetilde{f}(h_0^{-1}s) = \overline{f(s^{-1}h_0)} = \overline{\rho(h_0)^{1/2}\overline{\chi(h_0)}f(s^{-1})} \\ &= \rho(h_0)^{1/2}\chi(h_0)\widetilde{f}(s) = \rho(h_0)^{1/2}\chi(h_0)f(s), \end{aligned} \tag{7.3}$$

so $U^\chi(h_0)f = \rho(h_0)^{1/2}\chi(h_0)f$. But U^χ is a unitary representation, so the coefficient of f on right hand side of (7.3) must have absolute value 1. Hence $\rho(h_0) = 1$. The computation (7.3) then also proves the second statement of the lemma. $\square$

7.2 Applying the general theory to U^χ

We described in Subsection 7.1 how the induced representation U^χ fits into the set up of Section 3. Keeping that set up we shall in this Subsection 7.2 derive explicit formulas for the operators $\overline{R_{f_0}}$ and $T_{x_0'}$ from Section 4 in terms of matrix-elements, and apply this to find subrepresentations of U^χ.

We start by interpreting the condition (4.3) in the present context of induced representations.

Lemma 7.6. *Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$, and let $x_0', y' \in \mathcal{H}'$. The following two statements are equivalent:*

(a) There exists a constant C such that $|\langle\sigma(\phi)x_0', y'\rangle| \le C\,\|i(\phi)\|_{\mathcal{H}^\chi}$ for all $\phi \in C_c(G)$, i.e. the inequality (4.3) *holds.*

(b) The matrix-element $s \mapsto \langle y', \sigma(s)x_0'\rangle$ is in $\mathcal{H}^\chi$.

Proof. (a) $\Rightarrow$ (b): The inequality shows that the map $i(\phi) \mapsto \langle\sigma(\phi)x_0', y'\rangle$ is a continuous linear functional on $\mathcal{H}^\chi$. Thus there exists an element $f \in \mathcal{H}^\chi$ such that $\langle\sigma(\phi)x_0', y'\rangle = \langle i(\phi), f\rangle_{\mathcal{H}^\chi}$ for all $\phi \in C_c(G)$. From the formula (7.2) we then get that

$$\int_G \phi(s)\langle\sigma(s)x_0', y'\rangle\,ds = \int_G \phi(s)\overline{f(s)}\,ds \text{ for all } \phi \in C_c(G),$$

which implies that $\langle y', \sigma(s)x_0'\rangle = f(s)$ for almost all $s \in G$.

(b) $\Rightarrow$ (a): By assumption $f := \langle y', \sigma(\cdot)x_0'\rangle \in \mathcal{H}^\chi$ so we get by (7.2) that

$$\begin{aligned} |\langle\sigma(\phi)x_0', y'\rangle| &= \left|\int_G \phi(s)\langle\sigma(s)x_0', y'\rangle\,ds\right| = \left|\int_G \phi(s)\overline{\langle y', \sigma(s)x_0'\rangle}\,ds\right| \\ &= \left|\int_G \phi(s)\overline{f(s)}\,ds\right| = |\langle i(\phi), f\rangle_{\mathcal{H}^\chi}| \le \|f\|_{\mathcal{H}^\chi}\,\|i(\phi)\|_{\mathcal{H}^\chi}, \end{aligned}$$

which is the estimate of (a) with $C = \|f\|_{\mathcal{H}^\chi}$. $\square$

Proposition 7.7. *Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$ and fix $x_0' \in \mathcal{H}'$. Using the notation of Definition 4.3 we have that*

(a)
$$\mathcal{D}(T_{x_0'}) = \{x' \in \mathcal{H}' \mid \langle x', \sigma(\cdot)x_0'\rangle \in \mathcal{H}^\chi\}, \text{ and} \tag{7.4}$$
$$T_{x_0'}x' = \langle x', \sigma(\cdot)x_0'\rangle \text{ for } x' \in \mathcal{D}(T_{x_0'}). \tag{7.5}$$

(b) Assume furthermore that $x_0' \in \mathcal{H}'$ is cyclic and that the estimate (4.3) *holds for some cyclic vector $x_1' \in \mathcal{H}'$.*
Then $f_0 = Wx_0' \in \mathcal{H}^\chi$ from Theorem 4.4 is a function of positive type. The operator $R_{f_0} : i(C_c(G)) \to \mathcal{H}^\chi$ given by

$$R_{f_0}(i(\phi)) = U^\chi(\phi)f_0 = \phi * f_0 \text{ for } \phi \in C_c(G),$$

is well-defined. It is a positive, essentially self-adjoint operator, and

$$\mathcal{D}(\overline{R_{f_0}}) = \{f \in \mathcal{H}^\chi \mid \langle f, L(\cdot)f_0\rangle_{\mathcal{H}^\chi} \in \mathcal{H}^\chi\}, \text{ and} \tag{7.6}$$
$$\overline{R_{f_0}}f = \langle f, L(\cdot)f_0\rangle_{\mathcal{H}^\chi} \text{ for } f \in \mathcal{D}(\overline{R_{f_0}}). \tag{7.7}$$

Proof. Let for brevity $T = T_{x_0'}$.

(a) That $x' \in \mathcal{D}(T)$ means by Definition 4.3 that the inequality (4.3) holds, so (7.4) is a corollary of Lemma 7.6.

Let $x' \in \mathcal{D}(T)$. Then $\langle \sigma(\phi)x_0', x'\rangle = \langle i(\phi), Tx'\rangle_{\mathcal{H}^\chi}$ for all $\phi \in C_c(G)$, which by the formula (7.2) means that

$$\int_G \phi(s)\langle \sigma(s)x_0', x'\rangle\, ds = \int_G \phi(s)\overline{Tx'(s)}\, ds \text{ for all } \phi \in C_c(G).$$

Thus $Tx'(s) = \langle x', \sigma(s)x_0'\rangle$ for almost all $s \in G$, which is the formula (7.5).

(b) Theorem 4.4(e) tells us that f_0 is a positive element in $\mathcal{H}^\chi$, and hence a function of positive type, and that R_{f_0} is well-defined. Using this and (7.2) we find for any $\phi \in C_c(G)$ that

$$\begin{aligned}(\phi * f_0)(s) &= \int_G \phi(t)f_0(t^{-1}s)\, dt = \int_G \phi(t)\overline{f_0(s^{-1}t)}\, dt \\ &= \int_G \phi(t)\overline{(U^\chi(s)f_0)(t)}\, dt = \langle i(\phi), U^\chi(s)f_0\rangle_{\mathcal{H}^\chi},\end{aligned}$$

so $R_{f_0}(i(\phi)) = \phi * f_0 = \langle i(\phi), U^\chi(\cdot)f_0\rangle_{\mathcal{H}^\chi}$.

By Theorem 4.4(e) the operator $R_{f_0} : i(C_c(G)) \to \mathcal{H}^\chi$ is essentially self-adjoint and positive, and $\overline{R_{f_0}} = W|T|W^* = TW^*$, so that

$$\begin{aligned}\mathcal{D}(\overline{R_{f_0}}) &= \mathcal{D}(TW^*) = \{f \in \mathcal{H}^\chi \mid W^*f \in \mathcal{D}(T)\} \\ &= \{f \in \mathcal{H}^\chi \mid (W^*f, \sigma(\cdot)x_0') \in \mathcal{H}^\chi\} = \{f \in \mathcal{H}^\chi \mid \langle f, W\sigma(\cdot)x_0'\rangle_{\mathcal{H}^\chi} \in \mathcal{H}^\chi\} \\ &= \{f \in \mathcal{H}^\chi \mid \langle f, L(\cdot)f_0\rangle_{\mathcal{H}^\chi} \in \mathcal{H}^\chi\}, \text{ i.e. (7.6).}\end{aligned}$$

To prove (7.7) we let $f \in \mathcal{D}(\overline{R_{f_0}}) \subseteq \mathcal{H}^\chi$. This means that there exists a sequence $\{\phi_n\}$ from $C_c(G)$ such that $i(\phi_n) = \phi_n^\chi \to f$ and $R_{f_0}(i(\phi_n)) \to \overline{R_{f_0}}f$ in $\mathcal{H}^\chi$. For any

$s \in G$ we find that $R_{f_0}(i(\phi_n))(s) = \langle \phi_n^\chi, L(s)f_0\rangle_{\mathcal{H}^\chi} \to \langle f, L(s)f_0\rangle_{\mathcal{H}^\chi}$ with bounded convergence with respect to $s \in G$ because

$$|R_{f_0}(i(\phi_n))(s)| = |\langle \phi_n^\chi, L(s)f_0\rangle_{\mathcal{H}^\chi}| \leq \|\phi_n^\chi\|_{\mathcal{H}^\chi}\|L(s)f_0\|_{\mathcal{H}^\chi} = \|\phi_n^\chi\|_{\mathcal{H}^\chi}\|f_0\|_{\mathcal{H}^\chi},$$

so by the dominated convergence theorem we get for any $\phi \in C_c(G)$ that

$$\langle i(\phi), R_{f_0}(i(\phi_n))\rangle_{\mathcal{H}^\chi} = \int_G \phi(s)\overline{R_{f_0}(i(\phi_n))(s)}\,ds \to \int_G \phi(s)\overline{\langle f, L(s)f_0\rangle_{\mathcal{H}^\chi}}\,ds.$$

On the other hand,

$$\langle i(\phi), R_{f_0}(i(\phi_n))\rangle_{\mathcal{H}^\chi} \to \langle i(\phi), \overline{R_{f_0}}f\rangle_{\mathcal{H}^\chi} = \int_G \phi(s)\overline{(\overline{R_{f_0}}f)(s)}\,ds.$$

Since this holds for all $\phi \in C_c(G)$ we infer that $\langle f, L(s)f_0\rangle_{\mathcal{H}^\chi} = (\overline{R_{f_0}}f)(s)$ for almost all $s \in G$, so that $\overline{R_{f_0}}f = \langle f, L(\cdot)f_0\rangle_{\mathcal{H}^\chi}$, which is (7.7). □

Definition 7.8. Let σ be a representation of G on a Hilbert space $\mathcal{H}'$. We say that a vector $x_0' \in \mathcal{H}'$ is *spherical*, if $\sigma(h)x_0' = \chi(h)\rho(h)^{1/2}x_0'$ for all $h \in H$.

All vector in $\mathcal{H}'$ are spherical, if $H = \{e\}$.

The character χ is implicit in the word spherical. The terminology agrees when $\rho = 1$ with the one of Definition 7.4 in which $\sigma = U^\chi$.

If σ is unitary and x_0' a non-zero spherical vector, then $\rho \equiv 1$. That is why we in our discussion of spherical vectors mainly shall restrict our attention to the pairs G, H for which $\rho \equiv 1$.

The simple relation in the following Lemma 7.9 between spherical vectors and the condition (II) in the definition of $\mathcal{H}^\chi$ shows that we only have to take the square integrability condition (III) into account when working with a spherical vector $x_0' \in \mathcal{H}'$.

Lemma 7.9. *Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$, and let $x_0', x_1' \in \mathcal{H}'$.*

If x_0' is a spherical vector, then the matrix-element $s \mapsto \langle x_1', \sigma(s)x_0'\rangle$ has the properties (I) *and* (II) *from the definition of $\mathcal{H}^\chi$.*

Conversely, if x_1' is cyclic and the function $s \mapsto \langle x_1', \sigma(s)x_0'\rangle$ has the property (II), *then x_0' is a spherical vector.*

We shall apply Theorem 4.4 to the induced representation U^χ:

Theorem 7.10. *Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$. Let $x_0', x_1' \in \mathcal{H}'$ be cyclic vectors with x_0' spherical. Assume furthermore that*

$$\int_G |\langle x_1', \sigma(s)x_0'\rangle|^2\Theta(s)\,ds < \infty.$$

Then σ is a sub-representation of U^χ.

Proof. By Lemma 7.6 there exists a constant C such that the inequality (4.3) is true, so the result follows from Corollary 4.5. □

Theorem 7.10 generalizes the result [14, Theorem 4.6] that a unitary representation σ with a cyclic vector x_0' such that $\langle x_0', \sigma(\cdot)x_0'\rangle \in L^2(G)$ is a sub-representation of the left regular representation (the case of $H = \{e\}$).

Theorem 7.11 provides further information when σ is irreducible and not just cyclic, by giving formulas for how σ is realized as a subrepresentation of U^χ. A partial inverse of it can be found as Theorem 9.4.

Theorem 7.11. *Let σ be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}'$. Let $x_0' \in \mathcal{H}'$ be a non-zero spherical vector. Assume furthermore that $\int_G |\langle x_1', \sigma(s)x_0'\rangle|^2 \Theta(s)\, ds < \infty$ for some non-zero vector $x_1' \in \mathcal{H}'$. Then*

(a) $\langle x', \sigma(\cdot)x_0'\rangle \in \mathcal{H}^\chi$ for all $x' \in \mathcal{H}'$.

(b) $c = \|x_0'\|^{-1} \|\langle x_0', \sigma(\cdot)x_0'\rangle\|_{\mathcal{H}^\chi} > 0$.

(c) The map $W : x' \mapsto c^{-1}\langle x', \sigma(\cdot)x_0'\rangle$ is an isometry of $\mathcal{H}'$ into $\mathcal{H}^\chi$, intertwining σ and U^χ, so σ is a sub-representation of U^χ.

(d) $W(\mathcal{H}')$ consists of continuous functions.

(e) $f_0 = Wx_0' \in W(\mathcal{H}')$ is a non-zero, continuous, positive definite and spherical function, and $f_0 = c^{-1}\langle f_0, L(\cdot)f_0\rangle_{\mathcal{H}^\chi}$.

(f) The orthogonal projection P of $\mathcal{H}^\chi$ onto $W(\mathcal{H}')$ is $Pf = c^{-1}\langle f, L(\cdot)f_0\rangle_{\mathcal{H}^\chi}$, $f \in \mathcal{H}^\chi$.

Proof. Due to Lemma 7.6 there is a constant such that the inequality (4.3) is true, which means that the hypotheses of Theorem 4.4 hold. We shall apply the notation and the results of Theorem 4.4 and Corollary 4.6.

(a) Since σ is irreducible we know that $\mathcal{D}(T_{x_0'}) = \mathcal{H}'$ (Corollary 4.6(b)). This means by Lemma 7.7(a) that (a) holds.

(b) Since $T_{x_0'}x' = \langle x', \sigma(\cdot)x_0'\rangle$ by Lemma 7.7(a) we shall prove that $c = \|x_0'\|^{-1} \cdot \|T_{x_0'}x_0'\|_{\mathcal{H}^\chi} > 0$. But this is stated in Corollary 4.6(b).

(c) This is Theorem 4.4(g) combined with the fact from Corollary 4.6(b) that $T_{x_0'} = cW$.

(d) This is immediate from the formula for W in (c).

(f) Combining Corollary 4.6(d) with Lemma 7.7(b) we have that

$$Pf = \tfrac{1}{c}\overline{R_{f_0}}f = \tfrac{1}{c}\langle f, L(\cdot)f_0\rangle_{\mathcal{H}^\chi}.$$

(e) $f_0 = Wx_0'$ is non-zero, because $x_0' \neq 0$ and W is an isometry. The continuity of $f_0 = Wx_0'$ was established under (d). From the general theory (Theorem 4.4(e)) we know that $f_0 = Wx_0'$ is positive, which here by by the discussion in Section 5 means that it is of positive type. Being also continuous, f_0 is positive definite. By assumption x_0' is spherical. It follows that so is $f_0 = Wx_0'$, because W intertwines σ and $\pi = U^\chi$ by Theorem 4.4(d). Finally, $f_0 = Pf_0$, since $f_0 = Wx_0' \in W(\mathcal{H}')$, so the formula for P in (f) provides the last statement. □

Remark 7.12. In the Theorems 7.10 and 7.11 we saw that a spherical representation σ under certain conditions was a subrepresentations of U^χ.

Subrepresentations σ of an induced representation U^τ, where τ is a representation of H, were described by Cassinelli and De Vito [3] in terms of admissible maps modulo (H, τ) (defined in [3, Definition 3.1]).

Using our results we will now relate [3] to our work by finding admissible maps modulo (H, χ) for the representations that we consider. We let as usual σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$ with a spherical, cyclic vector $x_0' \in \mathcal{H}' \setminus \{0\}$. We assume that there exists a cyclic vector $x_1' \in \mathcal{H}'$ such that $\int_G |\langle x_1', \sigma(s)x_0'\rangle|^2 \Theta(s)\, ds < \infty$.

For $S \in \sigma(H)'$ we consider the bounded map $A : \mathcal{H}' \to \mathbb{C}$ defined by $Ax' = \langle Sx', x_0'\rangle$, $x' \in \mathcal{H}'$. It satisfies the condition [3, (5)]. The corresponding wavelet map $s \mapsto A\sigma(s^{-1})x' = (W_A x')(s)$ from [3] is the matrix-coefficient $(W_A x')(s) = \langle Sx', \sigma(s)x_0'\rangle$. Comparing with Proposition 7.7(a) we see that $W_A x' = T_{x_0'} S x'$, if $Sx' \in \mathcal{D}(T_{x_0'})$.

The condition [3, (6)] is that there exists a constant β such that

$$\int_G |A\sigma(s^{-1})x'|^2 \Theta(s)\, ds \le \beta\, \|x'\|^2 \text{ for all } x' \in \mathcal{D}_\sigma, \text{ i.e. that}$$

$$\int_G |\langle Sx', \sigma(s)x_0'\rangle|^2 \Theta(s)\, ds \le \beta\, \|x'\|^2 \text{ for all } x' \in \mathcal{D}_\sigma, \tag{7.8}$$

where $\mathcal{D}_\sigma$ denotes the Gårding domain of σ. (7.8) means according to Proposition 7.7(a) that the operator $T_{x_0'} S$ is bounded. When [3, (5) and (6)] are satisfied, then A is in the terminology of [3] an admissible map for σ modulo (H, χ).

If σ is irreducible we may take $S = I$. We get by Theorem 7.11 that

$$\int_G |\langle x', \sigma(s)x_0'\rangle|^2 \Theta(s)\, ds = c^2 \|x'\|^2.$$

This implies that the conditions [3, (5) and (6)] hold, so that A is an admissible map for σ modulo (H, χ). By [3, Corollary 1] we even get that σ is square integrable modulo (H, χ), i.e. it is a subrepresentation of U^χ. This result is, however, already contained in Theorem 7.11.

If σ is only cyclic we must be a bit more careful in our choice of admissible map for σ modulo (H, χ). We choose $S = (I + |T_{x_0'}|)^{-1}$. Since $T_{x_0'}(I + |T_{x_0'}|)^{-1}$ is bounded we know from the discussion above that the conditions [3, (5) and (6)] hold. Noting that W_A is injective, because x_0' is cyclic, we read from [3, Comment on p. 1453] that σ is a subrepresentation of U^χ. This conclusion is also stated in Theorem 7.10.

We finish this subsection by extending Kunze's result on discrete decomposition of a representation with square integrable matrix-coefficients (stated in Section 6 above) to the induced representation U^χ.

Corollary 7.13. *Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$. Assume that there exists a vector $x_0' \in \mathcal{H}'$ which is both spherical and cyclic.*

(a) If $\int_G |\langle y', \sigma(s)x'\rangle|^2 \Theta(s)\, ds < \infty$ for all $x' \in \overline{\sigma(C_c(G))' x_0'}$ and $y' \in \mathcal{H}'$, then σ is discretely decomposable into irreducible subrepresentations, each of which has a non-zero spherical vector.

(b) In particular, if $\int_G |\langle y', \sigma(s)x'\rangle|^2 \Theta(s)\, ds < \infty$ for all $x', y' \in \mathcal{H}'$ for which x' is spherical, then σ is discretely decomposable into irreducible subrepresentations, each of which has a non-zero spherical vector.

Proof. Theorem 6.1 combined with Lemma 7.6 gives us that we can write $\mathcal{H}' = \bigoplus_\alpha \mathcal{H}_\alpha'$, where each $\mathcal{H}_\alpha'$ is a σ-invariant, non-zero, irreducible subspace of $\mathcal{H}'$. If P_α denotes the orthogonal projection of $\mathcal{H}'$ onto $\mathcal{H}_\alpha'$ then $P_\alpha x_0'$ is a spherical non-zero vector in $\mathcal{H}_\alpha'$ (If $P_\alpha x_0' = 0$ then x_0' cannot be cyclic). $\square$

8 On the condition $\|i(a^*)\| = \|i(a)\|$

We shall in this section discuss

- a kind of inverse to Corollary 4.6(a), and
- Schur's orthogonality relations,

in the general set up of Section 3. The topic of induced representations will be taken up again in Section 9 and discussed by help of the results in this section.

8.1 General considerations

We have not imposed the seemingly natural condition $\|i(a^*)\| = \|i(a)\|$, $a \in \mathfrak{A}$, as a standard rule, simply because it is not satisfied in various interesting examples. What we will do is to replace it by the weaker requirement that $\|i(a_0^*)\| = \|i(a_0)\|$ for all $a_0 \in \mathfrak{A}_0$, where $\mathfrak{A}_0$ is a self-adjoint sub-algebra of $\mathfrak{A}$. This procedure works for Schur's orthogonality relations and sub-representations of induced representations. A price to pay for dealing with $\mathfrak{A}_0$ instead of with all of $\mathfrak{A}$ is that we cannot derive the orthogonality relations for all vectors of $\mathcal{H}'$, but only for spherical vectors.

To get an example of such an $\mathfrak{A}_0$ let H be a closed subgroup of G. We may as sub-$*$-algebra $\mathfrak{A}_0$ of the convolution algebra $\mathfrak{A} = C_c(G)$ choose

$$\mathfrak{A}_0 = \{\phi \in C_c(G) \mid \phi(hsh^{-1}) = \Delta_G(h)\phi(s),\ \forall s \in G \text{ and } \forall h \in H\}. \tag{8.1}$$

The condition $\|i(a^*)\| = \|i(a)\|$, $a \in \mathfrak{A}$, is satisfied for any Hilbert-algebra ([14, Proposition 1.2]). In Example 3.3 it holds if and only if G is unimodular. For induced representations the situation is more complicated. See Lemma 9.1 for a positive result.

Standard considerations based on the polarization identity prove the following lemma.

Lemma 8.1. *Let $\mathfrak{A}_0$ be a sub-$*$-algebra of $\mathfrak{A}$ such that $\|i(a_0^*)\| = \|i(a_0)\|$ for all $a_0 \in \mathfrak{A}_0$. Then*

(a) There exists exactly one conjugate-linear isometry $x \mapsto x^*$ of the closed subspace $\overline{i(\mathfrak{A}_0)}$ of $\mathcal{H}$ onto itself, such that $i(a_0)^* = i(a_0^*)$ for all $a_0 \in \mathfrak{A}_0$.

(b) The isometry from (a) satisfies that $\langle x^*, y^*\rangle = \langle y, x\rangle$ for all $x, y \in \overline{i(\mathfrak{A}_0)}$.

Let $\mathfrak{A}_0$ be a sub-$*$-algebra of $\mathfrak{A}$. It turns out to be technically important that there exists a map $P : \mathfrak{A} \to \mathfrak{A}_0$ such that

$$\langle i(P(a)), i(a_0)\rangle = \langle i(a), i(a_0)\rangle \text{ for all } a \in \mathfrak{A} \text{ and all } a_0 \in \mathfrak{A}_0. \tag{8.2}$$

P may be thought of as a kind of projection map. An example of such a map P will be studied in subsection 9.1 (see the formula (9.1)).

(8.2) forces P to have certain algebraic and topological properties as for example:

Lemma 8.2. *Let $P : \mathfrak{A} \to \mathfrak{A}_0$ be a map such that (8.2) holds. Then*

$$i(P(a_0 a)) = i(a_0 Pa) \text{ for all } a_0 \in \mathfrak{A}_0 \text{ and } a \in \mathfrak{A}, \text{ and} \tag{8.3}$$

$$\|i(Pa)\| \le \|i(a)\| \text{ for all } a \in \mathfrak{A}. \tag{8.4}$$

Proof. For all $a \in \mathfrak{A}$ and $a_0, b_0 \in \mathfrak{A}_0$ we find using (8.2) twice that

$$\begin{aligned}\langle i(P(a_0a)), i(b_0)\rangle &= \langle i(a_0a), i(b_0)\rangle = \langle \pi(a_0)i(a), i(b_0)\rangle \\ &= \langle i(a), \pi(a_0^*)i(b_0)\rangle = \langle i(a), i(a_0^*b_0)\rangle = \langle i(Pa), i(a_0^*b_0)\rangle \\ &= \langle i(Pa), \pi(a_0^*)i(b_0)\rangle = \langle \pi(a_0)i(Pa), i(b_0)\rangle = \langle i(a_0Pa), i(b_0)\rangle,\end{aligned}$$

which proves (8.3), $b_0 \in \mathfrak{A}_0$ being arbitrary.

It is easy to see that $i(Pa_0) = i(a_0)$ for $a_0 \in \mathfrak{A}_0$. Putting $a_0 = Pa$ in (8.2) we get (8.4). □

Another property of P, that we shall need, is the following:

$$i(P(aa_0)) = i(P(a)a_0) \text{ for all } a \in \mathfrak{A} \text{ and all } a_0 \in \mathfrak{A}_0. \tag{8.5}$$

The formula (8.5) is the same as (8.3), except that the order of a and a_0 is reversed. Of course both (8.2) and (8.5) are satisfied, if $\mathfrak{A} = \mathfrak{A}_0$ so that P can be taken as the identity map.

Lemma 8.3. *Let $\mathfrak{A}_0$ be a sub-$*$-algebra of $\mathfrak{A}$ and $P : \mathfrak{A} \to \mathfrak{A}_0$ a map such that* (8.2) *and* (8.5) *hold. Then*

$$\langle \pi(Pa)x_0, y_0\rangle = \langle \pi(a)x_0, y_0\rangle \text{ for all } a \in \mathfrak{A},\ x_0, y_0 \in \overline{i(\mathfrak{A}_0)}. \tag{8.6}$$

Proof. For any $a \in \mathfrak{A}$ and $a_0, b_0 \in \mathfrak{A}_0$ we find that

$$\begin{aligned}\langle \pi(Pa)i(a_0), i(b_0)\rangle &= \langle i((Pa)a_0), i(b_0)\rangle = \langle i(P(aa_0)), i(b_0)\rangle \\ &= \langle i(aa_0), i(b_0)\rangle = \langle \pi(a)i(a_0), i(b_0)\rangle,\end{aligned}$$

which implies (8.6). □

8.2 Irreducible subrepresentations of π

In this subsection we look for an inverse to Corollary 4.6, so we want to deduce that the inequality (4.3) holds when σ is the restriction of π to an irreducible, invariant, closed subspace of $\mathcal{H}$, π being the representation from the general set up of Section 3. For a unimodular group G our result (Theorem 8.4) reduces to the well known fact that any irreducible subrepresentation of the left regular representation on $L^2(G)$ is square integrable, but Theorem 9.4, that applies it to representations induced from a character, seems to be new. Theorem 8.4 works in a more general setting than [14] and [13]. However, it deals with irreducible representations where [14, Theorem 4.6] manages with cyclic ones, and (at least in examples) with unimodular groups which is not necessary in [13, Theorem 1.2].

Theorem 8.4. *Let $\mathfrak{A}_0$ be a sub-$*$-algebra of $\mathfrak{A}$ and $P : \mathfrak{A} \to \mathfrak{A}_0$ a map such that* (8.2) *and* (8.5) *hold. Assume furthermore that $\|i(a_0^*)\| = \|i(a_0)\|$ for all $a_0 \in \mathfrak{A}_0$.*

If $\mathcal{H}'$ is an irreducible, π-invariant, closed subspace of $\mathcal{H}$, then there exists a constant $C \in [0, \infty[$ such that

$$|\langle \pi(a)x_0', x\rangle| \le C\, \|i(a)\|\, \|x_0'\|\, \|x\| \tag{8.7}$$

for all $a \in \mathfrak{A}$, $x_0' \in \mathcal{H}' \cap \overline{i(\mathfrak{A}_0)}$ and $x \in \mathcal{H}$.

We shall in Proposition 8.9(b) see that $\mathcal{H}' \cap \overline{i(\mathfrak{A}_0)}$ under certain conditions is the set of spherical vectors in $\mathcal{H}'$.

Proof. By the uniform boundedness principle it suffices to prove that there to any given $x_0' \in \mathcal{H}' \cap \overline{i(\mathfrak{A}_0)}$ exists a constant $C(x_0') \in [0, \infty[$ such that

$$|\langle \pi(a)x_0', x\rangle| \leq C(x_0')\, \|i(a)\|\, \|x\| \text{ for all } a \in \mathfrak{A} \text{ and } x \in \mathcal{H}. \tag{8.8}$$

Let P' denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}' \cap \overline{i(\mathfrak{A}_0)}$. Note that $\pi(a_0)$ leaves $\mathcal{H}' \cap \overline{i(\mathfrak{A}_0)}$ invariant for any $a_0 \in \mathfrak{A}_0$, so that $\pi(a_0)$ commutes with P'. The estimate (8.8) is trivially true if $x_0' = 0$, so we may assume that $x_0' \neq 0$. In that case $\mathcal{H}' \cap \overline{i(\mathfrak{A}_0)} \neq \{0\}$, and so there exists an $a_0 \in \mathfrak{A}_0$ such that $P'(i(a_0)) \neq 0$.

Now for any $a \in \mathfrak{A}$ we get by Lemma 8.3 that

$$\begin{aligned}\langle \pi(a)x_0', P'(i(a_0))\rangle &= \langle \pi(Pa)x_0', P'(i(a_0))\rangle = \langle P'\pi(Pa)x_0', i(a_0)\rangle \\ &= \langle \pi(Pa)x_0', i(a_0)\rangle = \langle x_0', \pi(Pa)^* i(a_0)\rangle = \langle x_0', i((Pa)^* a_0)\rangle.\end{aligned}$$

Using Lemma 8.1 we continue as follows

$$= \langle i(a_0^* Pa), (x_0')^*\rangle = \langle \pi(a_0^*) i(Pa), (x_0')^*\rangle = \langle i(Pa), \pi(a_0)((x_0')^*)\rangle,$$

from which we derive the inequality

$$\begin{aligned}|\langle \pi(a)x_0', P'(i(a_0))\rangle| &= |\langle i(Pa), \pi(a_0)((x_0')^*)\rangle| \\ &\leq \|\pi(a_0)((x_0')^*)\|\, \|i(Pa)\| \leq \|\pi(a_0)((x_0')^*)\|\, \|i(a)\|,\end{aligned}$$

where the last inequality sign is justified by (8.4).

Using the assumption of irreducibility of $\mathcal{H}'$ we get from Theorem 4.6 that the operator $T_{x_0'}$ from $\mathcal{H}'$ to $\mathcal{H}$ is everywhere defined, which means that there to each $x' \in \mathcal{H}'$ exists a constant $C(x')$ such that

$$|\langle \pi(a)x_0', x'\rangle| \leq C(x')\, \|i(a)\| \text{ for all } a \in \mathfrak{A}.$$

We get the same estimate for an arbitrary $x \in \mathcal{H}$ by decomposing x as $x = x' + x^\perp$, where $x' \in \mathcal{H}'$ and $x^\perp \in (\mathcal{H}')^\perp$, and so we see that there to each $x \in \mathcal{H}$ exists a constant $C(x)$ such that

$$|\langle \pi(a)x_0', x\rangle| \leq C(x)\, \|i(a)\| \text{ for all } a \in \mathfrak{A}.$$

Finally we apply the uniform boundedness principle to those linear functionals $x \mapsto \langle x, \pi(a)x_0'\rangle$ for which $\|i(a)\| \leq 1$. □

8.3 The orthogonality relations for spherical vectors

The key to the proof of the orthogonality relations for spherical vectors (Theorem 8.5) is Schur's lemma, just like it is for the classical orthogonality relations for matrix elements of a square integrable representation.

In this subsection we do not need to specify what the set $\mathcal{S}'$ of spherical vectors is, except that it must satisfy the identity (8.9) below. $\mathcal{S}'$ will be specified later in special cases.

In Theorem 8.5 and its proof we use the notation from Definition 4.3.

Theorem 8.5 (The orthogonality relations)**.** *Let $\mathfrak{A}_0$ be a sub-$*$-algebra of $\mathfrak{A}$ such that $\|i(a_0^*)\| = \|i(a_0)\|$ for all $a_0 \in \mathfrak{A}_0$ and such that there exists a map $P : \mathfrak{A} \to \mathfrak{A}_0$ satisfying (8.2).*

Let σ be an irreducible $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}'$. Let $\mathcal{S}'$ be a subset of $\mathcal{H}'$ such that

$$\langle \sigma(P(a))x', y' \rangle = \langle \sigma(a)x', y' \rangle \quad \textit{for all } a \in \mathfrak{A},\ x', y' \in \mathcal{S}'. \tag{8.9}$$

Assume finally that there exist non-zero vectors $x_1' \in \mathcal{S}'$, $x_2' \in \mathcal{H}'$ and a constant C such that $|\langle \sigma(a)x_1', x_2' \rangle| \leq C\|i(a)\|$ for all $a \in \mathfrak{A}$.

(a) $T_{x_0'} \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ and $T_{x_0'}\mathcal{S}' \subseteq \overline{i(\mathfrak{A}_0)}$ for any $x_0' \in \mathcal{S}'$.

(b) There exists a constant $c > 0$ such that

$$\langle T_{x_0'}x', T_{y_0'}y' \rangle = c\, \langle x', y' \rangle\, \overline{\langle x_0', y_0' \rangle} \tag{8.10}$$

for all $x_0', y_0' \in \mathcal{S}'$ and $x', y' \in \mathcal{H}'$.

(c) $c = \|T_{z_0'}z_0'\|^2/\|z_0'\|^4$, where z_0' is any non-zero vector in $\mathcal{S}'$.

Proof. (a) Let $x_0' \in \mathcal{S}'$. Since $T_{x_1'} \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ (by the proof of Corollary 4.6) we get from (8.9) for any $a \in \mathfrak{A}$ that

$$\langle \sigma(a)x_0', x_1' \rangle = \langle \sigma(Pa)x_0', x_1' \rangle = \overline{\langle \sigma((Pa)^*)x_1', x_0' \rangle} = \overline{\langle i((Pa)^*), T_{x_1'}x_0' \rangle}.$$

Since $Pa \in \mathfrak{A}_0$, we get, using (8.4) for the last inequality sign, that

$$|\langle \sigma(a)x_0', x_1' \rangle| \leq \|T_{x_1'}x_0'\|\|i((Pa)^*)\| = \|T_{x_1'}x_0'\|\|i(Pa)\| \leq \|T_{x_1'}x_0'\|\|i(a)\|.$$

We then see from the proof of Corollary 4.6 that $T_{x_0'} \in \mathcal{B}(\mathcal{H}', \mathcal{H})$.

Let $x' \in \mathcal{S}'$. From the formula

$$\langle \sigma(a)x_0', x' \rangle = \langle i(a), T_{x_0'}x' \rangle, \text{ valid for all } a \in \mathfrak{A}, \tag{8.11}$$

it follows that the map $i(a_0) \mapsto \langle \sigma(a_0)x_0', x' \rangle$ is a continuous linear functional on $i(\mathfrak{A}_0) \subseteq \mathcal{H}$. Thus there exists a vector $y \in \overline{i(\mathfrak{A}_0)}$ such that

$$\langle \sigma(a_0)x_0', x' \rangle = \langle i(a_0), y \rangle \text{ for all } a_0 \in \mathfrak{A}_0.$$

Now, by the assumption (8.9) we get for all $a \in \mathfrak{A}$ that

$$\langle \sigma(a)x_0', x' \rangle = \langle \sigma(Pa)x_0', x' \rangle = \langle i(Pa), y \rangle = \langle i(a), y \rangle,$$

where the last equality sign comes from the assumption (8.2) that can be used because $y \in \overline{i(\mathfrak{A}_0)}$. Comparing with the formula (8.11) we see that $T_{x_0'}x' = y \in \overline{i(\mathfrak{A}_0)}$.

(b) Let $x_0', y_0' \in \mathcal{S}'$. We shall below need the formula

$$T_{x_0'}y_0' = (T_{y_0'}x_0')^*, \tag{8.12}$$

in which the $*$ refers to the involution in $\overline{i(\mathfrak{A}_0)}$ from Lemma 8.1. The formula, which makes sense because $T_{y_0'}x_0' \in \overline{i(\mathfrak{A}_0)}$ by (a), is a consequence of the following computation which is valid for any $a_0 \in \mathfrak{A}$:

$$\begin{aligned}\langle i(a_0), T_{x_0'}y_0'\rangle &= \langle \sigma(a_0)x_0', y_0'\rangle = \overline{\langle \sigma(a_0^*)y_0', x_0'\rangle} \\ &= \overline{\langle i(a_0^*), T_{y_0'}x_0'\rangle} = \langle T_{y_0'}x_0', i(a_0^*)\rangle = \langle i(a_0), (T_{y_0'}x_0')^*\rangle.\end{aligned}$$

The operator $T_{y_0'}^* T_{x_0'} \in \mathcal{B}(\mathcal{H}')$ is an intertwining operator for the irreducible representation σ (Theorem 4.4(b)), so there exists by Schur's lemma a constant $k(y_0', x_0') \in \mathbb{C}$ such that $T_{y_0'}^* T_{x_0'} = k(y_0', x_0')I$. Thus

$$\langle T_{x_0'}x', T_{y_0'}y'\rangle = \langle T_{y_0'}^* T_{x_0'}x', y'\rangle = k(y_0', x_0')\langle x', y'\rangle \text{ for all } x', y' \in \mathcal{H}'. \tag{8.13}$$

We continue by determining $k(y_0', x_0')$. Using (8.12) we find

$$\begin{aligned}k(y_0', x_0')\langle x_1', x_1'\rangle &= \langle T_{x_0'}x_1', T_{y_0'}x_1'\rangle = \langle (T_{y_0'}x_1')^*, (T_{x_0'}x_1')^*\rangle \\ &= \langle T_{x_1'}y_0', T_{x_1'}x_0'\rangle = k(x_1', x_1')\langle y_0', x_0'\rangle,\end{aligned}$$

so that $k(y_0', x_0') = c\langle y_0', x_0'\rangle$, where $c = k(x_1', x_1')/\|x_1'\|^2$. Substituting this into (8.13) we get (8.10), except for the claim that $c > 0$. To establish the claim we put $x_0' = y_0' = x' = y' = x_1'$ in (8.10) and get that $\langle T_{x_1'}x_1', T_{x_1'}x_1'\rangle = c\,\langle x_1', x_1'\rangle\,\overline{\langle x_1', x_1'\rangle}$, which shows us that

$$c = \frac{\|T_{x_1'}x_1'\|^2}{\|x_1'\|^4} \geq 0.$$

Since $T_{x_1'}$ is injective (Theorem 4.4(g)) we conclude that $c > 0$.

(c) Put $x_0' = y_0' = x' = y' = z_0'$ in (8.10). □

8.4 Invariance under a group H

This subsection constitutes an intermediary between the general theory and the special case of induced representations. We introduce an action by a group H on the $*$-algebra $\mathfrak{A}$, let $\mathfrak{A}_0$ be the set of fixed point in $\mathfrak{A}$ and describe certain circumstances under which the conditions (8.2) and (8.5) hold. For induced representations $\mathfrak{A}$ will be the convolution algebra $C_c(G)$ of the group G and H will be the subgroup of G, from which we induce the character χ.

Let H be a transformation group of $\mathfrak{A}$ with the action by $h \in H$ on $a \in \mathfrak{A}$ denoted $h \cdot a$. We relate the group action to the algebraic structure by assuming that the map $a \mapsto h \cdot a$ is a $*$-algebra automorphism for each $h \in H$. Then $\mathfrak{A}_0 = \{a \in \mathfrak{A} \mid h \cdot a = a \text{ for all } h \in H\}$ is a sub-$*$-algebra of $\mathfrak{A}$.

We assume furthermore that $i : \mathfrak{A} \to \mathcal{H}$ and the action by the group H on $\mathfrak{A}$ are tied together by a unitary representation π_1 of H on $\mathcal{H}$ such that

$$i(h \cdot a) = \pi_1(h)(i(a)) \text{ for all } h \in H \text{ and } a \in \mathfrak{A}. \tag{8.14}$$

π_1 is uniquely determined by (8.14), $i(\mathfrak{A})$ being dense in $\mathcal{H}$.

By the *spherical vectors* in $\mathcal{H}$ we mean the set

$$\mathcal{S} = \{x \in \mathcal{H} \mid \pi_1(h)x = x \text{ for all } h \in H\}. \tag{8.15}$$

It is a closed subspace of $\mathcal{H}$. All vectors are spherical if $H = \{e\}$.

In the next lemma we relate π_1 and π.

Lemma 8.6. *(a)* $\pi(h \cdot a) = \pi_1(h)\pi(a)\pi_1(h)^{-1}$ *for all* $h \in H$ *and* $a \in \mathfrak{A}$.

(b) $\overline{i(\mathfrak{A}_0)} \subseteq \mathcal{S}$.

Proof. (a) Combining (8.14) with the basic relation $\pi(a)i(b) = i(ab)$ we find for any $h \in H$, $a, b \in \mathfrak{A}$ on the one hand that

$$\pi(h \cdot a)i(h \cdot b) = i((h \cdot a)(h \cdot b)) = i(h \cdot (ab)) = \pi_1(h)i(ab) = \pi_1(h)\pi(a)i(b)$$

and on the other hand that $\pi(h \cdot a)i(h \cdot b) = \pi(h \cdot a)\pi_1(h)i(b)$, so that $\pi_1(h)\pi(a)i(b) = \pi(h \cdot a)\pi_1(h)i(b)$. Since $i(\mathfrak{A})$ is dense in $\mathcal{H}$ we infer that the first statement holds.

(b) For any $a_0 \in \mathfrak{A}_0$ we get by (8.14) that $\pi_1(h)i(a_0) = i(h \cdot a_0) = i(a_0)$, so that $i(a_0) \in \mathcal{S}$. Hence $i(\mathfrak{A}_0) \subseteq \mathcal{S}$, and so $\overline{i(\mathfrak{A}_0)} \subseteq \mathcal{S}$, $\mathcal{S}$ being a closed subset of $\mathcal{H}$. □

We assume throughout the rest of this Subsection 8.4 that H is a subgroup of G which is generated by a compact subgroup K of G and a closed central subgroup A of G, where A acts trivially on $\mathfrak{A}$. The subgroup A will play a minor role, because its action is trivial. We need the compactness of K to be able to average over K's action. We skip the details of the proof of the next lemma, because they are straightforward.

Lemma 8.7. *Let* $H = \langle K, A \rangle$ *where* K *is a compact subgroup of* G *and* A *a closed central subgroup of* G.

(a) H *is a closed subgroup of* G.

(b) H *is unimodular with Haar integral* $f \mapsto \int_K \int_A f(ka)\, da\, dk$, $f \in C_c(H)$, *where* da *is a Haar measure on* A *and* dk *as usual denotes the normalized Haar measure on* K.

(c) $\Delta_G(h) = \Delta_H(h) = 1$ *for all* $h \in H$.

(d) If G *acts in the standard way on* $C_c(G)$ *(i.e. according to the formula* (2.1)*, then the center* $Z(G)$ *of* G *acts trivially on* $C_c(G)$*. In particular* A *acts trivially on* $C_c(G)$.

(e) The map $\Phi : K/(K \cap A) \times A/(K \cap A) \to H/(K \cap A)$*, given by*

$$\Phi(k(K \cap A), a(K \cap A)) = ka(K \cap A),\ k \in K,\ a \in A,$$

is a topological isomorphism. In other words, $H/(K \cap A)$ *is the direct topological product of its subgroups* $K/(K \cap A)$ *and* $A/(K \cap A)$.

(f) Let $f \in C(K/(K \cap A))$ *and* $g \in C(A/(K \cap A))$*. Then*

$$F(ka) = f(k)g(a), \quad k \in K,\ a \in A,$$

is a well-defined function in $C(H/(K \cap A))$.

Lemma 8.7(d) shows that the assumption that A acts trivially on $\mathfrak{A}$ is automatically satisfied for the standard action by G on $\mathfrak{A} = C_c(G)$. This will then be the case for all our examples of group representations.

We next discuss existence of a projection map $P : \mathfrak{A} \to \mathfrak{A}_0$. We would like to define Pa for $a \in \mathfrak{A}$ as the mean value $Pa := \int_K k \cdot a \, dk$ of the vector-valued function $k \mapsto k \cdot a$. However, $\mathfrak{A}$ does not come equipped with any topology, so the integral makes no sense in general. And even if it does make sense, we shall still need some kind of continuity of $i : \mathfrak{A} \to \mathcal{H}$ to ensure that the formulas (8.16) and (8.17) below hold. To proceed further in this subsection we assume that there exists a linear map $P : \mathfrak{A} \to \mathfrak{A}_0$ such that

$$\langle i(Pa), x\rangle = \int_K \langle i(k \cdot a), x\rangle \, dk \quad \text{for all } a \in \mathfrak{A} \text{ and } x \in \mathcal{H}, \text{ and} \tag{8.16}$$

$$\langle \pi(Pa)x, y\rangle = \int_K \langle \pi(k \cdot a)x, y\rangle \, dk \quad \text{for all } a \in \mathfrak{A} \text{ and } x, y \in \mathcal{H}. \tag{8.17}$$

Remark 8.8. (1) Noting that the map $k \mapsto i(k \cdot a) = \pi_1(k)(i(a))$ is continuous from K to $\mathcal{H}$ for each fixed $a \in \mathfrak{A}$ (because we have built strong continuity into the definition of a group representation, in particular of π_1), the integral on the right hand side of (8.16) makes sense.

(2) Noting that the map $k \mapsto \langle \pi(k \cdot a)x, y\rangle$ is continuous (use Lemma 8.6(a)) the integral on the right hand side of (8.17) makes sense.

Proposition 8.9. *If the linear map $P : \mathfrak{A} \to \mathfrak{A}_0$ exists and satisfies* (8.16) *and* (8.17)*, then*

(a) (8.2) *and* (8.5) *hold.*

(b) $\mathcal{S} = \overline{i(\mathfrak{A}_0)}$.

Proof. (a) Combining (8.16) with (8.14) we get for any $a \in \mathfrak{A}$ and $a_0 \in \mathfrak{A}_0$ that

$$\begin{aligned}\langle i(Pa), i(a_0)\rangle &= \int_K \langle \pi_1(k)i(a), i(a_0)\rangle dk = \int_K \langle i(a), \pi_1(k^{-1})i(a_0)\rangle dk \\ &= \int_K \langle i(a), i(k^{-1} \cdot a_0)\rangle dk = \int_K \langle i(a), i(a_0)\rangle dk = \langle i(a), i(a_0)\rangle,\end{aligned}$$

which means that (8.2) is satisfied.
For any $a \in \mathfrak{A}$, $a_0 \in \mathfrak{A}_0$ and $x \in \mathcal{H}$ we find that

$$\begin{aligned}\langle i(P(aa_0)), x\rangle &= \int_K \langle i(k \cdot [aa_0]), x\rangle \, dk = \int_K \langle i([k \cdot a][k \cdot a_0]), x\rangle \, dk \\ &= \int_K \langle i((k \cdot a)a_0), x\rangle \, dk = \int_K \langle \pi(k \cdot a)i(a_0), x\rangle \, dk \\ &= \langle \pi(Pa)i(a_0), x\rangle = \langle i((Pa)a_0), x\rangle,\end{aligned}$$

from which we conclude that $i(P(aa_0)) = i((Pa)a_0)$, which is (8.5).

(b) In Lemma 8.6 we proved that $\overline{i(\mathfrak{A}_0)} \subseteq \mathcal{S}$. To prove the converse we let $x_0 \in \mathcal{S}$ and $y \in \mathcal{H}$ be arbitrary. The identity $\langle i(Pa) - x_0, y\rangle = \int_K \langle \pi_1(k)[i(a) - x_0], y\rangle \, dk$ implies that

$$|\langle i(Pa) - x_0, y\rangle| \leq \int_K |\langle \pi_1(k)[i(a) - x_0], y\rangle| \, dk \leq \|i(a) - x_0\| \|y\|,$$

and so $\|i(Pa) - x_0\| \le \|i(a) - x_0\|$ for any $a \in \mathfrak{A}$, $y \in \mathcal{H}$ being arbitrary. Since $i(\mathfrak{A})$ is dense in $\mathcal{H}$, we get that $x_0 \in \overline{i(\mathfrak{A}_0)}$. □

9 Sub-representations of induced representations

9.1 More on induced representations

In this subsection we continue the discussion of induced representations from Section 7.1 by deriving some formulas that we shall need soon, and by relating the induced representations to the discussion in Subsection 8.4.

As in Section 7.1 we let H be a closed subgroup of the locally compact Hausdorff group G and χ a fixed unitary character of H. We recall that G, and hence also H, acts by $*$-automorphisms as a transformation group on the convolution algebra $\mathfrak{A} = C_c(G)$ according to the prescription (2.1), that $i(\phi) = \phi^\chi$, that $\pi = L$ and that $\mathfrak{A}_0 = \{\phi \in C_c(G) \mid h \cdot \phi = \phi \text{ for all } h \in H\} = \{\phi \in C_c(G) \mid \Delta_G(h)\phi(h^{-1}th) = \phi(t) \text{ for all } h \in H \text{ and } t \in G\}$.

Lemma 9.1. *(a) We have for any $h \in H$ that*

$$i(h \cdot \phi) = \overline{\chi(h)}\, \rho(h)^{1/2}\, \pi(h)(i(\phi)) \text{ for all } \phi \in C_c(G).$$

(b) If G is unimodular, then $\|i(\phi_0^*)\|_{\mathcal{H}^\chi} = \|i(\phi_0)\|_{\mathcal{H}^\chi}$ for all $\phi_0 \in \mathfrak{A}_0$.

Proof. (a) is proved by computing: For any $h_0 \in H$ and $s \in G$ we get that

$$\begin{aligned}
[i(h_0 \cdot \phi)](s) = (h_0 \cdot \phi)^\chi(s) &= \int_H (h_0 \cdot \phi)(sh)\rho(h)^{-\frac{1}{2}}\chi(h)dh \\
&= \Delta_G(h_0) \int_H \phi(h_0^{-1}shh_0)\rho(h)^{-\frac{1}{2}}\chi(h)dh \\
&= \Delta_G(h_0)\Delta_H(h_0^{-1}) \int_H \phi(h_0^{-1}sh)\rho(hh_0^{-1})^{-\frac{1}{2}}\chi(hh_0^{-1})dh \\
&= \rho(h_0)^{1/2}\overline{\chi(h_0)} \int_H \phi(h_0^{-1}sh)\rho(h)^{-\frac{1}{2}}\chi(h)dh \\
&= \rho(h_0)^{1/2}\overline{\chi(h_0)}[\pi(h_0)i(\phi)](s) = \rho(h_0)^{1/2}\overline{\chi(h_0)}[\pi(h_0)i(\phi)](s).
\end{aligned}$$

(b) is also proved by a computation. We skip the straightforward details. □

Our next objective is to produce a projection map $P : \mathfrak{A} \to \mathfrak{A}_0$, such that the conditions of Subsection 8.4 hold. We do so under our earlier assumption that $H = \langle K, A\rangle$, where K is a compact subgroup of G and A is a closed central subgroup of G. From Lemma 8.7 we see that then $\rho = 1$ and $\mathfrak{A}_0 = \{\phi \in C_c(G) \mid \phi(hsh^{-1}) = \phi(s) \text{ for all } h \in H \text{ and } s \in G\}$.

We define the map $P : \mathfrak{A} \to \mathfrak{A}_0$ by

$$(P\phi)(s) = \int_K \phi(k^{-1}sk)\, dk, \; s \in G, \; \phi \in \mathfrak{A}. \tag{9.1}$$

The definition of $P\phi$, $\phi \in \mathfrak{A}$, makes sense for each $s \in G$, because the map $k \mapsto \phi(k^{-1}sk)$ is a continuous function on the compact set K and hence integrable. We skip the proof (easy) that P is a linear map of $\mathfrak{A}$ onto $\mathfrak{A}_0$.

Lemma 9.2. *(a) The conditions* (8.2) *and* (8.5) *hold.*

(b) $\mathcal{S} = \{x \in \mathcal{H}^\chi \mid \pi(h)x = \chi(h)x \text{ for all } h \in H\} = \overline{i(\mathfrak{A}_0)}$.

Proof. (a): This is Proposition 8.9(a).

(b): Comparing definition (8.14) with Lemma 9.1(a) we see that $\pi_1(h) = \overline{\chi(h)}\,\pi(h)$ for all $h \in H$, so that the set $\mathcal{S}$ of spherical vectors defined by (8.15) becomes $\mathcal{S} = \{x \in \mathcal{H}^\chi \mid \pi(h)x = \chi(h)x$ for all $h \in H\}$. This is the first identity of (b). The other comes from Proposition 8.9(b). □

The formula for $\mathcal{S}$ in Lemma 9.2(b) shows that our general definition of spherical vectors agrees with the one in Definition 7.4.

Remark 9.3. We will in this remark show that there are non-zero spherical vectors in $\mathcal{H}^\chi$, if H has the form $H = \langle K, A\rangle$ where K is a compact subgroup of G and A a closed central subgroup of G, and if furthermore $\chi = 1$ on $K \cap A$.

We note that χ is a function in $C(H/(K \cap A))$ and choose a function $\psi_A \in C_c(A/(K \cap A))$ such that

$$\int_{A/K} \psi_A(\dot a)\overline{\chi(\dot a)}\, d\dot a = 1.$$

Here $d\dot a$ denotes a Haar measure on $A/(K \cap A)$. Such a function ψ_A can be found: If the integral vanishes for all $\psi_A \in C_c(A/(K \cap A))$ then $\chi(\dot a) = 1$ for all $\dot a \in A/K$ in contradiction with $|\chi| = 1$. It follows that

$$\int_A \psi_A(a)\overline{\chi(a)}\, da = 1.$$

It follows from Lemma 8.7(f) that the prescription $\psi(ka) = \chi(k)\psi_A(a)$, $k \in K$, $a \in A$, gives a well-defined function in $C_c(H/(K \cap A)) \subseteq C_c(G)$.

By Urysohn's lemma we find a function $\alpha \in C_c(G)$ such that $\alpha = 1$ on the support of ψ. By Tietze's extension theorem we find a function $\beta \in C(G)$ such that $\beta = \psi$ on H. Then $\phi := \alpha\beta \in C_c(G)$ and $\phi = \psi$ on H. Finally we introduce $\phi_0 := \int_K k \cdot \phi\, dk \in \mathfrak{A}_0$. Now

$$\begin{aligned} i(\phi_0)(e) = \phi^\chi(e) &= \int_H \phi_0(h)\overline{\chi(h)}\, dh = \int_H \Big\{ \int_K \phi(k^{-1}hk)\, dk \Big\} \overline{\chi(h)}\, dh \\ &= \int_K \Big\{ \int_H \phi(k^{-1}hk)\overline{\chi(h)}\, dh \Big\}\, dk \\ &= \int_K \Delta_H(k^{-1}) \Big\{ \int_H \phi(h)\overline{\chi(khk^{-1})}\, dh \Big\}\, dk. \end{aligned}$$

Since K is compact we get that $\Delta_H(k^{-1}) = 1$ for all $k \in K$, so

$$\begin{aligned} i(\phi_0)(e) &= \int_K \Big\{ \int_H \phi(h)\overline{\chi(khk^{-1})}\, dh \Big\}\, dk \\ &= \int_K \Big\{ \int_H \phi(h)\overline{\chi(k)}\chi(h)\chi(k^{-1})\, dh \Big\}\, dk \\ &= \int_K \Big\{ \int_H \phi(h)\overline{\chi(h)}\, dh \Big\}\, dk = \int_H \phi(h)\overline{\chi(h)}\, dh. \end{aligned}$$

We now use the description of the Haar measure on H from Lemma 8.7(b) and find

$$\begin{aligned} i(\phi_0)(e) &= \int_K \int_A \psi(ka)\overline{\chi(ka)}\,da\,dk \\ &= \int_K \int_A \chi(k)\psi_A(a)\overline{\chi(k)}\,\overline{\chi(a)}\,da\,dk = \int_A \psi_A(a)\overline{\chi(a)}\,da = 1. \end{aligned}$$

Thus $i(\phi_0) \neq 0$, so $i(\phi_0)$ is a non-zero spherical vector in $\mathcal{H}^\chi$.

9.2 On spherical sub-representations of an induced representation

Throughout this subsection we assume that G is second countable, and that H is a closed subgroup of G As earlier we fix a unitary character χ of H and let U^χ denote the corresponding induced representation.

We recall that the set of spherical vectors for a unitary representation σ of G on a Hilbert space $\mathcal{H}'$ is $\mathcal{S}' = \{x' \in \mathcal{H}' \mid \sigma(h)x' = \chi(h)x' \text{ for all } h \in H\}$ (Definition 7.8).

We have in Theorem 7.11(c) seen that any irreducible unitary representation σ of G such that

(a) σ has a non-zero χ-spherical vector x'_0, and

(b) $|\langle x'_1, \sigma(\cdot)x'_0\rangle| \in L^2(G/H, \mu)$ for some non-zero vector x'_1,

is a sub-representation of U^χ.

We shall address ourselves to the converse question. More precisely whether any irreducible unitary sub-representation of U^χ with a non-zero spherical vector occurs in this way, i.e. whether it automatically satisfies (b). The following Theorem 9.4 claims that so is the case, if G is unimodular and $H = \langle A, K\rangle$, where A is a closed, central subgroup of G and K is a compact subgroup of G. The classical case of square integrable representations of a unimodular group is covered by the special case of $H = \{e\}$. The theorem also covers the case of a Riemannian symmetric pair (G, H), in which G is a semi-simple Lie group and H is a compact subgroup of G. For the corresponding result for square integrable representations of groups, that need not be unimodular, see [2, p. 4] and [5, Theorem 2].

Theorem 9.4. *Assume that G is unimodular, and that $H = \langle A, K\rangle$, where A is a closed, central subgroup of G, and where K is a compact subgroup of G. Let σ be an irreducible sub-representation of U^χ on a subspace $\mathcal{H}'$ of $\mathcal{H}^\chi$ having a spherical vector $f'_0 \in \mathcal{H}'$.*

Then $\langle f', \sigma(\cdot)f'_0\rangle_{\mathcal{H}^\chi} \in \mathcal{H}^\chi$ for all $f' \in \mathcal{H}'$, so in particular

$$|\langle f', \sigma(\cdot)f'_0\rangle_{\mathcal{H}^\chi}| \in L^2(G/H, \mu) \text{ for all } f' \in \mathcal{H}'. \tag{9.2}$$

Proof. $\mathfrak{A}_0 = \{a \in C_c(G) \mid h \cdot a = a \text{ for all } h \in H\}$ is a sub-$*$-algebra of $\mathfrak{A} = C_c(G)$. Formula (9.1) defines according to Lemma 9.2(a) a linear map $P : \mathfrak{A} \to \mathfrak{A}_0$ such that the identities (8.2) and (8.5) hold.

That $\|i(a_0^*)\|_{\mathcal{H}^\chi} = \|i(a_0)\|_{\mathcal{H}^\chi}$ for all $a_0 \in \mathfrak{A}_0$, was derived in Lemma 9.1(b). The theorem is thus a corollary of Theorem 8.4. □

Example 9.5. Let H be a central, closed subgroup of a unimodular group G. If $\sigma \in \widehat{G}$ is an irreducible, unitary representation of G on a Hilbert space $\mathcal{H}'$, then $\sigma|_H$ is by Schur's lemma a unitary character of H, say $\sigma(h) = \chi(h)I$, $h \in H$. Let $\pi = U^\chi$. By definition of χ any vector in $\mathcal{H}'$ is χ-spherical. If $|\langle y_0', \sigma(\cdot)x_0'\rangle| \in L^2(G/H)$ for some non-zero vectors $y_0', x_0' \in \mathcal{H}'$, then σ is a sub-representation of π, and $\langle y', \sigma(\cdot)x'\rangle \in \mathcal{H}^\chi$ for all $y', x' \in \mathcal{H}'$.

A particular example is the Heisenberg group

$$G = \{(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R}\},$$

with H as its center $H = \{(0,0,z) \mid z \in \mathbb{R}\}$.

The Schrödinger representation σ of G on $L^2(\mathbb{R})$, given by

$$[\sigma(x,y,z)f](t) = e^{2\pi i z} e^{2\pi i y t} f(t+x) \qquad \text{for } f \in L^2(\mathbb{R}),\ (x,y,z) \in G,\ t \in \mathbb{R},$$

is irreducible as is well known. σ reduces on H to the character χ given by $\chi(0,0,z) = \exp(2\pi i z)$ for $z \in \mathbb{R}$.

Taking $f_0 \in \mathcal{S}(\mathbb{R})$ and using that the Fourier transformation is unitary on $L^2(\mathbb{R})$ we find that $|\langle f_0, \sigma(\cdot)f_0\rangle| \in L^2(G/H) = L^2(\mathbb{R}^2)$. It follows that $\sigma \subseteq U^\chi$ and that each matrix-coefficient $\langle f_1, \sigma(\cdot)f_2\rangle$ of σ is in $\mathcal{H}^\chi$.

9.3 The orthogonality relations

We get Schur's classical orthogonality relations for a unimodular group G by taking $H = \{e\}$ in the following Theorem 9.6, and the more general case of square integrability modulo a compact subgroup K of G by taking $H = K$ and $\chi = 1$ there. Square integrability modulo the center is discussed in Subsection 9.4.

Theorem 9.6. *Let G be unimodular, and let $H = \langle A, K\rangle$, where A is a closed, central subgroup of G, and where K is a compact subgroup of G. Let σ be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}'$. Assume that there exist non-zero vectors $x_1', x_0' \in \mathcal{H}'$ such that $\langle x_1', \sigma(\cdot)x_0'\rangle \in \mathcal{H}^\chi$. Then*

(a) $\langle x', \sigma(\cdot)y_0'\rangle \in \mathcal{H}^\chi$ for all $x', y_0' \in \mathcal{H}'$ such that y_0' is χ-spherical.

(b) There exists a constant $c > 0$, such that we for all $x', y', x_0', y_0' \in \mathcal{H}'$ where x_0' and y_0' are χ-spherical, have that

$$\begin{aligned} &\langle \langle x', \sigma(\cdot)x_0'\rangle, \langle y', \sigma(\cdot)y_0'\rangle\rangle_{\mathcal{H}^\chi} \\ &= \int_{G/H} \langle x', \sigma(s)x_0'\rangle\, \overline{\langle y', \sigma(s)y_0'\rangle}\, d\mu_{G/H}(sH) \\ &= c\,\langle x', y'\rangle_{\mathcal{H}'} \langle y_0', x_0'\rangle_{\mathcal{H}'}. \end{aligned} \tag{9.3}$$

Proof. We derive Theorem 9.6 as a corollary of Theorem 8.5:

$\mathfrak{A}_0 = \{a \in C_c(G) \mid h \cdot a = a \text{ for all } h \in H\}$ is a sub-$*$-algebra of $\mathfrak{A} = C_c(G)$. That $\|i(a_0^*)\|_{\mathcal{H}^\chi} = \|i(a_0)\|_{\mathcal{H}^\chi}$ for all $a_0 \in \mathfrak{A}_0$ was stated in Lemma 9.1(b). Formula (9.1) defines according to Lemma 9.2 a linear map $P : \mathfrak{A} \to \mathfrak{A}_0$ such that the identities (8.2) and (8.5) hold. It also says that $\mathcal{S} = \overline{i(\mathfrak{A}_0)}$.

Let us reintroduce the isometry $W : \mathcal{H}' \to \mathcal{H}^\chi$ in the polar decomposition of the operator $T_{x'_0}$ from Theorem 4.4. W intertwines σ and $\pi = L$, so $W\mathcal{S}' \subseteq \mathcal{S}$. Via Lemma 8.3 we get for any spherical vectors $x', y' \in \mathcal{H}'$ and $a \in \mathfrak{A}$ that

$$\begin{aligned}\langle \sigma(Pa)x', y'\rangle &= \langle W\sigma(Pa)x', Wy'\rangle = \langle \pi(Pa)Wx', Wy'\rangle \\ &= \langle \pi(a)Wx', Wy'\rangle = \langle W\sigma(a)x', Wy'\rangle = \langle \sigma(a)x', y'\rangle,\end{aligned}$$

which means that the hypothesis (8.9) of Theorem 8.5 holds.

The assumption $\langle x'_1, \sigma(\cdot)x'_0\rangle \in \mathcal{H}^\chi$ of Theorem 9.6 implies that $x'_0 \in \mathcal{S}'$. Indeed, using the transformation property (II) for elements of $\mathcal{H}^\chi$ we find for all $s \in G$ and $h \in H$ that

$$\langle \sigma(s)x'_1, \sigma(h)x'_0\rangle = \langle x'_1, \sigma(s^{-1}h)x'_0\rangle = \overline{\chi(h)}\langle x'_1, \sigma(s^{-1})x'_0\rangle = \langle \sigma(s)x'_1, \chi(h)x'_0\rangle,$$

from which we infer that $\sigma(h)x'_0 = \chi(h)x'_0$, x'_1 being cyclic.

Lemma 7.6 says that there is a constant C such that $|\langle \sigma(a)x'_0, x'_1\rangle| \leq C\|i(a)\|$ for all $a \in C_c(G)$, which is the last assumption of Theorem 8.5.

Finally $T_{y'}x' = \langle x', \sigma(\cdot)y'\rangle$ for all $x', y' \in \mathcal{H}'$ by Lemma 7.7. □

9.4 On representations that are square integrable modulo the center

Let σ be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}'$. By Schur's lemma the restriction of σ to the center $Z(G)$ of G is a unitary character χ of $Z(G)$, so $\sigma(z) = \chi(z)I$ for all $z \in Z(G)$. It follows that $s \mapsto \langle x'_1, \sigma(s)y'_1\rangle\overline{\langle x'_2, \sigma(s)y'_2\rangle}$ is a function on $G/Z(G)$ for any $x'_1, y'_1, x'_2, y'_2 \in \mathcal{H}'$. In particular that the absolute value $|\langle x', \sigma(\cdot)y'\rangle|$ of any matrix element $\langle x', \sigma(\cdot)y'\rangle$ is a continuous function on $G/Z(G)$ for any $x', y' \in \mathcal{H}'$.

Assuming G unimodular and that $|\langle x', \sigma(\cdot)y'\rangle| \in L^2(G/Z(G))$ for all $x', y' \in \mathcal{H}'$, A. Borel [1, §5.13] proved that σ is a sub-representation of U^χ, and that the following version of Schur's orthogonality relations holds: Let $d\dot{s}$ denote a Haar measure on $G/Z(G)$. Then there exists a constant $d_\sigma > 0$ such that

$$\int_{G/Z(G)} \langle x'_1, \sigma(s)y'_1\rangle\overline{\langle x'_2, \sigma(s)y'_2\rangle}\, d\dot{s} = d_\sigma^{-1}\langle x'_1, y'_1\rangle\overline{\langle x'_2, y'_2\rangle} \tag{9.4}$$

for all $x'_1, y'_1, x'_2, y'_2 \in \mathcal{H}'$.

Borel's results fit with our results on induced representations: As our subgroup $H = \langle A, K\rangle$ we take $H = Z(G) = \langle Z(G), \{e\}\rangle$. Then all vectors in $\mathcal{H}'$ are spherical. Indeed, for any $x' \in \mathcal{H}'$ we have that $\sigma(h)x' = \chi(h)x'$ for all $h \in H$, and this is the definition of x' being spherical.

We get from Theorem 7.11(c) that σ is a sub-representation of U^χ. For this we do not require G to be unimodular. To get (9.4) we put $A = Z(G)$ in Theorem 9.6(b); here G is assumed unimodular.

9.5 On Gilmore-Perelomov theory

Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}'$. For any non-zero vector $x'_0 \in \mathcal{H}'$ we define its stability subgroup up to a phase factor by

$$H_{x'_0} = \{h \in G \mid \exists\, \chi(h) \in \mathbb{C} \text{ such that } \sigma(h)x'_0 = \chi(h)x'_0\}. \tag{9.5}$$

Lemma 9.7. *$H_{x'_0}$ is a closed subgroup of G, and χ is a unitary character on $H_{x'_0}$. If σ is irreducible, then $Z(G) \subseteq H_{x'_0}$.*

Proof. The proof consists of straightforward elementary considerations, so we leave it out. □

Considering the induced representation U^χ we see that x'_0 is a non-zero spherical vector for the representation σ.

Let $x'_1 \in \mathcal{H}'$. The matrix-element $s \mapsto \langle x'_1, \sigma(s)x'_0\rangle$, $s \in G$, is in $\mathcal{H}^\chi$, if $\int_G |\langle x'_1, \sigma(s)x'_0\rangle|^2 \Theta(s)\, ds < \infty$. If furthermore x'_1 and x'_0 are cyclic vectors, then σ is a sub-representation of U^χ by Theorem 7.10.

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