UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

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Preprint Series No.: 4

April 2007
2007/04/25

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Generators of Jacobians of Hyperelliptic Curves

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Abstract

This paper provides a probabilistic algorithm to determine generators of the m-torsion subgroup of the Jacobian of a hyperelliptic curve of genus two.

1 Introduction

Let C be a hyperelliptic curve of genus two defined over a prime field $\mathbb{F}_p$, and $\mathcal{J}_C$ the Jacobian of C. Consider the rational subgroup $\mathcal{J}_C(\mathbb{F}_p)$. $\mathcal{J}_C(\mathbb{F}_p)$ is a finite abelian group, and

$$\mathcal{J}_C(\mathbb{F}_p) \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \mathbb{Z}/n_3\mathbb{Z} \oplus \mathbb{Z}/n_4\mathbb{Z},$$

where $n_i \mid n_{i+1}$ and $n_2 \mid p-1$. Frey and Rück (1994) shows that if $m \mid p-1$, then the discrete logarithm problem in the rational m-torsion subgroup $\mathcal{J}_C(\mathbb{F}_p)[m]$ of $\mathcal{J}_C(\mathbb{F}_p)$ can be reduced to the corresponding problem in $\mathbb{F}_p^\times$ (Frey and Rück, 1994, corollary 1). In the proof of this result it is claimed that the non-degeneracy of the Tate pairing can be used to determine whether r random elements of the finite group $\mathcal{J}_C(\mathbb{F}_p)[m]$ in fact is an independent set of generators of $\mathcal{J}_C(\mathbb{F}_p)[m]$. This paper provides an explicit, probabilistic algorithm to determine generators of $\mathcal{J}_C(\mathbb{F}_p)[m]$.

In short, the algorithm outputs elements γ_i of the Sylow-ℓ subgroup Γ_ℓ of the rational subgroup $\Gamma = \mathcal{J}_C(\mathbb{F}_p)$, such that $\Gamma_\ell = \bigoplus_i \langle \gamma_i \rangle$ in the following steps:

1. Choose random elements $\gamma_i \in \Gamma_\ell$ and $h_j \in \mathcal{J}_C(\mathbb{F}_p)$, $i, j \in \{1, \ldots, 4\}$.

2. Use the non-degeneracy of the tame Tate pairing τ to *diagonalize* the sets $\{\gamma_i\}_i$ and $\{h_j\}_j$ with respect to τ; i.e. modify the sets such that $\tau(\gamma_i, h_j) = 1$ if $i \neq j$ and $\tau(\gamma_i, h_i)$ is an ℓ^{th} root of unity.

3. If $\prod_i |\gamma_i| < |\Gamma_\ell|$ then go to step 1.

4. Output the elements γ_1, γ_2, γ_3 and γ_4.

The key ingredient of the algorithm is the diagonalization in step 2; this process will be explained in section 5.

We will write $\langle \gamma_i | i \in I \rangle = \langle \gamma_i \rangle_i$ and $\bigoplus_{i \in I} \langle \gamma_i \rangle = \bigoplus_i \langle \gamma_i \rangle$ if the index set I is clear from the context.

2 Hyperelliptic curves

A hyperelliptic curve is a smooth, projective curve $C \subseteq \mathbb{P}^n$ of genus at least two with a separable, degree two morphism $\phi : C \to \mathbb{P}^1$. In the rest of this paper, let C be a hyperelliptic curve of genus two defined over a prime field $\mathbb{F}_p$ of characteristic $p > 2$. By the Riemann-Roch theorem there exists an embedding $\psi : C \to \mathbb{P}^2$, mapping C to a curve given by an equation of the form

$$y^2 = f(x),$$

where $f \in \mathbb{F}_p[x]$ is of degree six and have no multiple roots (see Cassels and Flynn, 1996, chapter 1).

The set of principal divisors $\mathcal{P}(C)$ on C constitutes a subgroup of the degree zero divisors $\mathrm{Div}_0(C)$. The Jacobian $\mathcal{J}_C$ of C is defined as the quotient

$$\mathcal{J}_C = \mathrm{Div}_0(C)/\mathcal{P}(C).$$

Consider the subgroup $\mathcal{J}_C(\mathbb{F}_p) < \mathcal{J}_C$ of $\mathbb{F}_p$-rational elements. There exist numbers n_i, such that

$$\mathcal{J}_C(\mathbb{F}_p) \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \mathbb{Z}/n_3\mathbb{Z} \oplus \mathbb{Z}/n_4\mathbb{Z}, \tag{1}$$

where $n_i \mid n_{i+1}$ and $n_2 \mid p-1$ (see Frey and Lange, 2006, proposition 5.78, p. 111). We wish to determine generators of the m-torsion subgroup $\mathcal{J}_C(\mathbb{F}_p)[m] < \mathcal{J}_C(\mathbb{F}_p)$, where $m \mid |\mathcal{J}_C(\mathbb{F}_p)|$ is the largest number such that $\ell \mid p-1$ for every prime number $\ell \mid m$.

3 Finite abelian groups

Miller (2004) shows the following theorem.

Theorem 1. *Let G be a finite abelian group of torsion rank r. Then for $s \geq r$ the probability that a random s-tuple of elements of G generates G is at least*

$$\frac{C_r}{\log\log |G|}$$

if $s = r$, and at least C_s if $s > r$, where $C_s > 0$ is a constant depending only on s (and not on $|G|$).

Proof. (Miller, 2004, theorem 3, p. 251) □

Combining theorem 1 and equation (1), we expect to find generators of $\Gamma[m]$ by choosing 4 random elements $\gamma_i \in \Gamma[m]$ in approximately $\frac{\log\log|\Gamma[m]|}{C_4}$ attempts.

To determine whether the generators are independent, i.e. if $\langle\gamma_i\rangle_i = \bigoplus_i \langle\gamma_i\rangle$, we need to know the subgroups of a cyclic ℓ-group G. These are determined uniquely by the order of G, since

$$\{0\} < \langle \ell^{n-1} g\rangle < \langle \ell^{n-2} g\rangle < \cdots < \langle \ell g\rangle < G$$

are the subgroups of the group $G = \langle g\rangle$ of order ℓ^n. The following corollary is an immediate consequence of this observation.

Corollary 2. *Let U_1 and U_2 be cyclic subgroups of a finite group G. Assume U_1 and U_2 are ℓ-groups. Let $\langle u_i\rangle < U_i$ be the subgroups of order ℓ. Then*

$$U_1 \cap U_2 = \{e\} \iff \langle u_1\rangle \cap \langle u_2\rangle = \{e\}.$$

Here $e \in G$ is the neutral element.

4 The tame Tate pairing

Let $\Gamma = \mathcal{J}_C(\mathbb{F}_p)$ be the rational subgroup of the Jacobian. Consider a number $\lambda \mid \gcd(|\Gamma|, p-1)$. Let $g \in \Gamma[\lambda]$ and $h = \sum_i a_i P_i \in \Gamma$ be divisors with no points in common, and let

$$\overline{h} \in \Gamma/\lambda\Gamma$$

denote the class containing the divisor h. Furthermore, let $f \in \mathbb{F}_p(C)$ be a rational function on C with divisor $\operatorname{div}(f) = \lambda g$. Set $f(h) = \prod_i f(P_i)^{a_i}$. Then

$$e_\lambda(g, \overline{h}) = f(h)$$

is a well-defined pairing $\Gamma[\lambda] \times \Gamma/\lambda\Gamma \longrightarrow \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^\lambda$, the *Tate pairing*; cf. Galbraith (2005). Raising to the power $\frac{p-1}{\lambda}$ gives a well-defined element in the subgroup $\mu_\lambda < \mathbb{F}_p^\times$ of the λ^{th} roots of unity. This pairing

$$\tau_\lambda : \Gamma[\lambda] \times \Gamma/\lambda\Gamma \longrightarrow \mu_\lambda$$

is called the *tame Tate pairing.*

Since the class $\overline{h}$ is represented by the element $h \in \Gamma$, we will write $\tau_\lambda(g, h)$ instead of $\tau_\lambda(g, \overline{h})$. Furthermore, we will omit the subscript λ and just write $\tau(g, h)$, since the value of λ will be clear from the context.

Hess (2004) gives a short and elementary proof of the following theorem.

Theorem 3. *The tame Tate pairing τ is bilinear and non-degenerate.*

Corollary 4. *For every element $g \in \Gamma$ of order λ an element $h \in \Gamma$ exists, such that $\mu_\lambda = \langle \tau(g, h) \rangle$.*

Proof. (Silverman, 1986, corollary 8.1.1., p. 98) gives a similar result for elliptic curves and the Weil pairing. The proof of this result only uses that the pairing is bilinear and non-degenerate. Hence it applies to corollary 4. $\square$

Remark 5. In the following we only need the existence of the element $h \in \Gamma$, such that $\mu_\lambda = \langle \tau(g, h) \rangle$; we do not need to find it.

5 Generators of $\Gamma[m]$

As in the previous section, let $\Gamma = \mathcal{J}_C(\mathbb{F}_p)$ be the rational subgroup of the Jacobian. We are searching for elements $\gamma_i \in \Gamma[m]$ such that $\Gamma[m] = \bigoplus_i \langle \gamma_i \rangle$. As an abelian group, $\Gamma[m]$ is the direct sum of its Sylow subgroups. Hence, we only need to find generators of the Sylow subgroups of $\Gamma[m]$.

Set $N = |\Gamma|$ and let $\ell \mid \gcd(N, p-1)$ be a prime number. Choose four random elements $\gamma_i \in \Gamma$. Let $\Gamma_\ell < \Gamma$ be the Sylow-ℓ subgroup of Γ, and set $N_\ell = |\Gamma_\ell|$. Then $\frac{N}{N_\ell}\gamma_i \in \Gamma_\ell$. Hence, we may assume that $\gamma_i \in \Gamma_\ell$. If all the elements γ_i are equal to zero, then we choose other elements $\gamma_i \in \Gamma$. Hence, we may assume that some of the elements γ_i are non-zero.

Let $|\gamma_i| = \lambda_i$, and re-enumerate the γ_i's such that $\lambda_i \leq \lambda_{i+1}$. Since some of the γ_i's are non-zero, we may choose an index $\nu \leq 4$, such that $\lambda_\nu \neq 1$ and $\lambda_i = 1$ for $i < \nu$. Choose λ_0 minimal such that $\lambda = \frac{\lambda_\nu}{\lambda_0} \mid p-1$. Then $\mathbb{F}_p$ contains an element ζ of order λ.

Now set $g_i = \frac{\lambda_i}{\lambda}\gamma_i$, $\nu \le i \le 4$. Then $g_i \in \Gamma[\lambda]$, $\nu \le i \le 4$. Finally, choose four random elements $h_i \in \Gamma$.

Let

$$\tau : \Gamma[\lambda] \times \Gamma/\lambda\Gamma \longrightarrow \langle\zeta\rangle$$

be the tame Tate pairing. Define remainders α_{ij} modulo λ by

$$\tau(g_i, h_j) = \zeta^{\alpha_{ij}}.$$

By corollary 4, for any of the elements g_i we can choose an element $h \in \Gamma$, such that $|\tau(g_i, h)| = \lambda$. Assume that $\Gamma/\lambda\Gamma = \langle \overline{h}_1, \overline{h}_2, \overline{h}_3, \overline{h}_4\rangle$. Then $\overline{h} = \sum_i q_i \overline{h}_i$, and so

$$\tau(g_i, h) = \zeta^{\alpha_{i1}q_1 + \alpha_{i2}q_2 + \alpha_{i3}q_3 + \alpha_{i4}q_4}.$$

If $\alpha_{ij} \equiv 0 \pmod{\ell}$, $1 \le j \le 4$, then $|\tau(g_i, h)| < \lambda$. Hence, if $\Gamma/\lambda\Gamma = \langle \overline{h}_1, \overline{h}_2, \overline{h}_3, \overline{h}_4\rangle$, then for all $i \in \{\nu, \ldots, 4\}$ we can choose a $j \in \{1, \ldots, 4\}$, such that $\alpha_{ij} \not\equiv 0 \pmod{\ell}$.

Enumerate the h_i such that $\alpha_{44} \not\equiv 0 \pmod{\ell}$. Now assume a number $j < 4$ exists, such that $\alpha_{4j} \not\equiv 0 \pmod{\lambda}$. Then $\zeta^{\alpha_{4j}} = \zeta^{\beta_1 \alpha_{44}}$, and replacing h_j with $h_j - \beta_1 h_4$ gives $\alpha_{4j} \equiv 0 \pmod{\lambda}$. So we may assume that

$$\alpha_{41} \equiv \alpha_{42} \equiv \alpha_{43} \equiv 0 \pmod{\lambda} \qquad \text{and} \qquad \alpha_{44} \not\equiv 0 \pmod{\ell}.$$

Assume similarly that a number $j < 4$ exists, such that $\alpha_{j4} \not\equiv 0 \pmod{\lambda}$. Now set $\beta_2 \equiv \alpha_{44}^{-1}\alpha_{j4} \pmod{\lambda}$. Then $\tau(g_j - \beta_2 g_4, h_4) = 1$. So we may also assume that

$$\alpha_{14} \equiv \alpha_{24} \equiv \alpha_{34} \equiv 0 \pmod{\lambda}.$$

Repeating this process recursively, we may assume that

$$\alpha_{ij} \equiv 0 \pmod{\lambda} \qquad \text{and} \qquad \alpha_{44} \not\equiv 0 \pmod{\ell}.$$

Again $\nu \le i \le 4$ and $1 \le j \le 4$.

The discussion above is formalized in the following algorithm.

Algorithm 1. As input we are given a hyperelliptic curve C of genus two defined over a prime field $\mathbb{F}_p$, the number $N = |\Gamma|$ of $\mathbb{F}_p$-rational elements of the Jacobian, and a prime factor $\ell \mid \gcd(N, p-1)$. The algorithm outputs elements $\gamma_i \in \Gamma_\ell$ of the Sylow-ℓ subgroup Γ_ℓ of Γ, such that $\langle\gamma_i\rangle_i = \bigoplus_i \langle\gamma_i\rangle$ in the following steps.

1. Compute the order N_ℓ of the Sylow-ℓ subgroup of Γ.
2. Choose elements $\gamma_i \in \Gamma$, $i \in I := \{1, 2, 3, 4\}$. Set $\gamma_i := \frac{N}{N_\ell}\gamma_i$.
3. Choose elements $h_j \in \Gamma$, $j \in J := \{1, 2, 3, 4\}$.
4. Set $K := \{1, 2, 3, 4\}$.
5. For k' from 0 to 3 do the following:
   (a) Set $k := 4 - k'$.
   (b) If $\gamma_i = 0$, then set $I := I \setminus \{i\}$. If $|I| = 0$, then go to step 2.

(c) Compute the orders $\lambda_\kappa := |\gamma_\kappa|$, $\kappa \in K$. Re-enumerate the γ_κ's such that $\lambda_\kappa \leq \lambda_{\kappa+1}$, $\kappa \in K$. Set $I := \{5-|I|, 6-|I|, \ldots, 4\}$.

(d) Set $\nu := \min(I)$, and choose λ_0 minimal such that $\lambda := \frac{\lambda_\nu}{\lambda_0} \mid p-1$. Set $g_\kappa := \frac{\lambda_\kappa}{\lambda}\gamma_\kappa$, $\kappa \in I \cap K$.

i. If $g_k = 0$, then go to step 6.

ii. If $\tau(g_k, h_j)^{\lambda/\ell} = 1$ for all $j \leq k$, then go to step 3.

(e) Choose a primitive λ^{th} root of unity $\zeta \in \mathbb{F}_p$. Compute α_{kj} and $\alpha_{\kappa k}$ from $\tau(g_k, h_j) = \zeta^{\alpha_{kj}}$ and $\tau(g_\kappa, h_k) = \zeta^{\alpha_{\kappa k}}$, $1 \leq j < k$, $\kappa \in I \cap K$. Re-enumerate $h_1, \ldots, h_k$ such that $\alpha_{kk} \not\equiv 0 \pmod{\ell}$.

(f) For $1 \leq j < k$, set $\beta \equiv \alpha_{kk}^{-1}\alpha_{kj} \pmod{\lambda}$ and $h_j := h_j - \beta h_k$.

(g) For $\kappa \in I \cap K \setminus \{k\}$, set $\beta \equiv \alpha_{kk}^{-1}\alpha_{\kappa k} \pmod{\lambda}$ and $\gamma_\kappa := \gamma_\kappa - \beta\frac{\lambda_k}{\lambda_\kappa}\gamma_k$.

(h) Set $K := K \setminus \{k\}$.

6. Output γ_1, γ_2, γ_3 and γ_4.

Remark 6. Algorithm 1 consists of a small number of

1. calculations of orders of elements $\gamma \in \Gamma_\ell$,
2. multiplications of elements $\gamma \in \Gamma$ with numbers $a \in \mathbb{Z}$,
3. additions of elements $\gamma_1, \gamma_2 \in \Gamma$,
4. evaluations of pairings of elements $\gamma_1, \gamma_2 \in \Gamma$ and
5. solving the discrete logarithm problem in $\mathbb{F}_p$, i.e. to determine α from ζ and $\xi = \zeta^\alpha$.

By (Miller, 2004, proposition 9), the order $|\gamma|$ of an element $\gamma \in \Gamma_\ell$ can be calculated in time $O(\log^3 N_\ell)\mathcal{A}_\Gamma$, where $\mathcal{A}_\Gamma$ is the time for adding two elements of Γ. A multiple $a\gamma$ or a sum $\gamma_1 + \gamma_2$ is computed in time $O(\mathcal{A}_\Gamma)$. By Frey and Rück (1994), the pairing $\tau(\gamma_1, \gamma_2)$ of two elements $\gamma_1, \gamma_2 \in \Gamma$ can be evaluated in time $O(\log N_\ell)$. Finally, by Pohlig and Hellmann (1978) the discrete logarithm problem in $\mathbb{F}_p$ can be solved in time $O(\log p)$. We may assume that addition in Γ is easy, i.e. that $\mathcal{A}_\Gamma < O(\log p)$. Hence algorithm 1 runs in expected time $O(\log p)$.

Careful examination of algorithm 1 gives the following lemma.

Lemma 7. *Let Γ_ℓ be the Sylow-ℓ subgroup of Γ, $\ell \mid p-1$. Algorithm 1 determines elements $\gamma_i \in \Gamma_\ell$ and $h_i \in \Gamma$, $1 \leq i \leq 4$, such that one of the following cases holds.*

1. $\alpha_{11}\alpha_{22}\alpha_{33}\alpha_{44} \not\equiv 0 \pmod{\ell}$ *and* $\alpha_{ij} \equiv 0 \pmod{\lambda}$, $i \neq j$, $i, j \in \{1,2,3,4\}$.
2. $\gamma_1 = 0$, $\alpha_{22}\alpha_{33}\alpha_{44} \not\equiv 0 \pmod{\ell}$ *and* $\alpha_{ij} \equiv 0 \pmod{\lambda}$, $i \neq j$, $i, j \in \{2,3,4\}$.
3. $\gamma_1 = \gamma_2 = 0$, $\alpha_{33}\alpha_{44} \not\equiv 0 \pmod{\ell}$ *and* $\alpha_{ij} \equiv 0 \pmod{\lambda}$, $i \neq j$, $i, j \in \{3,4\}$.
4. $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

If $|\gamma_i| = \lambda_i$, then $\lambda_i \leq \lambda_{i+1}$. Set $\nu = \min\{i|\lambda_i \neq 1\}$, and define λ_0 as the least number, such that $\lambda = \frac{\lambda_\nu}{\lambda_0} \mid p-1$. Set $g_i = \frac{\lambda_i}{\lambda}\gamma_i$, $\nu \leq i \leq 4$. Then the numbers α_{ij} above are determined by

$$\tau(g_i, h_j) = \zeta^{\alpha_{ij}},$$

where τ is the tame Tate pairing $\Gamma[\lambda] \times \Gamma/\lambda\Gamma \to \mu_\lambda = \langle\zeta\rangle$.

Theorem 8. *Algorithm 1 determines elements γ_1, γ_2, γ_3 and γ_4 of the Sylow-ℓ subgroup of Γ, $\ell \mid p-1$, such that $\langle\gamma_i\rangle_i = \bigoplus_i\langle\gamma_i\rangle$.*

Proof. Choose elements $\gamma_i, h_i \in \Gamma$ such that the conditions of lemma 7 are fulfilled. Set $\lambda_i = |\gamma_i|$, and let $\nu = \min\{i|\lambda_i \neq 1\}$. Define λ_0 as the least number, such that $\lambda = \frac{\lambda_\nu}{\lambda_0} \mid p-1$. Set $g_i = \frac{\lambda_i}{\lambda}\gamma_i$. Then the α_{ij}'s from lemma 7 are determined by

$$\tau(g_i, h_j) = \zeta^{\alpha_{ij}}.$$

We only consider case 1 of lemma 7, since the other cases follow similarly. We start by determining $\langle\gamma_3\rangle \cap \langle\gamma_4\rangle$. Assume that $g_3 = ag_4$. Then

$$1 = \tau(g_3, h_4) = \tau(ag_4, h_4) = \zeta^{a\alpha_{44}},$$

i.e. $a \equiv 0 \pmod{\lambda}$. Hence $\langle\gamma_3\rangle\cap\langle\gamma_4\rangle = \{0\}$. Then we determine $\langle\gamma_2\rangle\cap\langle\gamma_3,\gamma_4\rangle$. Assume $g_2 = ag_3 + bg_4$. Then

$$1 = \tau(g_2, h_3) = \tau(ag_3, h_3) = \zeta^{a\alpha_{33}},$$

i.e. $a \equiv 0 \pmod{\lambda}$. In the same way,

$$1 = \tau(g_2, h_4) = \zeta^{b\alpha_{44}},$$

i.e. $b \equiv 0 \pmod{\lambda}$. Hence $\langle\gamma_2\rangle \cap \langle\gamma_3,\gamma_4\rangle = \{0\}$. Similarly $\langle\gamma_1\rangle \cap \langle\gamma_2,\gamma_3,\gamma_4\rangle = \{0\}$. Hence $\langle\gamma_i\rangle_i = \bigoplus_i\langle\gamma_i\rangle$. □

$>$From theorem 8 we get the following probabilistic algorithm to determine generators of the m-torsion subgroup $\Gamma[m] < \Gamma$, where $m \mid |\Gamma|$ is the largest divisor of $|\Gamma|$ such that $\ell \mid p-1$ for every prime number $\ell \mid m$.

Algorithm 2. As input we are given a hyperelliptic curve C of genus two defined over a prime field $\mathbb{F}_p$, the number $N = |\Gamma|$ of $\mathbb{F}_p$-rational elements of the Jacobian, and the prime factors $p_1, \ldots, p_n$ of $\gcd(N, p-1)$. The algorithm outputs elements $\gamma_i \in \Gamma[m]$ such that $\Gamma[m] = \bigoplus_i\langle\gamma_i\rangle$ in the following steps.

1. Set $\gamma_i := 0$, $1 \le i \le 4$. For $\ell \in \{p_1, \ldots, p_n\}$ do the following:
   (a) Use algorithm 1 to determine elements $\tilde{\gamma}_i \in \Gamma_\ell$, $1 \le i \le 4$, such that $\langle\tilde{\gamma}_i\rangle_i = \bigoplus_i\langle\tilde{\gamma}_i\rangle$.
   (b) If $\prod_i |\tilde{\gamma}_i| < |\Gamma_\ell|$, then go to step 1a.
   (c) Set $\gamma_i := \gamma_i + \tilde{\gamma}_i$, $1 \le i \le 4$.
2. Output γ_1, γ_2, γ_3 and γ_4.

Remark 9. By remark 6, algorithm 2 has expected running time $O(\log p)$. Hence algorithm 2 is an efficient, probabilistic algorithm to determine generators of the m-torsion subgroup $\Gamma[m] < \Gamma$, where $m \mid |\Gamma|$ is the largest divisor of $|\Gamma|$ such that $\ell \mid p-1$ for every prime number $\ell \mid m$.

Remark 10. The strategy of algorithm 1 can be applied to *any* finite, abelian group Γ with bilinear, non-degenerate pairings into cyclic groups. For the strategy to be efficient, the pairings must be efficiently computable, and the discrete logarithm problem in the cyclic groups must be easy.

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