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C^*-ALGEBRAS OF HOMOCLINIC AND HETEROCLINIC STRUCTURE IN EXPANSIVE DYNAMICS

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Abstract

We unify various constructions of C^*-algebras from dynamical systems, specifically, the dimension group construction of Krieger for shift spaces, the corresponding constructions of Wagoner and Boyle, Fiebig and Fiebig for countable state Markov shifts and one-sided shift spaces, respectively, and the constructions of Ruelle and Putnam for Smale spaces. The general setup is used to analyze the structure of the C^*-algebras arising from the homoclinic and heteroclinic equivalence relations in expansive dynamical systems; in particular expansive group endomorphisms and automorphisms, and generalized 1-solenoids. For these dynamical systems it is shown that the C^*-algebras are inductive limits of homogeneous or sub-homogeneous algebras with one-dimensional spectra.

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Preface

The crossed product construction is the classical way of associating a C^*-algebra to a dynamical system, and it is sometimes interpreted as a non-commutative substitute for the often ill behaved space of orbits of the action. This view of the crossed product is most natural when the action is free, since the crossed product will otherwise depend on more than the orbit equivalence relation. Nonetheless it makes good sense to view the crossed product as an attempt to produce a non-commutative algebra, a C^*-algebra, from the orbit equivalence defined by the action, in order to capture important features of the orbit space which may be difficult or impossible to handle from a topological point of view.

In recent years there has been a growing interest in other equivalence relations arising from dynamics, in particular homoclinicity and heteroclinicity. These relations are in some sense transverse to orbit equivalence and the C^*-algebras which can be naturally associated to them are very different from the corresponding crossed product. This can be observed already in what seems to be the earliest of such constructions made by Wolfgang Krieger in [**Kr2**]: The C^*-algebras arising from the homoclinic and heteroclinic equivalence relations of a mixing topological Markov chain are simple AF-algebras while the corresponding crossed product is neither AF nor simple.

David Ruelle was the next to construct a C^*-algebra from the equivalence relation given by homoclinicity in dynamical systems, cf. [**Ru2**], and it is his approach I will consider in the present paper. What is crucial for the method of Ruelle is that in many dynamical systems, such as the Smale spaces considered by Ruelle, homoclinicity of two states can be extended to a 'uniform local homoclinicity'. See Condition C of [**Ru2**]. It is this strengthening of the relation which ensures that the topology on the graph of the equivalence relation defined by homoclinicity becomes what is nowadays called an étale equivalence relation, so that the construction of Renault, [**Re1**], can be used to construct the C^*-algebra of the relation. Here I take the stronger relation as point of departure and this allows the construction of an étale equivalence relation from the homoclinicity relation and the heteroclinicity relation in more general settings than the Smale spaces introduced by Ruelle.

Besides the work of Krieger and Ruelle the paper builds on, and is strongly influenced by the work of Ian Putnam and J. Wagoner. While Ruelle only considered homoclinicity, Putnam showed how one can construct the C^*-algebras of heteroclinic equivalence in Smale spaces. For this he used the concept of a Haar measure of the underlying groupoid, building again on the work of Renault. To let go of the étale condition is actually a weakening for many purposes, but through his work with J. Spielberg, [**PS**], his was able to partly remedy this defect. One of the major points of the present work is to show that an approach of Wagoner to the construction of a dimension group representation for countable state Markov shifts, [**Wa**], can be used to give a canonical construction of an étale equivalence relation whose C^*-algebra, when specialized to Smale spaces, is the stabilized version of what Putnam calls 'the stable algebra'. This illustrates one of the main purposes of the paper; to unify and simplify various constructions of C^*-algebras from dynamical systems, specifically, the dimension group construction of Krieger for shift spaces, the corresponding constructions of Wagoner and Boyle, Fiebig and Fiebig for countable state Markov shifts and one-sided shift spaces, respectively, besides the constructions of Ruelle and Putnam for Smale spaces. Another purpose is to use the general setup to improve our understanding of the structure of the C^*-algebras of the homoclinic and heteroclinic equivalence relations and initiate the study of their relation to the dynamical systems used for their construction.

Let me comment briefly on the content of each chapter of the paper. The first chapter contains the general construction of an étale equivalence relation from a (relatively) expansive dynamical system. It is this construction which is used in various settings in the following chapters.

The second chapter studies the functoriality properties of the C^*-algebras arising from the étale equivalence relations of the first chapter. As pointed out by Putnam (e.g. in [**Pu2**]) this issue is an important one, and many of the difficulties connected with the study of the C^*-algebras arise from the fact that the functoriality properties are very different from those of the crossed product construction.

Chapter 3 contains a study of the C^*-algebras arising from homoclinicity in various expansive dynamical systems. In particular, it is shown that for a two-sided shift space one obtains the AF-algebras whose dimension groups were constructed by Krieger in Section 2 of [**Kr2**]. For one sided shift spaces the dimension group of the resulting AF-algebra is what was called 'the images group' by Boyle, Fiebig and Fiebig in [**BFF**]. Further, it is shown that for positively expansive group endomorphisms and expansive group automorphisms, the homoclinic algebra is an AT-algebra; that is, a direct limit of a sequence of circle algebras. This conclusion is achieved by using some of the recent results from the classification program for simple C^*-algebras, in particular, results of Gong, Lin and Phillips. These results are combined with a thorough (but not complete) study of the homoclinic subgroup of expansive group automorphisms. Since there is recent work dedicated to the exhibition of an expansive automorphism of a compact connected group whose homoclinic group is not isomorphic to the dual of the group on which it acts, cf. [**CF**], we point out here, as an aside, that we obtain more examples of this kind, including examples where the homoclinic group contains torsion.

Chapter 4 describes the construction of the heteroclinic algebra alluded to above, and it is shown that the construction generalizes both some of the constructions of Wagoner from [**Wa**] as well as the construction of the stable algebra of a Smale space from [**Pu1**]. It is then shown that the heteroclinic algebra of Smale spaces arising from expanding maps, cf. [**Ru1**], are always AH-algebras.

In Chapter 5 it is shown that the heteroclinic algebra can be constructed for certain homeomorphisms that are not expansive, including general diffeomorphic automorphisms of a Lie group. Furthermore, it is shown that for an expansive group automorphism of a compact group the heteroclinic algebra is an AT-algebra, just as the homoclinic algebra is in this case. The study of these examples allow me to exhibit expansive automorphisms of the two-torus with the property that the heteroclinic algebra is not isomorphic to the heteroclinic algebra of its inverse; in Putnam's terminology, the stable and unstable algebras of these automorphisms are not (stably) isomorphic. It is tempting to conclude from these examples that the heteroclinic algebra is sensitive to properties of a dynamical system which the more conventional invariants, such as the topological entropy and the structure of periodic points, do not see.

Chapter 6 is devoted to the 1-solenoids of Yi, [**Y1**]. The study of the C^*-algebras arising from the heteroclinic structure of such spaces was started by Yi in [**Y3**] and we continue his attack by showing that the heteroclinic algebra of the 1-solenoids are simple, stable and can be realized as inductive limits of sub-homogenous C^*-algebras with one-dimensional spectra of a type introduced by the author in [**Th4**].

Finally, in Chapter 7, the heteroclinic algebra is used to remedy a defect of the dimension group representation for a countable state Markov shift as it was defined by Wagoner in [**Wa**]; namely that in Wagoner's approach only the automorphisms that are uniformly continuous with respect to a specific metric gives rise to automorphisms of the dimension group. This problem was pointed out to me by Michael Schraudner, and I am able to show that the heteroclinic algebra does give rise to a dimension group without the mentioned flaw, provided the Markov shift is locally compact and satisfies a certain condition ('finitely many edge-disjoint double paths') which was introduced by Schraudner himself in [**Sch**], and shown to be equivalent to countability of the automorphism group. However, I am only able to give a satisfying description of the dimension group and show that it is generally very different from that of Wagoner for locally compact Markov shifts whose one-point compactification is expansive. The methods used for this purpose may have independent interest; for example they allow me to obtain necessary and sufficient conditions for the one-point compactification of a countable state Markov shift to be sofic or of finite type.

It has been suggested by Putnam, e.g. in [**Pu2**], that the stable (and unstable) algebra of a Smale space might always be what is nowadays called an AH-algebra, and many of our results support this suspicion. However, the results on the Smale spaces of one-solenoids suggest that it may be necessary to allow more complicated building blocks.

Finally, it should be observed that both the homoclinic and the heteroclinic algebra of an expansive homeomorphism carries a natural automorphism which extends the given homeomorphism. The crossed product of this non-commutative dynamical system generalizes the Ruelle algebras of Putnam, [**Pu2**], [**PS**], and hence also the Cuntz-Krieger algebra, [**CuK**]. However, the present paper is devoted to the study of the homoclinic and heteroclinic algebras and the natural automorphisms arising from the underlying dynamical systems will mostly be ignored. In terms of equivalence relations this means that the focus is on homoclinicity and heteroclinicity in their pure form, without interference from orbit equivalence.

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Klaus Thomsen

CHAPTER 1

The Ruelle algebra of a relatively expansive system

1.1. Relatively expansive systems

We describe the input from dynamical systems which we need for the construction of the étale equivalence relations and C^*-algebras we are going to study.

Let X be a topological space and d a metric for the topology of X. Let $E \subseteq X$ be a subset of X equipped with a locally compact topology which is finer than the topology inherited from X. That is, $U \cap E$ is open in E when $U \subseteq X$ is open in X, but there may be open sets in E which are not of this form. Note that E is a locally compact Hausdorff space since the topology inherited from X is Hausdorff.

Let S be a countable set and for each $s \in S$, let $f_s : X \to X$ be a continuous map. Thus $f = (f_s)_{s\in S}$ is simply a collection of continuous self-maps of X, indexed by the set S. S may be a group or a semi-group, and $s \to f_s$ a homomorphism, but this is not necessary for the basic construction we describe below. We will assume that f is *relatively expansive* on E in the sense that there is a dense subset E_0 of E with the following two properties.[1]

1) E_0 is asymptotically stable in the sense that when $x \in E, y \in E_0$, and
$$\lim_{s\to\infty} d\left(f_s(x), f_s(y)\right) = 0,$$
then $x \in E_0$.
2) f is locally expansive on E_0 in the sense that for each $x \in E$ there is an open neighborhood U_x of x in E and a $\delta_x > 0$ such that
$$z, y \in E_0 \cap U_x,\ d\left(f_s(z), f_s(y)\right) \leq \delta_x\ \forall s \in S \Rightarrow\ z = y. \tag{1.1}$$

We call then the pair (U_x, δ_x) *an expansive pair at* x, and $\delta_x > 0$ is called *a local expansive constant at* x. We say that E is *an expansive region* for the action f. The tuple (X, d, S, f, E, E_0) will be called *a relatively expansive system* in the following.

1.1.1. Examples.

Example 1.1. Let (X, d) be a locally compact metric space, $\operatorname{End} X$ the semi-group of continuous maps from X to itself. Let Γ be a discrete semi-group. An *action* of Γ on X is a semi-group homomorphism $\Gamma \ni \gamma \mapsto f_\gamma \in \operatorname{End} X$. The action is called *expansive* when there is a $\delta > 0$ such that
$$x, y \in X,\ \sup_{\gamma\in\Gamma} d\left(f_\gamma(x), f_\gamma(y)\right) \leq \delta\ \Rightarrow\ x = y.$$
Then (X, d, Γ, f, X, X) is a relatively expansive system. More generally with E any open or closed subset of X, and E_0 any dense asymptotically stable subset of E, the tuple $(X, d, \Gamma, f, E, E_0)$ is a relatively expansive system. The most familiar examples of expansive actions are actions of $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Z}^n$ on compact metric spaces. $\square$

[1] Given a function $G : S \to [0, \infty)$ we write $\lim_{s\to\infty} G(s) = 0$ to mean that for every $\epsilon > 0$ there is a finite set $F \subseteq S$ such that $G(s) \leq \epsilon$ when $s \notin F$.

EXAMPLE 1.2. Let $G = (V, E)$ be a countable strongly connected, locally finite directed graph with vertex set V, edge set E together with the maps $i, t : E \to V$, where $i(e)$ is the initial and $t(e)$ the terminal vertex of an edge $e \in E$. Then

$$X_G = \left\{(x_i)_{i\in\mathbb{Z}} \in E^{\mathbb{Z}} : \ t(x_j) = i(x_{j+1}) \ \forall j \in \mathbb{Z}\right\}$$

is a locally compact subset of $E^{\mathbb{Z}}$ and the shift σ acts as a homeomorphism of X_G in the standard way: $\sigma(x)_j = x_{j+1}$. Furthermore, the locally compact topology of X_G is given by a metric, called the Gurevich metric, cf. [**Sch**]. Unless G is finite the shift is rarely expansive with respect to the Gurevich metric, but there is a natural class of graphs for which the shift acts expansively on a canonical dense subset: Assume that G only has finitely many pairwise edge-disjoint double paths, as defined by Schraudner in [**Sch**]. As shown by Schraudner in Theorem 3.4 of [**Sch**] there is then a constant $c > 0$ such that $\sup_{n\in\mathbb{Z}} d(\sigma^n(x), \sigma^n(y)) \geq c$ whenever x and y are different points in X_G and at least one of them is doubly transitive, meaning that both the forward and the backward orbit is dense in X_G. Note that the doubly transitive points are dense in X_G since G is strongly connected. Hence, if we follow [**Sch**] and let $DT(X_G)$ denote the doubly transitive points of X_G, the tuple $(X_G, d, \mathbb{Z}, \sigma, X_G, DT(X_G))$ will be a relatively expansive system. $\square$

1.2. The étale equivalence relation of local conjugacy

We recall the definition of an étale equivalence relation, cf. [**Re1**], [**GPS**].

Let X be a set and $R \subseteq X \times X$ an equivalence relation. We say that R is a *topological equivalence relation* when R is equipped with a topology (possibly different from the topology inherited from $X \times X$) such that the inversion $R \ni (x, y) \mapsto (y, x) \in R$ is a homeomorphism and the composition

$$R^{(2)} \ni ((x, y), (y, z)) \mapsto (x, z) \in R$$

is continuous, where the set of *composable pairs*

$$R^{(2)} = \{((x, y), (u, v)) \in R \times R : \ u = y\}$$

has the relative topology inherited from $R \times R$. In this setting we call $r(x, y) = x$ the *range map* and $s(x, y) = y$ the *source map*.

DEFINITION 1.3. Let X be a locally compact Hausdorff space and $R \subseteq X \times X$ a topological equivalence relation. R is an *étale equivalence relation* when the range map $r : R \to X$ is a local homeomorphism in the sense that every element $\gamma \in R$ has an open neighborhood U_γ of γ such that $r(U_\gamma)$ is open in X and $r : U_\gamma \to r(U_\gamma)$ is a homeomorphism.

We come now to the basic construction of the paper; the construction of an étale equivalence relation from a relatively expansive system (X, d, S, f, E, E_0).

Two elements $x, y \in E$ are said to be *locally conjugate*, written $x \sim y$, when there are open neighborhoods U and V of x and y in E, and a homeomorphism $\chi : U \to V$ such that $\chi(x) = y$ and

$$\lim_{s\to\infty} \sup_{z\in U} d(f_s(z), f_s(\chi(z))) = 0.$$

The triple (U, V, χ) is called *a local conjugacy from x to y*, or just *a local conjugacy* or *a conjugacy* for short when it is not necessary to emphasize the points x

and y. Note that local conjugacy is an equivalence relation on E. The graph of this equivalence relation is the set

$$R_f(X,E) = \{(x,y) \in E \times E : \ x \sim y\}.$$

We call $R_f(X,E)$ the *local conjugacy relation* on E.

Define a topology on $R_f(X,E)$ by declaring sets of the form

$$\{(x, \chi(x)) : \ x \in U\}, \tag{1.2}$$

where (U,V,χ) is a conjugacy, to be a subbase for the topology, i.e. a subset of $R_f(X,E)$ is open if and only if it is the union of finite intersections of sets of the form (1.2). It is then easy to see that the range and source maps, r and s, are continuous.

LEMMA 1.4. *Let (U,V,χ) and (U',V',χ') be local conjugacies from x to y in E. There are then open neighborhoods U_0 of x and V_0 of y in E such that $x \in U_0 \subseteq U \cap U'$, $y \in V_0 \subseteq V \cap V'$, $\chi|_{U_0} = \chi'|_{U_0}$ and $\chi(U_0) = V_0$.*

PROOF. Let (U_y, δ_y) be an expansive pair at y. By shrinking U_y we can arrange that $U_y \subseteq V \cap V'$. There is a finite set $F \subseteq S$ such that

$$\sup_{x' \in U} d\left(f_s(x'), f_s\left(\chi\left(x'\right)\right)\right) \leq \frac{\delta_y}{2}$$

and

$$\sup_{x' \in U'} d\left(f_s(x'), f_s\left(\chi'\left(x'\right)\right)\right) \leq \frac{\delta_y}{2}$$

when $s \notin F$. It follows that $\sup_{x' \in U \cap U'} d\left(f_s\left(\chi(x)\right), f_s\left(\chi'(x)\right)\right) \leq \delta_y$ when $s \notin F$. Since the topology of E is finer than the relative topology inherited from X, there is an open neighborhood W of y in E such that $y \in W \subseteq U_y$ and $\sup_{s \in F} d\left(f_s(y'), f_s(y'')\right) \leq \delta_y$ for all $y', y'' \in W$. Set $U_0 = \chi^{-1}(W) \cap \chi'^{-1}(W)$. Let $x' \in U_0 \cap E_0$. Note that $\chi(x'), \chi'(x') \in E_0 \cap U_y$ since E_0 is asymptotically stable. Since $d\left(f_s\left(\chi(x')\right), f_s\left(\chi'(x')\right)\right) \leq \delta_y$ for all $s \in S$ it follows from (1.1) that $\chi'(x') = \chi(x')$. Since $U_0 \cap E_0$ is dense in U_0 the continuity of χ and χ' implies that $\chi|_{U_0} = \chi'|_{U_0}$. Set $V_0 = \chi(U_0)$. □

COROLLARY 1.5. *The sets of the form (1.2) is a base for the topology of $R_f(X,E)$.*

LEMMA 1.6. *Let Ω be an open subset of the topological product $E \times E$. It follows that $R_f(X,E) \cap \Omega$ is open in $R_f(X,E)$.*

PROOF. Let $(x,y) \in R_f(X,E) \cap \Omega$. There is a local conjugacy (U,V,χ) from x to y, and there are open sets $W_x, W_y \subseteq E$ such that $(x,y) \in W_x \times W_y \subseteq \Omega$. Set $U_0 = W_x \cap \chi^{-1}\left(W_y \cap V\right)$. Then $(x,y) \in \{(z,\chi(z)) : \ z \in U_0\} \subseteq R_f(X,E) \cap \Omega$, proving that $R_f(X,E) \cap \Omega$ is indeed open in $R_f(X,E)$. □

THEOREM 1.7. *$R_f(X,E)$ is an étale equivalence relation.*

PROOF. It follows from Lemma 1.6 that the topology is Hausdorff since the topology of E is. To see that $R_f(X,E)$ is locally compact, consider an element $\xi = (x,y) \in R_f(X,E)$, and let (U,V,χ) be a conjugacy from x to y. Let $U_0 \subseteq U$ be an open neighborhood of x such that $\overline{U_0} \subseteq U$ is compact. Set

$$K = \left\{(z, \chi(z)) : \ z \in \overline{U_0}\right\}.$$

To show that K is compact in $R_f(X,E)$ it suffices, either by Alexander's subbase theorem or by Corollary 1.5, to show that any cover of K by open sets of the form (1.2) has a finite sub-cover. Let therefore $(U_\alpha, V_\alpha, \chi_\alpha), \alpha \in I$, be a collection of conjugacies such that

$$K \subseteq \bigcup_{\alpha \in I} \{(z, \chi_\alpha(z)) : \ z \in U_\alpha\}.$$

It follows from Lemma 1.4 that for any $z \in \overline{U_0}$ there is an $\alpha(z) \in I$ and an open neighborhood $W_{\alpha(z)} \subseteq U_\alpha \cap U$ of z such that $\chi|_{W_{\alpha(z)}} = \chi_{\alpha(z)}|_{W_{\alpha(z)}}$. By compactness of $\overline{U_0}$ there is a finite set $z_1, z_2, \dots, z_N$ in $\overline{U_0}$ such that

$$\overline{U_0} \subseteq \bigcup_{i=1}^{N} W_{\alpha(z_i)}.$$

When $z \in \overline{U_0} \cap W_{\alpha(z_i)}$ we have that $(z, \chi(z)) = \left(z, \chi_{\alpha(z_i)}(z)\right)$, so we conclude that

$$K \subseteq \bigcup_{i=1}^{N} \left\{ \left(z, \chi_{\alpha(z_i)}(z)\right) : \ z \in U_{\alpha(z_i)} \right\},$$

completing the proof of local compactness of $R_f(X,E)$.

To prove that $R_f(X,E)$ is a topological equivalence relation, observe first that the inversion $(x,y) \to (y,x)$ is clearly a homeomorphism. It suffices therefore to show that the composition is continuous. So let $((x,y),(y,z)) \in R_f(X,E)^{(2)}$, and let (U,V,χ) be a conjugacy from x to z. We must show that there is a open neighborhood Ω of $((x,y),(y,z))$ in $R_f(X,E) \times R_f(X,E)$ such that $(x',z') \in \{(v,\chi(v)) : \ v \in U\}$ when $(x',y'),(y',z')$ is a composable pair in Ω. To this end let (U_1,V_1,χ_1) be a conjugacy from x to y and (U_2,V_2,χ_2) a conjugacy from y to z. It follows from Lemma 1.4 that $\chi_2 \circ \chi_1$ agrees with χ in a neighborhood W of x. Hence

$$\Omega = \{(x', \chi_1(x')) : \ x' \in W\} \times \{(y', \chi_2(y')) : \ y' \in \chi_1(W)\}$$

has the required property.

To prove that r is a local homeomorphism, let (U,V,χ) be a conjugacy. Then $r : \{(x,\chi(x)) : \ x \in U\} \to U$ is a homeomorphism since its inverse, the map $U \ni z \mapsto (z,\chi(z))$, is continuous by Lemma 1.4. $\square$

In many important cases the expansive region E is X itself and $E_0 = E = X$. We denote then the local conjugacy relation by $R_f(X)$.

1.2.1. Miscellaneous observations. In this section we have gathered a series of observations on the construction of $R_f(X,E)$ that we are going to need later on.

LEMMA 1.8. *$R_f(X,E)$ is second countable if and only if E is. In this case there is a countable base for $R_f(X,E)$ consisting of sets of the form (1.2).*

PROOF. Since the range map r is continuous and open it follows immediately that $R_f(X,E)$ can only be second countable when E is. So assume that E is second countable and let $\mathcal{B}$ be a countable base for the topology of E. Since S is countable we can write $S = \bigcup_{n \in \mathbb{N}} F_n$ where each F_n is a finite subset of S. Let $\delta > 0$ be rational, $U_0, V_0 \in \mathcal{B}$ and $N \in \mathbb{N}$. We require that (V_0, δ) is an expansive pair at y for some $y \in V_0$, and $d(f_s(z), f_s(z')) \leq \delta$ when $z, z' \in V_0$ and $s \in F_N$. A local conjugacy (U,V,χ) is will be said to be of type (U_0,V_0,δ,N) when $U = U_0$, $V \subseteq V_0$, and $\sup_{z \in U} d(f_s(z), f_s(\chi(z))) \leq \frac{\delta}{2}$ when $s \notin F_N$. If (U',V',χ') is also of type (U_0, V_0, δ, N) we have that $U' = U_0 = U$, and that $d\left(f_s\left(\chi(z)\right), f_s\left(\chi'(z)\right)\right) \leq \delta$ for all $s \in S$. It follows then from the first condition on (V_0, δ) that $\chi(z) = \chi'(z)$ when $z \in E_0 \cap U$. By density of E_0 and continuity of χ and χ' it follows that $\chi = \chi'$. Thus $V = \chi(U) = \chi'(U') = V'$. This shows that there is at most one local conjugacy of type (U_0, V_0, δ, N). There are therefore only countably many local conjugacies that are of some type and it remains now only to show that they give rise to a base for the topology of $R_f(X, E)$. So we consider a local conjugacy (U, V, χ) and a point $(x, y) \in \{(z, \chi(z)) : \ z \in U\}$. There is then an $N \in \mathbb{N}$ such that $\sup_{z \in U} d\left(f_s(z), f_s\left(\chi(z)\right)\right) \leq \frac{\delta}{2} \leq \frac{\delta_y}{2}$ for all $s \notin F_N$, where δ_y is a local expansive constant at y and $\delta > 0$ is rational. Since $\mathcal{B}$ is a base for the topology of E there are elements $U_0, V_0 \in \mathcal{B}$ such that $x \in U_0 \subseteq U$, $\chi\left(U_0\right) \subseteq V_0$, and $d\left(f_s(z), f_s(z')\right) \leq \delta$ for all $z, z' \in V_0$ and all $s \in F_N$. Then $\left(U_0, \chi\left(U_0\right), \chi\right)$ is of type (U_0, V_0, δ, N) and $(x, y) \in \{(z, \chi(z)) : \ z \in U_0\} \subseteq \{(z, \chi(z)) : \ z \in U\}$. $\square$

Lemma 1.9. *A subset $D \subseteq R_f(X, E)$ is pre-compact in $R_f(X, E)$ if and only if there is a finite collection $\left(U_i, V_i, \chi_i\right), i = 1, 2, \ldots, N$, of local conjugacies in E and compact subsets $L_i \subseteq U_i$ such that*

$$D \subseteq \bigcup_{i=1}^{N} \left\{(x, \chi_i(x)) : \ x \in L_i\right\}. \tag{1.3}$$

Proof. From the proof of Theorem 1.7 it follows that each $\{(x, \chi_i(x)) : \ x \in L_i\}$ is compact so D is certainly pre-compact when the condition holds. For the converse assume that D is pre-compact. Then the closure $\overline{D}$ of D can be covered by a finite collection of sets from the subbase. Thus we have a finite collection $\left(U_i, V_i, \chi_i\right), i = 1, 2, \ldots, N$, of local conjugacies in E and inclusions

$$D \subseteq \overline{D} \subseteq \bigcup_{i=1}^{N} \left\{(x, \chi_i(x)) : \ x \in U_i\right\}.$$

Since $R_f(X, E)$ is locally compact and $\overline{D}$ is compact there is a partition of unity $\varphi_i, i = 1, 2, \ldots, N$, on $\overline{D}$ such that $\operatorname{supp} \varphi_i \subseteq \{(x, \chi_i(x)) : \ x \in U_i\}$. Set $L_i = r\left(\operatorname{supp} \varphi_i\right)$. Then (1.3) holds. $\square$

Let $C_c\left(R_f(X, E)\right)$ be the space of continuous complex functions on $R_f(X, E)$ of compact support. We say that a function $f \in C_c\left(R_f(X, E)\right)$ is *localized* when its support is contained in the set $\{(z, \mu(z)) : \ z \in U\}$ for some local conjugacy (U, V, μ).

Lemma 1.10. *Every element of $C_c\left(R_f(X, E)\right)$ is the sum of finitely many localized functions.*

Proof. This follows from Lemma 1.9 and an obvious partition of unity argument. $\square$

We now make an additional assumption which will allow us to give an alternative description of $R_f(X, E)$ and its topology. Specifically, we will assume that there is a $\delta > 0$ such that

$$z, y \in E_0, \ d\left(f_s(z), f_s(y)\right) \leq \delta \ \forall s \in S \Rightarrow \ z = y. \tag{1.4}$$

There is an increasing sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ of finite subsets of S such that $S = \bigcup_{n=1}^{\infty} F_n$. Let $\delta > 0$ be a constant such that (1.4) holds. Set

$$A_n = \left\{ (x,y) \in E \times E : \ d\left(f_s(x), f_s(y)\right) < \frac{\delta}{2}, \ s \notin F_n \right\}. \tag{1.5}$$

LEMMA 1.11. *Let $\xi = (x,y) \in A_n \cap R_f(X,E)$, and consider a local conjugacy (U,V,χ) from x to y. There is then an open neighborhood W of x in E such that $x \in W \subseteq U$, $(W, \chi(W), \chi)$ is a conjugacy from x to y and*

$$\{(z,\chi(z)) : \ z \in W\} \subseteq A_n \cap R_f(X,E).$$

PROOF. There is an $m > n$ such that $\sup_{z \in U} d\left(f_s(z), f_s(\chi(z))\right) < \frac{\delta}{2}$ when $s \notin F_m$. Note that $d\left(f_s(x), f_s(\chi(x))\right) < \frac{\delta}{2}$ when $s \notin F_n$. There is therefore an open neighborhood $W \subseteq U$ of x in E such that $\sup_{z \in W} d\left(f_s(z), f_s(\chi(z))\right) < \frac{\delta}{2}$ for all $s \in F_m \backslash F_n$. Then W has the stated properties. □

It follows from Lemma 1.11 that $A_n \cap R_f(X,E)$ is open in $R_f(X,E)$. Note that

$$R_f(X,E) = \bigcup_{n=1}^{\infty} A_n \cap R_f(X,E). \tag{1.6}$$

LEMMA 1.12. *The topology of $A_n \cap R_f(X,E)$ inherited from $R_f(X,E)$ is the same as the topology inherited from $E \times E$.*

PROOF. By Lemma 1.6 it suffices to consider an open set Ω in $R_f(X,E)$ and show that $\Omega \cap A_n \cap R_f(X,E)$ is open in the topology inherited from $E \times E$. To this end we may take Ω to be of the form (1.2). Let $\xi = (x,y) \in \Omega \cap A_n \cap R_f(X,E)$. By Lemma 1.11 we can find an open neighborhood W of x in E and a conjugacy $(W, \chi(W), \chi)$ from x to y such that

$$\{(z,\chi(z)) : \ z \in W\} \subseteq \Omega \cap A_n \cap R_f(X,E). \tag{1.7}$$

Let W' be an open neighborhood of y contained in $\chi(W)$ such that

$$\sup_{y',y'' \in W'} d\left(f_s(y'), f_s(y'')\right) < \delta \tag{1.8}$$

for all $s \in F_n$. We claim that

$$\left(\chi^{-1}(W') \times W'\right) \cap A_n \cap R_f(X,E) \subseteq \{(z,\chi(z)) : \ z \in W\}. \tag{1.9}$$

To prove this let $(x',y') \in (\chi^{-1}(W') \times W') \cap A_n \cap R_f(X,E)$. There is then a conjugacy (U',V',χ') from x' to y' such that $U' \times V' \subseteq \chi^{-1}(W') \times W'$. Let $m \in \mathbb{N}$ be so large that $\sup_{z \in U'} d\left(f_s(z), f_s\left(\chi'(z)\right)\right) < \frac{\delta}{2}$ when $s \notin F_m$. Since $d\left(f_s(x'), f_s(y')\right) < \frac{\delta}{2}$ when $s \notin F_n$, we can shrink U' to achieve that

$$\sup_{z \in U'} d\left(f_s(z), f_s\left(\chi'(z)\right)\right) < \frac{\delta}{2}$$

when $s \notin F_n$. Let $x'' \in U' \cap E_0$ and note that

$$d\left(f_s(x''), f_s(\chi(x''))\right) < \frac{\delta}{2}$$

for all $s \notin F_n$ because $(x'', \chi(x'')) \in A_n$, cf. (1.7), and that

$$d\left(f_s(\chi'(x'')), f_s(\chi(x''))\right) < \delta$$

for all $s \in F_n$ because $\chi(x''), \chi'(x'') \in W'$. It follows that

$$d\left(f_s(\chi'(x'')), f_s(\chi(x''))\right) < \delta$$

for all $s \in S$, and hence that $\chi(x'') = \chi'(x'')$ because of (1.4). Since $U' \cap E_0$ is dense in U' we conclude that $\chi(x') = \chi'(x')$. Thus $(x', y') = (x', \chi(x')) \in \{(z, \chi(z)) : z \in W\}$, completing the proof of (1.9).

It follows from (1.7) and (1.9) that $\xi \in (\chi^{-1}(W') \times W') \cap A_n \cap R_f(X, E) \subseteq \Omega \cap A_n \cap R_f(X, E)$, proving that $\Omega \cap A_n \cap R_f(X, E)$ is open in the topology of $A_n \cap R_f(X, E)$ inherited from $E \times E$. □

REMARK 1.13. As observed in Example 1.1 an expansive homeomorphism φ of a compact metric space (X, d) gives rise to a relatively expansive system in a canonical way. For such a system it is clear that conjugacy of two points $x, y \in X$ implies that x and y are *homoclinic* in the sense that

$$\lim_{|k| \to \infty} d\left(\varphi^k(x), \varphi^k(y)\right) = 0.$$

In many cases, such as the Smale spaces of Ruelle, this condition is sufficient to ensure the existence of a local conjugacy between x and y; that is, points are conjugate if and only if they are homoclinic.

To give an example with two homoclinic points that are not locally conjugate, consider the even shift Y which is the two-sided shift in the alphabet $\{0, 1\}$ obtained by disallowing the words $\left\{10^{2k+1}1 : \ k = 0, 1, 2, \ldots\right\}$. Set $y_i = 0, i \in \mathbb{Z}$, and $x_i = 0, i \in \mathbb{Z}\backslash\{0\}$, $x_0 = 1$. Then $x = (x_i)_{i \in \mathbb{Z}}$ and $y = (y_i)_{i \in \mathbb{Z}}$ are both elements of Y, and x and y are homoclinic under the shift. To see that x and y are not conjugate, let $\delta > 0$ be an expansive constant for Y such that $z, z' \in Y, \ d(z, z') < \delta \Rightarrow z_0 = z'_0$. Assume to get a contradiction, that (U, V, χ) is a conjugacy from x to y. There is then a $K \in \mathbb{N}$ such that

$$\chi(z)_k = z_k, \ |k| \geq K, \ z \in U. \tag{1.10}$$

By definition of the topology of Y there is an open neighborhood V_0 of y such that $y \in V_0 \subseteq V$ and $z_{[-K,K]} = y_{[-K,K]} = 0^{2K+1}$ when $z \in V_0$. Consider the sequence $a_i, i \in \mathbb{N}$, where

$$a_i = \ldots 111110^{2i}10^{2i}111111 \ldots$$

Then $a_i \in Y$ for all i and $\lim_{i \to \infty} a_i = x$. In particular, $a_i \in \chi^{-1}(V_0)$ for all i large enough. For such an i, $\chi(a_i)_{[-K,K]} = 0^{2K+1}$ and $\chi(a_i)_k = (a_i)_k$ for all $|k| \geq K$, thanks to (1.10). In particular, for some $i \geq K$,

$$\chi(a_i) = 1^\infty 0^{4i+1} 1^\infty$$

which is not an element of Y. It follows that x and y are not conjugate. □

REMARK 1.14. This remark concerns the relationship between the construction of Section 1.2 and a construction of Renault, cf. p. 139 of [**Re1**], which has subsequently been developed further by himself as well as by Deaconu, Anantharaman-Delaroche and others. In the most general setup (with compact unit space) the input for Renaults construction is a compact Hausdorff space X equipped with a continuous surjection $\sigma : X \to X$ which is also a local homeomorphism. For each $n \in \mathbb{N}$, let

$$R_n^\sigma = \{(x, y) \in X \times X : \ \sigma^n(x) = \sigma^n(y)\}$$

have the relative topology inherited from $X \times X$. Since R^σ_n is open in R^σ_{n+1} the union $R^\sigma = \bigcup_n R^\sigma_n$ is a locally compact Hausdorff space in the inductive limit topology, and in fact an étale equivalence relation. For the point we want to make, note that the openness of σ is crucial for the construction; it does not suffice that σ is locally injective.

If we now also assume that X is a metric space (with metric d) and that σ is expansive, the tuple (X, d, σ, X, X) is a relatively expansive tuple and we can construct $R_\sigma(X)$ as above. It follows from Lemma 1.12 that the two constructions are identical in this situation, i.e. $R_\sigma(X) = R^\sigma$, with the same topology. However, when σ is merely expansive, and not necessarily open, Renault's construction does not work. Consequently Renault's construction does not work for a one-sided shift-space which is not of finite type since the latter condition is equivalent to openness of the shift, [**Pa**]. Specifically, Renault's groupoid will not be étale when the shift is note of finite type. In particular, Renault's construction does not apply to the even shift, as claimed on page 222 of [**A**]. In contrast $R_\sigma(X)$ makes sense for any one-sided shift space, but for the even shift the corresponding AF-algebra will not be simple, cf. Example 3.5 below. $\square$

1.3. The C^*-algebra of a local conjugacy relation

There is a general construction which produces a C^*-algebra from an étale equivalence relation R on a locally compact Hausdorff space X, cf. [**Re1**]: First observe that the space $C_c(R)$ of compactly supported continuous functions on R is a $*$-algebra with the product

$$(f \cdot g)(x, y) = \sum_{\{z:\ (x,z) \in R\}} f(x, z) g(z, y)$$

and the involution

$$f^*(x, y) = \overline{f(y, x)}.$$

To obtain a C^*-norm on $C_c(R)$ we introduce a family of representations in the following way. For every point $x \in X$ we let $[x]$ denote the set of points in X that are equivalent to x, and we denote by $l^2[x]$ the Hilbert space of square-summable complex functions on $[x]$. For each $f \in C_c(R)$ we define a bounded operator $\kappa_{[x]}(f)$ on $l^2[x]$ such that

$$\left(\kappa_{[x]}(f)\psi\right)(y) = \sum_{z \in [x]} f(y, z)\psi(z). \tag{1.11}$$

Each $\kappa_{[x]}$ is a $*$-representation of $C_c(R)$, and together they form a separating family so we get a C^*-norm by putting

$$\|f\| = \sup_{x \in X} \left\|\kappa_{[x]}(f)\right\|. \tag{1.12}$$

The completion of $C_c(R)$ in this norm is then a C^*-algebra $C^*_{\mathrm{red}}(R)$.

Applied to the local conjugacy relation $R_f(X, E)$ we obtain a C^*-algebra which we denote by $A_f(X, E)$ and call *the Ruelle algebra*. In the remaining part of this section we collect a few observations about the Ruelle algebra which we shall use later on.

Remark 1.15. Let $C_b(X)$ be the C^*-algebra of continuous bounded functions on X, and let $M\left(C^*_r(R)\right)$ denote the multiplier algebra of $C^*_r(R)$. There is an embedding $\Phi : C_b(X) \to M(C_r^*(R))$ defined such that $\Phi(C_b(X)) C_c(R) \subseteq C_c(R)$ and

$$(\Phi(f)g)(x,y) = f(x)g(x,y),$$

when $f \in C_b(X), g \in C_c(R)$. Note that $\Phi(C_0(X)) \subseteq C_r^*(R)$. In the following we suppress Φ from the notation and consider instead $C_0(X)$ and $C_b(X)$ as C^*-sub-algebras of $C_r^*(R)$ and $M(C_r^*(R))$, respectively. It is known that $C_0(X)$ is a maximal abelian C^*-algebra of $C_r^*(R)$ with other nice properties, cf. [**Re1**]. In particular, $C_0(X)$ has the (unique) extension property in $C_r^*(R)$, i.e. a pure state of $C_0(X)$ has a unique (pure) state extension to $C_r^*(R)$, cf. Lemma A.13. □

REMARK 1.16. Let R be an étale equivalence relation on X and R' an étale equivalence relation on X'. A map $\Lambda : R \to R'$ is an *isomorphism* when there is a homeomorphism $\varphi : X \to X'$ such that $\Lambda = \varphi \times \varphi$ and Λ is a homeomorphism. It is clear that such an isomorphism gives rise to a $*$-isomorphism $C_r^*(R') \to C_r^*(R)$ sending $f \in C_c(R')$ to $f \circ \Lambda$. This is the easy part of the following result which is proved in Appendix A. □

THEOREM 1.17. *Two étale equivalence relations, R on X and R' on X', are isomorphic if and only if there is a $*$-isomorphism $\psi : C_r^*(R) \to C_r^*(R')$ such that $\psi(C_0(X)) = C_0(X')$.*

PROOF. The theorem follows from a general construction which produces an étale equivalence relation from a pair $D \subseteq A$ where A is a C^*-algebra and D is an abelian C^*-sub-algebra with the extension property. See Theorem A.16 in Appendix A. □

REMARK 1.18. When $U \subseteq R$ is an open subrelation, the inclusion $C_c(U) \subseteq C_c(R)$ extends to an embedding $C_r^*(U) \subseteq C_r^*(R)$, cf. e.g. Proposition 1.9 of [**Ph1**]. □

LEMMA 1.19. *Let R be an étale equivalence relation on a locally compact Hausdorff space X. Let $V \subseteq X$ be an open subset, and set $U = r^{-1}(V) \cap s^{-1}(V)$. Then $C_r^*(U) \subseteq C_r^*(R)$ is the hereditary C^*-sub-algebra generated by $C_0(V) \subseteq C_r^*(R)$.*

PROOF. We must show that $C_r^*(U)$ is the closure of the span of elements of the form bab', where $a \in C_r^*(R)$, $b, b' \in C_0(V)$. It is easy to see that $g \cdot f \cdot g' \in C_c(U)$ when $f \in C_c(R), g, g' \in C_c(V)$, and this gives one of the required inclusions. For the other let $f \in C_c(U)$. Then $r(\operatorname{supp} f) \cup s(\operatorname{supp} f)$ is a compact subset of V and there is an element $h \in C_c(V)$ such that $h(t) = 1$ for all $t \in r(\operatorname{supp} f) \cup s(\operatorname{supp} f)$. Since $f = h \cdot f \cdot h$ this implies the other inclusion. □

LEMMA 1.20. *Let R be an étale equivalence relation on a locally compact Hausdorff space X. Let $R_1 \subseteq R_2 \subseteq R_3 \subseteq \dots$ be an increasing sequence of open subrelations of R such that $R = \bigcup_{n=1}^{\infty} R_n$. It follows that*

$$C_r^*(R) = \overline{\bigcup_{n=1}^{\infty} C_r^*(R_n)}.$$

PROOF. This follows from Remark 1.18 and the observation that $C_c(R) = \bigcup_{n=1}^{\infty} C_c(R_n)$. □

LEMMA 1.21. *Let $A_f(X, E)$ be the Ruelle algebra of a relatively expansive system. Then $A_f(X, E)$ is separable if and only if E is second countable.*

PROOF. The inclusion $C_0(E) \subseteq A_f(X,E)$ shows that $C_0(E)$ is separable when $A_f(X,E)$ is. Hence E is second countable in this case. Assume that E is second countable. By Lemma 1.8 there is a countable collection of local conjugacies such that the corresponding sets (1.2) form a base for the topology of $R_f(X,E)$. Therefore every element of $C_c\left(R_f(X,E)\right)$ is a finite sum of functions localized on these sets. It suffices then to show that the set of elements of $C_c\left(R_f(X,E)\right)$ that are localized on a set of the form (1.2) is a separable subset of $A_f(X,E)$. This follows from the fact that $C_0(U)$ is separable for any open subset U of E because E is second countable, combined with the observation that

$$\|f\| = \sup_{z \in R_f(X,E)} |f(z)|,$$

when f is localized. □

1.4. Products and unions

When R and R' are two étale equivalence relations on the locally compact Hausdorff spaces X and X', respectively, there is a natural way to define the product $R \times R'$, namely as the equivalence relation in $X \times X'$ given by

$$R \times R' = \left\{((x,x'),(y,y')) \in (X \times X')^2 : \ (x,y) \in R, \ (x',y') \in R'\right\}.$$

By transferring the topology from the topological product of R and R' to $R \times R'$ by use of the map $((x,y),(x',y')) \mapsto ((x,x'),(y,y'))$ we turn $R \times R'$ into an étale equivalence relation.

LEMMA 1.22. *$C^*_{red}\left(R \times R'\right) \simeq C^*_{red}(R) \otimes C^*_{red}(R')$, where $\otimes$ is the minimal (or spatial) tensor product of C^*-algebras.*

PROOF. This follows from the identification $l^2\left[x,x'\right] = l^2[x] \otimes l^2[x']$. □

Consider now two relatively expansive systems,

$$(X,d,S,f,E,E_0) \quad \text{and} \quad (X',d',S,f',E',E_0'),$$

where only the index-set S for the continuous transformations are the same. We can then form a product of the two systems in the following way: On the product space $X \times X'$ we use the metric $d \times d'$ given by

$$d \times d'\left((a,a'),(b,b')\right) = d(a,b) + d'(a',b').$$

In the product topology $E \times E'$ is finer than the topology inherited from the metric space $(X \times X', d \times d')$ and $E_0 \times E_0'$ is of course dense in $E \times E'$. For $s \in S$ we set $(f \times f')_s(x,y) = (f_s(x), f_s'(y))$. Then

$$(X \times X', d \times d', S, f \times f', E \times E', E_0 \times E_0')$$

is a relatively expansive system.

PROPOSITION 1.23. *There is an isomorphism of étale equivalence relations*

$$R_{f \times f'}(X \times X', E \times E') \simeq R_f(X,E) \times R_{f'}(X',E'),$$

and hence an isomorphism

$$A_{f \times f'}(X \times X', E \times E') \simeq A_f(X,E) \otimes A_{f'}(X',E')$$

of C^*-algebras.

PROOF. If (U, V, χ) is a conjugacy in E and (U', V', χ') a conjugacy in E', it follows that $(U \times U', V \times V', \chi \times \chi')$ is a conjugacy in $E \times E'$. We can therefore define a map

$$\Lambda : R_f(X, E) \times R_f(X', E') \rightarrow R_f(X \times X', E \times E')$$

such that

$$\Lambda(((x, x'), (y, y')) = ((x, y), (x', y')),$$

and Λ is clearly open and injective. By Lemma 1.22 and Remark 1.16 it remains now only to show that Λ is surjective and continuous. To this end, let $(x, x'), (y, y') \in E \times E'$ be conjugate, and let (U, V, χ) be a conjugacy from (x, x') to (y, y') in $E \times E'$. There is then an open neighborhood A of x in E such that $A \times \{x'\} \subseteq U$. Define $\mu : A \rightarrow E$ such that $\mu(z) = p_1(\chi(z, x'))$, where $p_1 : X \times X' \rightarrow X$ is the projection. Similarly, there is an open neighborhood A' of x' and a map $\nu : A' \rightarrow E'$ defined such that $\nu(z) = p_2(\chi(x, z))$ where $p_2 : X \times X' \rightarrow X'$ is the projection to the second coordinate. Then $\mu(x) = y,\ \nu(x') = y'$ and

$$\begin{aligned} &\lim_{s\to\infty} \sup_{z \in A} d(f_s(\mu(z)), f_s(z)) \\ &\qquad \leq \lim_{s\to\infty} \sup_{z\in A} d(f_s \times f'_s(\chi(z, x')), f_s \times f'_s(z, x')) = 0. \end{aligned}$$

In the same way we find that $\lim_{s\to\infty} \sup_{z\in A'} d(f'_s(\nu(z)), f'_s(z)) = 0$. By shrinking A and A' if necessary we can assume that $A \times A' \subseteq U$. Since

$$\begin{aligned} &d \times d'(f_s \times f'_s(\mu(z), \nu(z')), f_s \times f'_s(\chi(z, z'))) \\ &\quad \leq d \times d'(f_s \times f'_s(\mu(z), \nu(z')), (f_s(z), f'_s(z'))) \\ &\qquad + d \times d'((f_s(z), f'_s(z')), f_s \times f'_s(\chi(z, z'))) \end{aligned}$$

we see that $d \times d'(f_s \times f'_s(\mu(z), \nu(z')), f_s \times f'_s(\chi(z, z')))$ tends to zero uniformly in $A \times A'$ as s leaves every finite subset of S. Therefore we can shrink A and A' further to arrange that $d \times d'(f_s \times f'_s(\mu(z), \nu(z')), f_s \times f'_s(\chi(z, z')))$ is smaller than an expansive constant at (y, y') for all $s \in S$. It follows that when A and A' are sufficiently small we have that $\chi = \mu \times \nu$ on $A \times A'$. By using the same reasoning to χ^{-1} in place of χ we conclude that there are conjugacies (A, B, μ) and (A', B', ν) from x to y and from x' to y', respectively, such that

$$\Lambda(\{(z, \mu(z)) :\ z \in A\} \times \{(z', \nu(z')) :\ z' \in A'\}) \subseteq \{((z, z'), \chi(z, z')) :\ (z, z') \in U\}.$$

$\square$

LEMMA 1.24. *Let (X, d, S, f, E, E_0) be a relatively expansive system. Assume that $E^1 \subseteq E^2 \subseteq E^3 \subseteq \dots$ is a sequence of open subsets of E such that $E = \bigcup_{n=1}^{\infty} E^n$. There is then a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of hereditary C^*-sub-algebras of $A_f(X, E)$ and $*$-isomorphisms $\psi_n : A_n \rightarrow C^*_r(R_f(X, E^n))$ such that*

$$\begin{array}{ccc} A_n & \xrightarrow{\psi_n} & C^*_r(R_f(X, E^n)) \\ \big\downarrow & & \big\downarrow \\ A_{n+1} & \xrightarrow{\psi_{n+1}} & C^*_r(R_f(X, E^{n+1})) \end{array}$$

commutes and $A_f(X, E) = \overline{\bigcup_n A_n}$.

PROOF. Note that $R_f(X, E^n)$ is an open sub-relation of $R_f(X, E)$ and that $R_f(X, E) = \bigcup_{n=1}^{\infty} R_f(X, E_n)$. Apply Lemma 1.20 and Lemma 1.19. $\square$

CHAPTER 2

On the functoriality of the Ruelle algebra

2.1. Contravariant functoriality

Let (X, d, S, E, f, E_0) and $(X', d', E', S', f', E'_0)$ be two relatively expansive systems. Assume that

$$\pi : E \to E'$$

is a continuous map. We seek to identify conditions that ensure that $\pi \times \pi$ gives rise to a map from $R_f(X, E) \to R_{f'}(X', E')$ and in turn to a $*$-homomorphism $\pi^\bullet : A_{f'}(X', E') \to A_f(X, E)$ between the Ruelle algebras of the two systems.

Define $m_\pi : E \to \mathbb{N} \cup \{\infty\}$ such that

$$m_\pi(x) = \# \left\{ y \in \pi^{-1}(\pi(x)) : \ y \sim x \right\}.$$

We consider the following conditions:

Condition 1: When (U, V, χ) is a conjugacy in E from x to y, there is a conjugacy (U', V', χ') from $\pi(x)$ to $\pi(y)$ in E' and an open neighborhood $U_0 \subseteq U$ of x such that

$$\chi' \circ \pi(z) = \pi \circ \chi(z) \tag{2.1}$$

for all $z \in \pi^{-1}(U') \cap U_0$.

Condition 2: $m_\pi(x)$ is finite for all $x \in E$ and m_π is locally constant.

Condition 3: π is surjective, and $\pi : \{z \in E : \ z \sim x\} \to \{z' \in E' : \ z' \sim \pi(x)\}$ is surjective for all $x \in E$.

Lemma 2.1. *Assume that condition 1 holds. Then*

$$(\pi \times \pi)(R_f(X, E)) \subseteq R_{f'}(X', E')$$

and $\pi \times \pi : R_f(X, E) \to R_{f'}(X', E')$ is continuous.

Proof. Only the continuity of $\pi \times \pi$ is not obvious. To prove it, let (W, V, μ) be a conjugacy in E' and $(x, y) \in R_f(X, E)$ an element such that $(\pi(x), \pi(y)) \in W \times V$ and $\mu(\pi(x)) = \pi(y)$. Since $(x, y) \in R_f(X, E)$ there is a conjugacy (U, V, χ) in E from x to y, and then by condition 1 also a conjugacy (U', V', χ') from $\pi(x)$ to $\pi(y)$ in E' such that (2.1) holds for some open neighborhood $U_0 \subseteq U$ of x. It follows from Lemma 1.4 that there is an open neighborhood $W' \subseteq U' \cap W$ of $\pi(x)$ such that $\chi' = \mu$ on W'. Then $\Omega = \{(z, \chi(z)) : \ z \in \pi^{-1}(W') \cap U_0\}$ is an open neighborhood of (x, y) in $R_f(X, E)$ such that $(\pi \times \pi)(\Omega) \subseteq \{(y, \mu(y)) : \ y \in W\}$. $\square$

Lemma 2.2. *Assume that condition 1 and condition 2 both hold. For every $x \in E$ there is an open neighborhood U_x of x and conjugacies (U_x, V_i, χ_i), $i = 1, 2, \dots, m_\pi(x) - 1$, such that $V_i \cap U_x = \emptyset$, $V_i \cap V_j = \emptyset, i \neq j$, and*

$$\{v \in E : \ v \sim z, \ \pi(v) = \pi(z)\} = \left\{ z, \chi_1(z), \chi_2(z), \dots, \chi_{m_\pi(x)-1}(z) \right\}$$

for all $z \in U_x$.

PROOF. Set $k = m_\pi(x)$. It follows from condition 2 that there is an open neighborhood W of x such that $m_\pi(z) = k$ for all $z \in W$. Let $y_1, y_2, \dots, y_{k-1}$ be the elements of $\{z \in \pi^{-1}(\pi(x)) \backslash \{x\} : z \sim x\}$. By shrinking W we may assume that there are open neighborhoods U_i of y_i such that $U_i \cap U_j = \emptyset$ when $i \neq j$ and $U_i \cap W = \emptyset$ for all i. There is then an open neighborhood $U_x \subseteq W$ of x and conjugacies (U_x, V_i, χ_i), from x to y_i, such that $V_i \subseteq U_i$, $i = 1, 2, \dots, k-1$. It follows from condition 1 and Lemma 1.4 that we can arrange, by shrinking U_x, that $\pi \circ \chi_i(z) = \pi(z), z \in U_x, i = 1, 2, \dots, k-1$. It follows then that for every $z \in U_x$, the elements $z, \chi_1(z), \dots, \chi_{k-1}(z)$ are mutually conjugate and distinct elements of $\pi^{-1}(\pi(z))$, which implies that

$$\{v \in E : v \sim z,\ \pi(v) = \pi(z)\} = \{z, \chi_1(z), \chi_2(z), \dots, \chi_{k-1}(z)\}$$

since $m_\pi(z) = k$. □

Recall that a continuous map between topological spaces is *proper* when the pre-image of any compact set of the target space is compact in the domain space.

LEMMA 2.3. *Assume that conditions 1,2 and 3 all hold, and that π is proper. It follows that $\pi \times \pi : R_f(X, E) \to R_{f'}(X', E')$ is a proper surjection.*

PROOF. It follows from condition 3 that $\pi \times \pi$ is surjective. Let (U', V', χ') be a conjugacy in E'. Let $K \subseteq U'$ be a compact subset. By Lemma 1.9 it suffices to show that

$$(\pi \times \pi)^{-1} (\{(z, \chi'(z)) : z \in K\}) \tag{2.2}$$

is compact in $R_f(X, E)$. Let $x \in \pi^{-1}(U')$. By condition 3 there is an element $y \in E$ such that $x \sim y$ and $\pi(y) = \chi'(\pi(x))$. Let (U, V, μ) be a conjugacy from x to y. It follows from condition 1 and Lemma 1.4 that we can arrange that $U \subseteq \pi^{-1}(U')$ and $\pi \circ \mu = \chi' \circ \pi$ on U. By Lemma 2.2 there is an open neighborhood $V_0 \subseteq V$ of y and conjugacies $(V_0, V_j, \chi_j), j = 1, 2, \dots, m_\pi(y)$, such that

$$\{v \in E : v \sim z,\ \pi(v) = \pi(z)\} = \left\{\chi_1(z), \chi_2(z), \dots, \chi_{m_\pi(y)}(z)\right\}$$

for all $z \in V_0$. Set $k_x = m_\pi(y)$, $W_x = \mu^{-1}(V_0)$ and $\mu_j^x = \chi_j \circ \mu$. Then

$$(\pi \times \pi)^{-1} (\{(z, \chi'(z)) : z \in U'\}) \cap r^{-1}(W_x) = \bigcup_{j=1}^{k_x} \left\{\left(z, \mu_j^x(z)\right) : z \in W_x\right\}.$$

Since π is proper there is a finite sub-cover $W_{x_1}, W_{x_2}, \dots, W_{x_N}$ of the cover $W_x, x \in \pi^{-1}(K)$, of $\pi^{-1}(K)$. Since E is locally compact there are compact subsets $L_i \subseteq W_{x_i}$ such that $\pi^{-1}(K) \subseteq \bigcup_{i=1}^N L_i$. Then

$$(\pi \times \pi)^{-1} (\{(z, \chi'(z)) : z \in K\}) \subseteq \bigcup_{i=1}^N \bigcup_{j=1}^{k_{x_i}} \left\{\left(z, \mu_j^{x_i}(z)\right) : z \in L_i\right\}. \tag{2.3}$$

By Lemma 1.9 and Lemma 2.1 we can conclude from (2.3) that (2.2) is compact, as desired. □

THEOREM 2.4. *Assume that π is proper, and that condition 1, condition 2 and condition 3 all hold. It follows that there is a $*$-homomorphism $\pi^\bullet : A_{f'}(X', E') \to A_f(X, E)$ such that*

$$\pi^\bullet(f)(x, y) = m_\pi(x)^{-\frac{1}{2}} m_\pi(y)^{-\frac{1}{2}} f(\pi(x), \pi(y))$$

when $f \in C_c(R_{f'}(X', E'))$.

PROOF. $\pi^\bullet : C_c(R_{f'}(X', E')) \to C_c(R_f(X, E))$ is defined by Lemma 2.1 and Lemma 2.3. Let $f, g \in C_c(R_f(X, E))$ and observe that

$$\begin{aligned}
&(\pi^\bullet(f) \cdot \pi^\bullet(g))(x, y) \\
&= m_\pi(x)^{-\frac{1}{2}} m_\pi(y)^{-\frac{1}{2}} \sum_{z \sim x} m_\pi(z)^{-1} f(\pi(x), \pi(z)) g(\pi(z), \pi(y)) \\
&= m_\pi(x)^{-\frac{1}{2}} m_\pi(y)^{-\frac{1}{2}} \sum_{v \sim \pi(x)} \sum_{\{z \in \pi^{-1}(v):\ z \sim x\}} m_\pi(z)^{-1} f(\pi(x), v) g(v, \pi(y)) \\
&= m_\pi(x)^{-\frac{1}{2}} m_\pi(y)^{-\frac{1}{2}} \sum_{v \sim \pi(x)} f(\pi(x), v) g(v, \pi(y)) = \pi^\bullet(f \cdot g)(x, y),
\end{aligned}$$

proving that $\pi^\bullet$ is a $*$-homomorphism. Let $\psi \in l^2[x]$. We define $\varphi : [\pi(x)] \to \mathbb{C}$ such that

$$\varphi(v) = \sum_{\{z \in \pi^{-1}(v):\ z \sim x\}} m_\pi(z)^{-\frac{1}{2}} \psi(z).$$

Then

$$\begin{aligned}
|\varphi(v)|^2 &\leq \Big(\sum_{\{z \in \pi^{-1}(v):\ z \sim x\}} m_\pi(z)^{-1} \Big) \Big(\sum_{\{z \in \pi^{-1}(v):\ z \sim x\}} |\psi(z)|^2 \Big) \\
&= \sum_{\{z \in \pi^{-1}(v):\ z \sim x\}} |\psi(z)|^2 .
\end{aligned}$$

It follows that $\varphi \in l^2[\pi(x)]$ and $\|\varphi\| \leq \|\psi\|$. Note that

$$\begin{aligned}
\left(\kappa_{[x]}(\pi^\bullet(f))\psi\right)(y) &= \sum_{z \sim x} m_\pi(z)^{-\frac{1}{2}} m_\pi(y)^{-\frac{1}{2}} f(\pi(y), \pi(z)) \psi(z) \\
&= \sum_{v \sim \pi(x)} m_\pi(y)^{-\frac{1}{2}} f(\pi(y), v) \varphi(v) = m_\pi(y)^{-\frac{1}{2}} \left(\kappa_{[\pi(x)]}(f)\varphi\right)(\pi(y))
\end{aligned}$$

and that

$$\begin{aligned}
&\sum_{y \sim x} \left| m_\pi(y)^{-\frac{1}{2}} \left(\kappa_{[\pi(x)]}(f)\varphi\right)(\pi(y)) \right|^2 \\
&\leq \sum_{v \sim \pi(x)} \sum_{\{y \in \pi^{-1}(v):\ y \sim x\}} m_\pi(y)^{-1} \left|\kappa_{[\pi(x)]}(f)\varphi(v)\right|^2 = \left\|\kappa_{[\pi(x)]}(f)\varphi\right\|^2
\end{aligned}$$

It follows first that $\left\|\kappa_{[x]}(\pi^\bullet(f))\right\| \leq \left\|\kappa_{[\pi(x)]}(f)\right\|$, and then that $\|\pi^\bullet(f)\| \leq \|f\|$. We conclude that $\pi^\bullet$ extends by continuity to a $*$-homomorphism $\pi^\bullet : A_{f'}(X', E') \to A_f(X, E)$. $\square$

COROLLARY 2.5. *Assume that*

- $S = S'$,
- $f'_s(E') \subseteq E'$ *and* $f_s(E) \subseteq E$ *for all* $s \in S$,
- $\pi : E \to E'$ *is a homeomorphism,*
- $\pi \circ f_s = f'_s \circ \pi$ *for all* $s \in S$, *and*
- π *and* π^{-1} *are uniformly continuous with respect to the metrics* d *and* d'.

It follows that there is a $*$-isomorphism $\pi^\bullet : A_{f'}(X', E') \to A_f(X, E)$ *such that*

$$\pi^\bullet(f)(x, y) = f(\pi(x), \pi(y))$$

when $f \in C_c(R_{f'}(X', E'))$.

PROOF. It is straightforward to check that conditions 1-3 hold with $m_\pi = 1$. $\square$

2.2. Covariant functoriality

The Ruelle-algebra construction is also functorial in a covariant way; at least into the KK-category of C^*-algebras and under appropriate conditions. We retain the setting from Section 2.1.

Set

$$\begin{aligned} E \times_\pi E' &= \{(x,e) \in E \times E' : \ \pi(x) \sim e\} \\ &= \{(x,e) \in E \times E' : \ (\pi(x), e) \in R_{f'}(X', E')\} . \end{aligned}$$

Sets of the form

$$\{(z, \mu \circ \pi(z)) : \ z \in U_0\} ,$$

where (U, V, μ) is a local conjugacy in E' and $U_0 \subseteq \pi^{-1}(U)$ is open, form a subbase for the topology of $E \times_\pi E'$ we consider in the following. We summarize the principal facts about $E \times_\pi E'$ in the next lemma.

A function $f \in C_c(E \times_\pi E')$ is *localized* when its support is contained in the set $\{(z, \mu \circ \pi(z)) : \ z \in U_0\}$ for some conjugacy (U, V, μ) in E' and some open subset $U_0 \subseteq \pi^{-1}(U)$.

LEMMA 2.6.

a) *$E \times_\pi E'$ is a locally compact Hausdorff space whose topology is finer than the relative topology inherited from $E \times E'$.*
b) *The map $E \times_\pi E' \ni (x, b) \mapsto x \in E$ is a local homeomorphism, i.e. open and locally injective.*
c) *A subset D of $E \times_\pi E'$ is pre-compact if and only if there are finitely many local conjugacies (U_i, V_i, μ_i), and compact subsets $L_i \subseteq \pi^{-1}(U_i), i = 1, 2, \ldots, N$, such that*

$$D \subseteq \bigcup_{i=1}^{N} \{(z, \mu_i \circ \pi(z)) : \ z \in L_i\} .$$

d) *Every element of $C_c(E \times_\pi E')$ is the sum of finitely many localized functions.*

PROOF. The relevant arguments from the proofs of Lemma 1.6, Theorem 1.7, Lemma 1.9 and Lemma 1.10 are straightforward to adopt. We omit the repetition. $\square$

Note that $C_c(E \times_\pi E')$ is a right $C_c(R_{f'}(X', E'))$-module defined such that

$$(f \cdot g)(x, b) = \sum_{a \sim b} f(x, a) g(a, b)$$

when $f \in C_c(E \times_\pi E')$, $g \in C_c(R_{f'}(X', E'))$. $C_c(E \times_\pi E')$ is also a left-module over $C_c(E)$:

$$h \cdot f(x, b) = h(x) f(x, b)$$

when $h \in C_c(E)$.

We consider now the following conditions.

Condition 4: $\pi : E \to E'$ is a local homeomorphism, i.e. π is open and locally injective.

Condition 5: For all $z \in E'$ there is an $x \in E$ such that $\pi(x) \sim z$.

Condition 6: $x, y \in E,\ \pi(x) \sim \pi(y) \ \Rightarrow\ x \sim y.$

LEMMA 2.7. *Assume that condition 4 holds. Let $f, g \in C_c\left(E \times_\pi E'\right)$. Then the function $\langle f, g\rangle : R_{f'}(X', E') \to \mathbb{C}$ defined by*

$$\langle f, g\rangle (a, b) \ = \sum_{\{x \in E:\ \pi(x) \sim a\}} \overline{f(x, a)} g(x, b), \tag{2.4}$$

is in $C_c\left(R_{f'}(X', E')\right)$.

PROOF. It suffices to consider the case where f and g are localized. So assume that (U, V, μ) and (U', V', μ') are local conjugacies in E' and $U_0 \subseteq \pi^{-1}(U)$, $U_0' \subseteq \pi^{-1}(U')$ open sets such that f is supported in $\{(z, \mu \circ \pi(z)) :\ z \in U_0\}$ and g is supported in $\{(z, \mu' \circ \pi(z)) :\ z \in U_0'\}$. Since π is locally injective we may restrict the attention to the case where π is injective on U_0 and U_0'. Then

$$\sum_{\{x \in E:\ \pi(x) \sim a\}} \overline{f(x, a)} g(x, b) = \begin{cases} h(z) & \text{when } (a, b) = (z, \mu' \circ \mu^{-1}(z)) \text{ for some } z \in \mu \circ \pi\left(U_0 \cap U_0'\right), \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

where $h(z) = \overline{f\left(\pi^{-1} \circ \mu^{-1}(z), z\right)} g\left(\pi^{-1} \circ \mu^{-1}(z), \mu' \circ \mu^{-1}(z)\right)$. Since π is open by assumption we see that $(a, b) \mapsto \sum_{\{x \in E:\ \pi(x) \sim a\}} \overline{f(x, a)} g(x, b)$ is a localized function on $R_{f'}\left(X', E'\right)$. □

By using the $\kappa_{[x]}$-representations, $x \in E'$, cf. (1.11), it is easy to check that $\langle f, f\rangle \geq 0$ in $A_{f'}(X', E')$ and therefore $\langle f, f\rangle = 0 \Rightarrow f = 0$. Thus $\langle \cdot, \cdot\rangle$ is a $C_c\left(R_{f'}(X', E')\right)$-valued inner product. An easy calculation confirms that

$$\langle f, g \cdot h\rangle = \langle f, g\rangle \cdot h \tag{2.6}$$

when $f, g \in C_c\left(E \times_\pi E'\right)$ and $h \in C_c\left(R_{f'}(X', E')\right)$. We get therefore a norm $\|\cdot\|_\pi$ on $C_c\left(E \times_\pi E'\right)$ when we set

$$\|f\|_\pi = \|\langle f, f\rangle\|^{\frac{1}{2}}.$$

The completion $C_c\left(E \times_\pi E'\right)$ in this norm is then a Hilbert $A_{f'}(X', E')$-module, cf. e.g. Lemma 1.1.2 of [**K-JT**]. We denote this Hilbert C^*-module by $\mathcal{E}_\pi$. [1]

LEMMA 2.8. *Assume that conditions 4 and 5 both hold. Then*

$$\left\{\langle f, g\rangle :\ f, g \in C_c\left(E \times_\pi E'\right)\right\}$$

spans a dense subspace in $A_{f'}(X', E')$. In particular, $\mathcal{E}_\pi$ is a full Hilbert $A_{f'}(X', E')$-module.

PROOF. Since $C_c(E') \cdot C_c\left(R_{f'}(X', E')\right)$ spans a dense subspace of $A_{f'}(X', E')$ it follows from (2.6) that it suffices to show that

$$C_c(E') \subseteq \operatorname{Span}\left\{\langle f, g\rangle :\ f, g \in C_c\left(E \times_\pi E'\right)\right\}. \tag{2.7}$$

Consider a function $\varphi \in C_c(E')$, $\varphi \geq 0$. Let $y \in \operatorname{supp} \varphi \subseteq E'$. By condition 5 there is an $x \in E$ and a local conjugacy (U, V, μ) from $\pi(x)$ to y. Since π is a local

[1]We refer to [**K-JT**] for the basic theory of Hilbert C^*-modules as well as the fundamentals of Kasparovs KK-theory which we will be using. However, our notation does deviate slightly from the notation in [**K-JT**] in that we here write $\mathbb{L}_B(F)$ for the C^*-algebra of adjointable operators of the Hilbert B-module F and $\mathbb{K}_B(F)$ for the ideal in $\mathbb{L}_B(F)$ consisting of the 'compact' operators.

homeomorphism we can shrink U to achieve that there is an open neighborhood U_x of x such that π is injective on U_x, $\overline{U_x}$ is compact and $\pi(U_x) = U$. By compactness of $\operatorname{supp}\varphi$ we have then a finite set $x_1, x_2, \ldots, x_N$ in E and for each i a local conjugacy $(\pi(U_{x_i}), V_i, \mu_i)$ such that $\operatorname{supp}\varphi \subseteq \bigcup_{i=1}^N V_i$. Let $\{g_i\}_{i=1}^N \subseteq C_c(E')$ be a partition of unity on $\operatorname{supp}\varphi$ subordinate to $\{V_i\}_{i=1}^N$, and define $f_i \in C_c(E \times_\pi E')$ as the function localized in $\{(z, \mu_i \circ \pi(z)) : z \in U_{x_i}\}$ such that $f_i(z, \mu_i \circ \pi(z)) = \sqrt{g_i(\mu_i \circ \pi(z))\, \varphi(\mu_i \circ \pi(z))}$. Then $\sum_{i=1}^N \langle f_i, f_i \rangle = \varphi$. Since every element of $C_c(E')$ is a linear combination of four non-negative elements of $C_c(E')$, we obtain (2.7). $\square$

When condition 1 holds we can also make $C_c(E \times_\pi E')$ into a left $C_c(R_f(X, E))$-module such that

$$(h \cdot f)(x, b) = \sum_{y \sim x} h(x, y) f(y, b), \tag{2.8}$$

when $f \in C_c(E \times_\pi E')$, $h \in C_c(R_f(X, E))$. Let $\varphi \in l^2[c]$ for some $c \in E'$. The estimate

$$\begin{aligned}
&\sum_{a,b} \overline{\varphi(a)} \langle h \cdot f, h \cdot f \rangle (a, b) \varphi(b) \\
&\quad = \sum_{a,b} \overline{\varphi(a)} \sum_{\{x:\ \pi(x) \sim a\}} \overline{h \cdot f(x, a)}\, h \cdot f(x, b) \varphi(b) \\
&\quad = \sum_{a,b} \sum_{\{x:\ \pi(x) \sim a\}} \sum_{\substack{y \sim x \\ z \sim x}} \overline{\varphi(a) h(x, y) f(y, a)}\, h(x, z) f(z, b) \varphi(b) \\
&\quad = \sum_{\{x:\ \pi(x) \sim a\}} \overline{\Big(\sum_{a, y \sim x} h(x, y) f(y, a) \varphi(a)\Big)} \Big(\sum_{b, z \sim x} h(x, z) f(z, b) \varphi(b)\Big) \\
&\quad \le \sum_{x} \overline{\Big(\sum_{a, y \sim x} h(x, y) f(y, a) \varphi(a)\Big)} \Big(\sum_{b, z \sim x} h(x, z) f(z, b) \varphi(b)\Big) \\
&\quad = \sum_{a,b} \sum_{x} \sum_{\substack{y \sim x \\ z \sim x}} \overline{\varphi(a) h(x, y) f(y, a)}\, h(x, z) f(z, b) \varphi(b) \\
&\quad = \sum_{a,b} \sum_{y \sim z} \overline{\varphi(a) f(y, a)}\, (h^* \cdot h)(y, z) f(z, b) \varphi(b) \\
&\quad \le \|h\|^2 \sum_{a,b} \sum_{y} \overline{\varphi(a) f(y, a)} f(y, b) \varphi(b)
\end{aligned}$$

shows that

$$\langle h \cdot f, h \cdot f \rangle \le \|h\|^2 \langle f, f \rangle$$

for all $f \in C_c(E \times_\pi E')$. It follows that we obtain a $*$-homomorphism

$$\pi_\bullet : A_f(X, E) \to \mathbb{L}_{A_{f'}(X', E')}(\mathcal{E}_\pi).$$

defined such that $\pi_\bullet(h) f = h \cdot f$ when $f \in C_c(E \times_\pi E')$, $h \in C_c(R_f(X, E))$.

LEMMA 2.9. *Assume that conditions 1 and 4 hold. Then*

$$\pi_\bullet(A_f(X, E)) \subseteq \mathbb{K}_{A_{f'}(X', E')}(\mathcal{E}_\pi),$$

and $\pi_\bullet$ is injective.

PROOF. Since $C_c(E)$ contains an approximate unit for $A_f(X,E)$, and using that $\mathbb{K}_{A_{f'}(X',E')}(\mathcal{E}_\pi)$ is a closed two-sided ideal in $\mathbb{L}_{A_{f'}(X',E')}(\mathcal{E}_\pi)$ it suffices to show that $\pi_\bullet(C_c(E)) \subseteq \mathbb{K}_{A_{f'}(X',E')}(\mathcal{E}_\pi)$. To this end it suffices to show that $\pi_\bullet(\varphi) \in \mathbb{K}_{A_{f'}(X',E')}(\mathcal{E}_\pi)$ when $\varphi \in C_c(E)$ is supported in an open set $V \subseteq E$ on which π is injective. We write φ as a product $\varphi = \varphi_1\overline{\varphi_2}$, where $\varphi_1, \varphi_2 \in C_c(V)$, and define $f, g \in C_c(E \times_\pi E')$ to be the functions with supports in $\{(x, \pi(x)) :\ x \in V\}$ such that $f(x,\pi(x)) = \varphi_1(x)$ and $g(x,\pi(x)) = \varphi_2(x)$. We claim that

$$\pi_\bullet(\varphi) = \Theta_{f,g}; \tag{2.9}$$

a fact that will finish the proof.[2] To check (2.9), consider an open neighborhood W in E such that π is injective on W, and let $h \in C_c(E \times_\pi E')$ have support in $\{(x, \mu\circ\pi(x) :\ x \in W\}$ for some local conjugacy μ defined on $\pi(W)$. It suffices to check (2.9) on h since functions of this sort span a dense subspace of $\mathcal{E}_\pi$, cf. d) of Lemma 2.6. It is straightforward to check that

$$\begin{aligned}
\pi_\bullet(\varphi)h(x,b) &= \begin{cases} \varphi(x)h(x,\mu\circ\pi(x)) & \text{when } x \in V\cap W \text{ and } b = \mu\circ\pi(x), \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \varphi_1(x)\overline{\varphi_2(x)}h(x,\mu\circ\pi(x)) & \text{when } x \in V\cap W \text{ and } b = \mu\circ\pi(x), \\ 0, & \text{otherwise} \end{cases} \\
&= \Theta_{f,g}(h)(x,b).
\end{aligned}$$

To establish the injectivity of $\pi_\bullet$ it suffices, by Proposition 4.6 of [**Re1**], to show that $\pi_\bullet(\varphi) = 0 \Rightarrow \varphi = 0$, which is easy. $\square$

Lemma 2.9 implies that $(\pi_\bullet, \mathcal{E}_\pi, 0)$ is a Kasparov $A_f(X,E) - A_{f'}(X',E')$ module, and hence the triple defines an element

$$[\pi] \in KK\left(A_f(X,E), A_{f'}(X',E')\right).$$

LEMMA 2.10. *Assume that conditions 1 and 4 hold, and that E is second countable. Then the element $[\pi] \in KK\left(A_f(X,E), A_{f'}(X',E')\right)$ is represented by a $*$-homomorphism $A_f(X,E) \to A_{f'}(X',E') \otimes \mathbb{K}$.*

PROOF. To simplify notation, set $A = A_f(X,E)$ and $B = A_{f'}(X',E')$. Using the notation from [**K-JT**] we have that $[\pi]$ is represented by $(\pi_\bullet, \mathcal{E}_\pi, 0) \oplus (0, H_B, 0)$. It follows then from Kasparov's stabilization theorem, cf. e.g. Theorem 1.1.24 in [**K-JT**], that $[\pi]$ is represented by a triple $(\psi, H_B, 0)$. Since $\mathbb{K}(H_B) \simeq B \otimes \mathbb{K}$ by another result of Kasparov, cf. e.g. Lemma 1.2.7 of [**K-JT**], the result follows from this. $\square$

The main point of the last lemma is that it shows that the map of K-theory induced by $[\pi]$ is positive on K_0.

Assume that we have a third relatively expansive system $(X'', d'', S'', , f'', E'', E''_0)$ and let $\pi' : E' \to E''$ be a continuous map.

PROPOSITION 2.11. *Assume that both π and π' satisfy conditions 1 and 4, and that E, E' and E'' are second countable. It follows that*

$$[\pi' \circ \pi] = [\pi'] \bullet [\pi]$$

[2]If necessary, see page 5 of [**K-JT**] for the definition of $\Theta_{f,g}$.

in $KK\left(A_f(X,E), A_{f''}(X'',E'')\right)$*, where* $\bullet$ *is the Kasparov-product.*

PROOF. The reader can find the definition of the Kasparov product in (e.g.) [**K-JT**]. The present case is greatly simplified by the fact that the degree 1 operator of our Kasparov triples are both zero. Thus the product $[\pi'] \bullet [\pi]$ is represented by the triple $\left(\mathcal{E}_\pi \otimes_{\pi'_\bullet} \mathcal{E}_{\pi'}, \pi_\bullet \otimes_{\pi'_\bullet} \mathrm{id}_{\mathcal{E}_{\pi'}}, 0\right)$, in the notation from [**K-JT**]. Define $\Phi : C_c\left(E \times_\pi E'\right) \otimes C_c\left(E' \times_{\pi'} E''\right) \to C_c\left(E \times_{\pi' \circ \pi} E''\right)$ such that

$$\Phi(f \otimes g)(x,b) = \sum_{z \sim b} f(x,z) g(z,b).$$

It is then straightforward to check that $\langle \Phi(f \otimes g), \Phi(f' \otimes g')\rangle = \langle g, \pi_\bullet\left(\langle f, f'\rangle\right) g'\rangle$ and $\Phi\left((f \cdot h) \otimes g\right) = \Phi\left(f \otimes (h \cdot g)\right)$ when $f, f' \in C_c\left(E \otimes_\pi E'\right)$, $h \in C_c\left(R_{f'}(X',E')\right)$ and $g, g' \in C_c\left(E' \otimes_{\pi'} E''\right)$, which is what is required to see that Φ falls to a map $\Phi : \mathcal{E}_\pi \otimes_{\pi'} \mathcal{E}_{\pi'} \to \mathcal{E}_{\pi' \circ \pi}$. To conclude that Φ is an isomorphism of Hilbert $A_{f''}\left(X'', E''\right)$-modules it remains only to show that Φ is surjective. Let $k \in C_c\left(E \times_{\pi' \circ \pi} E''\right)$ be a localized and non-negative function supported in $\{(t, \mu \circ \pi' \circ \pi(t)) : \ t \in U\}$ for some local conjugacy μ in E''. Since π' and π are local homeomorphisms we may assume that $\pi(U)$ and $\pi' \circ \pi(U)$ are open and that $\pi : U \to \pi(U)$ and $\pi' : \pi(U) \to \pi' \circ \pi(U)$ are both homeomorphisms. We can then define $f \in C_c\left(E \times_\pi E'\right)$ and $g \in C_c\left(E' \times_\pi E''\right)$ such that f is supported in $\{(t, \pi(t)) : \ t \in U\}$ and satisfies that $f(t, \pi(t)) = \sqrt{k(t, \mu \circ \pi' \circ \pi(t))}$ while g is supported in $\{(s, \mu \circ \pi'(s)) : \ s \in \pi(U)\}$ and satisfies that $g(s, \mu \circ \pi'(s)) = \sqrt{k\left(\pi^{-1}(s), \mu \circ \pi'(s)\right)}$. Then $\Phi(f \otimes g) = k$. Since functions with the properties we have required by k span all of $C_c\left(E \times_\pi E''\right)$ we conclude that Φ is indeed an isomorphism of $A_{f''}\left(X'', E''\right)$-modules. The proof of the proposition is then completed by the trivial observation that $\Phi \circ \left(\pi_\bullet \otimes_{\pi'_\bullet} \mathrm{id}_{\mathcal{E}_{\pi'}}\right) = \left(\pi' \circ \pi\right)_\bullet$. ∎

LEMMA 2.12. *Assume conditions 1,4 and 6 all hold.*
It follows that $\pi_\bullet : A_f(X,E) \to \mathbb{K}_{A_{f'}(X',E')}\left(\mathcal{E}_\pi\right)$ *is a* $*$*-isomorphism.*

PROOF. Let $f, g \in C_c\left(E \times_\pi E'\right)$. In view of Lemma 2.9 it suffices to show that $\Theta_{f,g} \in \pi_\bullet\left(A_f(X,E)\right)$. Since $\Theta_{f,g}$ is sesqui-linear in (f,g) we may assume that f and g are both localized. Let (U, V, μ) and (U', V', μ') be local conjugacies in E' and $U_0 \subseteq \pi^{-1}(U), U_0' \subseteq \pi^{-1}(U')$ open sets such that f is supported in $\{(t, \mu \circ \pi(t)) : \ t \in U_0\}$ and g is supported in $\{(t, \mu' \circ \pi(t)) : \ t \in U_0'\}$. Since condition 4 holds we may assume, in addition, that π is injective on U_0 and U_0'. Observe that

$$\Theta_{f,g}(h)(x,b) = \sum_{c \sim b} f(x,c) \langle g, h\rangle (c,b) = \sum_{c \sim b} \sum_{\{y : \pi(y) \sim c\}} f(x,c) \overline{g(y,c)} h(y,b). \tag{2.10}$$

There are compact subsets $K \subseteq U_0$ and $K' \subseteq U_0'$ such that

$$\operatorname{supp} f \subseteq \{(t, \mu \circ \pi(t)) : \ t \in K\} \text{ and } \operatorname{supp} g \subseteq \{(t, \mu' \circ \pi(t)) : \ t \in K'\}.$$

Set $L = K \cap \pi^{-1} \circ \mu^{-1}\left(\mu' \circ \pi(K')\right)$. There is then for each $t \in L$ a unique element $\gamma(t) \in K'$ such that $\mu \circ \pi(t) = \mu' \circ \pi(\gamma(t))$. Since condition 6 holds we see that $t \sim \gamma(t)$ in E. By condition 1 there is then an open neighborhood U of t and a conjugacy (U, V, ν) from t to $\gamma(t)$ such that $U \subseteq U_0, V \subseteq U_0'$ and $\pi \circ \nu(s) = \mu'^{-1} \circ \mu \circ \pi(s)$ for all $s \in U$. By compactness of L we get a finite collection $(U_i, V_i, \nu_i), i = 1, 2, \dots, N$, of local conjugacies in E with $U_i \subseteq U_0, \ L \subseteq \bigcup_{i=1}^N U_i, \ V_i \subseteq U_0'$ and $\pi \circ \nu_i(s) = \mu'^{-1} \circ \mu \circ \pi(s), \ s \in U_i$ for all i. Let $h_i, i = 1, 2, \dots, N$, be a partition of unity on L subordinate to the U_i's. For each i we can then define an element $\varphi_i \in C_c(R_f(X,E))$ such that $\operatorname{supp}\varphi_i \subseteq \{(t,\nu_i(t)) : t \in U_i\}$ and $\varphi_i(t,\nu_i(t)) = h_i(t)f(t,\mu\circ\pi(t))\overline{g(\nu_i(t),\mu'\circ\pi\circ\nu_i(t))}$. It is straightforward to check that $\Theta_{f,g} = \pi_\bullet\left(\sum_{i=1}^N \varphi_i\right)$, yielding the desired conclusion. □

THEOREM 2.13. *Assume conditions 1,4,5 and 6 all hold. It follows that $A_f(X,E)$ and $A_{f'}(X',E')$ are strongly Morita equivalent in the sense of Rieffel,* [**Ri1**].

PROOF. It follows from Lemma 2.12 and Lemma 2.8 that $\mathcal{E}_\pi$ is an imprimitivity bi-module for $A_{f'}(X',E')$ and $A_f(X,E)$. □

COROLLARY 2.14. *Assume that $U \subseteq E$ is an open subset such that every element of E is conjugate to an element from U. It follows that $A_f(X,E)$ is strongly Morita equivalent to $A_f(X,U)$.*

PROOF. Apply Theorem 2.13 to the inclusion of U into E. □

Both $A_f(X,E)$ and $A_{f'}(X',E')$ are separable C^*-algebras when E and E' are second countable, cf. Lemma 1.21. In this case strong Morita equivalence is the same as stable isomorphism, cf. [**BGR**].

REMARK 2.15. In the setting of Corollary 2.14 we know from Lemma 1.19 that $A_f(X,U)$ is a hereditary C^*-sub-algebra of $A_f(X,E)$. Using the description of the ideals in a C^*-algebra of an étale equivalence relation given in [**Re1**] it follows that $A_f(X,U)$ is full in $A_f(X,E)$ in the sense of [**Br**]. In this way Corollary 2.14 follows from [**Re1**] and [**Br**], at least in the separable case.

Similarly Theorem 2.13 can be deduced from Theorem 2.8 of [**MRW**] by showing that the conditions 1,4,5 and 6 are sufficient to make $E \times_\pi E'$ into a $(R_f(X,E), R_{f'}(X',E'))$-equivalence in the sense of [**MRW**]. □

CHAPTER 3

The homoclinic algebra of expansive actions

Let S be a countable set and (X, d) a locally compact metric space. Let $\operatorname{End} X$ denote the semi-group of continuous maps $X \to X$, and let $\alpha : S \to \operatorname{End} X$ a map. We assume here that α is an *expansive action* in the sense that there is $\delta > 0$ such that

$$\sup_{s \in S} d\left(\alpha_s(x), \alpha_s(y)\right) \leq \delta \ \Rightarrow \ x = y. \tag{3.1}$$

In this case (X, d, S, α, X, X) is a relatively expansive system in the sense of Section 1.1 and the Ruelle algebra $A_\alpha(X)$ is defined as described in Chapter 1. We will call it *the homoclinic algebra* of α. In the following sections we study this C^*-algebra is more detail for certain classes of expansive actions.

Let $\alpha : S \to \operatorname{End} X$ and $\beta : S \to \operatorname{End} Y$ be expansive actions of the same countable set S on (X, d) and (Y, d'), respectively. A uniformly continuous homeomorphism $\pi : X \to Y$, with a uniformly continuous inverse, is an *asymptotic conjugacy* when

$$\lim_{s \to \infty} \sup_{x \in X} d'\left(\pi \circ \alpha_s(x), \beta_s \circ \pi(x)\right) = 0.$$

Theorem 3.1. *Let $\pi : X \to Y$ be an asymptotic conjugacy between the expansive actions $\alpha : S \to \operatorname{End} X$ and $\beta : S \to \operatorname{End} Y$. It follows that there is a $*$-isomorphism $\pi^\bullet : A_\beta(Y) \to A_\alpha(X)$ such that $\pi^\bullet\left(C_c\left(R_\beta(X)\right)\right) = C_c\left(R_\alpha(X)\right)$ and*

$$\pi^\bullet(f)(x, y) = f\left(\pi(x), \pi(y)\right)$$

when $f \in C_c\left(R_\beta(Y)\right)$.

Proof. It is straightforward to check that $\pi \times \pi$ is a homeomorphism from $R_\alpha(X)$ onto $R_\beta(Y)$ and hence an isomorphism of étale equivalence relations. As observed in Remark 1.16 this implies the result. $\square$

When $\alpha : S \to \operatorname{End} X$ takes values in the group $\operatorname{Aut} X$ of uniformly continuous homeomorphisms of X, the action α extends to a map $\alpha : S \to \operatorname{Aut} A_\alpha(X)$ such that

$$\alpha_s(f)(x, y) = f\left(\alpha_s^{-1}(x), \alpha_s^{-1}(y)\right).$$

We call $\alpha : S \to \operatorname{Aut} A_\alpha(X)$ *the canonical action* of S on the homoclinic algebra.

3.1. Shift spaces

In [**Kr1**] and Section 2 of [**Kr2**] Wolfgang Krieger introduced the dimension group for a general shift space. This dimension group is the K_0-group of an AF-algebra which we now describe before we go on to show that it agrees with the homoclinic algebra of the shift-space.

Let $\mathcal{A}$ be a finite set, sometimes called the *alphabet*, and $X \subseteq \mathcal{A}^{\mathbb{Z}}$ a *shift space*. Thus X is a closed subset of $\mathcal{A}^{\mathbb{Z}}$ which is shift-invariant in the sense that $\sigma(X) = X$, where σ is the shift on $\mathcal{A}^{\mathbb{Z}}$, viz. $\sigma\left((x_i)_{i \in \mathbb{Z}}\right)_j = x_{j+1}$ for all $j \in \mathbb{Z}$. The *words* in X

are the elements $w \in \bigcup_{n=0}^{\infty} \mathcal{A}^n$ which occur in some element of X in the sense that $w = x_{i+1}x_{i+2}\dots x_{i+|w|}$ for some $i \in \mathbb{Z}$ and some $x \in X$. Here $|w|$ is the *length* of w, i.e. the number of letters in w. The empty word is then the word of length 0. The set of words in X of length k is denoted by $\mathbb{W}_k(X)$, so that the set $\mathbb{W}(X)$ of all words in X is $\mathbb{W}(X) = \bigcup_{k=0}^{\infty} \mathbb{W}_k(X)$.

Let $w \in \mathbb{W}(X)$. The *context* $E(w)$ of w consists of the pair $(a,b) \in \mathbb{W}(X)^2$ with the property that $awb \in \mathbb{W}(X)$. A simple but crucial observation is the following: When $w, v \in \mathbb{W}(X)$, $E(w) = E(v)$ and $(a,b) \in E(w)$, then $E(awb) = E(avb)$. This observation is used tacitly (and often) in the following.

Let $n \in \mathbb{N}$. We denote by $R(n,X)$ the set of pairs $(w,v) \in \mathbb{W}_{2n+1}(X)^2$ for which $E(w) = E(v)$. The free complex vector space $A_X(n)$ with basis $R(n,X)$ is a $*$-algebra with involution $*$ and product defined such that

$$(w,v)^* = (v,w)$$

and

$$(w,v)\,(s,t) = \begin{cases} (w,t) \text{ when } v = s \\ 0 \text{ when } v \neq s. \end{cases}$$

Thus

$$A_X(n) \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C}),$$

where $i = 1, 2, \dots, N$ numbers the equivalence classes in $\mathbb{W}_{2n+1}(X)$ of words with the same context, and n_i is the number of elements in the i'th equivalence class. Let $\varphi_n : A_X(n) \to A_X(n+1)$ be the linear map which satisfies that

$$\varphi_n(w,v) = \sum_{\{(a,b)\in\mathcal{A}^2:\ (a,b)\in E(w)\}} (awb, avb).$$

Then φ_n is an injective unital $*$-homomorphism. Let A_X be the resulting AF-algebra,

$$A_X = \varinjlim_n \left(A_X(n), \varphi_n\right),$$

which we will call the *Krieger algebra* of the shift space X. We denote by $\varphi_{\infty,n} : A_X(n) \to A_X$ the canonical $*$-homomorphism associated with the inductive limit construction, and for $j \geq i$ by $\varphi_{j,i}$ the composite $*$-homomorphism $\varphi_{j-1} \circ \varphi_{j-2} \circ \cdots \circ \varphi_i : A_X(i) \to A_X(j)$.

The shift gives rise to an automorphism of A_X in the following way. Define a unital $*$-homomorphism $\sigma'_n : A_X(n) \to A_X(n+1)$ such that

$$\sigma'_n(w,v) = \sum_{\{x\in\mathbb{W}_2(X):\ (wx,vx)\in\mathbb{W}_{2n+3}(X)^2\}} (wx, vx).$$

Then the diagram

$$\begin{tikzcd} A_X(n) \arrow[r, "\sigma'_n"] \arrow[d, "\varphi_n"'] & A_X(n+1) \arrow[d, "\varphi_{n+1}"] \\ A_X(n+1) \arrow[r, "\sigma'_{n+1}"'] & A_X(n+2) \end{tikzcd} \tag{3.2}$$

commutes, giving us a $*$-endomorphism $\sigma' : A_X \to A_X$ such that $\sigma' \circ \varphi_{\infty,n} = \varphi_{\infty,n+1} \circ \sigma'_n$ for all n. Define ${\sigma'}_n^{-1} : A_X(n) \to A_X(n+1)$ such that

$$ {\sigma'}_n^{-1}(w,v) = \sum_{\{x \in \mathbb{W}_2(X):\ (xw,xv) \in \mathbb{W}_{2n+3}(X)^2\}} (xw,xv), $$

and note that the diagram

$$ \begin{tikzcd} A_X(n) \arrow[r, "\varphi_{n+2,n}"] \arrow[d, "\sigma'_n"'] & A_X(n+2) \arrow[d, "{\sigma'}_{n+2}"] \\ A_X(n+1) \arrow[r, "\varphi_{n+3,n+1}"'] \arrow[ur, "{\sigma'}_{n+1}^{-1}"'] & A_X(n+3) \end{tikzcd} \tag{3.3} $$

commutes, showing that there is $*$-endomorphism ${\sigma'}^{-1} : A_X \to A_X$ such that ${\sigma'}^{-1} \circ \varphi_{\infty,n} = \varphi_{\infty,n+1} \circ {\sigma'}_n^{-1}$. ${\sigma'}^{-1}$ is the inverse of σ', i.e. σ' is an automorphism of A_X with inverse ${\sigma'}^{-1}$. We call σ' the *shift automorphism* of the Krieger algebra A_X.

We show next that the Krieger algebra is the same as the homoclinic algebra of X. When $x \in X$ and $i \leq j$ in $\mathbb{Z}$, we denote the word $x_i x_{i+1} x_{i+2} \dots x_j \in \mathbb{W}(X)$ by $x_{[i,j]}$. When $w \in \mathbb{W}_{2n+1}(X)$, set

$$ C_w = \left\{ (x_i)_{i \in \mathbb{Z}} \in X :\ x_{[-n,n]} = w \right\}. $$

Then $C_w, w \in \bigcup_{n=0}^{\infty} \mathbb{W}_{2n+1}(X)$, is a base for the topology of X.

LEMMA 3.2. *Two points $x, y \in X$ are locally conjugate if and only if there is an $N \in \mathbb{N}$ such that $x_i = y_i$ and $E\left(x_{[-i,i]}\right) = E\left(y_{[-i,i]}\right)$ for all $i \geq N$. In fact, we can then define a local conjugacy $\left(C_{x_{[-N,N]}}, C_{y_{[-N,N]}}, \chi\right)$ from x to y such that*

$$ \chi(z)_i = \begin{cases} z_i, & |i| > N, \\ y_i, & |i| \leq N \end{cases}. \tag{3.4} $$

PROOF. Assume first that x and y are conjugate. Then x and y are forward and backward asymptotic under the shift so there is an $M \in \mathbb{N}$ such that $x_i = y_i, |i| \geq M$. Choose $\epsilon > 0$ such that $z, z' \in X, d(z,z') \leq \epsilon \Rightarrow z_0 = z'_0$. Since x is conjugate to y there is then a $K \in \mathbb{N}$ with the following property: When V is an open neighborhood of y there is an open neighborhood U of x such that

$$ \sup_{x' \in U} \inf_{y' \in V} \left(\sup_{|i| \geq K} d\big(\sigma^i(x'), \sigma^i(y')\big) \right) \leq \epsilon. $$

Choose now $N_0 > \max\{K, M\}$. It follows then that there is an $N_1 \in \mathbb{N}$ such that $N_1 > N_0$ and when $z \in C_{x_{[-N_1,N_1]}}$ there is a $z' \in C_{y_{[-N_0,N_0]}}$ such that $z_i = z'_i$ for all $|i| \geq K$. Since $x_i = y_i$ when $|i| \geq N_0$ because $N_0 > M$, it follows that in fact $z' \in C_{y_{[-N_1,N_1]}}$. This shows that $E\left(x_{[-N_1,N_1]}\right) \subseteq E\left(y_{[-N_1,N_1]}\right)$, and it follows that $x_i = y_i$ and $E\left(x_{[-i,i]}\right) \subseteq E\left(y_{[-i,i]}\right)$ for $i \geq N_1$. By symmetry there is also an $N_2 \in \mathbb{N}$ such that $x_i = y_i$ and $E\left(y_{[-i,i]}\right) \subseteq E\left(x_{[-i,i]}\right)$ for $i \geq N_2$. Set $N = \max\{N_1, N_2\}$.

The converse is straightforward. $\square$

THEOREM 3.3. *There is a $*$-isomorphism $\psi : A_X \to A_\sigma(X)$ such that $\sigma \circ \psi = \psi \circ \sigma'$ when $\sigma \in \operatorname{Aut} A_\sigma(X)$ is the canonical automorphism of the homoclinic algebra.*

Proof. Let $(w,v) \in R(n,X)$. By Lemma 3.2 we define $1_w \times_n 1_v \in C_c\left(R_\sigma(X)\right)$ such that

$$1_w \times_n 1_v(x,y) = \begin{cases} 1, & \text{when } (x,y) \in C_w \times C_v \text{ and } x_i = y_i,\ |i| \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$$

We can then define a $*$-homomorphism $\psi_n : A_X(n) \to C_c\left(R_\sigma(X)\right)$ such that $\psi_n(w,v) = 1_w \times_n 1_v$. Then $\psi_{n+1} \circ \varphi_n = \psi_n$ and it follows that there is a $*$-homomorphism $\psi : A_X \to A_\sigma(X)$ such that $\psi \circ \varphi_{\infty,n} = \psi_n$. Note that ψ is injective since each ψ_n is. To see that ψ is also surjective it suffices, by Lemma 1.10, to show that every localized function $f \in C_c\left(R_\sigma(X)\right)$ is in the range of ψ. Let $\epsilon > 0$ and let (U,V,χ) be a conjugacy such that $\operatorname{supp} f \subseteq \{(x,\chi(x)) :\ x \in U\}$. Since f has compact support and X is totally disconnected there is a compact and open subset $L \subseteq U$ such that $\operatorname{supp} f \subseteq \{(x,\chi(x)) :\ x \in L\}$. It follows from Lemma 3.2 and Lemma 1.4 that there is an $n \in \mathbb{N}$ and elements $w_1, w_2, \ldots, w_N, v_1, v_2, \ldots, v_N \in \mathbb{W}_n(X)$ such that $L = \bigcup_{i=1}^N C_{w_i}$ is a partition, $(w_i, v_i) \in R(n,X)$ and $\chi(z)_{[-n,n]} = v_i,\ \chi(z)_j = z_j, |j| > n$, when $z \in C_{w_i}$, for all i. Furthermore, we can arrange that $|f(x,\chi(x)) - f(y,\chi(y))| \leq \epsilon$ for all $x, y \in C_{w_i}$.

For each i we choose an element $x_i \in C_{w_i}$. To estimate the distance in $A_\sigma(X)$ between f and $\sum_{i=1}^N f\left(x_i, \chi\left(x_i\right)\right) 1_{w_i} \times_n 1_{v_i}$ we define functions $h, k : X \to \mathbb{C}$ such that

$$h(x) = f\left(x, \chi(x)\right)$$

and

$$k(x) = \sum_{i=1}^N f\left(x_i, \chi\left(x_i\right)\right) 1_{C_{w_i}}(x)$$

when $x \in U$ and $h(x) = k(x) = 0$ when $x \notin U$. $1_{C_{w_i}}$ is here the characteristic function of the set C_{w_i}. Then $h, k \in C(X) \subseteq C_c\left(R_\sigma(X)\right)$ and $\|h - k\| < \epsilon$. Since $f = h \cdot \sum_{i=1}^N 1_{w_i} \times_n 1_{v_i}$ and $\sum_{i=1}^N f\left(x_i, \chi\left(x_i\right)\right) 1_{w_i} \times_n 1_{v_i} = k \cdot \sum_{i=1}^N 1_{w_i} \times_n 1_{v_i}$ we find that

$$\left\| f - \sum_{i=1}^N f\left(x_i, \chi\left(x_i\right)\right) 1_{w_i} \times_n 1_{v_i} \right\| \leq \|h - k\| \left\| \sum_{i=1}^N 1_{w_i} \times_n 1_{v_i} \right\|$$

in $A_\sigma(X)$. Note that $\left(\sum_{i=1}^N 1_{w_i} \times_n 1_{v_i}\right) \cdot \left(\sum_{i=1}^N 1_{w_i} \times_n 1_{v_i}\right)^* = 1_L$, and that $1_L \in C(X)$ is projection in $A_\sigma(X)$. It follows that $\left\|\sum_{i=1}^N 1_{w_i} \times_n 1_{v_i}\right\| \leq 1$ and we conclude that

$$\left\| f - \sum_{i=1}^N f\left(x_i, \chi\left(x_i\right)\right) 1_{w_i} \times_n 1_{v_i} \right\| \leq \|h - k\| \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary and $\sum_{i=1}^N f\left(x_i, \chi\left(x_i\right)\right) 1_{w_i} \times_n 1_{v_i}$ is in the range of ψ, it follows that so is f.

For the equivariance part of the theorem it suffices to check that $\psi \circ \sigma' \circ \varphi_{\infty,n}(w,v) = \sigma \circ \psi \circ \varphi_{\infty,n}(w,v)$ when $(w,v) \in R(n,X)$. We leave this to the reader. $\square$

3.1.1. One-sided shift-spaces. There is an analogous version of the preceding for one-sided shift spaces. The only difference is that there is no natural extension of the one-sided shift to an endomorphism of the homoclinic algebra. Apart from this the key definitions can be adopted with the obvious modifications. We outline the constructions.

Let Z be a closed subset of $\mathcal{A}^{\mathbb{N}}$ which is shift-invariant in the sense that $\sigma(Z) = Z$, where $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is now the one-sided shift. As above the point of departure is the collection of words $\mathbb{W}(Z)$ occurring in the shift space. The *follower set* $F(w)$ of a word $w \in \mathbb{W}(Z)$ is the set

$$F(w) = \{v \in \mathbb{W}(Z) : \ wv \in \mathbb{W}(Z)\} .$$

Let $n \in \mathbb{N}$. We denote by $F(n, Z)$ the set of pairs $(w, v) \in \mathbb{W}_n(Z)^2$ for which $F(w) = F(v)$. The free complex vector space $A_Z(n)$ with basis $F(n, Z)$ is then a finite dimensional C^*-algebra in the same way as above, cf. (3.1) and (3.1).

We can define a unital $*$-homomorphism $\varphi_n : A_Z(n) \to A_Z(n+1)$ such that

$$\varphi_n\left((w, v)\right) = \sum_{\{b \in \mathcal{A} : b \in F(w)\}} (wb, vb).$$

Note that φ_n is injective and unital, and let A_Z be the resulting AF-algebra,

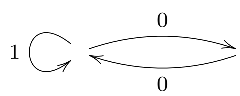
$$A_Z = \varinjlim_n \left(A_Z(n), \varphi_n\right) .$$

The K_0-group of A_Z has appeared before in the work of Boyle, Fiebig and Fiebig, [**BFF**], as 'the images group'. The coincidence of the two follows from the description of the images group given in Section 10 of [**BFF**]. In particular, it follows that $K_0\left(A_Z\right)$ is the dimension group of Z, cf. [**LM**], when Z is of finite type. As stated in [**BFF**] the "construction of the images group is very much in the tradition of Krieger's construction of a dimension group for a two-sided SFT", and we take this as justification for adopting the terminology from the two-sided case and call A_Z the *Krieger-algebra* of Z.

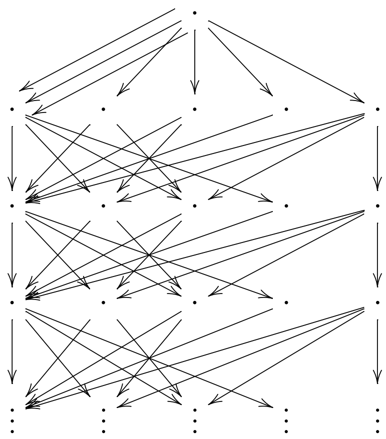
It is straightforward to prove the analogue of Lemma 3.2 and use it to prove the following

THEOREM 3.4. *The Krieger-algebra A_Z is $*$-isomorphic to the homoclinic algebra $A_\sigma(Z)$.*

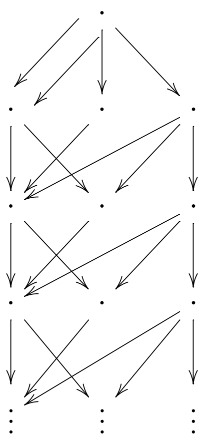
EXAMPLE 3.5. One virtue of Theorem 3.3 and Theorem 3.4 is that they make it possible to write down Bratteli diagrams for the homoclinic algebra of a shift space. For sofic shift spaces one can use the Fischer cover for this purpose. To illustrate this consider the even shift which was mentioned in Remark 1.13. The even shift can be presented by a labeling of its Fischer cover, cf. e.g. [**LM**]:



The Bratteli diagram for the homoclinic algebra of the two-sided even shift becomes



For the one-sided even shift the Bratteli diagram becomes



3.2. Expansive actions by affine maps

Assume now that X is a compact metric group with neutral element e. An *affine endomorphism* of X is a map $\alpha : X \to X$ such that

$$\alpha(x) = \lambda\alpha_0(x), \ x \in X,$$

where $\lambda \in X$ and α_0 is a continuous group-endomorphism. We say that λ is the *translation part* of α and that α_0 is the *endomorphism part* of α. Let (X, d, S, α) be an expansive action as in (3.1). Assume that α is an *expansive affine action* in the sense that each α_s is an affine endomorphism of X. Slightly generalizing a definition of Lind and Schmidt, [**LS**], we call

$$\Delta_\alpha = \left\{x \in X : \lim_{s\to\infty} d\left(\alpha_s(x), \alpha_s(e)\right) = 0\right\}.$$

the *homoclinic group* of α. Note that since d is equivalent to a left-invariant metric the homoclinic group only depends on the endomorphism parts. Specifically, if we let α_s^0 denote the endomorphism part of α_s,

$$\Delta_\alpha = \left\{x \in X : \lim_{s\to\infty} d\left(\alpha_s^0(x), e\right) = 0\right\}.$$

In particular, Δ_α is indeed a subgroup of X.

LEMMA 3.6. *Let $x, y \in X$. The following are equivalent:*

a) *x and y are conjugate.*
b) $\lim_{s\to\infty} d(\alpha_s(x), \alpha_s(y)) = 0$.
c) $xy^{-1} \in \Delta_\alpha$.

PROOF. a) $\Rightarrow$ b) is trivial. b) $\Rightarrow$ c):

$$\begin{gathered}\lim_{s\to\infty} d(\alpha_s(x), \alpha_s(y)) = 0 \\ \Updownarrow \\ \lim_{s\to\infty} d\left(\alpha_s^0(x), \alpha_s^0(y)\right) = 0 \\ \Updownarrow \\ \lim_{s\to\infty} d\left(\alpha_s^0(xy^{-1}), e\right) = 0.\end{gathered}$$

c) $\Rightarrow$ a): Define $\chi : X \to X$ such that $\chi(z) = yx^{-1}z$ and observe that (X, X, χ) is then a conjugacy from x to y. □

LEMMA 3.7. *The map*

$$(x, y) \mapsto \left(yx^{-1}, y\right)$$

is an isomorphism of topological groupoids from $R_\alpha(X)$ onto the transformation groupoid $\Delta_\alpha \times X$ corresponding to left-translation by Δ_α on X.

PROOF. It follows from Lemma 3.6 that the map is an algebraic isomorphism of groupoids. (A description of the groupoid $\Delta_\alpha \times X$ can be found in [**Ph1**], for example.) We leave the reader to check that Γ is a homeomorphism. □

To describe the homoclinic algebra $A_\alpha(X)$ we introduce some (standard) notation regarding crossed products of C^*-algebras. Let B be a C^*-algebra with an automorphic action $\beta : H \to \operatorname{Aut} B$ of a discrete group H. The Hilbert B-module $l^2(H, B)$ carries a unitary representation u of H given by

$$(u_h\psi)(g) = \psi\left(h^{-1}g\right),$$

and there is an embedding $B \subseteq \mathbb{L}_B\left(l^2(H, B)\right)$ defined such that

$$(b\psi)(h) = \beta_{h^{-1}}(b)\psi(h).$$

By definition the reduced crossed product $B \rtimes_\beta H$ is the C^*-sub-algebra of $\mathbb{L}_B\left(l^2(H, B)\right)$ generated by B and the unitaries $\{u_h : h \in H\}$.

When G is a compact group and $H \subseteq G$ is a subgroup we denote by τ the action of H on $C(G)$ given by left-translation:

$$\tau_h(f)(g) = f(h^{-1}g).$$

THEOREM 3.8. *Let α be an expansive affine action on X. The homoclinic algebra $A_\alpha(X)$ and the crossed-product $C(X) \rtimes_\tau \Delta_\alpha$ are related by a $*$-isomorphism ψ such that the diagram*

$$\begin{tikzcd} A_\alpha(X) \arrow[r, "\psi"] & C(X) \rtimes_\tau \Delta_\alpha \\ C(X) \arrow[u, hook] \arrow[r, equal] & C(X) \arrow[u, hook] \end{tikzcd}$$

commutes.

PROOF. This follows from Lemma 3.7 since the reduced C^*-algebra of the groupoid $\Delta_\alpha \times X$ is $C(X) \rtimes_\tau \Delta_\alpha$, cf. e.g. Proposition 1.8 of [**Ph1**]. □

COROLLARY 3.9. *Let α be an expansive affine action on X. The following are equivalent.*

i) *The homoclinic algebra $A_\alpha(X)$ is simple.*
ii) *The trace state of $A_\alpha(X)$ is unique.*
iii) *The homoclinic group Δ_α is dense in X.*

PROOF. This can be deduced from the identification $A_\alpha(X) = C(X) \rtimes_\tau \Delta_\alpha$ as follows. Since Δ_α acts freely on X it follows from a classical result of Zeller-Meyer, cf. 4.20 of [**Z**], that there is a bijective correspondance between ideals in $C(X) \rtimes_\tau \Delta_\alpha$ and closed τ-invariant subsets of X. It follows that i) is equivalent to iii). Furthermore, it follows from Theorem 4.5 of [**Th3**] that there is a bijective correspondence between trace states of $C(X) \rtimes_\tau \Delta_\alpha$ and τ-invariant Borel probability measures on X. If iii) holds the normalized Haar measure of X is the only τ-invariant Borel probability measures on X. In this way iii) implies ii). Conversely, if ii) holds the compact group $\overline{\Delta_\alpha}$ must be all of X since its Haar measure would otherwise give rise to a τ-invariant Borel probability measure on X different from the Haar measure of X. This would produce two different trace states of $C(X) \rtimes_\tau \Delta_\alpha$ contradicting ii). In this way ii) implies iii). □

3.2.1. Expansive algebraic actions on connected groups. Given a C^*-algebra B one can define the *dual groupoid*. The idea of the construction is apparently due to Alain Connes and one can find a description of it in [**Re2**]. The elements of the dual groupoid are the extremal elements of the unit ball in the dual space B^* of B. Let $D \subseteq B$ be an abelian C^*-sub-algebra of B with the unique extension property for pure states. That is, every pure state of D has a unique (pure) state extension to B. The reduction of the dual groupoid of B to the pure state space of D is then a locally compact topological groupoid by [**Re2**]. This is relevant for us here because the canonical copy of $C(X)$ inside the crossed product $C(X) \rtimes G$ coming from a free action of the discrete group G on X is known to have the extension property by Corollary 6.2 of [**Ba**] and Remark 2.8 (i) of [**ABG**]. Hence the inclusion $C(X) \subseteq C(X) \rtimes G$ gives rise to a locally compact groupoid via a fairly general construction, and it was shown in [**Th1**] that this groupoid is the product of the transformation groupoid coming from the action of G on X with the circle group $\mathbb{T}$. When G is connected (and only in this case), it follows that the inclusion $C(X) \subseteq C(X) \rtimes G$ determines the action of G on X modulo a conjugacy and an automorphism of G, cf. Theorem 9 of [**Th1**]. In particular, this is the case in the setting of Theorem 3.8 when X is connected. In this section we gather some consequences of this.

THEOREM 3.10. *Let X and Y be compact connected abelian metric groups and $\alpha : S \to \operatorname{End} X$ and $\beta : S \to \operatorname{End} Y$ expansive affine actions of the countable set S on X and Y, respectively. Assume that the homoclinic group Δ_α of α is dense in X. Consider a homeomorphism $\psi : X \to Y$ such that $\psi(0) = 0$.*

The following conditions are equivalent:

1) *$\psi \times \psi$ is an isomorphism between the étale equivalence relations $R_\alpha(X)$ and $R_\beta(Y)$.*
2) *ψ is a topological group isomorphism $\psi : X \to Y$ such that $\psi(\Delta_\alpha) = \Delta_\beta$.*
3) *There is a $*$-isomorphism $\mu : A_\alpha(X) \to A_\beta(Y)$ such that $\mu(f) = f \circ \psi^{-1}$ for $f \in C(X)$.*

PROOF. 1) $\Rightarrow$ 3) follows from Remark 1.16. 3) $\Rightarrow$ 2): We adopt the notation from [**Th1**]. By Theorem 3.8 we may consider μ as a $*$-isomorphism from $C(X) \rtimes_\tau \Delta_\alpha$ to $C(Y) \rtimes_\tau \Delta_\beta$. As such μ defines a topological groupoid isomorphism $\mu^* : G(C(Y) \rtimes_\tau \Delta_\beta, C(Y)) \to G(C(X) \rtimes_\tau \Delta_\alpha, C(X))$ whose action on the unit space Y is ψ^{-1}. By (the proof of) Theorem 9 of [**Th1**] there is an isomorphism $\varphi : \Delta_\alpha \to \Delta_\beta$ such that $\psi(x - \varphi(h)) = \psi(x) - h$ for all $x \in X$ and all $h \in \Delta_\alpha$. It follows then from the density of Δ_α in X that $\psi - \psi(0)$ is a group isomorphism taking Δ_α onto Δ_β. Since $\psi(0) = 0$, 2) follows.

The implication 2) $\Rightarrow$ 1) follows from Lemma 3.7. □

The equivalence of 1) and 3) in Theorem 3.10 is a special case of a much more general result. See Appendix A.

By inspection of the last proof it becomes clear that the implication 3) $\Rightarrow$ 2) is a result on topological isomorphism rigidity. This is a new approach to this type of results and it seems worthwhile to pause a little to develop this point. For previous work on topological isomorphism rigidity we refer, without any claim of completeness, to [**W**], [**B1**], [**B2**], [**KS**], [**BW**], [**EW**] and [**BS**].

THEOREM 3.11. *Let X and Y be compact connected abelian metric groups and $\alpha : S \to \operatorname{End} X$ and $\beta : S \to \operatorname{End} Y$ expansive affine actions of the countable set S on X and Y, respectively. Assume that $\psi : X \to Y$ is an asymptotic conjugacy.*

It follows that there is a group isomorphism $\varphi : \Delta_\beta \to \Delta_\alpha$ such that

$$\psi(x) - h = \psi(x - \varphi(h))$$

for all $x \in X$ and all $h \in \Delta_\beta$.

PROOF. ψ induces a topological isomorphism $R_\alpha(X) \to R_\beta(Y)$ of étale equivalence relations, and hence by Lemma 3.7 also a homeomorphism of the corresponding transformation groupoids arising form the actions of the homoclinic groups. Since X (and Y) are connected Theorem 9 of [**Th1**] gives a group isomorphism $\varphi : \Delta_\beta \to \Delta_\alpha$ such that $f(\psi(x) - h) = f(\psi(x - \varphi(h)))$ for all $f \in C(Y), x \in X, h \in \Delta_\beta$. The conclusion follows from this. □

REMARK 3.12. It follows from Theorem 3.11 that a topological conjugacy between expanding affine actions of the same semi-group on connected groups is a group-isomorphism on the homoclinic group, and hence on the entire group when the homoclinic subgroup is dense. For actions by a group this follows from [**B1**]. □

3.2.2. Positively expansive endomorphisms of compact groups. Recall that a continuous map $\psi : X \to X$ on a metric space (X, d) is said to be *positively expansive* when the corresponding action of the semi-group $\mathbb{N}$ is expansive, i.e. when there is a $\delta > 0$ such that

$$\sup_{n \in \mathbb{N}} d(\psi^n(x), \psi^n(y)) \leq \delta \;\Rightarrow\; x = y.$$

LEMMA 3.13. *Let X be a compact metric group. Let $\psi : X \to X$ be a positively expansive and surjective group endomorphism. Then ψ is open. If in addition X is connected or ψ is ergodic with respect to the Haar-meausure of X, then $\Delta_\psi = \bigcup_{k \geq 1} \ker \psi^k$ is dense in X.*

PROOF. It is obvious that $\Delta_\psi = \bigcup_{k \geq 1} \ker \psi^k$. Let $H = \overline{\bigcup_{k \geq 1} \ker \psi^k}$ be the closure of $\bigcup_{k \geq 1} \ker \psi^k$ in X; clearly a normal subgroup. To prove that $H = X$ note first that ψ is constant-to-one. In fact, $\# \ker \psi < \infty$ by expansiveness and then $\#\psi^{-1}(x) = \# \ker \psi$ for all $x \in X$ since ψ is a surjective group endomorphism. Let $\delta > 0$ be so small that

$$a, b \in \psi^{-1}(x),\ a \neq b \ \Rightarrow\ d(a,b) \geq \delta. \tag{3.5}$$

It follows that for each $x \in X$ and each $\epsilon > 0$ there is an $\epsilon' > 0$ such that

$$d\left(\psi(x), z\right) \leq \epsilon' \ \Rightarrow\ \exists a \in \psi^{-1}(z):\ d\left(a, x\right) < \epsilon. \tag{3.6}$$

Indeed, if not there is a sequence $\{z_i\} \subseteq X$ such that $\lim_{i\to\infty} z_i = \psi(x)$ while $\operatorname{dist}\left(\psi^{-1}\left(z_i\right), x\right) \geq \epsilon$ for all i. Let $a_i^j, j = 1, 2, \ldots, \# \ker \psi$, be the elements of $\psi^{-1}\left(z_i\right)$. By compactness of X there is a sequence $\{n_i\}$ in $\mathbb{N}$ such that each sequence $\left\{a_{n_i}^j\right\}_{i=1}^{\infty}$ converges, say to $a^j \in X, j = 1, 2, \ldots, \# \ker \psi$. By combining (3.5) with the fact that $d\left(a_j, x\right) \geq \epsilon > 0$, we conclude that $x, a_1, a_2, \ldots, a_{\# \ker \psi}$ are distinct elements of $\psi^{-1}\left(\psi(x)\right)$, contradicting that $\#\psi^{-1}\left(\psi(x)\right) = \# \ker \psi$. Hence (3.6) holds. Note that it follows that ψ is open. In addition it follows also from (3.6) that

$$\psi(x) \in H \Rightarrow x \in H, \tag{3.7}$$

which implies that the continuous group endomorphism $\psi' : X/H \to X/H$ induced by ψ is injective. Since it is obviously also surjective we see that ψ' is an automorphism of the compact group X/H.

It follows from [**Rd**] that there are a metric d' for the topology of X, a $\lambda > 1$ and an $\epsilon_1 > 0$ such that $d'(x,y) \leq \epsilon_1 \Rightarrow d'(\psi(x), \psi(y)) \geq \lambda d'(x,y)$. By substituting $d'(x,y)$ with

$$\sup_{a \in X} d'(xa, ya)$$

we may assume that d' is right-invariant. We get then a metric $\tilde{d}$ for the topology on X/H such that

$$\tilde{d}\left(xH, yH\right) = \inf_{h,h' \in H} d'(xh, yh').$$

It follows from (3.6) that there is an $\epsilon_2 > 0$ such that

$$d'\left(\psi(x), z\right) \leq \epsilon_2 \ \Rightarrow\ \exists a \in \psi^{-1}(z):\ d'\left(a, x\right) < \epsilon_1 \tag{3.8}$$

for all $x \in X$. Let $\epsilon_3 > 0$ be so small that

$$\tilde{d}\left(xH, yH\right) \leq \epsilon_3 \Rightarrow \tilde{d}\left(\psi(x)H, \psi(y)H\right) \leq \frac{\epsilon_2}{2}.$$

Consider $x, y \in X$ such that $\tilde{d}\left(xH, yH\right) \leq \epsilon_3$. Let $\delta > 0$. There are elements $k, k' \in H$ such that $d'\left(\psi(x)k, \psi(y)k'\right) \leq \tilde{d}\left(\psi(x)H, \psi(y)H\right) + \delta$ and $d'\left(\psi(x)k, \psi(y)k'\right) < \epsilon_2$. It follows from (3.7) that $k = \psi(h)$ for some $h \in H$. It follows then from (3.8) that there is an element $z \in X$ such that $\psi(z) = \psi(y)k'$ and $d'(xh, z) < \epsilon_1$. Set $h' = y^{-1}z$ and note that $h' \in H$ by (3.7). Then

$$\begin{aligned} \tilde{d}\left(\psi(x)H, \psi(y)H\right) + \delta &\geq d'\left(\psi(xh), \psi(yh')\right) \\ &\geq \lambda d'(xh, yh') \geq \lambda \tilde{d}(xH, yH). \end{aligned}$$

Since $\delta > 0$ was arbitrary, we conclude that

$$\tilde{d}\left(xH, yH\right) \leq \epsilon_3 \Rightarrow \tilde{d}\left(\psi(x)H, \psi(y)H\right) \geq \lambda \tilde{d}(xH, yH). \tag{3.9}$$

It follows from (3.9) that ψ' is positively expansive. By a result of S. Schwartzman a compact metric space that supports a positively expansive homeomorphism is finite. See [**CK**] for a short proof of this. Hence we know now that X/H is finite.

Under any of the two additional assumptions it is easy to prove that $X/H = 0$, as desired. □

We pause again to point out a consequence concerning isomorphism rigidity which does not seem to have been noted before.

Theorem 3.14. *Let X and Y be compact connected abelian metric groups and $\alpha : X \to X$, $\beta : Y \to Y$ positively expansive and surjective affine maps.*
Assume that $\psi : X \to Y$ is an asymptotic conjugacy. It follows that ψ is affine.

Proof. Combine Theorem 3.10 and Lemma 3.13. □

We turn to the structure of the homoclinic algebra.

Definition 3.15. Let $\mathbb{T}$ denote the unit circle in the complex plane. A *circle algebra* is a C^*-algebra $*$-isomorphic to $C(\mathbb{T}) \otimes F$ for some finite-dimensional C^*-algebra F. An *AT-algebra* is a C^*-algebra A which contains an increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of circle algebras as C^*-sub-algebras such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$.

Theorem 3.16. *Let G be a compact metric group and $\psi : G \to G$ a positively expansive and surjective endomorphism. Assume that G is connected or that ψ is ergodic with respect to the Haar measure of G. It follows that $A_\psi(G)$ is a simple unital AT-algebra of real rank zero with a unique trace state.*

Proof. We argue first that

a) $\dim G < \infty$, and that
b) $K_*(C(G))$ is torsionfree.

Note first that G is homeomorphic to the topological product $G^0 \times (G/G^0)$, where G^0 denotes the connected component of the neutral element in G. This seems to be a well-known fact; a more general statement appears as Proposition 5.9 of [**KS**]. It follows from Lemma 3.13 and Theorem 7.12 of [**HeRo**] that ψ restricts to a positively expansive map of G^0 which is expanding in the sense of [**Ao**]. It follows then from Theorem 2 of [**Ao**] that G^0 is an inverse limit of tori of the same fixed dimension. Since G/G_0 is totally disconnected by Theorem 7.3 of [**HeRo**], both a) and b) follow easily.

By Lemma 3.13 the homoclinic group Δ_ψ is the union of an increasing sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ of finite subgroups of G. It follows therefore from Theorem 3.8 that $A_\psi(G)$ is the inductive limit of a sequence

$$C(G) \rtimes_\tau F_1 \to C(G) \rtimes_\tau F_2 \to C(G) \rtimes_\tau F_3 \to \dots$$

By Lemma B.5 in Appendix B $C(G) \rtimes_\tau F_n$ is stably isomorphic to $C(G/F_n)$. Set $A_k = C(G) \rtimes_\tau F_k$ and let $\varphi_k : A_k \to A_{k+1}$ be the connecting map in the above sequence. Since A_k is unital and $A_k \otimes \mathbb{K} \simeq C(G/F_n) \otimes \mathbb{K}$ it follows that there is an $l_k \in \mathbb{N}$ and a projection $p_k \in M_{l_k}(C(G/F_n))$ such that $A_k \simeq p_k M_{l_k}(C(G/F_n)) p_k$. It follows that there are $*$-isomorphisms $\iota_k : A_k \to p_k M_{l_k}(C(G/F_k)) p_k$ and unital $*$-homomorphisms $\psi_k : p_k M_{l_k}(C(G/F_k)) p_k \to p_{k+1} M_{l_{k+1}}(C(G/F_{k+1})) p_{k+1}$ such that the infinite diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "\iota_1"'] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\iota_2"'] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "\iota_3"'] & \cdots \\
p_1 M_{l_1}(C(G/F_1)) p_1 \arrow[r, "\psi_1"] & p_2 M_{l_2}(C(G/F_2)) p_2 \arrow[r, "\psi_1"] & p_3 M_{l_3}(C(G/F_3)) p_3 \arrow[r, "\psi_3"] & \cdots
\end{tikzcd}$$

commutes. Hence $A_\psi(G) \simeq \varinjlim_k (p_k M_{l_k}(C(G/F_k)) p_k, \psi_k)$. As observed above $C(G)$ has torsion-free K-theory and finite covering dimension. It follows that $C(G/F_k)$ has torsion-free K-theory and that $\dim G/F_k = \dim G < \infty$ for all k. Since $C(G/F_k)$ and $p_k M_{l_k}(C(G/F_k)) p_k$ have the same K-theory we conclude that $K_*(A_\psi(G))$ is torsion-Free. Since $A_\psi(G)$ is simple by Lemma 3.13 and Theorem 3.8, it follows now from Corollary 6.7 of [**G**] that $A_\psi(G)$ is in fact an AT-algebra. Being simple it is then approximately divisible in the sense of [**BKR**] by a result of Elliott, [**Ell1**]. Furthermore, it follows from Corollary 3.9 that $A_\psi(G)$ has a unique trace state (arising from the normalized Haar measure of G), and hence $A_\psi(G)$ has real rank zero by [**BKR**]. $\square$

3.3. Expansive group automorphisms

3.3.1. The shift of a solenoid. Let $n \in \mathbb{N}$ and let $\rho : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the quotient map. Let $\|\cdot\|$ be a vector space norm on $\mathbb{R}^n$ such that $\|z\| \geq 1$ for all $z \in \mathbb{Z}^n$. We will work with the metric d_0 on $\mathbb{T}^n$ given by

$$d_0(\rho(x), \rho(y)) = \inf_{z \in \mathbb{Z}^n} \|x - y - z\|.$$

Let $Q \in Gl_n(\mathbb{Q})$, and set

$$S_Q = \left\{ (z_i)_{i \in \mathbb{Z}} \in (\mathbb{T}^n)^{\mathbb{Z}} : \ (z_i, z_{i+1}) \in \{(\rho(t), \rho(Qt)) : \ t \in \mathbb{R}^n\} \ \forall i \in \mathbb{Z} \right\}. \tag{3.10}$$

We equip S_Q with the metric d given by

$$d\left((x_i)_{i \in \mathbb{Z}}, (y_i)_{i \in \mathbb{Z}}\right) = \sum_{i \in \mathbb{Z}} 2^{-|i|} d_0(x_i, y_i).$$

Let $\sigma_Q : S_Q \to S_Q$ be the shift of S_Q, i.e. $\sigma_Q\left((z_i)_{i \in \mathbb{Z}}\right)_j = z_{j+1}$.

LEMMA 3.17. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. There is a decomposition $\mathbb{R}^n = \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S}$ of $\mathbb{R}^n$ into a direct sum of the subspaces $\mathcal{U}$, $\mathcal{N}$ and $\mathcal{S}$, such that*

i) *$\mathcal{S} = \{x \in \mathbb{R}^n : \ \lim_{n \to \infty} L^n x = 0\}$,*
ii) *$\mathcal{U} = \{x \in \mathbb{R}^n : \ \lim_{n \to -\infty} L^n x = 0\}$, and*
iii) *$\{x \in \mathbb{R}^n : \ \sup_{k \in \mathbb{Z}} \|L^k x\| < \infty\} \subseteq \mathcal{N}$.*

Furthermore there are constants $\lambda, K > 0$, $\lambda < 1$, such that $\|L^n x\| \leq K\lambda^{|n|}\|x\|$ when $x \in \mathcal{U}$ and $n \leq 0$ or $x \in \mathcal{S}$ and $n \geq 0$.

PROOF. See e.g. pp. 23-26 in [**HK**]. $\square$

In the following we let

$$\mathbb{R}^n = \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S} \tag{3.11}$$

be the decomposition obtained by applying Lemma 3.17 to $Q : \mathbb{R}^n \to \mathbb{R}^n$. The hyperbolicity of Q is equivalent to expansiveness of (S_A, σ_Q) and means that $\mathcal{N} = \{0\}$, cf. Proposition 6.2 of [**KS**]. But we will not yet assume that the shift is expansive.

We seek to describe the homoclinic group. The following lemma is due to Brenken, cf. Proposition 3.6 and Proposition 3.7 of [**Bre**].

LEMMA 3.18. *(Brenken) There is a $\delta > 0$ such that for all $x = (x_i)_{i=-\infty}^{\infty} \in S_Q$,*

i) *if $d_0(x_i, 0) < \delta$ for all $i \leq 0$, there is a unique element $z \in \mathbb{R}^n$ such that $x_i = \rho(Q^i z)$ and $\|Q^i z\| < \delta$ for all $i \leq 0$,*
ii) *if $d_0(x_i, 0) < \delta$ for all $i \leq 0$ and $\lim_{i \to -\infty} d_0(x_i, 0) = 0$, there is a unique element $z \in \mathcal{U}$ such that $x_i = \rho(Q^i z)$ and $\|Q^i z\| < \delta$ for all $i \leq 0$,*

iii) *if* $d_0(x_i, 0) < \delta$ *for all* $i \geq 0$*, there is a unique element* $z \in \mathbb{R}^n$ *such that* $x_i = \rho(Q^i z)$ *and* $\|Q^i z\| < \delta$ *for all* $i \geq 0$*,*
iv) *if* $d_0(x_i, 0) < \delta$ *for all* $i \geq 0$ *and* $\lim_{i\to\infty} d_0(x_i, 0) = 0$*, there is a unique element* $z \in \mathcal{S}$ *such that* $x_i = \rho(Q^i z)$ *and* $\|Q^i z\| < \delta$ *for all* $i \geq 0$*,*

Proof. i) and iii) are reformulations of Proposition 3.6 and Proposition 3.7 in [**Bre**], respectively. ii) and iv) follow from i) and iii) since ρ is a local homeomorphism. $\square$

Let $N \in \mathbb{N}$ and set

$$H_N = \left\{ \left(x, (w_i)_{i=1}^{2N+1}, y\right) \in \mathcal{U} \oplus (\mathbb{Z}^n)^{2N+1} \oplus \mathcal{S} : \ Q^{2N} x + \sum_{i=1}^{2N+1} Q^{2N+1-i} w_i = y \right\}.$$

For $\xi = \left(x, (w_i)_{i=1}^{2N+1}, y\right) \in H_N$, define $\Lambda_N(\xi) \in S_Q$ such that

$$\Lambda_N(\xi)_k = \begin{cases} \rho\left(Q^{k+N} x\right), & k \leq -N \\ \rho\left(Q^{k+N} x + Q^{k+N} w_1 + Q^{k+N-1} w_2 + \cdots + Q w_{k+N}\right), & -N < k < N \\ \rho(Q^{k-N} y), & k \geq N. \end{cases} \tag{3.12}$$

Then Λ_N is a homomorphism and $\Lambda_N(H_N) \subseteq \Delta_{\sigma_Q}$. Define $\iota_N : H_N \to H_{N+1}$ such that

$$\iota_N\left(x, (w_1, w_2, \dots, w_{2N+1}), y\right) = \left(Q^{-1} x, (0, w_1, w_2, \dots, w_{2N+1}, 0), Q y\right). \tag{3.13}$$

Then ι_N is a homomorphism and $\Lambda_{N+1} \circ \iota_N = \Lambda_N$. We get therefore a homomorphism

$$\Lambda : \varinjlim_N (H_N, \iota_N) \to \Delta_{\sigma_Q}. \tag{3.14}$$

Lemma 3.19. Λ *is surjective.*

Proof. Let $z \in \Delta_{\sigma_Q}$. It follows from Lemma 3.18 that there is an $N \in \mathbb{N}$ and elements $x \in \mathcal{U}$, $y \in \mathcal{S}$ such that $z_{-N-j} = \rho(Q^{-j} x)$ and $z_{N+j} = \rho(Q^j y)$ for all $j \geq 0$. By successively applying the condition that $(z_j, z_{j+1}) \in \{(\rho(t), \rho(Qt)) : \ t \in \mathbb{R}^n\}$ for all $-N \leq j \leq N$, we get an element $(w_1, w_2, \dots, w_{2N+1}) \in (\mathbb{Z}^n)^{2N+1}$ such that $Q^{2N} x + \sum_{i=1}^{2N+1} Q^{2N+1-i} w_i = y$ and

$$\rho\left(Q^{k+N} x + Q^{k+N} w_1 + Q^{k+N-1} w_2 + \cdots + Q w_{k+N}\right) = z_k$$

when $-N < k < N$. $\square$

Let H be a finitely generated abelian group. Then $H \simeq \mathbb{Z}^m \oplus F$ for some $m \in \mathbb{N}$ and some finite abelian group F. We call m the *rank* of H and denote it by $\operatorname{Rank} H$.

Lemma 3.20. H_N *is finitely generated and torsion-free of rank* $\leq n$*. In particular,* $\operatorname{Rank} \Lambda_N(H_N) \leq n$ *for all* $N \in \mathbb{N}$*.*

Proof. Let $m \in \mathbb{N}$ be so large that $mQ^k \in M_n(\mathbb{Z})$ for all $k \in \{0, 1, 2, \dots, 2N+1\}$. Define $\Phi : H_N \to \mathbb{Z}^n$ such a way that $\Phi\left(x, (w_i)_{i=1}^{2N+1}, y\right) = m \sum_{i=1}^{2N+1} Q^{2N+1-i} w_i$. If $\Phi\left(x, (w_i)_{i=1}^{2N+1}, y\right) = 0$ it follows that $Q^{2N+1} x = y$ and hence that $x = y = 0$ since $\mathcal{U} \cap \mathcal{S} = \{0\}$. This shows that Φ is injective and the lemma follows. $\square$

LEMMA 3.21. *The solenoid S_Q is a projective limit of n-tori, i.e. there is a sequence $\varphi_k : \mathbb{T}^n \to \mathbb{T}^n, k = 1, 2, \ldots,$ of continuous surjective group endomorphisms such that S_Q is isomorphic, as a topological group, to the projective limit of the sequence*

$$\mathbb{T}^n \xleftarrow[\varphi_1]{} \mathbb{T}^n \xleftarrow[\varphi_2]{} \mathbb{T}^n \xleftarrow[\varphi_3]{} \mathbb{T}^n \xleftarrow[\varphi_4]{} \cdots$$

In particular, S_Q is connected and divisible.

PROOF. For each k, equip the subgroup

$$\Omega_k = \left(Q^t\right)^{-k}(\mathbb{Z}^n) + \left(Q^t\right)^{-k+1}(\mathbb{Z}^n) + \cdots + \left(Q^t\right)^{k-1}(\mathbb{Z}^n) + \left(Q^t\right)^{k}(\mathbb{Z}^n)$$

of $\mathbb{Q}^n$ with the discrete topology. The dual group $\widehat{S_Q}$ of S_Q can then be identified with the union $\bigcup_{k=1}^{\infty} \Omega_k$, cf. the proof of Proposition 6.2 of [**KS**]. It follows that S_Q is isomorphic to the corresponding projective limit

$$\widehat{\Omega_1} \longleftarrow \widehat{\Omega_2} \longleftarrow \widehat{\Omega_3} \longleftarrow \widehat{\Omega_4} \longleftarrow \cdots$$

Since $\Omega_k \simeq \mathbb{Z}^n$, and hence $\widehat{\Omega_k} \simeq \mathbb{T}^n$ for each k, this yields the lemma. $\square$

We denote in the following by $\langle M \rangle$ the subgroup of $\mathbb{R}^n$ generated by a subset $M \subseteq \mathbb{R}^n$.

LEMMA 3.22. *The homoclinic group Δ_{σ_Q} is dense in the solenoid S_Q if and only if*

$$\left(\mathcal{U} + \left\langle \bigcup_{j \geq k} Q^j(\mathbb{Z}^n) \right\rangle\right) \cap \left(\mathcal{S} + \left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right) \tag{3.15}$$

is dense in $\mathbb{R}^n$ for all $k \in \mathbb{N}$.

PROOF. Let $m \in \mathbb{N}$ be a natural number such that mQ has integer entries. Assume first that Δ_{σ_Q} is dense in S_Q. Let $a_0 \in \mathbb{R}^n$, $\epsilon > 0$ and $d \geq 2k$ be given. Set $a_i = Q^i a_0, i \in \mathbb{Z}$. Then $(\rho(a_i))_{i \in \mathbb{Z}} \in S_Q$ and it follows therefore from the density of Δ_{σ_Q} that there is an element $c = (c_i)_{i \in \mathbb{Z}} \in \Delta_{\sigma_Q}$ such that $d_0(\rho(a_0), c_0) \leq \frac{\epsilon}{2}$ and $d_0(\rho(a_i), c_i) \leq \frac{1}{4(\|Q\|+1)m}$ for $0 \leq i \leq d$. Choose $b_i \in \mathbb{R}^n$ such that $c_i = \rho(b_i)$, $\|a_0 - b_0\| \leq \epsilon$ and $\|a_i - b_i\| \leq \frac{1}{3(\|Q\|+1)m}, i = 0, 1, 2, \ldots, d$. Then $\|Qb_j - b_{j+1}\| \leq \|Qb_j - Qa_j\| + \|a_{j+1} - b_{j+1}\| \leq \frac{2}{3m}$, and hence $Qb_j = b_{j+1}$ for $j = 0, 1, 2, \ldots, d-1$. Since $\lim_{i \to \pm\infty} d_0(c_i, 0) = 0$ it follows from ii) and iv) of Lemma 3.18 that $b_0 \in \mathcal{U} + \left\langle \bigcup_{j \geq 0} Q^j(\mathbb{Z}^n) \right\rangle$ while $b_d \in \mathcal{S} + \left\langle \bigcup_{j \leq 0} Q^j(\mathbb{Z}^n) \right\rangle$. Since $b_d = Q^d b_0$ we conclude that

$$b_0 \in \left(\mathcal{U} + \left\langle \bigcup_{j \geq 0} Q^j(\mathbb{Z}^n) \right\rangle\right) \cap \left(\mathcal{S} + \left\langle \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n) \right\rangle\right).$$

Since

$$\begin{aligned} &\left(\mathcal{U} + \left\langle \bigcup_{j \geq k} Q^j(\mathbb{Z}^n) \right\rangle\right) \cap \left(\mathcal{S} + \left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right) \\ &\qquad \supseteq Q^k\left(\left(\mathcal{U} + \left\langle \bigcup_{j \geq 0} Q^j(\mathbb{Z}^n) \right\rangle\right) \cap \left(\mathcal{S} + \left\langle \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n) \right\rangle\right)\right), \end{aligned}$$

this proves the density of (3.15) in $\mathbb{R}^n$.

Conversely, assume that (3.15) is dense in $\mathbb{R}^n$ for each k. Fix $b \in S_Q$, and let $\epsilon > 0$ and $d \in \mathbb{N}$ be given. It follows from Lemma 3.21 that there is an element

$a \in S_Q$ such that $m^d a = b$. By definition of S_Q there are then $c_0 \in \mathbb{R}^n$ and $u_1, u_2, \dots, u_d \in \mathbb{Z}^n$ such that

$$\begin{aligned} a_0 &= \rho(c_0), \\ a_1 &= \rho(Qc_0 + Qu_1), \\ a_2 &= \rho\left(Q^2 c_0 + Q^2 u_1 + Qu_2\right), \\ &\ \ \vdots \\ a_d &= \rho\left(Q^d c_0 + Q^d u_1 + Q^{d-1} u_2 + \dots + Qu_d\right). \end{aligned}$$

Set $c = m^d c_0$ and note that $b_j = \rho(Q^j c), j = 0, 1, 2, \dots, d$. Thanks to the density of (3.15) for $k = d$, there is an element

$$\xi \in \left(\mathcal{U} + \left\langle \bigcup_{j \geq 1} Q^j(\mathbb{Z}^n) \right\rangle\right) \cap \left(\mathcal{S} + \left\langle \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n) \right\rangle\right)$$

such that $\|c - \xi\| \leq \epsilon$. There are an $M \in \mathbb{N}, M > d$, and elements

$$w_1, w_2, \dots, w_M, v_1, v_2, \dots, v_M \in \mathbb{Z}^n$$

and $u \in \mathcal{U}, s \in \mathcal{S}$ such that $\xi = u + Q^M w_1 + Q^{M-1} w_2 + \dots + Qw_M = s + Q^{-d} v_1 + Q^{-d-1} v_2 + \dots + Q^{-M} v_{M-d+1}$. Set

$$t_i = \begin{cases} w_i, & 1 \leq i \leq M, \\ 0, & M+1 \leq i \leq M+d, \\ -v_{i-M-d}, & M+d+1 \leq i \leq 2M+1. \end{cases}$$

Then

$$\xi' = \left(Q^{-M} u, (t_1, t_2, \dots, t_{2M+1}), Q^M s\right) \in H_M$$

and $d\left(b_j, \Lambda_M(\xi')_j\right) \leq \|Q\|^j \epsilon$ for all $j = 0, 1, 2, \dots, d$. Since the the homoclinic group is invariant under the shift this proves its density in S_Q. □

LEMMA 3.23. *Assume that σ_Q is expansive. i.e. that $\mathcal{N} = \{0\}$. It follows that the homoclinic group Δ_{σ_Q} is dense in S_Q.*

PROOF. Let $k \in \mathbb{N}$. Let $P_{\mathcal{U}} : \mathbb{R}^n \to \mathcal{U}$ and $P_{\mathcal{S}} : \mathbb{R}^n \to \mathcal{S}$ be the projections corresponding to the decomposition (3.11). Note that $H = P_{\mathcal{U}}\left(\left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right)$ is a Q^{-1}-invariant subgroup of $\mathcal{U}$ which spans $\mathcal{U}$ linearly. It follows that $\mathcal{U}/\overline{H}$ is a compact group on which Q^{-1} induces a surjective continuous group-endomorphism. Some power of this endomorphism is a strict contraction since some power of Q^{-1} is a strict contraction on $\mathcal{U}$ by Lemma 3.17. By compactness this implies that $\mathcal{U}/\overline{H} = \{0\}$, i.e. $P_{\mathcal{U}}\left(\left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right)$ is dense in $\mathcal{U}$. Similarly, we see that $P_{\mathcal{S}}\left(\left\langle \bigcup_{j \geq k} Q^j(\mathbb{Z}^n) \right\rangle\right)$ is dense in $\mathcal{S}$. Hence we conclude that

$$P_{\mathcal{U}}\left(\left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right) + P_{\mathcal{S}}\left(\left\langle \bigcup_{j \geq k} Q^j(\mathbb{Z}^n) \right\rangle\right)$$

is dense in $\mathbb{R}^n$. Since $x = P_{\mathcal{U}}(x) + P_{\mathcal{S}}(x)$ for all $x \in \mathbb{R}^n$ we have that

$$\begin{aligned} &\left(\mathcal{U} + \left\langle \bigcup_{j \geq k} Q^j(\mathbb{Z}^n) \right\rangle\right) \cap \left(\mathcal{S} + \left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right) \\ &\qquad = P_{\mathcal{U}}\left(\left\langle \bigcup_{j \leq -k} Q^j(\mathbb{Z}^n) \right\rangle\right) + P_{\mathcal{S}}\left(\left\langle \bigcup_{j \geq k} Q^j(\mathbb{Z}^n) \right\rangle\right), \end{aligned}$$

so we conclude that

$$\Big(\mathcal{U} + \Big\langle \bigcup_{j \geq k} Q^j\left(\mathbb{Z}^n\right) \Big\rangle\Big) \cap \Big(\mathcal{S} + \Big\langle \bigcup_{j \leq -k} Q^j\left(\mathbb{Z}^n\right) \Big\rangle\Big) \tag{3.16}$$

is dense in $\mathbb{R}^n$. This proves the density of Δ_{σ_Q} in S_Q by Lemma 3.22. □

REMARK 3.24. Another approach to Lemma 3.23 would be to show that σ_Q has completely positive entropy and then appeal to Theorem 4.2 of [**LS**]. □

By combining Theorem 3.8, Lemma 3.23 and some more or less wellknown results about crossed product C^*-algebras, which we relegate to Appendix B, we get a relatively detailed description of the homoclinic algebra $A_{\sigma_Q}\left(S_Q\right)$. To state the result, consider an abelian group H, a compact abelian group G and a homomorphism $p : H \to G$. We can then define an action $\tau \circ p$ of H on $C\left(G\right)$ such that

$$\left(\tau \circ p\right)_h (f)(x) = f\left(x - p(h)\right).$$

When p is injective we suppres it in the notation and write τ for this action. If, in addition, $H \simeq \mathbb{Z}^k$ and $G \simeq \mathbb{T}^n$ we will call the resulting crossed product C^*-algebra $C\left(\mathbb{T}^n\right) \rtimes_\tau H$ *a special non-commutative torus of rank* (n, k).

LEMMA 3.25. *Assume that $Q \in Gl_n(\mathbb{Q})$ is hyperbolic. Then the homoclinic algebra $A_{\sigma_Q}\left(S_Q\right)$ is simple and $*$-isomorphic to an inductive limit $\varinjlim_k \left(A_k, \varphi_k\right)$ where the φ_k's are unital $*$-homomorphisms and*

$$A_k \simeq C\left(\mathbb{T}^n\right) \rtimes_{\tau \circ p^k} H_k$$

for some finitely generated abelian group H_k and some homomorphism $p^k : H_k \to \mathbb{T}^n$. Furthermore, each A_k is stably isomorphic to a finite direct sum of copies of the same special non-commutative torus of rank (n_k, m_k), where $n_k + m_k \leq 2n$.

PROOF. Note that Δ_{σ_Q} is countable, e.g. by Lemma 3.19 and Lemma 3.20. The simplicty of $A_{\sigma_Q}\left(S_Q\right)$ follows from Corollary 3.9 and Lemma 3.23. The other statements follow from Lemma B.2 and Lemma B.6 in Appendix B, using the inverse limit decomposition of Lemma 3.21. The bound on the rank follows from Lemma B.6 and Lemma 3.20. □

REMARK 3.26. The simplicity of $A_{\sigma_Q}\left(S_Q\right)$ follows also from a combination of [**Br**] and [**PS**]. □

In the following we collect some information on the structure of the homoclinic algebra $A_{\sigma_Q}\left(S_Q\right)$ which follows by combining Lemma 3.25 with other mathematicians work on the structure of (simple) C^*-algebras.

PROPOSITION 3.27. *Assume that $Q \in Gl_n(\mathbb{Q})$ is hyperbolic.*

1) *$A_{\sigma_Q}\left(S_Q\right)$ is in the bootstrap category $\mathcal{N}$ of Rosenberg and Schochet. In particular, the UCT-theorem of [**RS**] holds for $A_{\sigma_Q}\left(S_Q\right)$ with respect to an arbitrary separable 'coefficient' C^*-algebra.*
2) *$A_{\sigma_Q}\left(S_Q\right)$ has a unique trace state.*
3) *$K_*\left(A_{\sigma_Q}\left(S_Q\right)\right)$ is torsionfree.*
4) *$A_{\sigma_Q}\left(S_Q\right)$ is approximately divisible (in the sense of [**BKR**]).*
5) *$A_{\sigma_Q}\left(S_Q\right)$ has stable rank 1.*
6) *$A_{\sigma_Q}\left(S_Q\right)$ has real rank 0.*

7) $K_0\left(A_{\sigma_Q}(S_Q)\right)_+ = \left\{x \in K_0\left(A_{\sigma_Q}(S_Q)\right) : \ \omega(x) > 0 \ \right\} \cup \{0\}$, *where* ω *is the trace state of* $A_{\sigma_Q}(S_Q)$.

Proof. 1) follows directly from Lemma 3.25 by using that $\mathcal{N}$ by definition contains all separable abelian C^*-algebras and is closed under the formation of crossed products by $\mathbb{Z}$, under (countable) inductive limits and and stable isomorphism.

2) follows from Corollary 3.9 and Lemma 3.23; the unique trace is the trace on the crossed product $C(S_Q) \rtimes_\tau \Delta_{\sigma_Q}$ coming from the Haar measure of S_Q.

3) follows from Lemma 3.25 and the wellknown fact that the K-theory of a non-commutative torus is torsion-free.

To prove 4) we proceed as follows: By Lemma 3.25 $A_{\sigma_Q}(S_Q)$ is $*$-isomorphic to the inductive limit of a sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

of unital C^*-algebras such that each φ_k is unital and such that $A_k \otimes \mathbb{K} \simeq B_k \otimes \mathbb{C}^{a_k} \otimes \mathbb{K}$, where $B_k \simeq C\left(\mathbb{T}^{b_k}\right) \rtimes_\tau \mathbb{Z}^{c_k}$ for some natural numbers a_k, b_k with $b_k + b_k \leq 2n$. We consider the following cases separately:

B_k is nonrational in the sense of [**BKR**] *for infinitely many k.* In this case we can assume that B_k is nonrational for all k. By standard C^*-algebra techniques, as in the proof of Theorem 3.16, we get then for each k a natural number l_k, a projection $p_k \in M_{l_k}(B_k \otimes \mathbb{C}^{a_k})$, a $*$-isomorphism $\iota_k : A_k \to p_k M_{l_k}(B_k \otimes \mathbb{C}^{a_k}) p_k$ and a unital $*$-homomorphism $\psi_k : p_k M_{l_k}(B_k \otimes \mathbb{C}^{a_k}) p_k \to p_{k+1} M_{l_{k+1}}(B_{k+1} \otimes \mathbb{C}^{a_{k+1}}) p_{k+1}$ such that the infinite diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "\iota_1"'] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\iota_2"'] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "\iota_3"'] & \cdots \\
p_1 M_{l_1}(B_1 \otimes \mathbb{C}^{a_1}) p_1 \arrow[r, "\psi_1"] & p_2 M_{l_2}(B_2 \otimes \mathbb{C}^{a_2}) p_2 \arrow[r, "\psi_1"] & p_3 M_{l_3}(B_3 \otimes \mathbb{C}^{a_3}) p_3 \arrow[r, "\psi_3"] & \cdots
\end{tikzcd} \tag{3.17}$$

commutes. Since approximate divisibility is preserved by taking tensor products we conclude from Theorem 1.5 and Corollary 2.9 of [**BKR**] that $p_k M_{l_k}(B_k \otimes \mathbb{C}^{a_k}) p_k$ is approximately divisible for all k. It follows that the inductive limit of the lower sequence in (3.17) is approximately divisible. But this inductive limit is $*$-isomorphic to $A_{\sigma_Q}(S_Q)$ by (3.17).

B_k is rational in the sense of [**BKR**] *except for finitely many k.* We may then assume that B_k is rational for all k. Note that $C\left(\mathbb{T}^{b_k}\right) \rtimes_\tau \mathbb{Z}^{c_k}$ is rational only when $c_k = 0$. Thus $B_k \simeq C\left(\mathbb{T}^{b_k}\right)$ for all k. We construct a diagram (3.17) as above with the only difference that now $B_k \simeq C\left(\mathbb{T}^{b_k}\right)$ for some $b_k \leq 2n$. Corollary 6.7 of [**G**] implies that the inductive limit of the lower sequence in (3.17) and hence also $A_{\sigma_B}(S_B)$ is an inductive limit of direct sums of circle algebras. It follows then from a result of Elliott, [**Ell1**], that $A_{\sigma_B}(S_B)$ is approximately divisible.

Having established 4), both 5) and 6) follow from Theorem 1.4 of [**BKR**]. 7) follows from Corollary 3.9 of [**BKR**]. □

By using recent results of H. Lin and N.C. Phillips we obtain the following.

Theorem 3.28. *Assume that $Q \in Gl_n(\mathbb{Q})$ is hyperbolic. Then $A_{\sigma_Q}(S_Q)$ is a simple unital AT-algebra of real rank zero with a unique trace state.*

PROOF. By combining Lemma 3.25 with Proposition B.7 from Appendix B we see that $A_{\sigma_Q}(S_Q)$ is locally AH. Hence, by Proposition 5.3 of [**Lin**], the trace state of $A_{\sigma_Q}(S_Q)$ is approximately AC in the sense of [**Lin**]. Therefore we can combine Proposition 3.27 with Theorem 4.15 of [**L**] to conclude that $A_{\sigma_Q}(S_Q)$ has tracial rank zero. It follows then from Proposition 3.7 of [**Ph2**] that $A_{\sigma_Q}(S_Q)$ is AT. $\square$

The previous analysis and its results can be slightly improved in the case where $Q \in M_n(\mathbb{Z})$. In this case Q induces a group endomorphism of the n-torus given by the tautologically looking formula

$$Q\rho(x) = \rho(Qx).$$

Define $\Lambda : \mathbb{R}^n \to S_Q$ such that $\Lambda(x)_i = \rho(Q^i x)$, $i \in \mathbb{Z}$, and set

$$W^u(0) = \Big\{x \in X : \lim_{n \to -\infty} d\left(\sigma_Q^n(x), 0\right) = 0\Big\}.$$

LEMMA 3.29. *Assume that $Q \in M_n(\mathbb{Z})$. Then $\Lambda(\mathcal{U}) = W^u(0)$ and Λ is injective on $\mathcal{U}$.*

PROOF. Clearly, $\Lambda(\mathcal{U}) \subseteq W^u(0)$. Let $z = (z_i)_{i\in\mathbb{Z}} \in W^u(0)$. It follows from Lemma 3.18 that there is an $x \in \mathcal{U}$ such that $\rho(Q^{-i}x) = z_{-N-i}$ for all $i \geq 0$. Then $Q^N x \in \mathcal{U}$ and $\Lambda\left(Q^N x\right) = z$. Hence $\Lambda(\mathcal{U}) = W^u(0)$. If $x \in \mathcal{U}$ and $\Lambda(x) = 0$, we see that $Q^i x \in \mathbb{Z}^n$ for all $i \in \mathbb{Z}$. Since some power of Q^{-1} is a contractive automorphism of the discrete group $\mathcal{U} \cap \bigcap_{k\in\mathbb{Z}} Q^k(\mathbb{Z}^n)$, this group must be $\{0\}$ and hence $x = 0$. $\square$

LEMMA 3.30. *Assume that $Q \in M_n(\mathbb{Z})$. The homoclinic group Δ_{σ_Q} is*

$$\Lambda\Big(\mathcal{U} \cap \Big(\mathcal{S} + \bigcup_{k\geq 0} Q^{-k}(\mathbb{Z}^n)\Big)\Big).$$

PROOF. Let $z = (z_i)_{i\in\mathbb{Z}} \in \Delta_{\sigma_Q}$. Since $\Delta_{\sigma_Q} \subseteq W^u(0)$ it follows from Lemma 3.29 that there is an $x \in \mathcal{U}$ such that $\rho(Q^i x) = z_i$ for all $i \in \mathbb{Z}$. On the other hand, it follows from Lemma 3.18 that there is an $N \in \mathbb{N}$ and a vector $y \in \mathcal{S}$ such that $\rho(Q^j y) = z_{N+j}$ for all $j \geq 0$. Since $\rho\left(Q^N x\right) = z_N = \rho(y)$, it follows that $Q^N x - y \in \mathbb{Z}^n$. Hence $x \in \mathcal{S} + Q^{-N}(\mathbb{Z}^n)$, proving that $\Delta_{\sigma_Q} \subseteq \Lambda\left(\mathcal{U} \cap \left(\mathcal{S} + \bigcup_{k\geq 0} A^{-k}(\mathbb{Z}^n)\right)\right)$. The reversed inclusion is trivial. $\square$

LEMMA 3.31. *The map*

$$\Lambda \circ P_{\mathcal{U}} : \bigcup_{j\geq 0} Q^{-j}(\mathbb{Z}^n) \cap (\mathcal{U} + \mathcal{S}) \to \Delta_{\sigma_A}$$

is an isomorphism.

PROOF. Surjectivity: Let $x \in \mathcal{U} \cap \left(\mathcal{S} + \bigcup_{k\geq 0} Q^{-k}(\mathbb{Z}^n)\right)$. There are then elements $s \in \mathcal{S}$ and $v \in \bigcup_{j\geq 0} Q^{-j}(\mathbb{Z}^n) \cap (\mathcal{U} + \mathcal{S})$ such that $x = s + v$. It follows that $P_{\mathcal{U}}(v) = x$ and hence that $\Lambda(x) = \Lambda \circ P_{\mathcal{U}}(v)$. By Lemma 3.30 this gives the surjectivity. Injectivity: The injectivity of Λ on $\mathcal{U}$ follows from Lemma 3.29. It suffices therefore to prove the injectivity of $P_{\mathcal{U}}$ on $Q^{-k}(\mathbb{Z}^n) \cap (\mathcal{U} + \mathcal{S})$ for any $k \geq 0$. To this end observe that some power of Q is an injective strict contraction on the discrete set $\mathcal{S} \cap \mathbb{Z}^n$ which implies that $\mathcal{S} \cap \mathbb{Z}^n = \{0\}$. Hence, if $x \in Q^{-k}(\mathbb{Z}^n) \cap (\mathcal{U} + \mathcal{S})$ and $P_{\mathcal{U}}(x) = 0$, we find that $Q^k(x) \in \mathcal{S} \cap \mathbb{Z}^n = \{0\}$. Hence $x = 0$. $\square$

THEOREM 3.32. *Assume that $Q \in M_n(\mathbb{Z}) \cap Gl(n,\mathbb{Q})$ is hyperbolic. The homoclinic algebra $A_{\sigma_Q}(S_Q)$ is simple and $*$-isomorphic to the crossed product $C(S_Q) \rtimes_\tau \bigcup_{k\geq 0} Q^{-k}(\mathbb{Z}^n)$.*

PROOF. Simplicity follows from Theorem 3.25. The crossed product description follows from Theorem 3.8 and Lemma 3.31. □

By specializing further we get also the following.

THEOREM 3.33. *Let $Q \in Gl_n(\mathbb{Z})$ be hyberbolic and let $\varphi : \mathbb{T}^n \to \mathbb{T}^n$ be the corresponding expansive automorphism. It follows that the homoclinic algebra $A_\varphi(\mathbb{T}^n)$ is a simple special non-commutative torus of the form $C(\mathbb{T}^n) \rtimes_\tau \mathbb{Z}^n$.*

REMARK 3.34. Let $Q \in M_n(\mathbb{Z}) \cap Gl(n,\mathbb{Q})$ be hyperbolic. It follows from Lemma 3.31 that Δ_{S_Q} is isomorphic to the inductive limit group of the sequence

$$\mathbb{Z}^n \xrightarrow{Q} \mathbb{Z}^n \xrightarrow{Q} \mathbb{Z}^n \xrightarrow{Q} \cdots \tag{3.18}$$

while $\widehat{S_Q}$ is isomorphic to the inductive limit group of the sequence

$$\mathbb{Z}^n \xrightarrow{Q^t} \mathbb{Z}^n \xrightarrow{Q^t} \mathbb{Z}^n \xrightarrow{Q^t} \cdots, \tag{3.19}$$

where Q^t is the transpose of Q. It can happen that these inductive limit groups are not isomorphic. This is for example the case when

$$Q = \begin{pmatrix} 65 & 7 \\ 24 & 67 \end{pmatrix},$$

cf. Example 3.6 of [**BJKR**]. In this case the homoclinic group of σ_Q is not isomorphic to $\widehat{S_Q}$ (as it is in the case where $Q \in Gl_n(\mathbb{Z})$). Since the argument in [**BJKR**] only shows that the two groups are not isomorphic as ordered groups (i.e. as dimension groups) the arguments for the stronger statement have been included in Appendix C. Presumably the phenomenon is not exceptional at all; the methods explored in Appendix C can be used to find other examples, albeit in a rather unsystematic way. It is, for example, easy to check by the same procedure that the matrix

$$\begin{pmatrix} 82 & 57 \\ 5 & 86 \end{pmatrix}$$

is another example of this sort.

A different example of an expansive automorphism of a compact connected group whose homoclinic group is not isomorphic to the dual of the group on which it acts has been exhibited before in [**CF**]. All these examples show that the statement (2) of Theorem 5.1 in [**KPS**] is incorrect.

In all the cases mentioned above, the homoclinic group is at least torsion free. Below, in Remark 6.9, we exhibit a hyperbolic matrix $Q \in M_2(\mathbb{Q})$ such that Δ_{σ_Q} has torsion. □

3.3.2. General expansive group automorphisms. We extend here Theorem 3.28 to a general expansive automorphism of a compact group. This is relatively straightforward thanks to the following result of Kitchens and Schmidt, cf. Theorem 6.7 of [**KS**].

THEOREM 3.35. *(Kitchens and Schmidt) Let $\psi : G \to G$ be an expansive automorphism of the compact group G. Then (G, ψ) is topologically conjugate to a product $(F \times \Sigma_m \times S_Q, \tau \times \sigma \times \sigma_Q)$, where τ is an automorphism of the finite group F, (Σ, σ) is the full m-shift for some $m \geq 1$ and σ_Q is the shift of the solenoid S_Q for some hyperbolic $Q \in Gl_n(\mathbb{Q})$.*

By using Proposition 1.23 it follows from Theorem 3.35 that

$$A_\psi(G) \simeq A_\tau(F) \otimes A_\sigma(\Sigma_m) \otimes A_{\sigma_Q}(S_Q).$$

Clearly $A_\tau(F) = \mathbb{C}^{\#F}$ and it follows from Theorem 3.3 that $A_\sigma(\Sigma_m)$ is the UHF-algebra of Glimm-type m^∞. Note that the finite group F must be trivial when ψ is mixing. The following theorem follows therefore from Theorem 3.28 and Proposition 3.27.

THEOREM 3.36. *Let $\psi : G \to G$ be an expansive automorphism of the compact group G. When ψ is mixing it follows that $A_\psi(G)$ is a simple AT-algebra of real rank zero with a unique trace state.*

In general, $A_\psi(G) \simeq \mathbb{C}^k \otimes Q$ for some $k \in \mathbb{N}$ where Q is a simple AT-algebra of real rank zero with a unique trace state.

CHAPTER 4

The heteroclinic algebra

In this chapter we introduce a canonical construction of a relatively expansive system for a class of invertible dynamical systems with periodic points. This allows us to define a C^*-algebra from these dynamical systems which we call the heteroclinic algebra. The construction generalizes the construction of Putnam, [**Pu1**], of what he calls the stable algebra.

4.1. Post-periodic points and the Wagoner topology

Let (X, d) be a metric space and $\varphi : X \to X$ a homeomorphism of X. For $x \in X$, set

$$W^u(x) = \left\{ y \in X : \lim_{k \to -\infty} d\left(\varphi^k(x), \varphi^k(y)\right) = 0 \right\}.$$

When $k \in \mathbb{Z}$ and $\epsilon > 0$, set

$$W^u(x, k, \epsilon) = \left\{ y \in W^u(x) : d\left(\varphi^i(x), \varphi^i(y)\right) \leq \epsilon, \ i \leq k \right\}.$$

To simplify notation, set $W^u(x, 0, \epsilon) = W^u(x, \epsilon)$.

In the following we let $\operatorname{Per} X$ denote the set of φ-periodic points and $|p|$ the minimal period of a φ-periodic point $p \in X$.

Definition 4.1. Let $p \in \operatorname{Per} X$. We say that φ is *locally expansive at* p when there is an $\epsilon_p > 0$ such that $W^u(p, \epsilon_p)$ is compact in X and

$$z, y \in W^u(p), \ d\left(\varphi^j(z), \varphi^j(y)\right) \leq \epsilon_p \ \forall j \in \mathbb{Z} \ \Rightarrow \ z = y. \tag{4.1}$$

Since $\varphi\left(W^u(p, \epsilon)\right) \cap \{x \in X : d(x, \varphi(p)) \leq \epsilon\} = W^u(\varphi(p), \epsilon)$ we see that φ is locally expansive at every element of the orbit of p when it is locally expansive at p. We say that φ is *locally expansive on post-periodic points* when it is locally expansive at every $p \in \operatorname{Per} X$.

Lemma 4.2. *Let K be a compact subset of X and $p \in \operatorname{Per} X$ such that*

- *$p \in K$,*
- *$\varphi^{-|p|}(K) \subseteq K$, and*
- *there is an $\epsilon > 0$ such that*

$$x \in K, \ d\left(\varphi^n(x), \varphi^n(p)\right) \leq \epsilon \ \forall n \in \mathbb{Z} \ \Rightarrow \ x = p.$$

It follows that there is a $\delta > 0$ such that

$$\lim_{n \to \infty} \sup_{x \in K^\delta} d\left(\varphi^{-n}(x), \varphi^{-n}(p)\right) = 0, \tag{4.2}$$

where

$$K^\delta = \left\{ x \in K : d\left(\varphi^j(x), \varphi^j(p)\right) \leq \delta, \ j \leq 0 \right\}.$$

Proof. Set $L = \bigcup_{j=0}^{|p|-1} \varphi^{-j}(K)$. Then L is compact, φ^{-1}-invariant and contains the orbit of p. Choose $\kappa > 0$ so small that

$$\kappa < \tfrac{1}{2} \min \left\{ \left| \varphi^i(p) - \varphi^j(p) \right| : \ i, j \in \{0, 1, 2, \dots, |p| - 1\}, \ i \neq j \right\} \tag{4.3}$$

and

$$\begin{aligned} &x, y \in L, \ d(x, y) \leq \kappa \ \Rightarrow \\ &\quad d\left(\varphi(x), \varphi(y)\right) \leq \tfrac{1}{2} \min \left\{ \left| \varphi^i(p) - \varphi^j(p) \right| : \ i, j \in \{0, 1, 2, \dots, |p| - 1\}, \ i \neq j \right\}. \end{aligned} \tag{4.4}$$

Let $\delta = \min\{\kappa, \epsilon\}$ and set

$$L' = \bigcup_{j=0}^{|p|-1} \varphi^{-j}\left(K^\delta\right).$$

Then L' is a compact φ^{-1}-invariant subset of L and contains the orbit of p. In particular,

$$\varphi^{-i-1}\left(L'\right) \subseteq \varphi^{-i}\left(L'\right) \tag{4.5}$$

for all $i \in \mathbb{N}$. We claim that

$$\bigcap_{i=0}^{\infty} \varphi^{-i}(L') = \left\{ p, \varphi(p), \varphi^2(p), \dots, \varphi^{|p|-1}(p) \right\}. \tag{4.6}$$

Let $z \in \bigcap_{i=0}^{\infty} \varphi^{-i}(L')$ and consider a $k \in \mathbb{Z}$. For each $m \in \mathbb{N}$ there is an element $x_m \in L'$ such that $\varphi^k(z) = \varphi^{k-m}(x_m)$. Note that for each m there is a $j \in \{0, 1, 2, \dots, |p| - 1\}$ such that $d\left(\varphi^n(x_m), \varphi^{n+j}(p)\right) \leq \delta$ for all $n \leq 0$. It follows that there is a sequence $m_1 < m_2 < m_3 < \cdots$ in $\mathbb{N}$ and a $j \in \{0, 1, 2, \dots, |p| - 1\}$ such that

$$d\left(\varphi^k(z), \varphi^{k-m_i}\left(\varphi^j(p)\right)\right) = d\left(\varphi^{k-m_i}\left(x_{m_i}\right), \varphi^{k-m_i}\left(\varphi^j(p)\right)\right) \leq \delta$$

for all i. Thus, for some $j' \in \{0, 1, 2, \dots, |p| - 1\}$ we have $d\left(\varphi^k(z), \varphi^k\left(\varphi^{j'}(p)\right)\right) \leq \delta$. Note that $\varphi^k(z) \in L' \subseteq L$. Since also $\varphi^{k+j'}(p)$ is in L it follows from (4.4) and (4.3) that the same j' works for all $k \in \mathbb{Z}$. That is,

$$d\left(\varphi^k(z), \varphi^{k+j'}(p)\right) \leq \delta \tag{4.7}$$

for all $k \in \mathbb{Z}$. Since $z \in L'$ there is a $z' \in K^\delta$ and a $j'' \in \mathbb{Z}$ such that $z = \varphi^{j''}(z')$ and $d\left(\varphi^i(z'), \varphi^i(p)\right) \leq \delta$ for all $i \leq 0$. It follows from (4.7) that

$$d\left(\varphi^k(z'), \varphi^k\left(\varphi^{j'-j''}(p)\right)\right) \leq \delta$$

for all $k \in \mathbb{Z}$. Since $\delta \leq \kappa$ it follows first that $j' - j'' = 0$ modulo $|p|$ and then, since $\delta \leq \epsilon$, that $z' = p$. Hence $z \in \left\{p, \varphi(p), \dots, \varphi^{|p|-1}(p)\right\}$, proving (4.6).

Let $\mu \in]0, \delta[$ be given. It follows from (4.6) and (4.5) that there is an N so big that

$$\varphi^{-n}\left(K^\delta\right) \subseteq \left\{ z \in X : \ \operatorname{dist}\left(z, \left\{p, \varphi(p), \varphi^2(p), \dots, \varphi^{|p|-1}(p)\right\}\right) \leq \mu \right\}$$

for all $n \geq N$. Since $2\delta \leq \min\left\{|\varphi^i(p) - \varphi^j(p)| : \ i, j \in \{0, 1, 2, \dots, |p| - 1\}, \ i \neq j\right\}$ we see that $d\left(\varphi^{-n}(u), \varphi^{-n}(p)\right) \leq \mu$ for all $u \in K^\delta$ and all $n \geq N$. This proves (4.2). $\square$

LEMMA 4.3. *Let (X,d) be a compact metric space and $\varphi : X \to X$ a homeomorphism of X. Let p be a φ-periodic point, and assume that there is an $\epsilon > 0$ such that*

$$x \in X,\ d\left(\varphi^j(x), \varphi^j(p)\right) \leq \epsilon \ \forall\ j \in \mathbb{Z} \ \Rightarrow\ x = p. \tag{4.8}$$

It follows that there is a $\delta > 0$ such that

$$W^u(p,\delta) = \left\{x \in X :\ d\left(\varphi^i(x), \varphi^i(p)\right) \leq \delta,\ \forall i \leq 0\right\}.$$

In particular, $W^u(p,\delta)$ is compact.

PROOF. Apply Lemma 4.2 with $K = \{x \in X :\ d\left(\varphi^j(x), \varphi^j(p)\right) \leq \epsilon,\ j \leq 0\}$. $\square$

LEMMA 4.4. *Let $p \in \operatorname{Per} X$ and assume φ is locally expansive at p. It follows that there is an $\epsilon'_p > 0$ such that*

$$\lim_{n\to\infty} \sup_{x \in W^u\left(p,\epsilon'_p\right)} d\left(\varphi^{-n}(x), \varphi^{-n}(p)\right) = 0.$$

PROOF. This follows by applying Lemma 4.2 to $K = W^u(p,\epsilon_p)$, where $\epsilon_p > 0$ is as in Definition 4.1. $\square$

LEMMA 4.5. *Let $p \in \operatorname{Per} X$ and assume φ is locally expansive at p. There is an $\eta'_p > 0$ such that*

$$\left\{x \in W^u(p) :\ d\left(\varphi^j(x), \varphi^j(p)\right) \leq \eta,\ j \leq k\right\}$$

is compact and

$$\left\{x \in W^u(p) :\ d\left(\varphi^j(x), \varphi^j(p)\right) < \eta,\ j \leq k\right\}$$

is locally compact in the topology inherited from (X,d) for all $k \in \mathbb{Z}$ and all $\eta \leq \eta'_p$.

PROOF. Set $\eta'_p = \min\{\epsilon'_{\varphi^n(p)} :\ n \in \mathbb{Z}\}$ and let $\eta \leq \eta'_p$. Let $k \in \mathbb{Z}$. It follows from Lemma 4.4 that there is an $N \in \mathbb{N}$ such that $d\left(\varphi^j(y), \varphi^j\left(\varphi^l(p)\right)\right) < \eta$ for all $j \leq -N$, all $y \in \left\{x \in W^u\left(\varphi^l(p)\right) :\ d\left(\varphi^j(x), \varphi^j\left(\varphi^l(p)\right)\right) \leq \eta,\ j \leq 0\right\}$ and all $l \in \mathbb{Z}$. Note that

$$\begin{aligned}
&\left\{x \in W^u(p) :\ d\left(\varphi^j(x), \varphi^j(p)\right) < \eta,\ j \leq k\right\} \\
&\quad = \varphi^{-k}\left(\left\{x \in W^u\left(\varphi^k(p)\right) :\ d\left(\varphi^j(x), \varphi^j\left(\varphi^k(p)\right)\right) < \eta,\ j \leq 0\right\}\right) \\
&\quad = \varphi^{-k}\left(E \cap F\right),
\end{aligned}$$

where

$$E = \left\{x \in W^u\left(\varphi^k(p)\right) :\ d\left(\varphi^j(x), \varphi^j\left(\varphi^k(p)\right)\right) \leq \eta,\ j \leq 0\right\}$$

and

$$F = \left\{x \in X :\ d\left(\varphi^j(x), \varphi^j\left(\varphi^k(p)\right)\right) < \eta,\ -N \leq j \leq 0\right\}.$$

Note that E is compact in X by the first condition of Definition 4.1. Hence $\varphi^{-k}(E) = \{x \in W^u(p) :\ d\left(\varphi^j(x), \varphi^j(p)\right) \leq \eta,\ j \leq k\}$ is also compact. Since F is open in X we conclude that $E \cap F$, and hence also $\varphi^{-k}(E \cap F)$, is locally compact in the relative topology. $\square$

To simplify the notation we let $W^u_<(y,k,\eta)$ denote the set

$$W^u_<(y,k,\eta) = \left\{x \in W^u(y) :\ d\left(\varphi^j(x), \varphi^j(p)\right) < \eta,\ j \leq k\right\}.$$

Assume now that φ is locally expansive on post-periodic points, and let $p \in \operatorname{Per} X$. It follows from Lemma 4.4 and Lemma 4.5 that there is an $\eta_p > 0$ such that for every $k \in \mathbb{Z}$ the set $W^u_<(p,k,\eta)$ is locally compact in the topology inherited from (X,d), and

$$\lim_{n\to-\infty} \sup_{x\in W^u(p,k,\eta)} d\left(\varphi^n(x),\varphi^n(p)\right) = 0. \tag{4.9}$$

when $\eta \leq \eta_p$. Since $W^u_<(p,k,\eta_p)$ is an open subset of $W^u_<(p,k-1,\eta_p)$, it follows from Lemma 4.5 that $W^u(p)$ is locally compact in the inductive limit topology corresponding to the union

$$W^u(p) = \bigcup_{k\in\mathbb{Z}} W^u_<(p,k,\eta_p)\,.$$

In this topology a subset $U \subseteq W^u(p)$ is open if and only if there are open sets W_k in X such that $U\cap W^u_<(p,k,\eta_p) = W_k \cap W^u_<(p,k,\eta_p)$ for all $k \in \mathbb{Z}$. It is easy to see, by use of (4.9), that this topology of $W^u(p)$ is independent of η_p in the sense that the inductive limit topology of $W^u(p)$ arising from a union $W^u(p) = \bigcup_{k\in\mathbb{Z}} W^u_<(p,k,\eta)$, where $0 < \eta \leq \eta_p$, will give the same topology.

By Definition 4.1 we can assume that $W^u(p,k,\eta)$ is compact for all $k \in \mathbb{Z}$ and all $0 < \eta \leq \eta_p$. Furthermore, by Lemma 4.3 we can also arrange, by taking a smaller η_p if necessary, that

$$W^u(p,k,\eta) = \left\{x \in X :\ d\left(\varphi^j(x),\varphi^j(p)\right) \leq \eta,\ j \leq k\right\} \tag{4.10}$$

for all $0 < \eta \leq \eta_p$ and all $k \in \mathbb{Z}$. We shall often tacitly assume that this holds.

The union

$$W_{X,\varphi} = \bigcup_{p\in \operatorname{Per} X} W^u(p),$$

or just W consists of the *post-periodic points*, and we equip $W = W_{X,\varphi}$ with a topology by declaring a subset $U \subseteq W$ to be open when $U \cap W^u(p)$ is open in $W^u(p)$ for all $p \in \operatorname{Per} X$. Thus W is the disjoint union of the $W^u(p)$'s, as a set as well as a topological space. It is a locally compact Hausdorff space which was first introduced by Wagoner in the setting of countable state Markov shifts, [**Wa**], and we will refer to its topology as the *Wagoner topology*. Note that the Wagoner topology is second countable if (and only if) there are only countably many periodic points in X. Note also that the Wagoner topology is finer than the topology inherited from X.

The following lemma describes a natural base for the Wagoner topology.

LEMMA 4.6. *Assume that (X,d,φ) is locally expansive on post-periodic points. Let $x \in W_{X,\varphi}$. There is then an ϵ_0 such that the sets*

$$\left\{y \in X :\ d\left(\varphi^j(y),\varphi^j(x)\right) < \epsilon,\ j \leq k\right\}, \quad k \in \mathbb{Z},\ \epsilon \in \left]0,\epsilon_0\right], \tag{4.11}$$

are all contained in $W^u(x)$ and form an open neighborhood base at x in the Wagoner topology.

PROOF. There is a periodic point p such that $x \in W^u(p)$. Since $W^u(p) = \bigcup_{n\in\mathbb{Z}} W^u_<\left(p,n,\frac{\eta_p}{2}\right)$ there is an $m \in \mathbb{Z}$ such that $x \in W^u_<\left(p,m,\frac{\eta_p}{2}\right)$. Set $\epsilon_0 = \frac{\eta_p}{2}$ and let $\epsilon \in]0,\epsilon_0[$. Then

$$\begin{aligned}&\left\{y \in X :\ d\left(\varphi^j(y),\varphi^j(x)\right) < \epsilon,\ j \leq k\right\}\\ &\qquad \subseteq \left\{y \in X :\ d\left(\varphi^j(y),\varphi^j(p)\right) < \eta_p,\ j \leq \min\{k,m\}\right\}\end{aligned}$$

and hence

$$\left\{y \in X : \ d\left(\varphi^j(y), \varphi^j(x)\right) < \epsilon, \ j \leq k\right\}$$
$$\subseteq W^u_<(p, \min\{k, m\}, \eta_p) \subseteq W^u(p) = W^u(x)$$

for all $k \in \mathbb{Z}$ by (4.10). It follows from (4.9) that there is an $N \in \mathbb{N}$ such that $d\left(\varphi^j(z), \varphi^j(p)\right) \leq \frac{\epsilon}{3}$ for all $z \in W^u_<(p, \min\{k, m\}, \eta_p)$ and all $j \leq -N$. Then

$$\left\{y \in X : \ d\left(\varphi^j(y), \varphi^j(x)\right) < \epsilon, \ j \leq k\right\}$$
$$= \left\{y \in X : \ d\left(\varphi^j(y), \varphi^j(x)\right) < \epsilon, \ -N \leq j \leq k\right\} \cap W^u_<(p, \min\{k, m\}, \eta_p),$$

showing that $\{y \in X : \ d\left(\varphi^j(y), \varphi^j(x)\right) < \epsilon, \ j \leq k\}$ is open in $W_{X,\varphi}$.

To show that the sets (4.11) form a local basis at x we consider an open neighborhood U of x in $W_{X,\varphi}$. Since $W^u(p) = \bigcup_{n \in \mathbb{Z}} W^u_<(p, n, \eta_p)$, there is a $k \in \mathbb{Z}$ such that $x \in U \cap W^u_<(p, k, \eta_p)$. Since $U \cap W^u_<(p, k, \eta_p)$ is open in the topology which $W^u_<(p, k, \eta_p)$ inherits from X, there is an $\epsilon \in]0, \epsilon_0]$

$$\left\{z \in X : \ d\left(\varphi^k(z), \varphi^k(x)\right) < \epsilon\right\} \cap W^u_<(p, k, \eta_p) \subseteq U \cap W^u_<(p, k, \eta_p).$$

Since $\epsilon \leq \epsilon_0 = \frac{\eta_p}{2}$ it follows from (4.10) that

$$\left\{y \in X : \ d\left(\varphi^j(y), \varphi^j(x)\right) < \epsilon, \ j \leq k\right\}$$
$$\subseteq \left\{z \in X : \ d\left(\varphi^k(z), \varphi^k(y)\right) < \epsilon\right\} \cap W^u_<(p, k, \eta_p) \ \subseteq U.$$

□

LEMMA 4.7. *Assume that (X, d, φ) is locally expansive on post-periodic points. Let $x \in W$. There is then an open neighborhood V_x of x in W and a $\delta_x > 0$ such that*

$$z, y \in V_x, \ d\left(\varphi^n(z), \varphi^n(y)\right) \leq \delta_x \ \forall n \in \mathbb{N} \ \Rightarrow \ z = y.$$

PROOF. There is a $p \in \operatorname{Per} X$ and a $k \in \mathbb{Z}$ such that $x \in W^u_<(p, k, \eta_p)$. Let $\epsilon_p > 0$ be as in Definition 4.1. It follows from (4.9) that there is an $N \in \mathbb{N}$ such that $-N \leq k$ and $d\left(\varphi^l(z), \varphi^l(p)\right) \leq \frac{\epsilon_p}{2}$ for all $z \in W^u(p, k, \eta_p)$ and all $l \leq -N$. Set

$$V_x = \left\{z \in X : \ d\left(\varphi^l(z), \varphi^l(x)\right) < \frac{\epsilon_p}{2}, \ -N \leq l \leq 0\right\} \cap W^u_<(p, k, \eta_p)$$

and $\delta_x = \frac{\epsilon_p}{2}$. □

It follows that $(X, d, \mathbb{N}, \varphi, W, W)$ is a relatively expansive system and we can define the local conjugacy relation $R_\varphi(X, W)$ and the corresponding Ruelle algebra $A_\varphi(X, W)$. Specifically, two elements $x, y \in W$ are locally conjugate if and only if there are open neighborhoods U and V of x and y in W and a homeomorphism $\chi : U \to V$ such that $\lim_{n \to \infty} \sup_{z \in U} d\left(\varphi^i(z), \varphi^i(\chi(z))\right) = 0$. In the following we will refer to $A_\varphi(X, W)$ as *the heteroclinic algebra* of (X, φ), and denote it by

$$B_\varphi(X).$$

4.2. Functoriality of the post-periodic points and the Wagoner topology

It is clear that the functorial properties of the heteroclinic algebra depends on the functoriality of the construction of W. We pause therefore to collect the facts on this issue which we are going to need later on.

LEMMA 4.8. *Let (X, d, φ) and (X', d', φ') be locally expansive on the post-periodic points. Let $f : X \to X'$ be a continuous map such that $\varphi \circ f = f \circ \varphi'$. It follows that $f(W_{X,\varphi}) \subseteq W_{X',\varphi'}$ and that $f : W_{X,\varphi} \to W_{X',\varphi'}$ is continuous for the Wagoner topology.*

Proof. It is obvious that $f(W_{X,\varphi}) \subseteq W_{X',\varphi'}$. To prove the continuity of $f : W_{X,\varphi} \to W_{X',\varphi'}$ with respect to the Wagoner topology, let p be a periodic point in X, of period $|p|$, and let $k \in \mathbb{Z}$ be given. Let $\eta > 0$ be sufficiently small. Since f is continuous and equivariant there is an $\epsilon > 0$ such that $d(y, \varphi^j(p)) < \epsilon \Rightarrow d'\left(f(y), \varphi'^j(f(p))\right) < \eta$ for all $j = 0, 1, 2, \dots, |p| - 1$. It follows from (4.9) that there is an $N \in \mathbb{N}$ such that $d\left(\varphi^{n|p|+j}(y), \varphi^j(p)\right) < \epsilon$ for all $y \in W^u(p, k, \eta_p)$, all $j \in \{0, 1, \dots, |p| - 1\}$ and all $n \leq -N$. Since $\varphi'^{|p|}(f(p)) = f(p)$, it follows that

$$d'\left(\varphi'^i(f(y)), \varphi'^i(f(p))\right) < \eta$$

for all $y \in W^u(p, k, \eta_p)$ when $i \leq -N|p|$. Hence

$$f\left(W^u_<(p, k, \eta_p)\right) \subseteq W^u_<\left(f(p), -N|p|, \eta_{f(p)}\right),$$

provided only that $\eta > 0$ is small enough. Now the continuity of f with respect to the Wagoner topology follows from the continuity of f with respect to the given topologies. $\square$

Lemma 4.9. *Let (X, d, φ) and (X', d', φ') be locally expansive on the post-periodic points. Let $f : X \to X'$ be continuous and proper map such that $\varphi \circ f = f \circ \varphi'$. Assume that $f^{-1}(p)$ is a finite set for every periodic point $p \in X'$. It follows that $f^{-1}(W_{X',\varphi'}) \subseteq W_{X,\varphi}$ and that $f|_{W_{X,\varphi}} : W_{X,\varphi} \to W_{X',\varphi'}$ is proper.*

Proof. It follows from the definition that a subset $K \subseteq W_{X',\varphi'}$ is compact in the Wagoner topology if and only it is compact in the original topology of X' and there are finite collections $p_1, p_2, \dots, p_N$ in $\operatorname{Per} X'$ and $n_1, n_2, \dots, n_N$ in $\mathbb{Z}$ such that $K \subseteq \bigcup_{i=1}^N W^u_<(p_i, n_i, \eta_{p_i})$. It suffices therefore to consider a compact subset K of X', contained in $W^u_<(p, n, \eta_p)$ for some $p \in \operatorname{Per} X'$ and some $n \in \mathbb{Z}$, and show that $f^{-1}(K)$ is compact in $W_{X,\varphi}$. It follows from (4.9) that there is an $m \leq 0$ such that

$$\varphi'^i\left(W^u(p, n, \eta_p)\right) \subseteq \bigcup_{j=0}^{|p|-1} W^u\left(\varphi'^j(p), \eta_{\varphi'^j(p)}\right)$$

for all $i \leq m$. Note that the set on the right-hand side is compact in X'. Then

$$\bigcup_{i \leq 0} \varphi'^i(K) \subseteq \varphi'^{-m}\left(\bigcup_{j=0}^{|p|-1} W^u\left(\varphi'^j(p), \eta_{\varphi'^j(p)}\right)\right) \tag{4.12}$$

and we conclude that the closure, L, of $\bigcup_{i \leq 0} \varphi'^i(K)$ is compact in X'. Note that L is φ'^{-1}-invariant and that

$$\bigcap_{i \leq 0} \varphi'^i(L) = \left\{p, \varphi'(p), \varphi'^2(p), \dots, \varphi'^{|p|-1}(p)\right\} \tag{4.13}$$

because (4.12) and (4.9) imply that

$$\lim_{k \to -\infty} \sup_{z \in L} \operatorname{dist}\left(\varphi'^k(z), \left\{p, \varphi'(p), \dots, \varphi'^{|p|-1}(p)\right\}\right) = 0$$

for all $z \in L$. Since f is proper and equivariant, $f^{-1}(L)$ is a compact φ^{-1}-invariant subset of X, and hence

$$f^{-1}(L) \supseteq \varphi^{-1}\left(f^{-1}(L)\right) \supseteq \varphi^{-2}\left(f^{-1}(L)\right) \supseteq \varphi^{-3}\left(f^{-1}(L)\right) \supseteq \dots \tag{4.14}$$

Since $\bigcap_{i\leq 0} \varphi^i\left(f^{-1}(L)\right) \subseteq f^{-1}\left(\bigcap_{i\leq 0} \varphi'^i(L)\right)$, it follows from (4.13) that

$$\bigcap_{i\leq 0} \varphi^i\left(f^{-1}(L)\right) \subseteq \bigcup_{j=0}^{|p|-1} f^{-1}\left(\varphi'^j(p)\right). \tag{4.15}$$

By assumption the right-hand side of (4.15) is a finite φ-invariant set, $q_1, q_2, \dots, q_M$, of periodic points. Let $\eta > 0$. Combining (4.14) and (4.15) we see that there in an $m \leq 0$ such that $\varphi^j\left(f^{-1}(L)\right) \subseteq \bigcup_{k=1}^{M} \{y \in X : \ d(y, q_k) \leq \eta\}$ for all $j \leq m$. It follows from this and Lemma 4.3 that if just η is small enough we have that

$$f^{-1}(L) \subseteq \bigcup_{k=1}^{M} W^u_<\left(q_k, m, \eta_{q_k}\right).$$

Hence $f^{-1}(K) \subseteq f^{-1}(L)$ is compact in $W_{X,\varphi}$. □

PROPOSITION 4.10. *Let (X, d, φ) and (X', d', φ') be locally expansive on post-periodic points. Let $\pi : X \to X'$ be uniformly continuous homeomorphism with a uniformly continuous inverse such that $\varphi' \circ \pi = \pi \circ \varphi$. There is then a $*$-isomorphism $f^\bullet : B_{\varphi'}(X') \to B_\varphi(X)$ such that*

$$\pi^\bullet(f)(x,y) = f(\pi(x), \pi(y))$$

when $f \in C_c\left(R_{\varphi'}\left(X', W_{X',\varphi'}\right)\right)$.

PROOF. Combine Lemma 4.8, Lemma 4.9 and Corollary 2.5. □

4.3. The heteroclinic algebra of a countable state Markov shift

Let $\mathbb{G}$ be a countable oriented graph with edge-set $\mathbb{E}$ and vertex set $\mathbb{V}$. We will assume that $\mathbb{G}$ has *finite out-degree* in the sense that there are only finitely many edges leaving each vertex. The terminal vertex of an edge $e \in \mathbb{E}$ will be denoted by $t(e)$, and the initial vertex of e by $i(e)$. The space

$$X_{\mathbb{G}} = \left\{(e_i)_{i\in\mathbb{Z}} \in \mathbb{E}^{\mathbb{Z}} : \ i(e_{i+1}) = t(e_i) \ \ \forall i \in \mathbb{Z}\right\}$$

consists of the bi-infinite paths in $\mathbb{G}$. $X_{\mathbb{G}}$ is a complete metric space with the metric

$$d\left((e_i)_{i\in\mathbb{Z}}, (f_i)_{i\in\mathbb{Z}}\right) = 2\sum_{i\in\mathbb{Z}} 8^{-|i|}\delta_{e_i,f_i}, \tag{4.16}$$

where

$$\delta_{e_i,f_i} = \begin{cases} 0, & \text{when } e_i = f_i \\ 1, & \text{when } e_i \neq f_i. \end{cases}$$

The shift σ acts as a uniformly continuous homeomorphism of $X_{\mathbb{G}}$ in the usual way: $\sigma\left((e_i)_{i\in\mathbb{Z}}\right)_j = e_{j+1}$, and we will refer to $X_{\mathbb{G}}$ as *a countable state Markov shift.*

The exact choice of metric d is not signifigant for the following constructions. In fact, it follows from Proposition 4.10 that only its equivalence class matters. But with the choice (4.16) we have that $\delta = 1$ is an expansive constant for σ, and that

$$d(x,y) \leq 1 \ \Leftrightarrow \ d(x,y) < 1 \ \Leftrightarrow \ x_0 = y_0.$$

It is then straightforward to see that σ is locally expansive on the post-periodic points. In particular, $W^u(p,1)$ is compact in $X_{\mathbb{G}}$ because we assume that $\mathbb{G}$ has finite out-degree.

LEMMA 4.11. *The sets $W^u(y,k,1), k \in \mathbb{Z},\ y \in W$, form a base for the topology of $W = W_{X_{\mathbb{G}},\sigma}$ consisting of open and compact sets.*

PROOF. Let $p \in \operatorname{Per} X_{\mathbb{G}}$ be a periodic point such that $y \in W^u(p,m,1)$ for some $m \in \mathbb{Z}$. Then

$$W^u(y,k,1) = W^u(p,m,1) \cap \{z \in X:\ z_i = y_i,\ m \le i \le k\}$$

if $m \le k$ and $W^u(y,k,1) = W^u(p,k,1)$ otherwise. In both cases we see that $W^u(y,k,1)$ is open and compact in W by definition of the Wagoner topology. That the collection $W^u(y,k,1),\ k \in \mathbb{Z}, y \in W$, is a base for the topology follows from Lemma 4.6. □

LEMMA 4.12. *Let $x, y \in W_{X_{\mathbb{G}},\sigma}$. Then the following are equivalent*

a) *x and y are conjugate.*
b) *There is an $N \in \mathbb{Z}$ such that $x_i = y_i, i \ge N$.*
c) *There is an $N \in \mathbb{Z}$ and a homeomorphism $\chi : W^u(x,N,1) \to W^u(y,N,1)$ such that $\chi(z)_i = z_i, i \ge N$.*

PROOF. Straightforward. □

Let $x \in X_{\mathbb{G}}$. We say that $x_{]-\infty,0]}$ is a *post-periodic past* when $x \in W^u(p)$ for some $p \in \operatorname{Per} X_{\mathbb{G}}$. Let $v \in \mathbb{V}$ be a vertex in $\mathbb{G}$. We denote by T the set of pairs (γ,μ) where γ and μ are post-periodic pasts terminating at the same vertex, and by T_v the set of pairs (γ,μ), where γ and μ are post-periodic pasts which terminate at v. Note that T is a countable set. The free complex vector space $\mathcal{M}_0$ with basis T is a $*$-algebra with involution $*$ and product defined such that

$$(\gamma,\mu)^* = (\mu,\gamma)$$

and

$$(\gamma,\mu)(\alpha,\beta) = \begin{cases} (\gamma,\beta), & \text{when } \alpha = \mu \\ 0, & \text{when } \alpha \ne \mu. \end{cases} \tag{4.17}$$

There is a unique C^*-norm on $\mathcal{M}_0$ and we denote by $\mathcal{M}$ the C^*-algebra completion of $\mathcal{M}_0$. For each $v \in \mathbb{V}$, let M_v denote the C^*-subalgebra of $\mathcal{M}_0$ generated by T_v. Then

$$M_v = \begin{cases} M_{n_v}(\mathbb{C}), & \text{when } n_v < \infty \\ \mathbb{K}, & \text{when } n_v = \infty. \end{cases}$$

where n_v is the number of post-periodic pasts terminating at v and $\mathbb{K}$ denotes the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. Furthermore,

$$\mathcal{M} \simeq \oplus_{v\in\mathbb{V}} M_v. \tag{4.18}$$

We define a $*$-homomorphism $\Phi_{v,w} : M_v \to M_w$ such that

$$\Phi_{v,w}(\gamma,\mu) = \sum_{\{e\in\mathbb{E}:\ s(e)=v,t(e)=w\}} (\gamma e, \mu e),$$

and then, using the identification (4.18), a $*$-homomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ such that

$$\Phi\left((x_v)_{v\in\mathbb{V}}\right) = \Big(\sum_{v\in\mathbb{V}} \Phi_{v,w}(x_v)\Big)_{w\in\mathbb{V}}.$$

In this way we get a stationary sequence of C^*-algebras,

$$\mathcal{M} \xrightarrow{\Phi} \mathcal{M} \xrightarrow{\Phi} \mathcal{M} \xrightarrow{\Phi} \cdots\cdots \tag{4.19}$$

We intend to prove that the inductive limit C^*-algebra of this sequence, (4.19), is $*$-isomorphic to the homoclinic algebra $B_\sigma(X_{\mathbb{G}})$.

Let $i \in \mathbb{N}$ and consider an element $(\gamma, \mu) \in T$. We let $1^i_{\gamma,\mu} \in C_c(R_\sigma(X_{\mathbb{G}}, W))$ be the characteristic function of the set

$$\left\{(x,y) \in W^2 : \ x_{]-\infty,i-1]} = \gamma, \ y_{]-\infty,i-1]} = \mu, \ x_j = y_j, \ j \geq i\right\}.$$

We can then define a $*$-homomorphism $\Psi_i : \mathcal{M} \to B_\sigma(X_{\mathbb{G}})$ such that

$$\Psi_i((\gamma,\mu)) = 1^i_{\gamma,\mu}.$$

It is easily seen that Ψ_i is an injective $*$-homomorphism and that $\Psi_{i+1} \circ \Phi = \Psi_i$. It follows that the Ψ_i's induce an injective $*$-homomorphism

$$\Psi : \ \varinjlim(\mathcal{M}, \Phi) \to B_\sigma(X_{\mathbb{G}}). \tag{4.20}$$

Proposition 4.13. *The $*$-homomorphism (4.20) is a $*$-isomorphism.*

Proof. The proof is essentially the same as the proof of Theorem 3.3. It remains only to establish the surjectivity of Ψ. By Lemma 1.10 it suffices to show that every localized element $g \in C_c(R_\sigma(X_{\mathbb{G}}, W))$ is in the range of Ψ. Assume therefore that g is supported in $\{(z,\chi(z)) : \ z \in U\}$ for some conjugacy (U, V, χ). It follows from Lemma 4.12 that we can asume, after an obvious partition of unity argument, that there are elements $x_0, y_0 \in W$ and an $N \in \mathbb{Z}$ such that $U = W^u(x_0, N, 1)$, $V = W^u(y_0, N, 1)$, $\chi(x_0) = y_0$, and $\chi(z)_i = z_i, i \geq N, z \in W^u(x_0, N, 1)$. Let $\epsilon > 0$. The map $W^u(x_0, N, 1) \ni z \mapsto g(z, \chi(z))$ is continuous. We can therefore find a finite open cover $V_j, j \in J$, of $r(\operatorname{supp} g)$ inside $W^u(x_0, N, 1)$ such that $|g(z,\chi(z)) - g(z',\chi(z'))| < \epsilon$ when $z, z' \in V_j, j \in J$. Since W is totally disconnected by Lemma 4.11 we can also arrange that the V_j's are compact and open, and mutually disjoint, and in fact that there is an $M \in \mathbb{N}$, $M \geq N$ and elements $x_j \in W^u(x_0, N, 1)$ such that

$$V_j = \left\{z \in W : \ z_{]-\infty,M]} = z_j|_{]-\infty,M]}\right\}$$

for all $j \in J$. Set $\lambda_j = g(x_j, \chi(x_j))$ for some choice of elements $x_j \in V_j$. Set

$$\gamma_j = x_j|_{]-\infty,M-1]}, \ \mu_j = \chi(x_j)|_{]-\infty,M-1]}$$

and note that $(\gamma_j, \mu_j) \in T$. It follows as in the proof of Theorem 3.3 that

$$\left\| g - \sum_{j \in J} \lambda_j 1^M_{\gamma_j,\mu_j} \right\| \leq \epsilon$$

in $B_\sigma(X_{\mathbb{G}})$. This completes the proof because $\sum_{j\in J} \lambda_j 1^M_{\gamma_j,\mu_j}$ is in the range of Ψ. $\square$

Note that $K_0(\mathcal{M}) = \oplus_{v\in\mathbb{V}}\mathbb{Z}$. This is an identification of partial ordered groups when $\oplus_{v\in\mathbb{V}}\mathbb{Z}$ has the natural ordering where an element $(x_v)_{v\in\mathbb{V}}$ is positive if and only if $x_v \geq 0$ for all v. It follows then from Proposition 4.13 that

$$K_0(B_\sigma(X_{\mathbb{G}})) \simeq \varinjlim(\oplus_{v\in\mathbb{V}}\mathbb{Z}, \Phi_*). \tag{4.21}$$

This is an isomorphism of partially ordered abelian groups when the inductive limit is taken in that category. Let $A = (A_{v,w})_{v,w\in\mathbb{V}}$ be the adjacency matrix of $\mathbb{G}$, i.e. $A_{v,w} \in \mathbb{N}\cup\{0\}$ is the number of edges in $\mathbb{G}$ with w as initial vertex and v as terminal vertex. It follows easily from the definition of Φ that Φ_* is given by A under the identification (4.21):

$$(\Phi_*(x))_v = \sum_{w \in \mathcal{V}} A_{v,w} x_w .$$

Thus $K_0(B_\sigma(X_{\mathbb{G}}))$ is isomorphic to the dimension group $K_0(X_A)$ introduced by Wagoner [**Wa**]. However, the C^*-algebra $B_\sigma(X_{\mathbb{G}})$ is not the same as the C^*-algebra $\mathcal{M}_A$ from [**Wa**] since the first is separable and the latter is not. But $B_\sigma(X_{\mathbb{G}})$ is, by construction, the analogue of the AF-algebra $\mathcal{MP}_A$ introduced for A finite in Section 5 of [**Wa**]. And it follows from Theorem 2.13 that $\mathcal{M}_A$ is Morita equivalent to $B_\sigma(X_{\mathbb{G}})$ in most of the interesting cases, in particular when $\mathbb{G}$ is strongly connected.

When G is finite we can combine with Section 3.1.1 and [**BFF**] to conclude that $K_0(B_\sigma(X_G))$ is isomorphic to $K_0(A_Z)$, where Z is the one-sided shift space defined by G. It follows in this way from the classification of AF-algebras that A_Z and $B_\sigma(X_G)$ are stably isomorphic. A more direct proof of this, in a much more general setting, will be given in Theorem 4.19.

4.4. The heteroclinic algebra of a Smale space

Let now (X,φ) be a Smale space, [**Ru1**],[**Ru2**] and [**Pu1**]. For simplicity we will consider only mixing Smale spaces. We adopt the notation and terminology from [**Pu1**], except that we use ϵ_X to denote the sufficiently small, but otherwise arbitrary positive number which was denoted by ϵ_0 in [**Pu1**].

It is straightforward to show that the homoclinic algebra $A_\varphi(X)$ is identical with the asymptotic algebra A of [**Pu1**]. It follows from the description of $V^U(x,\epsilon)$ given on p. 179 of [**Pu1**] that a Smale space is locally expansive on post-periodic points. In fact, this is the case for all expansive homeomorphisms on compact spaces by Lemma 4.3. The purpose of this section is to establish the relation between the heteroclinic algebra $B_\varphi(X)$ and the stable algebras of [**Pu1**].

LEMMA 4.14. *Let (X,φ) be a Smale space. Two post-periodic points $x, x' \in W_{X,\varphi}$ are conjugate (in $W_{X,\varphi}$) if and only if*

$$\lim_{k\to\infty} d\left(\varphi^k(x), \varphi^k(x')\right) = 0. \tag{4.22}$$

PROOF. Assume (4.22). Let $0 < \epsilon < \epsilon_X$ be so small that $[[x,y],z] = [x,z]$ when $d(x,y) < \epsilon$ and $d(x,z) < \epsilon$. Choose k so large that $d\left(\varphi^k(x), \varphi^k(x')\right) < \frac{\epsilon}{2}$ and $\varphi^k(x') \in V^S\left(\varphi^k(x), \epsilon\right)$. Then $d\left(\varphi^k(z), \varphi^k(x')\right) < \frac{\epsilon}{2}$ and $d\left(\varphi^k(z), \varphi^k(x)\right) < \frac{\epsilon}{2}$ for all z in an open neighborhood U of x. Set

$$\mu(z) = \varphi^{-k}\left[\varphi^k(z), \varphi^k(x')\right]$$

when $z \in U$. Then $\mu(x) = x'$. Note that $\mu(z) \in V^U(x')$ so that $\mu(z) \in W_{X,\varphi}$ since $x' \in W_{X,\varphi}$. Similarly, there is an open neighborhood U' of x' in X, defined in a similar way, such that we can define

$$\nu(z) = \varphi^{-k}\left[\varphi^k(z), \varphi^k(x)\right]$$

for $z \in U'$. When $z \in U' \cap \nu^{-1}(U) \cap \varphi^{-k}\left(V^U\left(\varphi^k(x'), \epsilon\right)\right)$ we find that

$$\mu \circ \nu(z) = \varphi^{-k}\left[\left[\varphi^k(z), \varphi^k(x)\right], \varphi^k(x')\right] = \varphi^{-k}\left[\varphi^k(z), \varphi^k(x')\right] = z.$$

Similarly, $\nu \circ \mu(z) = z$ for all $z \in U \cap \mu^{-1}(U') \cap \varphi^{-k}\left(V^U\left(\varphi^k(x), \epsilon\right)\right)$. Note that $U \cap \mu^{-1}(U') \cap \varphi^{-k}\left(V^U\left(\varphi^k(x), \epsilon\right)\right)$ and $U' \cap \nu^{-1}(U) \cap \varphi^{-k}\left(V^U\left(\varphi^k(x'), \epsilon\right)\right)$ are open sets in the Wagoner topology of $W_{X,\varphi}$. It follows that μ, suitably restricted, gives rise to a conjugacy from x to x' in $W_{X,\varphi}$.

The reversed implication, that conjugacy in $W_{X,\varphi}$ implies (4.22), is trivial. $\square$

LEMMA 4.15. *Let (X,φ) be a mixing Smale space. Then $B_\varphi(X)$ is a stable C^*-algebra.*

PROOF. Let a be a positive element in $B_\varphi(X)$. For any $\epsilon > 0$ there is an element $f \in C_c(R_\varphi(X,W))$ such that $\|a - f^*f\| < \epsilon$. Set $K = r(\operatorname{supp} f)$ and $K' = s(\operatorname{supp} f)$. Then K and K' are compact subsets of W and there are finite subsets $F, F' \subseteq \operatorname{Per} X$ such that $K' \subseteq \bigcup_{p \in F'} V^U(p)$ and $K \subseteq \bigcup_{p\in F} V^U(p)$. Choose $\delta' \in]0, \epsilon_X[$ so small that $[[x,y],z] = [x,z] = [x,[y,z]]$ and $d([x,y],y) + d([x,y],x) < \epsilon_X$ when $d(x,y) < 5\delta'$ and $d(x,z) < 5\delta'$. Choose then $\delta \in]0,\delta'[$ such that $d([x,y],y) < \delta'$ when $d(x,y) < 2\delta$. Since K is compact there is a finite cover $K \subseteq \bigcup_{j=1}^N V_j$ of K by non-empty open sets V_j in $W_{X,\varphi}$ such that $x_j \in V_j \subseteq V^U(x_j,\delta)$ for some $x_j \in \bigcup_{p\in F} V^U(p)$, cf. Lemma 4.6. Since the periodic points are dense in X, cf. [**Ru1**], and X is not finite, there are periodic points $q_1, q_2, \dots, q_N$ in X such that

$$F' \cap \{\varphi^n(q_i) : \ n \in \mathbb{Z}, \ i = 1,2,\dots,N\} = \emptyset,$$

and

$$\{\varphi^n(q_i) : \ n \in \mathbb{Z}\} \cap \{\varphi^n(q_j) : \ n \in \mathbb{Z}\} = \emptyset$$

when $i \neq j$. Let $i \in \{1,2,\dots,N\}$. Since (X,φ) is transitive, $\bigcup_{k=1}^{|q_i|} V^U\left(\varphi^k(q_i)\right)$ is dense in X, cf. [**Ru1**], and there is therefore an element $y_i \in \bigcup_{k=1}^{|q_i|} V^U\left(\varphi^k(q_i)\right)$ such that $d(x_i,y_i) < \delta$. Then $d(z,y_i) < 2\delta$ for all $z \in V_i$ and we can define a continuous map $\chi_i : V_i \to V^U(y_i,\epsilon_X)$ such that $\chi_i(z) = [z,y_i]$. Set $U_i = \chi_i(V_i)$ and note that $U_i \subseteq \left\{z \in V^U(y_i,\epsilon_X) : \ d(z,y_i) < \delta'\right\}$. Define $\mu_i : \left\{v \in V^U(y_i,\epsilon_X) : \ d(v,y_i) < \delta'\right\} \to V^U(x_i,\epsilon_X)$ such that $\mu_i(v) = [v,x_i]$. Then $\mu_i \circ \chi_i(z) = z$ for all V_i so that χ_i is a homeomorphism of V_i onto U_i. To see that U_i is open in $W_{X,\varphi}$, let $z \in U_i$. Then $z = \chi_i(z')$ for some $z' \in V_i$. Since $[z,x_i] = z' \in V_i$ there is an open neighborhood $\Omega \subseteq \left\{v \in V^U(y_i,\epsilon_X) : \ d(v,y_i) < \delta'\right\}$ of z in $W_{X,\varphi}$ such that $\mu_i(z'') \in V_i$ and $\chi_i \circ \mu_i(z'') = [[z'',x_i],y_i] = z''$ for all $z'' \in \Omega$. It follows that (V_i,U_i,χ_i) is a conjugacy in $W_{X,\varphi}$. Note that $U_i \subseteq \bigcup_{k=1}^{|q_i|} V^U\left(\varphi^k(q_i)\right)$ so that $U_i \cap U_j = \emptyset$ when $i \neq j$. Let $\{\varphi_i\}$ be a partition of unity on K subordinate to $\{V_i\}$ and define for each i a function $v_i \in C_c(R_\varphi(X,W_{X,\varphi}))$, localized in $\left\{\left(x,\chi_i^{-1}(x)\right) : \ x \in U_i\right\}$, such that $v_i\left(x,\chi_i^{-1}(x)\right) = \sqrt{\varphi_i(\chi_i^{-1}(x))}$. Set $v = \sum_{j=1}^n v_j$. Since $U_i \cap U_j = \emptyset$ we find that $v_j^* v_i = 0$ when $i \neq j$. It follows that $v^*vf = \sum_{i=j}^N v_i^* v_i f = f$ and, since $fv_i = 0$ for all i, we have also that $fv = 0$. Set $b = f^*f$ and $c = vff^*v^*$. Then $bc = 0$ and $b \perp c$ in the notation of [**HR**]. The stability of $B_\varphi(X)$ follows then by combining Proposition 2.2 and Theorem 2.1 of [**HR**]. $\square$

LEMMA 4.16. *Let (X,φ) be a mixing Smale space. Then the heteroclinic algebra $B_\varphi(X)$ is strongly Morita equivalent to the stable algebra S of Putnam,* [**Pu1**].

PROOF. Let $x_0 \in X$ be a periodic point. Then $V^U(x_0)$ is an open subset of $W_{X,\varphi}$. Let $x \in W_{X,\varphi}$. Since $V^U(x_0)$ is dense in X, [**Ru1**], there is an element $z \in V^U(x_0)$ such that $d(x,z) < \epsilon_X$. Then $[x,z] \in V^U(x_0) \cap V^S(x)$. It follows from Lemma 4.14 that x is conjugate to $[x,z]$ in $W_{X,\varphi}$. This shows that every element of $W_{X,\varphi}$ is conjugate to an element of $V^U(x_0)$ and Corollary 2.14 implies now that $B_\varphi(X)$ is strongly Morita equivalent to $A_\varphi\left(X, V^U(x_0)\right)$. By Theorem 3.7 of [**PS**] it suffices now to show that the étale equivalence relations $R_\varphi\left(X, V^U(x_0)\right)$ and $G_s(x_0)$ of [**PS**] are the same. It follows from Lemma 4.14 that $R_\varphi\left(X, V^U(x_0)\right)$ and $G_s(x_0)$ are identical as sets. To see that the identity map $G_s(x_0) \to R_\varphi\left(X, V^U(x_0)\right)$ is a homeomorphism, it suffices to show that the two topologies have the same converging sequences. This follows easily by combining the description given on page 287 of [**PS**] with Lemma 1.12 above. $\square$

THEOREM 4.17. *Let (X,φ) be a mixing Smale space. The heteroclinic algebra $B_\varphi(X)$ is $*$-isomorphic to the stabilized stable algebra S of* [**Pu1**]. *In symbols,*

$$B_\varphi(X) \simeq S \otimes \mathbb{K}.$$

PROOF. Use [**BGR**] in combination with Lemma 4.16 and Lemma 4.15. $\square$

This theorem makes it possible to give cleaner formulations of the results of Putnam from [**Pu3**] regarding the functoriality of heteroclinic algebras of Smale spaces with respect to resolving maps. For example it follows that the algebras obtained by taking the crossed product of the stable algebras of Putnam with respect to the canonical automorphism (these algebras are called Ruelle-algebras in [**PS**]) behave just as nicely with respect to resolving maps as the stable algebra; simply because the $*$-homomorphisms between the heteroclinic algebras induced by a resolving map are equivariant with respect to the canonical automorphisms.

4.5. More inductive limit decompositions

4.5.1. Smale spaces from expanding maps. Let X be a compact metric space and $\psi : X \to X$ a positively expansive map, cf. Section 3.2.2. The inverse limit space $\tilde{X} = \operatorname{proj\,lim}(X,\psi)$ of the sequence

$$X \xleftarrow[\psi]{} X \xleftarrow[\psi]{} X \xleftarrow[\psi]{} X \xleftarrow[\psi]{} \cdots$$

carries a homeomorphism $\tilde{\psi} : \tilde{X} \to \tilde{X}$ defined such that

$$\tilde{\psi}\left((x_i)_{i=0}^\infty\right) = \left(\psi\left(x_i\right)\right)_{i=0}^\infty .$$

Note that the inverse of $\tilde{\psi}$ is given by the shift, i.e. $\tilde{\psi}^{-1}\left((x_i)\right)_j = x_{j+1}$.

The dynamical system $\left(\tilde{X}, \tilde{\psi}\right)$ is *the natural invertible extension* of (X,ψ). If d is a metric for the topology of X there is a metric D for the topology of $\tilde{X}$ defined such that

$$D\big((x_i)_{i=0}^\infty, (y_i)_{i=0}^\infty\big) = \sum_{i=0}^\infty 2^{-i} d\left(x_i, y_i\right).$$

Note that $\tilde{\psi}$ is an expansive homeomorphism of $\tilde{X}$.

LEMMA 4.18. *Assume that ψ is surjective and open. It follows that $\left(\tilde{X}, \tilde{\psi}\right)$ is a Smale space.*

PROOF. We appeal first to [**Rd**] to get a metric d for the topology of X, a $\lambda > 1$ and an $\epsilon_0 > 0$ such that

$$d(x,y) \le \epsilon_0 \ \Rightarrow \ d\left(\psi(x),\psi(y)\right) \ge \lambda d(x,y). \tag{4.23}$$

By Lemma 1 of [**CV**] there is then a $\delta_0 \le \frac{\epsilon_0}{2}$ such that

$$x,y \in X,\ d\left(\psi(x),y\right) \le \delta_0 \ \Rightarrow \ \psi^{-1}(y) \cap \left\{ z \in X :\ d(z,x) \le \frac{\delta_0}{\lambda} \right\} \ne \emptyset. \tag{4.24}$$

It follows then that (X, ψ) satisfies condition (E) of Section 7.26 in [**Ru1**] and from the work of Ruelle it follows that $\left(\tilde{X}, \tilde{\psi}\right)$ is a Smale space. $\square$

A positively expansive map which is also surjective and open is said to be *expanding.*

THEOREM 4.19. *Assume that ψ is expanding and that the ψ-periodic points are dense in X. Then the homoclinic algebra $A_\psi(X)$ is stably $*$-isomorphic to the heteroclinic algebra $B_{\tilde{\psi}}(\tilde{X})$. In fact,*

$$B_{\tilde{\psi}}(\tilde{X}) \simeq A_\psi(X) \otimes \mathbb{K}.$$

PROOF. Define $p : \tilde{X} \to X$ such that $p\left((x_i)_{i=0}^\infty\right) = x_0$ and note that p is equivariant. We will show that $p : W_{\tilde{X},\tilde{\psi}} \to X$ satisfies conditions 1, 4, 5 and 6 of Chapter 2. For this purpose we first establish condition 4, i.e. that p is a local homeomorphism.

Let $x \in W_{\tilde{X},\tilde{\psi}}$. By Lemma 4.6 there is an ϵ_0 such that

$$U = \left\{y \in \tilde{X} : \ D\left(\tilde{\psi}^j(y), \tilde{\psi}^j(x)\right) < \epsilon, \ j \leq 0\right\}$$

is contained in $W^u(x)$ and is an open neighborhood of x for the Wagoner topology for all $\epsilon \in \,]0, \epsilon_0[$. We choose $\epsilon < \delta_0$ where δ_0 comes from (4.24). Then $d\left(y_i, x_i\right) < \delta_0$ for all $i \in \mathbb{N}$ when $y \in U$ and it follows from (4.23) that $p : U \to p(U)$ is injective. Let $y \in U$. Since $y \in W^u(x)$ we have that $\sup_{i\in\mathbb{N}} D\big(\tilde{\psi}^i(y), \tilde{\psi}^i(x)\big) < \epsilon$. Set $\delta = \epsilon - \sup_{i\in\mathbb{N}} D\big(\tilde{\psi}^i(y), \tilde{\psi}^i(x)\big)$ and

$$\delta' = \frac{\delta}{\sum_{i=0}^\infty 2^{-i}\lambda^{-i}},$$

where $\lambda > 1$ is the number from (4.23) and (4.24). Let $z_0 \in X$ such that $d\left(z_0, y_0\right) < \delta'$. By repeated use of (4.24) we construct $z_i \in X, i \geq 0$, such that $z = (z_i)_{i=0}^\infty \in \tilde{X}$ and $d\left(z_j, y_j\right) \leq \lambda^{-j}\delta'$ for all j. It follows that $D\big(\tilde{\psi}^j(z), \tilde{\psi}^j(y)\big) \leq \sum_{i=0}^\infty 2^{-i}\lambda^{-i}\delta' = \delta$ for all $j \leq 0$. By the choice of δ this shows that $z \in U$. Since $z_0 = p(z)$ we conclude that $p(U)$ is open in X. Inspection of the estimates show that in fact,

$$D\big(\tilde{\psi}^j(z), \tilde{\psi}^j(y)\big) \leq \sum_{i=0}^\infty 2^{-i}\lambda^{-i} d\left(z_0, y_0\right),$$

for all $j \leq 0$, and this proves that the inverse of $p : U \to p(U)$ is continuous, completing the proof that condition 4 holds.

Condition 1 follows from condition 4 since p is equivariant; the required local conjugacy χ' can be defined as $p \circ \chi \circ p^{-1}$ in a neighborhood of $p(x)$.

To establish condition 5 of Chapter 2 it suffices to show that $p : W_{\tilde{X},\tilde{\psi}} \to X$ is surjective. Since we assume that the ψ-periodic points are dense in X it suffices to consider a ψ-periodic point $x_0 \in X$ and show that $\{z_0 \in X : \ d\left(z_0, x_0\right) < \delta_0\} \subseteq p\big(W_{\tilde{X},\tilde{\psi}}\big)$. Let $x \in \tilde{X}$ be the $\tilde{\psi}$-periodic point such that $p(x) = x_0$. By repeated use of (4.24) in the same way as above we find $z \in \tilde{X}$ such that $d\left(z_i, x_i\right) \leq \lambda^{-i}\delta_0$ for all $i \in \mathbb{N}$. Then $z \in W^u(x)$ and $p(z) = z_0$.

To check that also condition 6 of Chapter 2 is fullfilled, let q, q' be periodic point for $\tilde{\psi}$ and $x \in W^u(q), y \in W^u(q')$ elements such that that $p(x) \sim p(y)$ in X. It follows that $\lim_{k\to\infty} d\left(\psi^k(x_0), \psi^k(y_0)\right) = 0$. Since ψ is positively expansive this implies that $\psi^k(x_0) = \psi^k(y_0)$ for some $k \in \mathbb{N}$. Since ψ^k is a local homeomorphism it follows from the preceding that there are open neighborhoods in the Wagoner topology, U_x of x and U_y of y, and a homeomorphism $\chi : U_x \to U_y$ such that $\psi^k(\chi(z)_0) = \psi^k(z_0)$ for all $z \in U_x$ and $\chi(x) = y$. It follows then that

$$\lim_{n\to\infty} \sup_{z\in U_x} D\left(\tilde{\psi}^n(\chi(z)), \tilde{\psi}^n(z)\right) = 0,$$

so that (U_x, U_y, χ) is a local conjugacy from x to y in $W_{\tilde{X},\tilde{\psi}}$. This shows that also condition 6 of Chapter 2 is fullfilled.

It follows now from Theorem 2.13 that $A_\psi(X)$ and $B_{\tilde{\psi}}(\tilde{X})$ are strongly Morita equivalent. Since both algebras are separable by Lemma 1.21 we conclude from [**BGR**] that the algebras are stably $*$-isomorphic. But $B_{\tilde{\psi}}(\tilde{X})$ is stable. When $\tilde{\psi}$ is mixing this follows from Lemma 4.18 and Lemma 4.15. Since $\tilde{\psi}$ is non-wandering, because we assume that the ψ-periodic points are dense, the stability of $B_{\tilde{\psi}}(\tilde{X})$ in the general case follows from the mixing case by use of Smales 'spectral decomposition', cf. [**Pu3**] or 7.4 in [**Ru1**]. ☐

Remark 4.20. It should be observed that the density of the ψ-periodic points is automatic, given the other two conditions, when X is connected. See Lemma 2 of [**Sa**]. Furthermore, the ψ-periodic points are dense in X whenever $\tilde{X}$ is mixing since the periodic points are dense in a mixing Smale space, cf. 7.19 of [**Ru1**]. ☐

Remark 4.21. As pointed out in Remark 1.14, the homoclinic algebra $A_\psi(X)$ is the same as the algebra coming from the construction of Renault. Hence Theorem 4.19 gives the answer to the question of Putnam from page 4.14 of [**Pu2**]. ☐

Corollary 4.22. *In the setting of Theorem 4.19 the heteroclinic algebra $B_{\tilde{\psi}}(\tilde{X})$ is the inductive limit of a sequence*

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$$

where $A_n \simeq C(X_n) \otimes \mathbb{K}$ for some compact metric space X_n of dimension $\operatorname{Dim} X_n = \operatorname{Dim} X$.

Proof. It follows from Corollary 2.2 of [**Re3**] and Proposition 2.2 of [**MW**] that $A_\psi(X) \otimes \mathbb{K}$ can be realized as the inductive limit of such a sequence. In particular, the fact that $\operatorname{Dim} X_n = \operatorname{Dim} X$ for all n follows from (e.g.) Theorem 1.12.7 on page 138 of [**En**]. ☐

Remark 4.23. By Proposition 2.1 of [**KuR**] $A_\psi(X)$ and $B_{\tilde{\psi}}(\tilde{X})$ are simple if and only if ψ is exact in the sense that for every non-empty open subset $U \subseteq X$ there is an $m \in \mathbb{N}$ such that $\psi^m(U) = X$. By Theorem 6.1 of [**Re3**] $A_\psi(X)$ will then have a unique trace state and $B_{\tilde{\psi}}(\tilde{X})$ a densely defined lower semi-continuous trace which is unique up to scalar multiplication. It follows from Corollary 4.22 and the work of Gong [**G**] that when $\operatorname{Dim} X$ is finite, $B_{\tilde{\psi}}(\tilde{X})$ is simple and the K-groups of $B_{\tilde{\psi}}(\tilde{X})$ are torsion-free, then $B_{\tilde{\psi}}(\tilde{X})$ is an AT-algebra, cf. Definition 3.15, necessarily of real rank zero. ☐

CHAPTER 5

One-dimensional generalized solenoids

R.F. Williams has developed a theory of expanding attractors for a dynamical system, cf. [**Wi1**] and [**Wi2**]. These can be modeled as shift maps of generalized n-solenoids which are defined as inverse limits of immersions of n-dimensional branched manifolds satisfying certain axioms. In the one-dimensional case these axioms were given a purely topological formulation by I.Yi, [**Y1**], and he called them 1-dimensional generalized solenoids or just 1-solenoids. See also [**Y2**],[**Y3**],[**Y4**],[**Y5**]. Under an additional assumption about orientability he subsequently made a first study of the stable and unstable algebras, in the sense of Putnam, which 1-dimensional generalized solenoids give rise to, cf. [**Y4**]. In this section we will carry this investigation further by showing that one-dimensional generalized solenoids are Smale spaces quite generally and that the corresponding heteroclinic algebra is a simple, stable C^*-algebra which can be realized as the inductive limit of certain sub-homogenous algebras with one-dimensional spectrum of a type which were originally introduced in the classification program to demonstrate the richness of the Elliott-invariant, cf. [**Th4**].

5.1. The Smale-space structure of 1-solenoids

Let Γ be a finite (unoriented) graph with edgeset $\mathbb{E}$ and vertex set $\mathbb{V}$. We consider Γ as a compact metric space with metric d such that the edges are isometrically homeomorphic to $[0,1]$, and d is the corresponding arclength metric. When $x, y \in \Gamma$ lie on the same edge $e \in \mathbb{E}$ of Γ we denote by $[x, y]$ the closed interval in e between x and y. The open interval $]x, y[$ is then $[x, y] \setminus \{x, y\}$, and similarly, $[x, y[= [x, y]\setminus\{y\}$, $]x, y] = [x, y]\setminus\{x\}$. When $e \in \mathbb{E}$ we denote by $\mathrm{int}(e)$ the 'interior' of e, i.e. $\mathrm{int}(e) = e\setminus\mathbb{V}$.

Let $f : \Gamma \to \Gamma$ be a continuous map. Set

$$\overline{\Gamma} = \left\{(x_i)_{i=0}^{\infty} \in \Gamma^{\mathbb{N}} : \ f(x_{i+1}) = x_i, \ i = 0, 1, 2, \dots \right\}.$$

We consider $\overline{\Gamma}$ as a compact metric space with the metric

$$D\big((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}\big) = \sum_{i=0}^{\infty} 2^{-i} d(x_i, y_i).$$

Define $\overline{f} : \overline{\Gamma} \to \overline{\Gamma}$ such that $\overline{f}(x)_i = f(x_i)$ for all $i \in \mathbb{N}$. This is clearly a homeomorphism and we seek to identify conditions on f that make $\left(\overline{\Gamma}, \overline{f}\right)$ a Smale space.

We assume that f has the following properties.

a) (Flattening) All $x \in \Gamma$ have an open neighborhood U_x such that $f(U_x)$ is homeomorphic to $]-1, 1[$.
b) (Expansion) There is a constant $\lambda > 1$ such that

$$d(f(x), f(y)) \geq \lambda d(x, y)$$

when $x, y \in e \in \mathbb{E}$ and there is an edge $e' \in \mathbb{E}$ with $f([x, y]) \subseteq e'$.

c) (Nonfolding) f and f^2 are locally injective on e for each $e \in \mathbb{E}$.
d) (Markov) $f(\mathbb{V}) \subseteq \mathbb{V}$.

LEMMA 5.1. *There is an $\epsilon > 0$ with the following property: When*

$$x_0, x_1, x_0', x_1', y_0, y_1, y_2 \in \Gamma$$

satisfy that $f(y_2) = y_1, f(y_1) = y_0,\ f(x_1) = x_0,\ f(x_1') = x_0'$, and

$$d(x_i, y_i) \leq \epsilon, \quad d(x_i', y_i) \leq \epsilon, \quad i = 0, 1, \tag{5.1}$$

then there are elements $z_1, z_2, z_1', z_2' \in \Gamma$ such that

- $f(z_2) = z_1,\ f(z_1) = x_0$,
- $f(z_2') = z_1',\ f(z_1') = x_0'$,
- $d(z_1, y_1) \leq \lambda^{-1} d(x_0, y_0)$, $d(z_2, y_2) \leq \lambda^{-1} d(z_1, y_1)$,
- $d(z_1', y_1) \leq \lambda^{-1} d(x_0', y_0)$, $d(z_2', y_2) \leq \lambda^{-1} d(z_1', y_1)$,
- $d(z_1, z_1') \leq \lambda^{-1} d(x_0', x_0)$, $d(z_2, z_2') \leq \lambda^{-1} d(z_1, z_1')$.

PROOF. Note that every vertex $v \in \mathbb{V}$ has a neighborhood Ω_v with the property that $\Omega_v \cap \mathbb{V} = \{v\}$ and

$$d(x, y) = d(x, v) + d(y, v) \tag{5.2}$$

when $x, y \in \Omega_v$ and x, y are not contained in the same edge of Γ. We arrange that $\Omega_v \cap \Omega_w = \emptyset$ when $w \neq v$, and by the nonfolding condition we can also arrange that f is injective on $e \cap \Omega_v$ for any $e \in \mathbb{E}$ and any $v \in \mathbb{V}$.

For $\delta > 0$ and $v \in \mathbb{V}$, set

$$U_v = \{x \in \Gamma : \ d(x, v) < \delta\},$$

and $U = \bigcup_{v \in \mathbb{V}} U_v$. We claim that when δ is small enough we have

i) $\operatorname{Dist}(U_v, U_w) > 0$ when $v \neq w$.
ii) $U_v \subseteq \Omega_v$ for all $v \in \mathbb{V}$.
iii) $f(U_w) \cap U_v = \emptyset$ unless $f(w) = v$.
iv) $f(U_v) \subseteq \Omega_{f(v)}$ for all $v \in \mathbb{V}$.
v) $f(U_v)$ and $f^2(U_v)$ contain exactly one vertex for each $v \in \mathbb{V}$.
vi) $f(U_v)$ is homeomorphic to $]-1, 1[$ for each $v \in \mathbb{V}$.
vii) $f(U_v \cap f(U_w)) \supseteq U_{f(v)} \cap f(U_v)$ when $v, w \in \mathbb{V}$ and $f(w) = v$.
viii) When $x \in \Gamma \backslash U$ and $f(x) \in U_w$ for some $w \in \mathbb{V}$, there is an open interval $i_x =]x', x''[$ containing x such that $f(I_x) \subseteq \Omega_w$, $f^2(I_x) \subseteq \Omega_{f(w)}$, f^2 is injective on I_x, and

$$f^2(I_x) \supseteq U_{f(w)} \cap f(U_w).$$

i), ii), iii), iv), v) and vi) will hold for all sufficiently small δ. It follows from the expansion and flattening conditions that the same is true for vii).

It remains to check that we can arrange viii) by choosing δ sufficiently small. Note that f is finite-to-one. In particular, $f^{-1}(\mathbb{V})$ is a finite set. For each $x \in f^{-1}(\mathbb{V}) \backslash \mathbb{V}$ there is an open interval I_x containing x such that $f^2(I_x) =]a_x, f^2(x)] \cup [f^2(x), b_x[\subseteq \Omega_{f^2(x)}$ for some $a_x, b_x \in \Gamma$ close to $f^2(x)$ and $f(I_x) \subseteq \Omega_{f(x)}$. (The first property uses the nonfolding condition.) By shrinking I_x we can ensure that f^2 is injective on I_x. Then

$$\cap_{x \in f^{-1}(w)}]a_x, f(w)] \cup [f(w), b_x[\ \supseteq \ ]a_v, f(w)] \cup [f(w), b_v[$$

for some b_v, a_v sufficiently close to $v = f(w)$. (This uses the flattening condition.) It suffices then to take the $\delta > 0$ so small that

$$f^{-1}(U_w) \backslash U \subseteq \bigcup_{x \in f^{-1}(w)} I_x$$

and

$$U_{f(w)} \cap f(U_w) \subseteq]a_v, f(w)] \cup [f(w), b_v[$$

for all $w \in \mathbb{V}$.

To proceed with the proof take $\epsilon > 0$ so small that

$$2\epsilon \leq \min_{v \neq w} \operatorname{Dist}(U_v, U_w), \tag{5.3}$$

$$2\epsilon \leq \operatorname{dist}(\Gamma \backslash U, \mathbb{V}), \tag{5.4}$$

and such that

$$x, y \in \Gamma,\ d(x,y) \leq \epsilon \ \Rightarrow\ 2d(f(x), f(y)) \leq \min_{w \neq v} \operatorname{Dist}(U_v, U_w). \tag{5.5}$$

Consider then $x_0, x_1, x_0', x_1', y_0, y_1, y_2 \in \Gamma$ as in the statement of the lemma.

Assume first that there is an edge $e \in \mathbb{E}$ such that $x_0, x_0', y_0 \in \operatorname{int}(e)$. Then the Markov condition implies that $y_1 \in \operatorname{int}(e_1)$ for some $e_1 \in \mathbb{E}$, and it follows from the Markov and nonfolding conditions that $f(e_1) \supseteq e$. There is then an element $z_1 \in \operatorname{int}(e_1)$ such that $f(z_1) = x_0$ and $f([z_1, y_1]) = [x_0, y_0]$. Note that $d(z_1, y_1) \leq \lambda^{-1} d(x_0, y_0)$ by the expansion condition. Similarly, there is an element $z_1' \in \operatorname{int}(e_1)$ such that $f(z_1') = x_0'$, $f([z_1', y_1]) = [x_0', y_0]$ and $d(z_1', y_1) \leq \lambda^{-1} d(x_0', y_0)$. Then $f([z_1, z_1']) \subseteq f([z_1, y_1]) \cup f([z_1', y_1]) \subseteq e$ and hence $\lambda d(z_1, z_1') \leq d(x_0, x_0')$, thanks to the Markov condition. Since $z_1, z_1', y_1 \in \operatorname{int}(e_1)$ we can repeat the construction to obtain z_2, z_2' with the required properties.

Assume that there is no edge containing x_0, x_0' and y_0 in its interior, but an edge $e \in \mathbb{E}$ with $x_1, x_1', y_1 \in \operatorname{int}(e)$. It follows from (5.1),(5.3) and (5.4) that there is a vertex $v \in \mathbb{V}$ such that $x_0, x_0', y_0 \in U_v$. Assume first that x_0 and y_0 do not lie on the same edge. Note that (5.5) implies that $2d(f(t), f(x_1)) \leq \min_{w \neq v} \operatorname{Dist}(U_v, U_w)$ for all $t \in [x_1, y_1]$. Since $f(x_1) = x_0$, $f(y_1) = y_0$ and x_0, y_0 are not contained in the interior of a common edge there must be an element $z \in [x_1, y_1] \subseteq e$ such that $f(z) = v$. It follows then from the expansion condition and (5.2) that $\lambda d(x_1, y_1) \leq \lambda d(x_1, z) + \lambda d(z, y_1) \leq d(x_0, v) + d(y_0, v) = d(x_0, y_0)$. The same estimate, $\lambda d(x_1, y_1) \leq d(x_0, y_0)$, follows from the expansion condition when x_0 and y_0 do lie on the same edge since $f([x_1, y_1]) = [x_0, y_0]$. Similarly, we find that $\lambda d(x_1', y_1) \leq d(x_0, y_0)$ and $\lambda d(x_1, x_1') \leq d(x_0, x_0')$, regardless of the position of x_0, x_0' and y_0 in U_v. We set $z_1 = x_1, z_1' = x_1'$ in this case. Then $z_1, z_1', y_1 \in \operatorname{int}(e)$ and we can construct z_2 and z_2' by the method of the first case above.

Assume then that neither $\{x_0, x_0', y_0\}$ nor $\{x_1', x_1, y_1\}$ is contained in the interior of the same edge. By combining (5.1), (5.3) and (5.4) it follows that $x_1, x_1', y_1 \in U_w$ and $x_0, x_0', y_0 \in U_v \cap f(U_w)$ for some $v, w \in \mathbb{V}$ with $f(w) = v$. We split the considerations into two cases:

$y_2 \notin U$: Then $y_2 \in \operatorname{int}(e)$ for some $e \in \mathbb{E}$. It follows from condition viii) above that there is an interval $I \subseteq \operatorname{int}(e)$ such that $y_2 \in I$, $f(I) \subseteq \Omega_w$ and $f^2(I) \supseteq U_v \cap f(U_w)$. Furthermore, the sets $I, f(I)$ and $f^2(I)$ contain at most one vertex each and f^2 is injective on I. Consider $a, b \in I$. If $f(I)$ does not contain a vertex, neither does I and the expansion property ensures that $\lambda d(a,b) \leq d(f(a), f(b))$. If $f(I)$ contains a vertex it must be w and we choose $c \in I$ such that $f(c) = w$.

Then $\lambda d(a,b) \leq \lambda d(a,c) + \lambda d(c,b) \leq d(f(a),w) + d(f(b),w) = d(f(a),f(b))$ when $[a,b]$ contains c. If not, we get the estimate $\lambda d(a,b) \leq d(f(a),f(b))$ from the expansion condition. Since f is injective on $f(I)$ we can use the same arguments to show that $\lambda d(f(a),f(b)) \leq d(f^2(a),f^2(b))$. It follows that when we choose $z_2, z_2' \in I$ such that $f^2(z_2) = x_0$ and $f^2(z_2') = x_0'$ and set $z_1 = f(z_2), z_1' = f(z_2')$, we will have obtained the desired properties.

$y_2 \in U$: Then $y_2 \in U_{v'}$ for some $v' \in \mathbb{V}$ with $f(v') = w$. Let then I be an interval, possibly in a slightly extended sense to allow a single vertex in I, containing y_2 such that $f(I) = f(U_{v'})$, f^2 is injective on I and the sets I, $f(I)$ and $f^2(I)$ contain at most one vertex. To construct such an interval I, note first that there are intervals $(a,v']$ and $[v',b)$ such that $y_2 \in (a,v']$ and $f((a,v'] \cup [v',b)) = f(U_{v'})$. If f is injective on $f(U_{v'})$ we take $I = (a,v'] \cup [v',b)$. If not we find the point a' on the edge containing $(a,v']$ such that $f^2((a,v']) = f^2(U_{v'})$. We take then $I = (a,v']$. In both cases we have that $f^2(I) = f^2(U_{v'}) \supseteq f(U_w \cap f(U_{v'})) \supseteq U_v \cap f(U_w) \supseteq \{x_0, x_0', y_0\}$ and we repeat the construction which was used in the case where $y_2 \notin U$. □

The conditions used by Yi in the definition of 1-solenoids were not exactly a)–d) above, but the following, cf. [**Y1**]:

a') (Flattening) There is a $d \in \mathbb{N}$ such that for all $i \geq d$ and all $x \in \Gamma$ there is a neighborhood U_x^i of x with $f^i(U_x^i)$ homeomorphic to $]-1,1[$.

b') (Expansion) There are constants $C > 0$ and $\lambda > 1$ such that

$$d(f^n(x), f^n(y)) \geq C\lambda^n d(x,y)$$

for every $n \in \mathbb{N}$ when $x, y \in e \in \mathbb{E}$ and there is an edge $e' \in \mathbb{E}$ with $f^n([x,y]) \subseteq e'$.

c') (Nonfolding) f^n is locally injective on e for each $e \in \mathbb{E}$ and each $n \in \mathbb{N}$.

d') (Markov) $f(\mathbb{V}) \subseteq \mathbb{V}$.

Condition a') is stronger than the flattening condition used by Yi, [**Y1**], but only in the absence of the indecomposability and nonwandering conditions emposed in [**Y1**].

From [**Y1**] we take the following

LEMMA 5.2. *(Lemma 2.9 of* [**Y1**]*.) Assume that f satisfies conditions a'), b'), c') and d'). There is an $l \in \mathbb{N}$ and an $\kappa > 0$ such that*

$$d\left(f^k(x), f^k(y)\right) \leq \kappa \ \forall k \in \mathbb{N} \ \Rightarrow \ f^l(x) = f^l(y).$$

In [**Y2**] this lemma is used to conclude that $\left(\overline{\Gamma}, \overline{f}\right)$ is a Smale space under some additional assumptions. However, the sketch of proof in [**Y2**] works only when f is both open and expansive. Furthermore, the additional assumptions are redundant. In fact, the following holds.

THEOREM 5.3. *Assume that (Γ, f) satisfies conditions a'), b'), c') and d'). Then $\left(\overline{\Gamma}, \overline{f}\right)$ is a Smale space.*

PROOF. It follows easily from Lemma 5.2 that $\overline{f}$ is expansive, cf. Proposition 2.11 of [**Y1**]. Let $\delta > 0$ be an expansive constant for $\overline{f}$. To define the local product structure note that for some sufficiently large $m \in \mathbb{N}$, the map f^m will satisfy conditions a),b),c) and d), possibly with a different $\lambda > 1$. We use this new $\lambda > 1$

to define a new metric D_1 for the topology of $\overline{\Gamma}$ such that

$$D_1(x,y) = \sum_{j=0}^{\infty} \lambda^{-j} d\left(x_{jm}, y_{jm}\right). \tag{5.6}$$

Let $\delta' > 0$ be such that

$$x,y \in \overline{\Gamma}, D_1(x,y) \le \delta' \ \Rightarrow \ D\left(\overline{f}^j(x), \overline{f}^j(y)\right) < \delta, \tag{5.7}$$

for $j = 0,1,2,\ldots,m$. It follows from Lemma 5.1 that there is an $\epsilon > 0$ with the property that when $x, y \in \overline{\Gamma}$ and $D_1(x,y) \le \epsilon$ there is an element $z \in \overline{\Gamma}$ such that

$$z_0 = x_0 \tag{5.8}$$

and

$$d\left(z_{(j+1)m}, y_{(j+1)m}\right) \le \lambda^{-1} d\left(z_{jm}, y_{jm}\right). \tag{5.9}$$

for all $j \in \mathbb{N}$. We may assume that $\frac{2\epsilon\lambda}{\lambda-1} \le \delta'$.

If $z' \in \overline{\Gamma}$ is another element of $\overline{\Gamma}$ with the properties (5.8) and (5.9) we find that

$$\begin{aligned} D_1\left(\overline{f}^{jm}(z), \overline{f}^{jm}(z')\right) &= \sum_{i=j+1}^{\infty} \lambda^{-i} d\left(z_{mi-mj}, z'_{mi-mj}\right) = \lambda^{-j} D_1\left(z, z'\right) \\ &\le D_1\left(z, z'\right) \le D_1\left(z, y\right) + D_1\left(y, z'\right) \\ &\le \frac{2\epsilon\lambda}{\lambda - 1} \le \delta' \end{aligned}$$

and

$$\begin{aligned} D_1\left(\overline{f}^{-jm}(z), \overline{f}^{-jm}(z')\right) &= \sum_{i=0}^{\infty} \lambda^{-i} d\left(z_{mi+mj}, z'_{mi+mj}\right) \\ &\le \sum_{i=0}^{\infty} \lambda^{-i} d\left(z_{mi+mj}, y_{mi+mj}\right) + \sum_{i=0}^{\infty} \lambda^{-i} d\left(z'_{mi+mj}, y_{mi+mj}\right) \\ &\le \sum_{i=0}^{\infty} \lambda^{-i-j} d\left(z_{mi}, y_{mi}\right) + \sum_{i=0}^{\infty} \lambda^{-i-j} d\left(z'_{mi}, y_{mi}\right) \\ &\le D_1\left(z, y\right) + D_1\left(z', y\right) \le \frac{2\epsilon\lambda}{\lambda - 1} \le \delta' \end{aligned}$$

for all $j \ge 0$. It follows then from (5.7) that $D\left(\overline{f}^k(z), \overline{f}^k(z')\right) < \delta$ for all $k \in \mathbb{Z}$ and hence that $z = z'$ since δ is an expansive constant for $\left(\overline{\Gamma}, D, \overline{f}\right)$.

We can therefore define

$$[\cdot,\ \cdot] : \left\{(x,y) \in \overline{\Gamma} \times \overline{\Gamma} :\ D_1(x,y) < \epsilon\right\} \to \overline{\Gamma}$$

such that $[x,y]$ is the unique element z of $\overline{\Gamma}$ for which (5.8) and (5.9) hold.

To verify that this gives $\left(\overline{\Gamma}, \overline{f}\right)$ the structure of a Smale space we prove first that there is an $\epsilon_0 \le \epsilon$ such that

i) $[\cdot,\ \cdot] :\ \left\{(x,y) \in \overline{\Gamma} \times \overline{\Gamma} :\ D_1(x,y) < \epsilon_0\right\} \to \overline{\Gamma}$ is continuous.

ii) $[x,x] = x$ for all $x \in \overline{\Gamma}$.

iii) $[[x,y],z] = [x,z] = [x,[y,z]]$ when $D_1(x,y) \le \epsilon_0$, $D_1([x,y],z) \le \epsilon_0$ and $D_1(x,[y,z]) \le \epsilon_0$.

iv) $\overline{f}([x,y]) = \left[\overline{f}(x), \overline{f}(y)\right]$ when $D_1(x,y) \le \epsilon_0$.

To check i) we use (5.9) to find that

$$\begin{aligned} D_1\left([x',y'],[x,y]\right) &\le D_1\left([x',y'],y'\right) + D_1\left(y',y\right) + D_1\left([x,y],y\right) \\ &\le \frac{1}{1-\lambda} d\left(x_0',y_0'\right) + \frac{1}{1-\lambda} d\left(x_0,y_0\right) + D_1(y',y) \end{aligned}$$

when $D_1(x,y) \le \epsilon$ and $D_1\left(x',y'\right) \le \epsilon$. Hence i) holds for any choice of $0 < \epsilon_0 \le \epsilon$.

ii) follows from the uniqueness of the element z satisfying (5.8) and (5.9).

To obtain iii) note that $\left[[x,z],z\right]_0 = [x,z]_0 = \left[x,[y,z]\right]_0 = x_0$, that

$$\begin{aligned} &d\left(\left[[x,y],z\right]_{im+m}, [x,z]_{im+m}\right) \\ &\quad \le d\left(\left[[x,y],z\right]_{im+m}, z_{im+m}\right) + d\left([x,z]_{im+m}, z_{im+m}\right) \\ &\quad \le \lambda^{-1} d\left(\left[[x,y],z\right]_{im}, z_{im}\right) + \lambda^{-1} d\left([x,z]_{im}, z_{im}\right) \end{aligned}$$

and that

$$\begin{aligned} &d\left([x,z]_{im+m}, [x,[y,z]]_{im+m}\right) \\ &\quad \le d\left([x,z]_{im+m}, z_{im+m}\right) + d\left(z_{im+m}, [y,z]_{im+m}\right) + d\left([y,z]_{im+m}, [x,[y,z]]_{im+m}\right) \\ &\quad \le \lambda^{-1} d\left([x,z]_{im}, z_{im}\right) + \lambda^{-1} d\left(z_{im}, [y,z]_{im}\right) + \lambda^{-1} d\left([y,z]_{im}, [x,[y,z]]_{im}\right) \end{aligned}$$

for all $i \in \mathbb{N}$. As above it follows from this that

$$D_1\left(\overline{f}^{jm}\left(\left[[x,y],z\right]\right), \overline{f}^{jm}\left([x,z]\right)\right) \le \frac{4\epsilon_1\lambda}{\lambda-1}$$

and

$$D_1\left(\overline{f}^{jm}\left([x,z]\right), \overline{f}^{jm}\left([x,[y,z]]\right)\right) \le \frac{8\epsilon_1\lambda}{\lambda-1}$$

for all $j \in \mathbb{Z}$, where

$$\begin{aligned} \epsilon_1 = \max\bigl\{ &D_1\left(\left[[x,y],z\right],z\right), D_1\left([x,z],z\right), \\ &D_1\left(y,[y,z]\right), D_1\left(y,[y,z]\right), D_1\left([y,z],[x,[y,z]]\right)\bigr\}. \end{aligned}$$

It follows from i) that $\frac{8\epsilon_1\lambda}{\lambda-1} \le \delta'$ when ϵ_0 is sufficiently small. Then iii) will hold with such a choice of ϵ_0.

To establish iv) let $\kappa > 0$ be the constant from Lemma 5.2. Note that

$$\begin{aligned} &d\left(\overline{f}\left([x,y]\right)_i, \left[\overline{f}(x),\overline{f}(y)\right]_i\right) \le d\left(\overline{f}\left([x,y]\right)_i, \overline{f}(y)_i\right) + d\left(\overline{f}(y)_i, \left[\overline{f}(x),\overline{f}(y)\right]_i\right) \\ &\quad = d\left([x,y]_{i-1}, y_{i-1}\right) + d\left(\overline{f}(y)_i, \left[\overline{f}(x),\overline{f}(y)\right]_i\right) \end{aligned}$$

for all $i \ge 1$. Since $d\left([x,y]_{jm}, y_{jm}\right) \le \frac{\lambda}{\lambda-1} D_1(x,y)$ and $d\bigl(\overline{f}(y)_{jm}, \left[\overline{f}(x),\overline{f}(y)\right]_{jm}\bigr) \le \frac{\lambda}{\lambda-1} D_1\left(\overline{f}(x),\overline{f}(y)\right)$ for all $j \in \mathbb{N}$ it follows that

$$d\left(\overline{f}\left([x,y]\right)_i, \left[\overline{f}(x),\overline{f}(y)\right]_i\right) \le \kappa$$

for all $i \ge 1$ provided ϵ_0 is so small that $\frac{\lambda}{\lambda-1} \max\left\{D_1(x,y), D_1\left(\overline{f}(x),\overline{f}(y)\right)\right\} \le \kappa$ when $D_1(x,y) \le \epsilon_0$. Since $\overline{f}\left([x,y]\right)_0 = f\left(x_0\right) = \left[\overline{f}(x),\overline{f}(y)\right]_0$ it follows from Lemma 5.2 that $\overline{f}\left([x,y]\right) = \left[\overline{f}(x),\overline{f}(y)\right]$ when $D_1(x,y) \le \epsilon_0$.

Define now a new metric D_2 on $\overline{\Gamma}$ such that

$$D_2(x,y) = \sum_{i=0}^{\infty} \lambda^{-i} d\left(x_i,y_i\right).$$

When $[x,y]=y$ and $[x,z]=z$ for some $x\in\overline{\Gamma}$ with $D_1(y,x)\le\epsilon_0$ and $D_1(z,x)\le\epsilon_0$ we find that

$$D_2\left(\overline{f}(y),\overline{f}(z)\right)=\sum_{i=1}^{\infty}\lambda^{-i}d\left(y_{i-1},z_{i-1}\right)=\lambda^{-1}D_2(y,z) \tag{5.10}$$

since $z_0=y_0=x_0$. When $[y,x]=y$ and $[z,x]=z$ for some $x\in\overline{\Gamma}$ with $D_1(y,x)\le\epsilon_0$ and $D_1(z,x)\le\epsilon_0$ we find from Lemma 5.1 applied to f^m that

$$D_2\left(\overline{f}^{-m}(y),\overline{f}^{-m}(z)\right)=\sum_{i=0}^{\infty}\lambda^{-i}d\left([y,x]_{i+m},[z,x]_{i+m}\right)\le\lambda^{-1}D_2(y,z). \tag{5.11}$$

Set

$$D_3(x,y)=\sum_{j=0}^{m-1}\lambda^{\frac{j}{m}}D_2\left(\overline{f}^{-j}(x),\overline{f}^{-j}(y)\right)$$

and note that D_3 is a metric for the topology of $\overline{\Gamma}$ and hence equivalent to D_1. Thus i)–iv) hold with D_1 replaced by D_3, provided ϵ_0 is changed accordingly. Furthermore, we have now that there is an $\epsilon_0'>0$ such that

v) $D_3\left(\overline{f}(y),\overline{f}(z)\right)\le\lambda^{-\frac{1}{m}}D_3(y,z)$ when $[x,y]=y$ and $[x,z]=z$ for some $x\in\overline{\Gamma}$ with $D_3(y,x)\le\epsilon_0'$ and $D_3(z,x)\le\epsilon_0'$, and

vi) $D_3\left(\overline{f}^{-1}(y),\overline{f}^{-1}(z)\right)\le\lambda^{-\frac{1}{m}}D_3(y,z)$ when $[y,x]=y$ and $[z,x]=z$ for some $x\in\overline{\Gamma}$ with $D_3(y,x)\le\epsilon_0'$ and $D_3(z,x)\le\epsilon_0'$.

v) follows from (5.10) since $\lambda^{-1}\le\lambda^{-\frac{1}{m}}$ and vi) follows from (5.11). The properties i) through vi) are exactly what is required in a Smale space, cf. Section 7.1 of [**Ru1**] and [**Pu1**]. □

5.2. The heteroclinic algebra of 1-solenoids

Let (Γ,f) and $\left(\overline{\Gamma},\overline{f}\right)$ be as in Theorem 5.3. Let $d\in\mathbb{N}$ be the number from the flattening condition a').

LEMMA 5.4. *Let $x,y\in\overline{\Gamma}$ be two elements such that $x_i=y_i$ for some $i\in\mathbb{N}$ and such that $[x,y]$ is defined. Assume that there is a $j>i$, $j-i\ge d$, and open neighborhoods U_{x_j} and U_{y_j} of x_j and y_j such that $f^{j-i}\left(U_{x_j}\right)=f^{j-i}\left(U_{y_j}\right)\simeq]-1,1[$. It follows that $[z,y]_i=z_i$ for all z in an open neighborhood of x.*

PROOF. Since $f^{j-i}\left(U_{y_j}\right)\simeq\,]-1,1[$ and $j-i\ge d$ it follows from the expansion and flattening axioms that there is an open neighborhood Ω of $y_i=x_i$ such that f^i is injective on $\Omega\cap f^{j-i}\left(U_{y_j}\right)$. By construction of $[\cdot,\cdot]$ there is a $K>0$ such that $d\left([z,y]_k,y_k\right)\le Kd\left(z_0,y_0\right)$ for all $k\in\mathbb{N}$ whenever $[z,y]$ is defined. Since $x_0=y_0$ this implies that there is an open neighborhood V of x such that $z_i\in\Omega\cap f^{j-i}\left(U_{x_j}\right)$ and $[z,y]_j\in U_{y_j}\cap f^{i-j}\left(\Omega\right)$ when $z\in V$. Then $z_i,[z,y]_i\in\Omega\cap f^{j-i}\left(U_{y_j}\right)=\Omega\cap f^{j-i}\left(U_{x_j}\right)$ when $z\in V$. Since $f^i\left([z,y]_i\right)=z_0=f^i\left(z_i\right)$ we conclude that $[z,y]_i=z_i$ for all z in V. □

Set $W=W_{\overline{\Gamma},\overline{f}}$. For each $i\in\mathbb{N}$ we let R_i denote the subequivalence relation of $R_{\overline{f}}\left(\overline{\Gamma},W\right)$ consisting of the pair $(x,y)\in W^2$ such that $f^i\left(x_0\right)=f^i\left(y_0\right)$ and $f^i\left(U_{x_0}\right)=f^i\left(U_{y_0}\right)\simeq]-1,1[$ for open neighborhoods U_{x_0} and U_{y_0} of x_0 and y_0, respectively.

LEMMA 5.5. *R_i is open in $R_{\overline{f}}\left(\overline{\Gamma}, W\right)$ when $i \geq d$.*

PROOF. Consider an element $(x,y) \in R_i$. It follows from the proof of Lemma 4.14 that there is a $k > i$ and a local conjugacy μ from x to y in W such that $\mu(z) = \left(\overline{f}\right)^{-k}\left[\overline{f}^k(z), \overline{f}^k(y)\right]$ in an open neighborhood of x in W. Since we have that $\overline{f}^k(x)_{k-i} = f^i(x_0) = f^i(y_0) = \overline{f}^k(y)_{k-i}$ and $f^i(U_{x_0}) = f^i(U_{y_0})$ for some open neighborhoods U_{x_0} of $x_0 = \overline{f}^k(x)_k$ and U_{y_0} of $y_0 = \overline{f}^k(y)_k$, respectively, we conclude from Lemma 5.4 that there is an open neighborhood V of x in W such that $\left[\overline{f}^k(z), \overline{f}^k(y)\right]_{k-i} = \overline{f}^k(z)_{k-i} = f^i(z_0)$ for all $z \in V$. It follows that $f^i(\mu(z)_0) = \left[\overline{f}^k(z), \overline{f}^k(y)\right]_{k-i} = f^i(z_0)$ for all z in a neighborhood of x in W. This shows that R_i is open in $R_{\overline{f}}\left(\overline{\Gamma}, W\right)$. □

It follows from Lemma 5.5 that R_i is an étale equivalence relation when $i \geq d$.

LEMMA 5.6. *Assume that $f : \Gamma \to \Gamma$ is surjective and let $i \geq d$, where $d \in \mathbb{N}$ is the number from the flattening condition a'). Then $C_r^*(R_i) \otimes \mathbb{K}$ is $*$-isomorphic to an extension E of the form*

$$0 \longrightarrow C_0(\Gamma \backslash \mathbb{V}) \otimes \mathbb{K} \longrightarrow E \longrightarrow \oplus_{v \in \mathbb{V}} F_v^i \otimes \mathbb{K} \longrightarrow 0, \tag{5.12}$$

where the $F_v^i, v \in \mathbb{V}$, are finite-dimensional.

PROOF. Set $\Gamma_0 = f^{-i}(\mathbb{V})$ and $\Gamma_1 = \Gamma \backslash \Gamma_0$. Then

$$W_1 = \{x \in W : \ x_0 \in \Gamma_1\}$$

is an open R_i-invariant subset of W and it follows from Proposition 4.5 of [**Re1**] that $C_r^*(R_i)$ is an extension of $C_r^*(R_i|_{W_0})$ by $C_r^*(R_i|_{W_1})$, where $W_0 = W \backslash W_1$. Note that it follows from the nonfolding condition that two elements x, x' of W_1 are R_i-equivalent if and only if $f^i(x_0) = f^i(x'_0)$. Therefore $R_i|_{W_1}$ is a second countable proper principal groupoid in the sense of [**MW**] and it follows from Proposition 2.2 of [**MW**] that $C_r^*(R_i|_{W_1}) \otimes \mathbb{K} \simeq C_0(W_1/R_i) \otimes \mathbb{K}$. Since f^i is a local homeomorphism on W_1 and maps W_1 onto $\Gamma \backslash \mathbb{V}$ we find that W_1/R_i is homeomorphic to $\Gamma \backslash \mathbb{V}$ and conclude therefore that $C_r^*(R_i|_{W_1}) \otimes \mathbb{K} \simeq C_0(\Gamma \backslash \mathbb{V}) \otimes \mathbb{K}$.

Note that Γ_0 is finite since f is finite-to-one. By definition of the Wagoner topology each element of W_0 is isolated in W and hence W_0 is countable. For each $v \in \mathbb{V}$, set $Y_v = \{x \in W : \ f^i(x_0) = v\}$. Then W_0 is the disjoint union of the Y_v and elements of W_0 can only be R_i-equivalent if they belong to the same Y_v. It follows that $C_r^*(R_i|_{W_0}) \simeq \oplus_{v \in \mathbb{V}} C_r^*(R_i|_{Y_v})$. Let $R_{i,v}$ be the equivalence relation on $f^{-1}(v)$ defined such that two elements $s, t \in f^{-1}(v)$ are equivalent if and only if there are neighborhoods U_s and U_t of s and t such that $f^i(U_s) = f^i(U_t) \simeq]-1,1[$. Then two elements, x and x', of Y_v are R_i-equivalent if and only if x_0 and x'_0 are $R_{i,v}$-equivalent. It follows that $C_r^*(R_i|_{Y_v}) \otimes \mathbb{K} \simeq C_r^*(R_{i,v}) \otimes \mathbb{K}$. Since $C_r^*(R_{i,v})$ is finite-dimensional, this completes the proof. □

PROPOSITION 5.7. *Assume that (Γ, f) satisfies conditions a'), b'), c') and d'), and that f is surjective. It follows that $B_{\overline{f}}(\overline{\Gamma}) \otimes \mathbb{K}$ is the inductive limit of a sequence*

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$$

where each A_i is an extension of the form (5.12).

Proof. Note that $R_k \subseteq R_{k+1}$ when $k \geq d$. Furthermore, it follows from Lemma 5.2 and the flattening axiom that $R_{\overline{f}}\left(\overline{\Gamma}, W\right) = \bigcup_{k=1}^{\infty} R_k$. It follows then from Lemma 1.24 that $B_{\overline{f}}(\overline{\Gamma})$ is the limit of the sequence

$$C_r^*(R_d) \longrightarrow C_r^*(R_{d+1}) \longrightarrow C_r^*(R_{d+2}) \longrightarrow \cdots$$

By Lemma 5.6 this yields the proposition. □

Consider the following condition:

e') (Mixing) For every edge $e \in \mathbb{E}$ there is an $m \in \mathbb{N}$ such that $\Gamma \subseteq f^m(e)$.

It follows from the work of Williams and Yi, cf. 1.6 of [**Wi1**] and Lemma 2.14 of [**Y1**], that condition e') holds when (Γ, f) satisfies conditions a'),b'),c'), d') plus the following two:

f') (Indecomposability) Γ is not the union of two nonempty, closed f-invariant subsets.

g') (Nonwandering) No point in Γ is wandering under f.

Presumably e') is equivalent to f') and g') in the presence of conditions a')–d'), and the reason I prefer e') over f') and g') is that e') is easiest to check in examples.

Lemma 5.8. *Assume that (Γ, f) satisfies conditions a'), b'), c'), d') and e'). It follows that for every non-degenerate interval $I \subseteq \Gamma$ there is an $m \in \mathbb{N}$ such that $f^m(I) \supseteq \Gamma$. Furthermore, $\left(\overline{\Gamma}, \overline{f}\right)$ is mixing in this case.*

Proof. Let $I \subseteq \Gamma$ be non-degenerate interval. It follows from the expansion and nonfolding conditions that there is an $n \in \mathbb{N}$ such that $f^n(I)$ contains an edge. By condition e') this implies that $\Gamma = f^m(I)$ for some $m \geq n$. To show that $\left(\overline{\Gamma}, \overline{f}\right)$ is mixing we take an arbitrary $\epsilon > 0$ and an open non-empty subset $U \subseteq \overline{\Gamma}$. By definition of D and compactness of Γ there is a $k \in \mathbb{N}$ such that

$$x, y \in \overline{\Gamma},\ x_j = y_j, j \leq k \ \Rightarrow\ D(x,y) \leq \epsilon.$$

Furthermore, there is an open non-degenerate interval $I \subseteq \Gamma$ and an $i \in \mathbb{N}$ such that $\left\{x \in \overline{\Gamma} :\ x_i \in I\right\} \subseteq U$. As we have just seen there is then an $n \in \mathbb{N}$ such that $f^n(I) = \Gamma$. Let $j \geq k - i$ and consider an element $z \in \overline{\Gamma}$. Since $f^n(I) = \Gamma$ there is an $a \in I$ such that $f^n(a) = z_{i+j}$. Since f is surjective there is an element $y \in \overline{\Gamma}$ such that $y_{i+j} = a$. Note that $f^n(y_l) = f^{n+i+j-l}(y_{i+j}) = f^{i+j-l}(z_{i+j}) = z_l$ when $l \leq i + j$. Since $i + j \geq k$ it follows that $D\left(\overline{f}^n(y), z\right) \leq \epsilon$. Since $y_{i+j} \in I$ we see that $y \in \overline{f}^{-j}(U)$. Hence we have shown that $\operatorname{dist}\left(\overline{f}^{n-j}(U), z\right) \leq \epsilon$ for all $z \in \overline{\Gamma}$ and all $j \geq k - i$. It follows that $\overline{f}$ is mixing. □

Let $d \in \mathbb{N}$ and $l \in \mathbb{N}$ be the numbers from the flattening condition a') and Lemma 5.2, respectively. Furthermore, we let $m \in \mathbb{N}$ and $\lambda_0 > 1$ be such that

$$d\left(f^m(x), f^m(y)\right) \geq \lambda_0 d(x,y) \tag{5.13}$$

when $x, y \in e \in \mathbb{E}$ and $f^m([x,y]) \subseteq e'$ for some edge $e' \in \mathbb{E}$.

Let $p = (p_0, p_1, \dots) \in \overline{\Gamma}$ be $\overline{f}$-periodic. For $\delta > 0$, set

$$\overline{\Gamma}_{p,\delta} = \left\{x \in \overline{\Gamma} :\ d\left(x_{imd|p|}, p_0\right) < \delta,\ i \in \mathbb{N}\right\},$$

and

$$I_{p_0,\delta} = \left\{z \in \Gamma :\ z = x_0 \text{ for some } x \in \overline{\Gamma}_{p,\delta}\right\}.$$

Lemma 5.9. *Let $\epsilon > 0$ be given. There is a $\delta > 0$ such that*

i) $\overline{\Gamma}_{p,\delta} \subseteq W^u(p,\epsilon)$,
ii) $I_{p_0,\delta} \simeq]-1,1[$,
iii) $x \mapsto x_0$ *is a homeomorphism from* $\overline{\Gamma}_{p,\delta}$ *onto* $I_{p_0,\delta}$.

PROOF. Let U_{p_0} be an open neighborhood of p_0 such that $f^{md|p|}(U_{p_0}) \simeq]-1,1[$. We assume, as we may, that both U_{p_0} and $f^{md|p|}(U_{p_0})$ at most contains one vertex, which must then be p_0. Then $f^{md|p|}\left(U_{p_0} \cap f^{md|p|}(U_{p_0})\right)$ is an open interval contained in $f^{md|p|}(U_{p_0})$. There is a $\delta > 0$ such that $\{z \in \Gamma : \ d(z,p_0) < \delta\} \subseteq U_{p_0}$,

$$x,y \in \Gamma,\ d(x,y) < \delta \ \Rightarrow\ d\left(f^j(x), f^j(y)\right) < \kappa,\ j = 0,1,2,\dots,md|p|, \tag{5.14}$$

where $\kappa < \frac{\epsilon}{2}$ and κ is smaller than an expansive constant for $\left(\overline{\Gamma},\overline{f}\right)$, and

$$\{z \in \Gamma : \ d(z,p_0) < \delta\} \cap f^{md|p|}(U_{p_0}) \subseteq f^{md|p|}\left(U_{p_0} \cap f^{md|p|}(U_{p_0})\right). \tag{5.15}$$

It follows then from (5.14) that $\overline{\Gamma}_{p,\delta} \subseteq W^u(p,\epsilon)$, i.e. i) holds. To establish ii) and iii) it suffices now to show that

$$\overline{\Gamma}_{p,\delta} \ni x \mapsto x_0 \in f^{md|p|}(U_{p_0}) \cap \{z \in \Gamma : \ d(z,p_0) < \delta\} \tag{5.16}$$

is a homeomorphism. The map is injective because we chose κ smaller than an expansive constant for $\left(\overline{\Gamma},\overline{f}\right)$. To show that the map is surjective, let $z' \in f^{md|p|}(U_{p_0}) \cap \{z \in \Gamma : \ d(z,p_0) < \delta\}$. It follows from (5.15) that

$$z' \in f^{md|p|}\left(U_{p_0} \cap f^{md|p|}(U_{p_0})\right),$$

i.e. there is an element $z_1 \in U_{p_0} \cap f^{md|p|}(U_{p_0})$ such that $f^{md|p|}(z_1) = z'$. Since $d(z_1,p_0) \leq \lambda_0^{-1} d(z,p_0) < \delta$ it follows from (5.15) that $z_1 \in f^{md|p|}\left(U_{p_0} \cap f^{md|p|}(U_{p_0})\right)$. We can therefore continue by induction to construct an element $x \in \overline{\Gamma}$ such that $x_0 = z'$ and $d\left(x_{imd|p|}, p_0\right) < \delta$ for all $i \in \mathbb{N}$. Then $x \in \overline{\Gamma}_{p,\delta}$ and it follows that (5.16) is surjective. Note that the preceding argument shows that when $z', z'' \in f^{md|p|}\left(U_{p_0} \cap f^{md|p|}(U_{p_0})\right)$ are sufficiently close, there are elements $x', x'' \in \overline{\Gamma}_{p,\delta}$ such that $x'_0 = z', x''_0 = z''$ and $d\left(x'_{imd|p|}, x''_{imd|p|}\right) \leq 2d(z',z'')$ for all $i \in \mathbb{N}$. This shows that the inverse of (5.16) is continuous. □

Let $g : [-1,1] \to \Gamma$ be a locally injective continuous map. We define an equivalence relation $\sim$ on $]-1,1[$ such that $t \sim s$ if and only if $g(t) = g(s)$ and there are open neighborhoods U_s and U_t of s and t in $]-1,1[$, respectively, such that $g(U_s) = g(U_t) \simeq]-1,1[$. Set

$$R = \left\{(s,t) \in]-1,1[^2 : \ s \sim t\right\}.$$

Give R the topology inherited from $]-1,1[^2$.

LEMMA 5.10. *R is an étale equivalence relation.*

PROOF. It is trivial that R is a topological equivalence relation. To prove that R is locally compact we will argue that R is the intersection of a closed and an open subset of $]-1,1[^2$. To this end consider $s \in]-1,1[\setminus g^{-1}(\mathbb{V})$. Since g is locally injective on $[-1,1]$ it is also finite-to-one and hence $g^{-1}(\mathbb{V})$ is finite. If $t \in]-1,1[\setminus g^{-1}(\mathbb{V})$ and $g(s) = g(t)$, there are neighborhood U_s and U_t in $]-1,1[$ of s and t, respectively, such that g is injective on both U_s and U_t, and $(U_s \cup U_t) \cap g^{-1}(\mathbb{V}) = \emptyset$. Then $g(U_s)$ and $g(U_t)$ are non-degenerate intervals such that $g(s) = g(t)$ is in the interior of $g(U_s) \cap g(U_t)$. It follows that if we shrink U_s and U_t we can ensure that $g(U_s) = g(U_t)$. This shows that

$$R = \left\{(s,t) \in]-1,1[^2 : \ g(s) = g(t)\right\} \setminus \left\{(s,t) \in g^{-1}(\mathbb{V})^2 : \ s \nsim y\right\};$$

clearly the intersection of a closed and an open subset.

To see that $r : R \to]-1,1[$ is a local homeomorphism, consider an element $(s,t) \in R$. Let U_s and U_t be open neighborhoods in $]-1,1[$ of s and t such that $g(U_s) = g(U_t) \simeq]-1,1[$. By shrinking U_s and U_t we may assume that g is injective on U_t. It follows that $r : (U_s \times U_t) \cap R \to U_s$ is a homeomorphism. □

We are going to use the étale equivalence relations of Lemma 5.10 in the special case where $g(-1), g(1) \in \mathbb{V}$ and $g(]-1,1[) = \Gamma$. When this holds we say that R is an *open interval-graph relation*.

DEFINITION 5.11. A C^*-algebra A is called *an interval building block* when there are finite-dimensional C^*-algebras F_1 and F_2 and $*$-homomorphisms $\varphi_0, \varphi_1 : F_1 \to F_2$ such that $A \simeq \{(a,f) \in F_1 \oplus (C([0,1], F_2)) : \ \varphi_0(a) = f(0), \ \varphi_1(a) = f(1)\}$.

LEMMA 5.12. *Let R be an open interval-graph relation. Then $C_r^*(R)$ is an interval building block.*

PROOF. Note that we can add vertices to Γ without affecting R. In this way we can arrange that Γ has no loops. Let $A_{\mathbb{V}}$ be the finite-dimensional C^*-algebra generated by matrix units $e_{x,y}$ where $x, y \in g^{-1}(\mathbb{V})$ are such that $g(]x-\epsilon, x+\epsilon[) \cap g(]y-\epsilon, y+\epsilon[) \simeq]-1,1[$ for all small $\epsilon > 0$. Note that $]-1,1[\setminus g^{-1}(\mathbb{V})$ is a collection $\mathcal{C}$ of open disjoint subintervals. For each $e \in \mathbb{E}$ we let M_e be the full matrix algebra generated by the matrix units $f^e_{\gamma,\gamma'}$, where $\gamma, \gamma' \in \mathcal{C}$ and $\overline{g(\gamma)} = \overline{g(\gamma')} = e$. When $e \in \mathbb{E}$ and $v \in \mathbb{V}$ are such that $v \in e$ we define $\varphi_{e,v} : A_{\mathbb{V}} \to M_e$ such that

$$\varphi_{e,v}(e_{x,y}) = \begin{cases} 0, & \text{when } x, y \notin g^{-1}(v), \\ f^e_{\gamma,\gamma'} \text{ where } x \in \overline{\gamma} \text{ and } y \in \overline{\gamma'}, & \text{when } x, y \in g^{-1}(v). \end{cases}$$

This is well-defined since Γ has no loops. When $h \in C_c(R)$ we define $a_h \in A_{\mathbb{V}}$ such that

$$a_h = \sum_{x,y \in g^{-1}(\mathbb{V})} h(x,y) e_{x,y}$$

and $h^e \in C(e, M_e)$ such that

$$h^e(s) = \sum_{(\gamma,\gamma') \in \mathcal{C}^2} h\left((g|_\gamma)^{-1}(s), (g|_{\gamma'})^{-1}(s)\right) f^e_{\gamma,\gamma'}, \ s \in \operatorname{int}(e).$$

It is now not difficult to see that the map $h \mapsto \left(a_h, (h^e)_{e\in\mathbb{E}}\right)$ extends to a $*$-isomorphism from $C_r^*(R)$ onto

$$\left\{\left(a, (f^e)_{e\in\mathbb{E}}\right) \in A_{\mathbb{V}} \oplus (\oplus_{e\in\mathbb{E}} C(e, M_e)) : \ \varphi_{e,v}(a) = f^e(v) \text{ when } v \in e\right\}.$$

Since the latter is an interval building block this completes the proof. □

THEOREM 5.13. *Assume that (Γ, f) satisfies conditions a'), b'), c'), d') and e'). It follows that $B_{\overline{f}}(\overline{\Gamma})$ is a stable and simple C^*-algebra which is isomorphic to the inductive limit of a sequence of interval building blocks.*

PROOF. That $B_{\overline{f}}(\overline{\Gamma})$ is stable and simple follows from Lemma 5.8, Theorem 5.3 and Theorem 4.17.

Let J be a non-empty open subinterval of the set $I_{p,\delta}$ of Lemma 5.9, chosen such that the endpoints of J are in $f^{-i_0}(\mathbb{V})$ for some $i_0 \in \mathbb{N}$. Set $\Omega = \left\{x \in \overline{\Gamma}_{p,\delta} : \ x_0 \in J\right\}$. Then Ω is an open subset of $W_{\overline{\Gamma},\overline{f}}$ and it follows from Lemma 5.8 that every element of $W_{\overline{\Gamma},\overline{f}}$ is conjugate to an element of Ω since e') holds. By Corollary 2.14 this implies that $B_{\overline{f}}(\overline{\Gamma})$ is stably isomorphic to $A_{\overline{f}}\left(\overline{\Gamma}, \Omega\right)$. By Lemma 5.5 $R_{\overline{f}}\left(\overline{\Gamma}, \Omega\right) \cap R_i$ is open in $R_{\overline{f}}(\overline{\Gamma}, \Omega)$ when $i \geq d$ and it follows from Lemma 5.2 and the flattening condition a') that

$$R_{\overline{f}}(\overline{\Gamma}, \Omega) = \bigcup_{i=d}^{\infty} R_{\overline{f}}\left(\overline{\Gamma}, \Omega\right) \cap R_i.$$

By Lemma 1.20 we have therefore that

$$A_{\overline{f}}\left(\overline{\Gamma}, \Omega\right) = \overline{\bigcup_{i=d}^{\infty} C_r^*\left(R_i'\right)},$$

where $R_i' = R_{\overline{f}}\left(W, \Omega\right) \cap R_i$. It follows from Lemma 5.8 that $f^m\left(J\right) = \Gamma$ for some $m \in \mathbb{N}$. R_i' is then an open interval-graph relation for every $i \geq \max\left\{i_0, m, d\right\}$. Since $B_{\overline{f}}(\overline{\Gamma}) \simeq A_{\overline{f}}\left(\overline{\Gamma}, \Omega\right) \otimes \mathbb{K}$ it follows that $B_{\overline{f}}(\overline{\Gamma})$ can be realized as the inductive limit of a sequence of C^*-algebras of the form $M_n\left(C_r^*(R)\right)$, where R is an open interval-graph relation. By Lemma 5.12 these are all interval building blocks. □

REMARK 5.14. Interval building blocks are slight generalizations of the building blocks used by the author in [**Th4**]. The difference is that we here allow the $*$-homomorphisms φ_0 and φ_1 to be non-unital. To get an idea about the variety of simple C^*-algebras which can aris as inductive limits of interval building blocks we refer to [**Th4**], [**JS**] and [**Ell2**]. It is difficult not to wonder about which C^*-algebras of this class can be realized by one-dimensional solenoids.

CHAPTER 6

The heteroclinic algebra of a group automorphism

6.1. Automorphisms locally expansive on post-periodic points

Let G be a locally compact metric group such that the metric d is *subinvariant* in the sense that there is a constant $K > 0$ such that

$$d(ax, ay) \le Kd(x, y) \text{ and } d(xa, ya) \le Kd(x, y)$$

for all $a, x, y \in G$. Let $\varphi : G \to G$ a continuous group automorphism.

THEOREM 6.1. *Let G be a Lie group with subinvariant metric d and $\varphi : G \to G$ an automorphism of G (i.e. a diffeomorphic group automorphism). Then φ is expansive on post-periodic points.*

PROOF. Let LG be the Lie algebra of G and $L = d\varphi : LG \to LG$ the differential at the identity e of φ. Let $LG = \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S}$ be the decomposition of LG obtained by applying Lemma 3.17 to L. There is then a $\delta > 0$ such that the exponential map $\exp : LG \to G$ is a diffeomorphism of $\{x \in LG : \|x\| < \delta\}$ onto an open neighborhood of e. Let $\epsilon > 0$ be such that

$$\{x \in G : d(x, e) \le K\epsilon\} \subseteq \left\{\exp y : \|y\| \le \frac{\delta}{\|L^{-1}\| + \|L\| + 1}\right\}.$$

Let $p \in \operatorname{Per} G$. If $x \in W^u(p, \epsilon)$ there are elements $y_i \in LG, \|y_i\| \le \frac{\delta}{\|L^{-1}\|+\|L\|+1}$, such that $\varphi^i(p^{-1}x) = \exp y_i$ for all $i \le 0$. Since $\|L^{-1}y_i\| < \delta$ and $\exp L^{-1}y_i = \varphi^{-1}(\exp y_i) = \exp y_{i-1}$, we conclude that $L^{-1}y_i = y_{i-1}$. It follows that $y_i = L^i y_0$ for all $i \le 0$. Since $\lim_{i\to-\infty} \varphi^i(p^{-1}x) = e$, we conclude that $\lim_{i\to-\infty} L^i y_0 = 0$ which means that $y_0 \in \mathcal{U}$ by Lemma 3.17.

Define $F : \{x \in LG : \|x\| < \delta\} \to G$ such that $F(y) = p \exp y$, and note that F is a homeomorphism from $\{x \in LG : \|x\| < \delta\}$ onto a neighborhood of p. From what we have just shown it follows that $W^u(p, \epsilon) = F(\Omega)$, where

$$\Omega = \left\{y \in \mathcal{U} : \|y\| \le \frac{\delta}{\|L^{-1}\| + \|L\| + 1}\right\} \cap F^{-1}\left(\left\{x \in G : d\left(\varphi^i(x), \varphi^i(p)\right) \le \epsilon, \ i \le 0\right\}\right).$$

Since Ω is compact in LG we conclude that $W^u(p, \epsilon)$ is compact in G.

Let $x, y \in W^u(p)$ and assume that $d(\varphi^i(x), \varphi^i(y)) \le \epsilon$ for all $i \in \mathbb{Z}$. As above we get a sequence of vectors

$$y_i \in \left\{y \in LG : \|y\| \le \frac{\delta}{\|L^{-1}\| + \|L\| + 1}\right\}$$

such that $\varphi^i(x^{-1}y) = \exp y_i$ for all $i \in \mathbb{Z}$. The same argument as above now shows that $y_i = L^i y_0$ for all $i \in \mathbb{Z}$. Since $\|L^i y_0\| < \delta$ for all $i \in \mathbb{Z}$ it follows from Lemma 3.17 that $y_0 \in \mathcal{N}$. However, $\lim_{n\to-\infty} d(\varphi^i(x^{-1}y), e) = 0$ since both x and y lie in $W^u(p)$. Hence $\lim_{i\to-\infty} L^i y_0 = 0$, and we conclude that $y_0 \in \mathcal{U} \cap \mathcal{N} = \{0\}$ by Lemma 3.17. It follows that $x = y$, completing the proof. □

LEMMA 6.2. *Let $Q \in M_n(\mathbb{Q})$ be invertible. Then the shift σ_Q of the solenoid S_Q is locally expansive on post-periodic points.*

PROOF. Let $p \in S_Q$ be periodic under σ_Q. It follows from Lemma 3.18 that there is an $\epsilon_p > 0$ such that $W^u(p, \epsilon_p)$ is homeomorphic to a compact neighborhood of 0 in $\mathcal{U}$, and therefore compact. There is an $m \in \mathbb{N}$ such that mQ and mQ^{-1} both have integer entries. Set

$$\epsilon = \frac{1}{m(\|Q\| + 1)}.$$

Here $\|Q\|$ is the operator norm of Q with respect to a norm of $\mathbb{R}^n$ such that $\|x\| \geq 1$ when $x \in \mathbb{Z}^n$. We claim that

$$x, y \in S_Q, \ \lim_{n\to-\infty} d\left(\sigma_Q^n(x), \sigma_Q^n(y)\right) = 0, \ \sup_{j\in\mathbb{Z}} d\left(\sigma_Q^j(x), \sigma_Q^j(y)\right) \leq \epsilon \ \Rightarrow \ x = y.$$

This will show that σ_Q is locally expansive on post-periodic points. So let $x, y \in S_Q$ such that $\lim_{n\to-\infty} d\left(\sigma_Q^n(x), \sigma_Q^n(y)\right) = 0$ and $d\left(\sigma_Q^j(x), \sigma_Q^j(y)\right) \leq \epsilon$ for all $j \in \mathbb{Z}$. Let $\mathbb{R}^n = \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S}$ be the decomposition obtained from Lemma 3.17 applied to Q. Set $w = y^{-1}x$. It follows from Lemma 3.18 that there is an $N \in \mathbb{N}$ and a $u \in \mathcal{U}$ such that $w_i = \rho(Q^i u)$, $i \leq -N$. By increasing N if necessary we may assume that $\|Q^i u\| \leq \delta$ for all $i \leq -N$. Assume then that $j \in \mathbb{Z}$ is such that $\|Q^j u\| \leq \epsilon$ and $w_j = \rho(Q^j u)$. This holds when $j \leq -N$. To see that it is also true for $j+1$, note that there is a $z \in \mathbb{Z}^n$ such that $w_{j+1} = \rho(Q^{j+1}u + Qz)$. Since $d_0(w_{j+1}, 0) \leq \epsilon$ there is a $\xi \in \mathbb{Z}^n$ such that $\|Q^{j+1}u + Qz - \xi\| \leq \epsilon$. Hence $m\|Qz - \xi\| \leq m\|Q^{j+1}u\| + m\epsilon \leq m\|Q\|\,\|Q^j u\| + m\epsilon \leq (m\|Q\| + m)\epsilon < 1$. It follows that $\xi = Qz$. Since $\|Q^{j+1}u\| \leq \epsilon$ and $w_{j+1} = \rho(Q^{j+1}u)$, we can proceed by induction to conclude that $\|Q^i u\| \leq \epsilon$ and $w_i = \rho(Q^i u)$ for all $i \in \mathbb{Z}$. Hence $u \in \mathcal{N} \cap \mathcal{U} = \{0\}$ by Lemma 3.17, and we conclude that $w_j = 0$ for all $j \in \mathbb{Z}$, i.e. $x = y$. □

Despite the impression one may get from Lemma 6.2 and Theorem 6.1 not every automorphism of a compact abelian group is locally expansive on post-periodic points. The shift on $\mathbb{T}^{\mathbb{Z}}$ is a counterexample.

In the following we let G be a locally compact group with subinvariant metric d, and we let $\varphi : G \to G$ be an automorphism of G which is locally expansive on post-periodic points.

LEMMA 6.3. *The set $W_{G,\varphi}$ of post-periodic points is a locally compact group in the Wagoner topology.*

PROOF. The periodic points clearly form a subgroup of G and it follows from the joint continuity of the product in G that $W_{G,\varphi}$ is a subgroup of G. It remains to prove the continuity of the group operations in the Wagoner topology. This follows readily from Lemma 4.6. □

Let e be the neutral element of G. Set

$$P\Delta_\varphi = \left\{ g \in W_{G,\varphi} : \ \lim_{n\to\infty} d\left(\varphi^n(g), e\right) = 0 \right\}.$$

This is clearly a subgroup of $W_{G,\varphi}$ and we call it *the heteroclinic subgroup* of φ. We consider $P\Delta_\varphi$ as a discrete group.

LEMMA 6.4. *Let $x, y \in W_{G,\varphi}$. The following are equivalent*

a) *x and y are conjugate in $W_{G,\varphi}$.*
b) $\lim_{n\to\infty} d\left(\varphi^n(x), \varphi^n(y)\right) = 0$.
c) $xy^{-1} \in P\Delta_\varphi$.

PROOF. The proof of Lemma 3.6 works with only the obvious changes. □

Let $P\Delta_\varphi \times W_{G,\varphi}$ be the transformation groupoid corresponding to the left-translation of $P\Delta_\varphi$ on $W_{G,\varphi}$. We can then define an isomorphism $\Phi : P\Delta_\varphi \times W_{G,\varphi} \to R_\varphi\left(G, W_{G,\varphi}\right)$ of topological groupoids such that

$$\Phi(h, x) = \left(xh^{-1}, x\right).$$

In this way we obtain

THEOREM 6.5. *Let G be a locally compact group with subinvariant metric d, and let $\varphi : G \to G$ be an automorphism of G which is locally expansive on post-periodic points. Then*

$$B_\varphi(G) \simeq C_0\left(W_{G,\varphi}\right) \rtimes_\tau P\Delta_\varphi,$$

where $\alpha_h(f)(x) = f(h^{-1}x),\ h \in P\Delta_\varphi,\ f \in C_0\left(W_{G,\varphi}\right)$.

PROOF. This follows from the preceding observations since $C_0\left(WP_G\right) \rtimes_\tau P\Delta_\varphi \simeq C_r^*\left(P\Delta_\varphi \times W_{G,\varphi}\right)$, cf. [**Re1**] or [**Ph1**]. □

To clarify the structure of $B_\varphi(G)$ further we observe that there is a homomorphism $\Gamma : PW_{G,\varphi} \to \operatorname{Per} G$ such that $\Gamma\left(W^u(p)\right) = \{p\}$. Set $\operatorname{Per}_e G = \Gamma\left(P\Delta_\varphi\right)$ so that

$$\operatorname{Per}_e G = \left\{p \in \operatorname{Per} G :\ W^u(p) \cap W^s(e) \neq \emptyset\right\},$$

where $W^s(e) = \{g \in G :\ \lim_{n\to\infty} d\left(\varphi^n(g), e\right) = 0\}$. Then $\operatorname{Per}_e G$ is a normal subgroup of $\operatorname{Per} G$, and it follows from Lemma 6.4 that two elements $x, y \in W_{G,\varphi}$ only can be locally conjugate when $\Gamma(x)^{-1}\Gamma(y) \in \operatorname{Per}_e G$. Hence

$$B_\varphi(G) = A_\varphi\left(G, W_{G,\varphi}\right) \simeq \oplus_{\chi \in \operatorname{Per} G / \operatorname{Per}_e G} A_\varphi\left(G, W_\chi\right),$$

where $W_\chi = \bigcup_{p\in\chi} W^u(p)$. For each $\chi \in \operatorname{Per} G / \operatorname{Per}_e G$, fix a representative $p_\chi \in \operatorname{Per} G$ of χ. It follows from Corollary 2.14 that $A_\varphi\left(G, W_\chi\right)$ is strongly Morita equivalent to $A_\varphi\left(G, W^u\left(p_\chi\right)\right)$. Since $\Gamma^{-1}(e) \cap P\Delta_\varphi = \Delta_\varphi$, where

$$\Delta_\varphi = \left\{g \in G :\ \lim_{|n|\to\infty} d\left(\varphi^n(g), e\right) = 0\right\}$$

is the homoclinic group of φ, it follows that

$$R_\varphi\left(G, W^u\left(p_\chi\right)\right) \simeq \Delta_\varphi \times W^u\left(p_\chi\right).$$

Since $\Delta_\varphi \times W^u\left(p_\chi\right) \simeq \Delta_\varphi \times W^u(e)$ under the map $(h, x) \mapsto \left(h, xp_\chi^{-1}\right)$, we find that $A_\varphi\left(G, W_\chi\right)$ is stably isomorphic to $C_0\left(W^u(e)\right) \rtimes_\tau \Delta_\varphi$. Hence

THEOREM 6.6. *Let G be a locally compact group with subinvariant metric d, and let $\varphi : G \to G$ be an automorphism of G which is locally expansive on post-periodic points. Then the heteroclinic algebra $B_\varphi(G)$ is stably isomorphic to*

$$\oplus_{\operatorname{Per} G / \operatorname{Per}_e G} C_0\left(W^u(e)\right) \rtimes_\tau \Delta_\varphi.$$

REMARK 6.7. I have not found an example of a non-expansive automorphism of a locally compact group for which the heteroclinic algebra is defined and simple. But it can certainly happen that $C_0(W^u(e)) \rtimes_\tau \Delta_\varphi$ is simple although φ is not expansive. By using Lemma 3.29-Lemma 3.31 it is easy to construc matrices $Q \in M_n(\mathbb{Z})$ such that this happens for the shift on the corresponding solenoid $S_{\mathbb{Q}}$. $Q = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is such an example. □

6.2. The heteroclinic algebra of an expansive automorphism of a compact group

Let G be a compact group and $\varphi : G \to G$ an expansive automorphism of G. It follows from Lemma 4.3 that φ is locally expansive on post-periodic points. It is the purpose of this section to investigate the structure of the heteroclinic C^*-algebra $B_\varphi(G)$ is this case.

6.2.1. The shift of a solenoid. Let $Q \in Gl_n(\mathbb{Q})$ and consider the corresponding solenoid S_Q, cf. (3.10). We seek first to determine the unstable manifold of 0, i.e. $W^u(0)$, for the shift σ_Q acting on S_Q; not only as a set, but as a locally compact space in the Wagoner topology. Let

$$K_Q = \left\{ (z_i)_{i=1}^\infty \in (\mathbb{Z}^n)^{\mathbb{N}} : \ Q^j z_1 + Q^{j-1} z_2 + \cdots + Q z_j \in \mathbb{Z}^n \ \forall j \in \mathbb{N} \right\}$$

and

$$D_Q = (\mathbb{Z}^n)^{\mathbb{N}} / K_Q.$$

To make D_Q into a compact group, set

$$D_Q^m = (\mathbb{Z}^n)^m / K_Q^m,$$

where

$$K_Q^m = \left\{ (z_i)_{i=1}^m \in (\mathbb{Z}^n)^m : \ Q^j z_1 + Q^{j-1} z_2 + \cdots + Q z_j \in \mathbb{Z}^n \ \forall j \in \{1, 2, \ldots, m\} \right\}.$$

Then D_Q^m is a finite group; indeed, D_Q^m is a quotient of $\mathbb{Z}^{nm}/L^m\mathbb{Z}^{nm}$ when $L \in \mathbb{N}$ is so large that LQ has integer entries. The map

$$(z_1, z_2, \ldots, z_m) \mapsto (z_1, z_2, \ldots, z_{m-1})$$

induces a homomorphism $p_{m-1} : D_Q^m \to D_Q^{m-1}$. The inverse limit group

$$\operatorname{proj\,lim} \left(D_Q^m, p_{m-1} \right)$$

of the sequence

$$D_Q^1 \xleftarrow{p_1} D_Q^2 \xleftarrow{p_2} D_Q^3 \xleftarrow{p_3} D_Q^4 \xleftarrow{p_4} \cdots$$

is then a compact group in the topology inherited from the product topology of $\prod_{j=1}^\infty D_Q^j$. The maps $(\mathbb{Z}^n)^{\mathbb{N}} \to D_Q^m$ arising from the projection to the first m coordinates fit together to give an isomorphism

$$D_Q \to \operatorname{proj\,lim} \left(D_Q^m, p_{m-1} \right),$$

and we equip D_Q with the topology coming from this identification. Define $T : (\mathbb{Z}^n)^{\mathbb{N}} \to (\mathbb{Z}^n)^{\mathbb{N}}$ such that

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots).$$

This is an injection and since $Tz \in K_Q$ if and only if $z \in K_Q$ we obtain from it an embedding $T : D_Q \to D_Q$. Note that T is continuous on D_Q and that $T(D_Q)$ is open in D_Q. It follows that the inductive limit group

$$\mathcal{D}_Q = \varinjlim (D_Q, T)$$

of the stationary system

$$D_Q \xrightarrow{T} D_Q \xrightarrow{T} D_Q \xrightarrow{T} \cdots$$

is a locally compact topological group in the inductive limit topology.

Let $\mathbb{R}^n = \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S}$ be the decomposition obtained by applying Lemma 3.17 to Q. Let $i \in \mathbb{Z}$ and define a homomorphism $\Psi_i : \mathcal{U} \times D_Q \to S_Q$ such that

$$\begin{aligned}&\Psi_i\left(u, (z_1, z_2, z_3, \dots) + K_Q\right) \\ &\quad = \begin{cases} \rho\left(Q^j u\right), & j \leq i \\ \rho\left(Q^j u + Q^{j-i} z_1 + Q^{j-i-1} z_2 + \dots + Q z_{j-i}\right), & j \geq i+1. \end{cases}\end{aligned}$$

The infinite commuting diagram

$$\begin{tikzcd}
\mathcal{U} \times D_Q \arrow[r, "\mathrm{id}_{\mathcal{U}} \times T"] \arrow[dr, "\Psi_0"'] & \mathcal{U} \times D_Q \arrow[r, "\mathrm{id}_{\mathcal{U}} \times T"] \arrow[d, "\Psi_{-1}"'] & \mathcal{U} \times D_Q \arrow[r, "\mathrm{id}_{\mathcal{U}} \times T"] \arrow[dl, "\Psi_{-2}"'] & \cdots \arrow[dll, "\cdots"] \\
 & S_Q & &
\end{tikzcd}$$

gives rise to a homomorphism

$$\iota : \mathcal{U} \times \mathcal{D}_Q \to S_Q.$$

LEMMA 6.8. *ι is an isomorphism of topological groups from $\mathcal{U} \times \mathcal{D}_Q$ onto $W^u(0)$ equipped with the Wagoner topology.*

PROOF. Clearly, $\iota(\mathcal{U} \times \mathcal{D}_Q) \subseteq W^u(0)$ and it follows then from Lemma 3.18 that $\iota(\mathcal{U} \times \mathcal{D}_Q) = W^u(0)$.

Injectivity of ι: If $(u, (z_1, z_2, \dots) + K_Q) \in \mathcal{U} \times D_Q$ is send to 0 under Ψ_i for some $i \leq 0$, we have in particular that $\rho(Q^j u) = 0$, i.e. that $Q^j u \in \mathcal{U} \cap \mathbb{Z}^n$, for all $j \leq i$. Since $\|Q^j u\|$ converges to 0 as j goes to $-\infty$ there is a $j \leq i$ such that $\|Q^j u\| < 1$, forcing the conclusion that $u = 0$. Once this is established it is clear that $(z_1, z_2, \dots) \in K_Q$.

To show that ι is continuous it suffices to establish the continuity of each Ψ_i, $i \leq 0$. Let $\Omega \subseteq W^u(0)$ be an open subset, and $(u, (z_1, z_2, \dots) + K_Q) \in \mathcal{U} \times D_Q$ an element of $\Psi_i^{-1}(\Omega)$. Set $\xi = \Psi_i((u, (z_1, z_2, \dots) + K_Q))$. It follows from Lemma 4.6 and Lemma 4.3 that there is an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that

$$\left\{(x_j)_{j \in \mathbb{Z}} \in S_Q : \ d_0\left(x_j, \rho\left(Q^j u\right)\right) \leq \epsilon, \ j \leq i, \text{ and } d_0(x_j, \xi_j) \leq \epsilon, \ -N \leq j \leq N\right\}$$

is a subset of Ω. By Lemma 3.17 there is a $\delta > 0$ such that $\|Q^j u - Q^j u'\| \leq \epsilon$ for all $j \leq N$ when $u' \in \mathcal{U}$ and $\|u - u'\| < \delta$. Then the set V consisting of the elements $(u', (z_1', z_2', \dots) + K_Q) \in \mathcal{U} \times D_Q$ such that $\|u - u'\| < \delta$ and

$$\left(z_1 - z_1', z_2 - z_2', \dots, z_{N-i} - z_{N-i}'\right) \in K_Q^{N-i}$$

is open in $\mathcal{U} \times D_Q$ and $\xi \in V \subseteq \Psi_i^{-1}(\Omega)$.

To complete the proof it suffices to show that ι is open, and hence that each $\Psi_i, i \leq 0$, is open. To this end consider $(u, (z_1, z_2, \dots) + K_Q) \in \mathcal{U} \times D_Q$, an $\epsilon > 0$ and an $m \in \mathbb{N}$. Set $U = U_1 \times U_2$, where $U_1 = \{u' \in \mathcal{U} : \ \|u - u'\| < \epsilon\}$ and

$$U_2 = \left\{(z_1', z_2', \dots) + K_Q \in D_Q : (z_1 - z_1', z_2 - z_2', \dots, z_m - z_m') \in K_Q^m\right\}.$$

Such sets U form a base for the topology of $\mathcal{U} \times D_Q$ so it suffices to show that $\Psi_i(U)$ is open in $W^u(0)$. Let $u' \in U_1$ and $(z_1', z_2', \dots) + K_Q \in U_2$. If $x \in W^u(0)$ is sufficiently close to $\Psi_i(u', (z_1', z_2', \dots) + K_Q)$ in $W^u(0)$, it follows from Lemma 3.18 and Lemma 4.6 that there is an element $v \in \mathcal{U}$ such that $\|u' + v - u\| < \epsilon$ and $x_k = \rho\left(Q^k u + Q^k v\right)$ for all $k \leq i$. If we assume that x is sufficiently close to $\Psi_i(u', (z_1', z_2', \dots) + K_Q)$ we get that $x_{i+1} = \rho(Q^{i+1}u' + Q^{i+1}v + Qz_1'')$ for some $z_1'' \in \mathbb{Z}^n$ and that $Q^{i+1}u' + Q^{i+1}v + Qz_1'' + z$ is very close to $Q^{i+1}u' + Qz_1'$ for some $z \in \mathbb{Z}^n$. If this approximation is close enough, so that also v is small enough, we conclude that $Qz_1'' + z = Qz_1'$, i.e. that $Q(z_1'' - z_1') \in \mathbb{Z}^n$. If only x is sufficiently close to $\Psi_i(u', (z_1', z_2', \dots) + K_Q)$ in $W^u(0)$ we can repeat this argument m times to conclude that there are elements $z_1'', z_2'', \dots, z_m'' \in \mathbb{Z}^n$ such that $x_{i+j} = \rho\left(Q^{i+j}u' + Q^{i+j}v + Q^j z_1'' + Q^{j-1}z_2'' + \dots + Qz_j''\right)$ and $Q^j(z_1'' - z_1') + Q^{j-1}(z_2'' - z_2') + \dots + Q\left(z_j'' - z_j'\right) \in \mathbb{Z}^n$ for all $j = 1, 2, \dots, m$. This shows that $x \in \Psi_i(U)$ if x is sufficiently close to $\Psi_i(u', (z_1', z_2', \dots) + K_Q)$ in $W^u(0)$. □

REMARK 6.9. We are now in position to show that the homoclinic group of the shift of a solenoid can contain torsion. Let T_Q be the subgroup of

$$\oplus_{\mathbb{N}} \mathbb{Z}^n = \left\{(z_i) \in (\mathbb{Z}^n)^{\mathbb{N}} : \ z_i = 0 \text{ for all except finitely many } i\right\}$$

consisting of the elements with the property that.

$$\sum_{i=1}^{\infty} Q^{k+1-i} z_i \in \mathbb{Z}^n$$

for all large k. Then $\Psi_0(0, z) \in \Delta_{\sigma_Q}$ and it follows from Lemma 6.8 that the map $z \mapsto \Psi_0(0, z)$ puts a copy of T_Q/K_Q into Δ_{σ_Q}. Note that T_Q/K_Q is a torsiongroup. To give an example where this group is non-trivial, let

$$Q = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 2 \end{pmatrix}.$$

Set $z_1 = (0, 1) \in \mathbb{Z}^2$. Then $(z_1, 0, 0, 0, \dots) \in T_Q$ and its image in T_Q/K_Q has order two. □

LEMMA 6.10. *Assume that Q is hyperbolic. Then $P\Delta_{\sigma_Q}$ is dense in W_{S_Q,σ_Q}.*

PROOF. Let $p \in \operatorname{Per} S_Q$ and let $q_j \in \mathbb{R}^n$ be vectors such that $p_j = \rho(q_j)$ for all $j \in \mathbb{Z}$. Let $u \in \mathcal{U}$ and consider an element $x \in W_{S_Q,\sigma_Q}$ such that $x_j = \rho(q_j + Q^j u), j \leq 0$. By Lemma 3.18 an arbitrary element y of $W^u(p)$ has the form $\sigma_Q^k(x)$ for some $k \in \mathbb{N}$ and some x of this form. Since σ_Q restricts to a homeomorphism of W_{S_Q,σ_Q} by Lemma 4.8 it suffices to approximate such an element x of W_{S_Q,σ_Q} by an element from $P\Delta_{\sigma_Q}$. Let $d \in \mathbb{N}$ and $\epsilon > 0$ be given. Let $m \in \mathbb{N}$ be such that $mQ \in M_n(\mathbb{Z})$. Since S_Q is divisible by Lemma 3.21 there is an element $y \in S_Q$ such that $m^d y = x$. Then $y_0 = \rho(b_0), y_1 = \rho(Qb_0 + Qz_1), \dots, y_d =$

$\rho\left(Q^d b_0 + Q^d z_1 + Q^{d-1} z_2 + Q^{d-2} z_3 + \cdots + Q z_d\right)$ for some $b_0 \in \mathbb{R}^n$ and some $z_1, z_2, \ldots, z_d \in \mathbb{Z}^n$. Set $b = m^d b_0$ and note that

$$q_0 + u = b + v \tag{6.1}$$

for some $v \in \mathbb{Z}^n$. Furthermore,

$$x_j = \rho\left(Q^j b\right), j = 0, 1, 2, \ldots, d. \tag{6.2}$$

As argued in the proof of Lemma 3.23 the group $P_{\mathcal{U}}\big(\big\langle \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n)\big\rangle\big)$ is dense in $\mathcal{U}$, and hence so is the set

$$P_{\mathcal{U}}\Big(\Big\langle \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n)\Big\rangle + v - q_0\Big) = P_{\mathcal{U}}\Big(\Big\langle \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n)\Big\rangle\Big) + P_{\mathcal{U}}(v - q_0).$$

Since $P_{\mathcal{U}} + P_{\mathcal{S}} = 1$ it follows that there is an element $\xi \in \bigcup_{j \leq -d} Q^j(\mathbb{Z}^n)$ and an element $s \in \mathcal{S}$ such that $\|u - u'\|$ is as small as we like, where $u' = s + v - q_0 + \xi$. To begin to define the desired approximating element, set $x'_j = \rho(q_j + Q^j u')$ when $j \leq 0$. It follows from Lemma 3.17 that $\sup_{j \leq 0} d_0\left(x_j, x'_j\right)$ can be made arbitrarily small if only $\|u - u'\|$ is small enough. We set $x'_j = \rho\left(Q^j q_0 + Q^j u' - Q^j v\right), j = 1, 2, \ldots, d$. It follows from (6.1) and (6.2) that $\sup_{1 \leq j \leq d} d_0\left(x_j, x'_j\right)$ can be made arbitrarily small if only $\|u - u'\|$ is small enough. To define $x'_j, j > d$, note that there are $N \in \mathbb{N}$ and elements $z_1, z_2, \ldots, z_N \in \mathbb{Z}^n$ such that $Q^d \xi = z_1 + Q^{-1} z_2 + \cdots + Q^{-N} z_N$ and that $Q^d q_0 + Q^d u' - Q^d v = Q^d s + q_0 - q_0 + Q^d \xi = Q^d s + z_1 + Q^{-1} z_2 + \cdots + Q^{-N} z_N$. Hence

$$x'_d = \rho\left(Q^d s + Q^{-1} z_2 + Q^{-2} z_3 + \cdots + Q^{-N} z_N\right)$$

and we set

$$x'_{d+j} = \rho\left(Q^{d+j} s + Q^{-1} z_{j+2} + Q^{-2} z_{j+3} + \cdots + Q^{-N+j} z_N\right),$$

$j = 1, 2, \ldots, N$, and $x'_{d+N+j} = \rho\left(Q^{d+N+j} s\right), j \geq 0$. Then we have $x' \in P\Delta_{\sigma_Q}$ and $\sup_{j \leq d} d_0\left(x'_j, x_j\right)$ is as small as we need if only $\|u - u'\|$ is chosen small enough. By Lemma 4.6 this shows that we have obtained the desired approximation of x in $W^u(0)$. $\square$

LEMMA 6.11. *Assume that Q is hyperbolic. Then $B_{\sigma_Q}(S_Q)$ is simple, stable and has a lower-semicontinuous densely defined trace which is unique up to scalar multiplication.*

PROOF. This follows by combining Theorem 6.5 and Lemma 6.10 with Corollary B.11 from Appendix B. $\square$

REMARK 6.12. We have based the proof of Lemma 6.11 on the crossed product description of $B_{\sigma_Q}(S_Q)$, but it seems appropriate to point out that there is another proof which uses work of Brenken, Putnam and Spielberg: (S_Q, σ_Q) is a mixing Smale space by [**Bre**] and it follows then from [**PS**] that its stable algebra is simple. Thus $B_{\sigma_Q}(S_Q)$ is simple and stable by Theorem 4.17. That $B_{\sigma_Q}(S_Q)$ has an essentially unique lower-semicontinuous densely defined trace can be deduced from Theorem 3.1 of [**Pu1**] and the uniqueness of the trace state of the homoclinic algebra $A_{\sigma_Q}(S_Q)$, which follows from Corollary 3.9. $\square$

For each $i \in \mathbb{N}$, set $H_i = (\Psi_{-i})^{-1}\left(\Delta_{\sigma_Q}\right) \subseteq \mathcal{U} \times D_Q$. Note that $\mathrm{id}_{\mathcal{U}} \times T(H_i) \subseteq H_{i+1}$.

LEMMA 6.13. *There is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of hereditary C^*-subalgebras of $A_{\sigma_Q}(S_Q, W^u(0))$ and $*$-isomorphisms $\psi_n : A_n \to C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_n$ such that*

$$\begin{tikzcd} A_n \arrow[r, "\psi_n"] \arrow[d, hook] & C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_n \arrow[d, hook] \\ A_{n+1} \arrow[r, "\psi_{n+1}"] & C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_{n+1} \end{tikzcd} \tag{6.3}$$

commutes, and $A_{\sigma_Q}(S_Q, W^u(0)) = \overline{\bigcup_n A_n}$.

PROOF. It follows from Lemma 6.8 that $\Psi_{-i}(\mathcal{U} \times D_Q)$ is open in $W^u(0)$ and that $W^u(0) = \bigcup_{i\in\mathbb{N}} \Psi_{-i}(\mathcal{U} \times D_Q)$. By Lemma 1.24 there is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ of hereditary C^*-subalgebras of $A_{\sigma_Q}(S_Q, W^u(0))$ such that $A_{\sigma_Q}(S_Q, W^u(0)) = \overline{\bigcup_n A_n}$ and $A_n \simeq A_{\sigma_Q}(S_Q, \Psi_{-n}(\mathcal{U} \times D_Q))$. It follows from Lemma 6.4 that there are commuting diagrams

$$\begin{tikzcd} R_{\sigma_Q}(S_Q, \Psi_{-n}(\mathcal{U} \times D_Q)) \arrow[r] \arrow[d, hook, "\mathrm{id}_{\mathcal{U}} \times T"] & H_n \times (\mathcal{U} \times D_Q) \arrow[d, hook, "\mathrm{id}_{\mathcal{U}} \times T"] \\ R_{\sigma_Q}(S_Q, \Psi_{-n-1}(\mathcal{U} \times D_Q)) \arrow[r] & H_{n+1} \times (\mathcal{U} \times D_Q) \end{tikzcd}$$

of étale equivalence relations such that the horizontal arrows represent topological groupoid isomorphisms. By combining this diagram with the diagram from Lemma1.24 we obtain the diagram (6.3). □

Let $\tilde{p}_m : \mathcal{U} \times D_Q \to \mathcal{U} \times D_Q^m$ be the map

$$\tilde{p}_m(u, (z_1, z_2, z_3, \dots) + K_Q) = \left(u, (z_1, z_2, \dots, z_m) + K_Q^m\right).$$

Let $m_1 < m_2 < m_3 < \dots$ be a sequence in $\mathbb{N}$. There is then a commuting diagram

$$\begin{tikzcd} C_0\left(\mathcal{U} \times D_Q^{m_1}\right) \rtimes_{\tau\circ\tilde{p}_{m_1}} H_1 \arrow[r, "j_{m_1}"] \arrow[d] & C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_1 \arrow[d] \\ C_0\left(\mathcal{U} \times D_Q^{m_2}\right) \rtimes_{\tau\circ\tilde{p}_{m_2}} H_2 \arrow[r, "j_{m_2}"] \arrow[d] & C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_2 \arrow[d] \\ C_0\left(\mathcal{U} \times D_Q^{m_3}\right) \rtimes_{\tau\circ\tilde{p}_{m_3}} H_3 \arrow[r, "j_{m_3}"] \arrow[d] & C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_3 \arrow[d] \\ \vdots & \vdots \end{tikzcd} \tag{6.4}$$

Here the vertical maps are induced by $\mathrm{id}_{\mathcal{U}} \times T$ in the natural way, by use of Lemma B.1, while the horizontal maps, the j_{m_i}'s, are induced by the p_{m_i}'s. It follows from Lemma 6.13 that the inductive limit of the right column in (6.4) is a copy of $A_{\sigma_Q}(S_Q, W^u(0))$. Note that $j_k : C_0\left(\mathcal{U} \times D_Q^k\right) \rtimes_{\tau\circ p_k} H_n \to C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_n$ is injective for each k, n, and that

$$C_0(\mathcal{U} \times D_Q) \rtimes_\tau H_n = \overline{\bigcup_{k=1}^{\infty} j_k\left(C_0\left(\mathcal{U} \times D_Q^k\right) \rtimes_{\tau\circ\tilde{p}_k} H_n\right)}$$

for each n. Since H_k is a countable abelian group it is the union of an increasing sequence of finitely generated abelian groups. In fact, by Lemma 3.20 each H_i is the union of an increasing sequence of finitely generated abelian groups of rank no more than n. Therefore we can choose finitely generated subgroups H'_k of H_k, all or rank $\leq n$, such that $\Delta_{\sigma_Q} = \bigcup_{i \leq 0} \Psi_i(H_i) = \bigcup_{i \leq 0} \Psi_i(H'_i)$. By using the separability of $A_{\sigma_Q}(S_Q, W^u(0))$, we see that if the sequences $(m_i)_{i=1}^{\infty}$ and $H'_i, i \in \mathbb{N}$, increase fast enough the diagram (6.4) will give us an isomorphism

$$A_{\sigma_Q}(S_Q, W^u(0)) \simeq \varinjlim_k C_0\left(\mathcal{U} \times D_Q^{m_k}\right) \rtimes_{\tau \circ \tilde{p}_{m_k}} H'_k. \tag{6.5}$$

Lemma 6.14. *Assume that Q is hyperbolic. It follows that $B_{\sigma_Q}(S_Q)$ is $*$-isomorphic to an inductive limit $\varinjlim_k A_k$, where each A_k has the form*

$$A_k \simeq \mathbb{K} \otimes \mathbb{C}^{j_k} \otimes A'_k,$$

and A'_k is a special non-commutative torus of rank (n_k, m_k) with $n_k + m_k \leq n$.

Proof. Let $x \in W^u(p)$. It follows from Lemma 6.10 and Corollary 2.14 that $B_{\sigma_Q}(S_Q)$ is stably isomorphic to $A_{\sigma_Q}(S_Q, W^u(0))$. Hence

$$B_{\sigma_Q}(S_Q) \simeq \mathbb{K} \otimes A_{\sigma_Q}(S_Q, W^u(0)) \tag{6.6}$$

by Lemma 6.11. We seek an inductive limit decomposition of $A_{\sigma_Q}(S_Q, W^u(0))$, and the point of departure for this is (6.5).

Let $q : \mathcal{U} \times D_Q \to \mathcal{U}$ be the projection. It follows from Lemma 6.10 and Lemma 6.8 that $\bigcup_{i=1}^{\infty} q(H'_i)$ is dense in $\mathcal{U}$. Hence, for some k_0, $\mathcal{U} = \operatorname{Span} q\left(H'_{k_0}\right)$. Let $L \in \mathbb{N}$ be such that LQ has integer entries only. Then

$$L^j q\left(H'_{k_0}\right) \subseteq \left\{u \in \mathcal{U} : \ (u,0) \in \tilde{p}_j\left(H'_{k_0}\right)\right\}$$

for all $j \in \mathbb{N}$. Hence

$$\mathcal{U} = \operatorname{Span}\left\{u \in \mathcal{U} : \ (u,0) \in \tilde{p}_j(H'_k)\right\} \tag{6.7}$$

for all j and all $k \geq k_0$. Let $k \geq k_0$. It follows from Lemma B.3 that

$$\left[C_0\left(\mathcal{U} \times D_Q^{m_k}\right) \rtimes_{\tau \circ \tilde{p}_{m_k}} H'_k\right] \otimes \mathbb{K} \simeq \left[C_0\left(\mathcal{U} \times D_Q^{m_k}\right) \rtimes_{\tau} \tilde{p}_{m_k}(H'_k)\right] \otimes \mathbb{C}^{l_k} \otimes C\left(\mathbb{T}^{n_k}\right) \otimes \mathbb{K}$$

for some $l_k, n_k \in \mathbb{N}$ with $n_k + \operatorname{Rank} \tilde{p}_{m_k}(H'_k) \leq \operatorname{Rank} H'_k \leq n$. By Lemma B.8,

$$C_0\left(\mathcal{U} \times D_Q^{m_k}\right) \rtimes_{\tau} \tilde{p}_{m_k}(H'_k) \simeq \mathbb{C}^{l'_k} \otimes \left[C_0(\mathcal{U}) \rtimes_{\tau} H'\right]$$

where $H' \subseteq \mathcal{U}$ is the image of $\tilde{p}_{m_k}(H'_k)$ under the projection $\mathcal{U} \times D_Q^{m_k} \to \mathcal{U}$ and $l'_k \leq \# D_Q^{m_k}$. Note that H' spans $\mathcal{U}$ by (6.7). By Theorem B.12 of Appendix B, $C_0(\mathcal{U}) \rtimes_{\tau} H' \simeq \mathbb{K} \otimes A'_k$, where A'_k is a special non-commutative torus of rank (n'_k, m'_k), where $n'_k = \dim \mathcal{U} \leq n$ and $m'_k = \operatorname{Rank} \tilde{p}_{m_k}(H'_k) - \dim \mathcal{U}$. Hence

$$C_0\left(\mathcal{U} \times D_Q^{m_k}\right) \rtimes_{\tau \circ \tilde{p}_{m_k}} H'_k \simeq \mathbb{K} \otimes \mathbb{C}^{j_k} \otimes A'_k, \tag{6.8}$$

where $j_k = l_k l'_k$. Since $C\left(\mathbb{T}^{n_k}\right) \otimes A'_k$ is a special non-commutative torus of rank $(n_k + n'_k, m'_k)$ the conclusion of the lemma follows from (6.5), (6.6) and (6.8). $\square$

Theorem 6.15. *Assume that Q is hyperbolic. Then $B_{\sigma_Q}(S_Q)$ is a simple stable AT-algebra of real rank zero with a densely defined lower semi-continuous trace which is unique up to scalar multiplication.*

PROOF. It follows from Lemma 6.14 that there is a sequence B_i of special non-commutative tori of ranks (n_i, m_i), where $n_i + m_i \leq n$, sequences $k_i, l_i \in \mathbb{N}$ and projections $p_i \in M_{k_i}\left(\mathbb{C}^{l_i} \otimes B_i\right)$ such that $B_{\sigma_Q}(S_Q)$ is the limit of the sequence

$$p_1 M_{k_1}\left(\mathbb{C}^{l_1} \otimes B_{i_1}\right) p_1 \longrightarrow p_2 M_{k_2}\left(\mathbb{C}^{l_2} \otimes B_{i_2}\right) p_2 \longrightarrow \cdots \tag{6.9}$$

To show that $B_{\sigma_Q}(S_Q)$ is approximately divisible and has real rank zero we can use [**BKR**] and [**G**] in combination with (6.9) in the same way as in the proof of Proposition 3.27: Either infinitely many of the B_i's are non-rational non-commutative tori in which case $B_{\sigma_Q}(S_Q)$ is approximately divisible with real rank zero by [**BKR**] or else infinitely many of the B_i are $*$-isomorphic to $C\left(\mathbb{T}^{j_i}\right)$ for some $j_i \leq n$ and then Corollary 6.7 of [**G**] shows that $B_{\sigma_Q}(S_Q)$ is an AT-algebra. Being simple and stable by Lemma 6.11 it is then $*$-isomorphic to $\mathbb{K} \otimes B$, where B is a unital simple AT-algebra. B is approximately divisible by [**Ell1**]. Since the lower semi-continuous trace of $B_{\sigma_Q}(S_Q)$ is essentially unique by Lemma 6.11 B has exactly one trace state and it follows then from Theorem 1.4 of [**BKR**] that B has real rank zero.

All in all we conclude that $B_{\sigma_Q}(S_Q)$ is approximately divisible with real rank zero in all cases. Let $p \in B_{\sigma_Q}(S_Q)$ be a projection. Then $pB_{\sigma_Q}(S_Q)\, p$ is unital and approximately divisible with real rank zero by [**BKR**]. It follows that $pB_{\sigma_Q}(S_Q)\, p$ has all the properties 1)–7) which were stated for $A_{\sigma_Q}(S_Q)$ in Proposition 3.27. By Lemma 6.14 and Proposition B.7 of Appendix B $B_{\sigma_Q}(S_Q)$ is locally AH in the sense of [**Lin**] and then the same is the case of $pB_{\sigma_Q}(S_Q)\, p$. By using the work of Lin and Phillips as in the proof of Theorem 3.28 we conclude that $pB_{\sigma_Q}(S_Q)\, p$ is AT. It follows that $B_{\sigma_Q}(S_Q) \simeq \mathbb{K} \otimes \left(pB_{\sigma_Q}(S_Q)\, p\right)$ is AT. □

By using Theorem 3.35 we can extend Theorem 6.15 to cover general expansive group automorphisms. For this we need the following

PROPOSITION 6.16. *Let (X, d, φ) and (X', d', φ') be locally expansive on the post-periodic points. Let D be the metric on $X \times X'$ such that $D\left((x, x'), (y, y')\right) = \max\left\{d(x, y), d'\left(x', y'\right)\right\}$. Then $(X \times X', D, \varphi \times \varphi')$ is locally expansive on post-periodic points and*

$$B_{\varphi \times \varphi'}\left(X \times X'\right) \simeq B_{\varphi}(X) \otimes B_{\varphi'}\left(X'\right). \tag{6.10}$$

PROOF. Let $p \in \operatorname{Per} X, p' \in \operatorname{Per} X'$. Then

$$W^u\left((p, p'), \min\left\{\epsilon_p, \epsilon_{p'}\right\}\right) \subseteq W^u\left(p, \epsilon_p\right) \times W^u\left(p', \epsilon_{p'}\right).$$

Since $W^u\left((p, p'), \min\left\{\epsilon_p, \epsilon_{p'}\right\}\right)$ is closed in $X \times X'$ it follows from this inclusion that it is in fact compact, and it is then clear that $(X \times X', D, \varphi \times \varphi')$ is expansive on post-periodic points. It follows easily from Lemma 4.6 that the obvious identification

$$W_{X \times X', \varphi \times \varphi'} = W_{X, \varphi} \times W_{X', \varphi'}$$

is a homeomorphism and then the isomorphism (6.10) follows from Proposition 1.23. □

THEOREM 6.17. *Let $\psi : G \to G$ be an expansive automorphism of the compact group G. Assume that ψ is mixing. It follows that the heteroclinic algebra $B_\psi(G)$ is a simple stable AT-algebra of real rank zero with a lower-semicontinuous densely defined trace which is unique up to scalar multiplication.*

PROOF. It follows from Theorem 3.35 that (G, ψ) is conjugate to

$$(\Sigma_m \times S_Q, \sigma \times \sigma_Q),$$

where (Σ_m, σ) is the full m-shift and (S_Q, σ_Q) is a solenoid corresponding to an hyperbolic matrix $Q \in M_n(\mathbb{Q})$. The finite group factor in Theorem 3.35 is trivial because we assume that ψ is mixing. It follows then from Proposition 6.16 that $B_\psi(G) \simeq B_\sigma(\Sigma) \otimes B_{\sigma_Q}(S_Q)$. It follows straightforwardly from Proposition 4.13 that $B_\sigma(\Sigma_m)$ is a copy of the UHF-algebra of Glimm-type m^∞ tensored with $\mathbb{K}$. The tensor product of an AT-algebra of real rank zero with a UHF-algebra is clearly again AT of real rank zero, and hence $B_\psi(G)$ is AT of real rank zero. The remaining properties of $B_\psi(G)$ can also be deduced from this tensor-product decomposition, but they follow also from Corollary B.11 in Appendix B. □

For a general expansive automorphism of a compact group the heteroclinic algebra is a finite direct sum of the same simple stable AT-algebra of real rank zero with a essentially unique lower semi-continuous trace.

6.2.2. The heteroclinic algebra of a torus automorphism. In the special case where the hyperbolic matrix Q has integer entries, the shift σ_Q of S_Q is the natural invertible extension of the endomorphism of $\mathbb{T}^n$ induced by Q. The structure of the heteroclinic algebra $B_{\sigma_Q}(S_Q)$ simplifies quite a bit in this case. In fact, it follows from Lemma 6.10, Theorem 6.6, Lemma 3.29 and Lemma 3.31 that $B_{\sigma_{\mathbb{Q}}}(S_Q)$ is stably isomorphic to the crossed product

$$C_0(\mathcal{U}) \rtimes_\tau P_{\mathcal{U}}\Big(\bigcup_{j\geq 0} Q^{-j}(\mathbb{Z}^n)\Big).$$

In fact, since S_Q is a Smale space by [**Bre**], it follows from Lemma 4.15 that $B_{\sigma_Q}(S_Q)$ is actually isomorphic to this crossed product. If we specialize further to the case where $Q \in Gl_n(\mathbb{Z})$ we find that

$$B_{\sigma_Q}(S_{\mathbb{Q}}) \simeq C_0(\mathcal{U}) \rtimes_\tau P_{\mathcal{U}}(\mathbb{Z}^n).$$

Note that $P_{\mathcal{U}}(\mathbb{Z}^n) \simeq \mathbb{Z}^n$ since $\mathbb{Z}^n \cap \mathcal{S} = \{0\}$. We obtain therefore the following conclusion from Theorem B.12:

THEOREM 6.18. *Let $Q \in Gl_n(\mathbb{Z})$ be hyperbolic. It follows that the heteroclinic algebra of the corresponding automorphism of $\mathbb{T}^n$ is a stabilized special non-commutative torus of rank $(k, n-k)$, where $k = \operatorname{Dim}\{x \in \mathbb{R}^n : \lim_{j\to-\infty} Q^j x = 0\}$.*

In general a non-commutative torus of rank n is defined from an anti-symmetric real matrix $\theta = (\theta_{i,j}) \in M_n(\mathbb{R})$ as the universal C^*-algebra generated by n unitaries $u_1, u_2, \dots, u_n$ satisfying the relation

$$u_i u_j = e^{2\pi\sqrt{-1}\theta_{i,j}} u_j u_i$$

for all i, j. For a special non-commutative torus of rank $(k, n-k)$ the matrix θ has the from

$$\theta = \begin{pmatrix} 0 & \theta_0 \\ -\theta_0 & 0 \end{pmatrix}$$

where θ_0 is a real $k \times (n-k)$-matrix. For a hyperbolic $Q \in Gl_n(\mathbb{Z})$ the matrix θ_0 depends on the position of the subspace $\{x \in \mathbb{R}^n : \lim_{j\to-\infty} Q^j x = 0\}$ in $\mathbb{R}^n$ relative to $\mathbb{Z}^n \subseteq \mathbb{R}^n$.

To illustrate how our results can be used we shall now apply them to give examples of hyperbolic automorphisms of the two-torus with the property that the corresponding heteroclinic algebras are not isomorphic to the heteroclinic algebra of the inverse automorphism.

6.2.3. Expansive automorphisms of the two-torus. Let $Q = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in Gl_2(\mathbb{Z})$ be hyperbolic, and let φ_Q be the corresponding expanding automorphism of the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. There are then real numbers $\alpha, \beta \in \mathbb{R}$ such that $(1,\alpha)$ is an eigenvector for Q corresponding to the eigenvalue of absolute value > 1 while $(1,\beta)$ is an eigenvector for Q corresponding to the eigenvalue of absolute value < 1. Then

$$(1,0) = \frac{\beta}{\beta-\alpha}(1,\alpha) + \frac{\alpha}{\alpha-\beta}(1,\beta),$$

and

$$(0,1) = \frac{1}{\alpha-\beta}(1,\alpha) + \frac{1}{\beta-\alpha}(1,\beta),$$

so it follows from Theorem 6.18 and its proof that $B_{\varphi_Q}(\mathbb{T}^2)$ is $*$-isomorphic to $C_0(\mathbb{R}) \rtimes_\alpha \mathbb{Z}^2$, where $\alpha_{(1,0)}$ is translation by 1 and $\alpha_{(0,1)}$ is translation by $\left(\frac{\beta}{\beta-\alpha}\right)^{-1} \frac{1}{\alpha-\beta} = -\beta^{-1}$. Thus, by Theorem B.12, $B_{\varphi_Q}(\mathbb{T}^2) \simeq \mathbb{K} \otimes R_{-\beta^{-1}}$, where $R_{-\beta^{-1}}$ is the irrational rotation C^*-algebra obtained from rotation by $e^{2\pi i(-\beta^{-1})}$. Now we recall the result of M. Rieffel, [**Ri2**], on stable isomorphism of irrational rotation C^*-algebras:

THEOREM 6.19. *(Rieffel) Let $\alpha, \beta \in \mathbb{R}\backslash\mathbb{Q}$. Then the corresponding irrational rotation C^*-algebras R_α and R_β are stably isomorphic if and only if there is a*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{Z})$$

such that $\frac{a\alpha+b}{c\alpha+d} = \beta$.

Now this result can of course also be deduced from the more general results of Phillips, [**Ph2**]. It follows, in particular, that $R_{-\beta^{-1}}$ is stably isomorphic to R_β. Thus we see that

$$B_{\varphi_Q}\left(\mathbb{T}^2\right) \simeq \mathbb{K} \otimes R_\beta. \tag{6.11}$$

Furthermore, it follows from Theorem 6.19 that when $Q' \in Gl_2\mathbb{Z})$ is another hyperbolic automorphism then

$$B_{\varphi_Q}\left(\mathbb{T}^2\right) \simeq B_{\varphi_{Q'}}\left(\mathbb{T}^2\right) \Leftrightarrow \frac{a\beta+b}{c\beta+d} = \beta' \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{Z}),$$

when $(1,\beta)$ and $(1,\beta')$ are eigenvectors of Q and Q', respectively, corresponding to the eigenvalue of smallest numerical value.

Calculations of β: Let $\chi_Q(t) = t^2 + Bt + C$ be the characteristic polynomium of Q. Set $D = B^2 - 4C$. The roots of χ_Q are $\frac{-B\pm\sqrt{D}}{2}$. It follows that

$$a_0 + b_0\beta = \frac{-B-\sqrt{D}}{2},$$

and hence that

$$\beta = \frac{-B-\sqrt{D}-2a_0}{2b_0}.$$

(If $b_0 = 0$, the roots of χ_Q are both integers which is not possible since Q is hyperbolic.) Similarly, we find that

$$\alpha = \frac{-B + \sqrt{D} - 2a_0}{2b_0}.$$

(Note that $\sqrt{D}$ must be irrational since otherwise the image of $(1, \alpha)$ in $\mathbb{T}^2$ will be periodic under φ_Q. But it is also asymptotic to 0 under the iteration of φ_Q, and hence it must be 0 in $\mathbb{T}^2$, i.e. $\alpha \in \mathbb{Z}$. For the same reason we have that $\beta \in \mathbb{Z}$, which is not possible since Q is hyperbolic.) By using Theorem 6.19 we see that $B_{\varphi_Q}(\mathbb{T}^2)$ and $B_{\varphi_Q^{-1}}(\mathbb{T}^2) = B_{\varphi_{Q^{-1}}}(\mathbb{T}^2)$ are isomorphic if and only there is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{Z})$$

such that

$$\frac{a\alpha + b}{c\alpha + d} = \beta. \tag{6.12}$$

Now (6.12) is equivalent to

$$a\alpha + b = \beta (c\alpha + d),$$

or

$$a \left(\frac{-B - 2a_0 + \sqrt{D}}{2b_0} \right) + b = \frac{(B + 2a_0)^2 - D}{4b_0^2} c + \frac{-B - 2a_0 - \sqrt{D}}{2b_0} d.$$

Since $\sqrt{D}$ is irrational this equation holds if and only if

$$a = -d \tag{6.13}$$

and

$$a \frac{-B - 2a_0}{b_0} + b = \frac{(B + 2a_0)^2 - D}{4b_0^2} c.$$

Since $B = -a_0 - d_0$ this can also be written

$$b = \frac{(d_0 - a_0)^2 - D}{4b_0^2} c + a \frac{a_0 - d_0}{b_0}. \tag{6.14}$$

Now note that $\chi_Q(t) = (t - a_0)(t - d_0) - c_0 b_0 = t^2 - (a_0 + d_0)t + a_0 d_0 - c_0 b_0 = t^2 - (a_0 + d_0)t + \operatorname{Det} A$. Hence

$$D = B^2 - 4C = (a_0 + d_0)^2 - 4 \operatorname{Det} Q.$$

Thus (6.14) becomes

$$b = \frac{-b_0 c_0}{b_0^2} c + a \frac{a_0 - d_0}{b_0} = a \frac{a_0 - d_0}{b_0} - \frac{c_0}{b_0} c.$$

Hence we see that $B_{\varphi_Q}(\mathbb{T}^2)$ and $B_{\varphi_Q^{-1}}(\mathbb{T}^2)$ are isomorphic if and only there are integers a, b, c such that

$$-a^2 - bc = \pm 1, \quad b_0 b = (a_0 - d_0)a - c_0 c. \tag{6.15}$$

By Proposition 17 of [**BR1**] there are integer solutions a, b, c to (6.15) if and only of the elements of the projective group $PGl_2(\mathbb{Z}) = Gl_2(\mathbb{Z}) / \{\pm 1\}$ represented by Q and Q^{-1} are conjugate in $PGl_2(\mathbb{Z})$. This should be compared with the fact that φ_Q and $\varphi_{Q^{-1}}$ are conjugate dynamical systems if and only if Q and Q^{-1} are conjugate in $Gl_2(\mathbb{Z})$. There are many cases where it is easy to find integer solutions to (6.15).

For example when $a_0 = d_0$, there is always the solution $a = a_0,\ b = -c_0, c = b_0$, and when $c_0|a_0 - d_0$, we have the solution $a = 1, b = 0, c = \frac{a_0-d_0}{c_0}$. Similarly, when $b_0|a_0 - d_0$ we have the solution $a = 1, b = \frac{a_0-d_0}{c_0}$ and $c = 0$, and when $b_0 = \pm c_0$, we have the solution $a = 0, b = 1, c = \mp 1$. Thus in all these cases the two algebras $B_{\varphi_Q}(\mathbb{T}^2)$ and $B_{\varphi_Q^{-1}}(\mathbb{T}^2)$ are isomorphic. But in fact there are many cases where there are no integer solutions to the equations (6.15). To see this note that there is a systematic way of deciding whether or not there are integer solutions to (6.15) which is a based on results going back to Gauss and described in Section 3.1.2 of [**BR2**]. By using this it can be seen that there are no integer solutions to (6.15) when Q is, for example, any of the following elements of $Gl_2(\mathbb{Z})$:

$$\begin{pmatrix} 11 & 4 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 10 & 3 \\ 7 & 2 \end{pmatrix}, \begin{pmatrix} 17 & 6 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 9 \\ 7 & 16 \end{pmatrix}.$$

According to [**BR2**] the last of these matrices has the the least possible absolute value of the trace among the elements Q of $Sl_2(\mathbb{Z})$ for which $B_{\varphi_Q}(\mathbb{T}^2)$ and $B_{\varphi_Q^{-1}}(\mathbb{T}^2)$ are not isomorphic. The interested reader can easily find more examples by combining the method described in [**BR2**] with an effective calculator to solve quadratic diophantine equations which may by found at

`http://www.alpertron.com.ar/QUAD.HTM`

CHAPTER 7

A dimension group for certain countable state Markov shifts

7.1. Markov shifts with finitely many edge-disjoint doublepaths

We shall now consider a countable state Markov shift coming from a countable oriented graph $\mathbb{G}$ as described in Section 4.3. If $\mathbb{G}$ has both finite out-degree and finite in-degree, the space

$$X_{\mathbb{G}} = \left\{ (e_i)_{i\in\mathbb{Z}} \in \mathbb{E}^{\mathbb{Z}} : \ i\left(e_{i+1}\right) = t\left(e_i\right) \ \forall i \in \mathbb{Z} \right\}$$

will be locally compact in the product topology. The shift σ acts as a uniformly continuous homeomorphism of $X_{\mathbb{G}}$ and it is obviously expansive with respect to the metric (4.16). Thus it gives rise to a relatively expansive system and we can define the corresponding homoclinic algebra $A_\sigma\left(X_{\mathbb{G}}\right)$ which can be shown to be an AF-algebra in essentially the same way it was done for shift spaces in Section 3.1. Likewise the heteroclinic algebra $B_\sigma\left(X_{\mathbb{G}}\right)$ can be defined, even when $\mathbb{G}$ only has finite out-degree, as described in Section 4.3. Both constructions provides us with a dimension group, namely the K_0-group of the resulting AF-algebra. Furthermore, the group $K_0\left(B_\sigma\left(X_{\mathbb{G}}\right)\right)$ generalizes the dimension-group which plays a prominent role in the study of shifts of finite type, see e.g. [**LM**], [**K**]. However, since the dimension group is particularly important and powerfull for the study of the automorphism group of a shift of finite type it is annoying that a shift-commuting homeomorphism of $X_{\mathbb{G}}$ only gives rise to an automorphism of the homoclinic algebra or the heteroclinic algebra when it is uniformly continuous with respect to the metric (4.16). As we shall now show this problem can be resolved for a class of graphs introduced by Michael Schraudner in [**Sch**].

Recall that a *path* γ in $\mathbb{G}$ is an ordered tuple $\gamma = (e_1, e_2, \ldots, e_n)$ (or $\gamma = e_1 e_2 \ldots e_n$) of edges in $\mathbb{G}$ such that $i\left(e_{k+1}\right) = t\left(e_k\right)$ for all $k = 1, 2, \ldots, n-1$. The number of edges, n, in γ is *the length* of γ which we denote by $|\gamma|$, and we set $i(\gamma) = i\left(e_1\right),\ t(\gamma) = t\left(e_n\right)$. A *doublepath*, [**Sch**], is an unordered tuple (γ, γ') of different paths, γ and γ', of the same length in $\mathbb{G}$ such that $i\left(\gamma\right) = i\left(\gamma'\right)$ and $t\left(\gamma\right) = t\left(\gamma'\right)$. A pair of doublepaths, (γ, γ') and (μ, μ'), are *edge-disjoint* when the set of edges making up γ and γ' is disjoint from the set of edges making up μ and μ'. Following [**Sch**] we say that $\mathbb{G}$ has *finitely many pairwise edge-disjoint doublepaths* when every collection of pairwise edge-disjoint doublepaths in $\mathbb{G}$ is finite. At first sight this condition may seem artificial, but as shown by Schraudner in Theorem 2.3 of [**Sch**] the property reflects an important intrinsic property of the countable state Markov shift defined by the graph. Specifically, it follows from Theorem 2.3 of [**Sch**] that $\mathbb{G}$ has finitely many pairwise edge-disjoint doublepaths if and only if the automorphism group of $(X_{\mathbb{G}}, \sigma)$ is countable.

We shall impose the following conditions on $\mathbb{G}$:

i) $\mathbb{G}$ has both finite in-degree and finite out-degree at each vertex,
ii) $\mathbb{G}$ is strongly connected in the sense that for every pair v, w of vertices in $\mathbb{G}$ there is a path γ in $\mathbb{G}$ with $i(\gamma) = v$ and $t(\gamma) = w$, and
iii) $\mathbb{G}$ has finitely many pairwise edge-disjoint doublepaths.

Since $\mathbb{E}$ is countable there is an injection $h : \mathbb{E} \to \mathbb{N}\backslash\{0\}$ and we define a metric d_G on $X_{\mathbb{G}}$ such that

$$d_G\left((e_i)_{i\in\mathbb{Z}}, (e'_i)_{i\in\mathbb{Z}}\right) = \sum_{n\in\mathbb{Z}} 2^{-|n|}\left|h\left(e_n\right)^{-1} - h\left(e'_n\right)^{-1}\right|.$$

This metric is known as *the Gurevich metric*, and it is characterized, up to equivalence, by the fact that it is the restriction to $X_{\mathbb{G}}$ of a metric for the topology of the one-point compactification $X_{\mathbb{G}}^+$ of $X_{\mathbb{G}}$. As in Section 4.3 we consider now the topological space $W_{X_{\mathbb{G}},\sigma}$ of post-periodic points in the Wagoner topology. To simplify the notation we denote this space by $W_{\mathbb{G}}$ in the following. With respect to the Gurevich metric the shift is no longer locally expansive on $W_{\mathbb{G}}$, but we shall nonetheless now construct a relatively expansive system in this setup. This is where condition iii) comes in. Let $W_{\mathbb{G}}^0$ denote the set of post-periodic points that are forward transitive; that is

$$W_{\mathbb{G}}^0 = \left\{x \in W_{\mathbb{G}} : \ \forall y \in X_{\mathbb{G}} \ \forall \epsilon > 0 \ \exists j \in \mathbb{N} : \ d_G\left(y, \sigma^j(x)\right) < \epsilon\right\}.$$

Note that $W_{\mathbb{G}}^0$ is dense in $X_{\mathbb{G}}$ as well as in $W_{\mathbb{G}}$ since $\mathbb{G}$ is countable and strongly connected.

LEMMA 7.1. *For each $p \in \operatorname{Per} X_{\mathbb{G}}$ there is $\delta_p > 0$ such that*

$$x, y \in W^u(p), \ y \in W_{\mathbb{G}}^0, \ \sup_{j\in\mathbb{Z}} d_G\left(\sigma^j(x), \sigma^j(y)\right) \le \delta_p \ \Rightarrow \ x = y.$$

PROOF. We simply mimic a part of the proof of Theorem 3.4 of [**Sch**]. Let F be the set consisting of the edges from a maximal collection of pair-wise edge-disjoint doublepaths in $\mathbb{G}$. By assumption this is a finite set. Set

$$\delta_p = \frac{1}{2}\min\left\{\frac{1}{h(e)} - \frac{1}{h(e)+1} : \ e \in F\right\},$$

where h is the injection used to define d_G. Assume that $x, y \in W^u(p), y \in W_{\mathbb{G}}^0$ and that $d_G\left(\sigma^j(x), \sigma^j(y)\right) \le \delta_p$ for all $j \in \mathbb{Z}$. Note first of all that $x_{-j} = y_{-j}$ for all sufficiently large j since $x, y \in W^u(p)$. Assume to get a contradiction that $x_{j_0} \neq y_{j_0}$ for some $j_0 \in \mathbb{Z}$. Since y is forward transitive there is a $j > j_0$ such that $y_j \in F$. By definition of δ_p this implies that $y_j = x_j$. It follows that both

$$a = \max\{i \le j_0 : \ x_i = y_i\}$$

and

$$b = \min\{i \ge j_0 : \ x_i = y_i\}$$

exist. Then the pair (γ, γ'), where

$$\gamma = (x_{a+1}, x_{a+2}, \dots, x_{b-1})$$

and

$$\gamma' = (y_{a+1}, y_{a+2}, \dots, y_{b-1}),$$

is a doublepath which does not contain any edge from F. In particular, (γ, γ') is edge-disjoint from any doublepath used to define F. This contradicts the maximality of that collection. It follows that $x = y$. ☐

It follows from Lemma 7.1 that $(X_{\mathbb{G}}, d_G, \mathbb{Z}, \sigma, W_{\mathbb{G}}, W_{\mathbb{G}}^0)$ is a relatively expansive system. In the following we denote the resulting Ruelle algebra $A_\sigma(X_{\mathbb{G}}, W_{\mathbb{G}})$ by $B_{\mathbb{G}}$.

Let $\operatorname{Aut} X_{\mathbb{G}}$ be the group of shift-commuting homeomorphisms of $X_{\mathbb{G}}$. Elements of $\operatorname{Aut} X_{\mathbb{G}}$ can fail to be uniformly continuous with respect to the metric d, but they are all uniformly continuous with respect to the Gurevich-metric and this is the reason that we can prove the following

THEOREM 7.2. *For every $\alpha \in \operatorname{Aut} X_{\mathbb{G}}$ there is a unique $*$-automorphism $\alpha^\bullet$ of $B_{\mathbb{G}}$ such that*

$$\alpha^\bullet(f)(x,y) = f\left(\alpha^{-1}(x), \alpha^{-1}(y)\right),$$

when $f \in C_c(R_\sigma(X_{\mathbb{G}}, W_{\mathbb{G}}))$.

PROOF. α induces a homeomorphism of $W_{\mathbb{G}}$ by Lemma 4.9. Since α and α^{-1} are uniformly continuous with respect to the Gurevich-metric d_G it follows that we can apply Corollary 2.5. ☐

Let α_* be the automorphism of $K_0(B_{\mathbb{G}})$ induced by $\alpha^\bullet$. It follows from Theorem 7.2 that the map

$$\operatorname{Aut} X_{\mathbb{G}} \ni \alpha \mapsto \alpha_* \in \operatorname{Aut} K_0(B_{\mathbb{G}})$$

is a homomorphism of groups. Furthermore, it follows from Section 4.3 that $K_0(B_{\mathbb{G}})$ is isomorphic, as an ordered group, to the dimension group of $X_{\mathbb{G}}$ when $\mathbb{G}$ is finite. In this case $\alpha \mapsto \alpha_*$ agrees with the dimension group representation of $\operatorname{Aut} X_{\mathbb{G}}$.

In the following we seek to determine the structure of $B_{\mathbb{G}}$. Let $x, y \in W_{\mathbb{G}}$. A conjugacy (U, V, χ) from x to y in $W_{\mathbb{G}}$ is *normal* when there is an $i_0 \in \mathbb{Z}$ such that $x_{i_0} = y_{i_0}$, $U = \{z \in W_{\mathbb{G}} : \ z_i = x_i, i \leq i_0\}$, $V = \{z \in W_{\mathbb{G}} : \ z_i = y_i, i \leq i_0\}$ and

$$\chi(z)_k = \begin{cases} y_k, & k \leq i_0, \\ z_k, & k \geq i_0 + 1. \end{cases}$$

We say that an element $x = (x_i)_{i \in \mathbb{Z}} \in X_{\mathbb{G}}$ *tends to infinity* when $\lim_{i \to \infty} h(x_i) = \infty$.

LEMMA 7.3. *Let $x, y \in W_{\mathbb{G}}$, and let (U, V, χ) be a conjugacy from x to y in $W_{\mathbb{G}}$. Assume that x does not tend to infinity. It follows that y does not tend to infinity and that there is a normal conjugacy (U_0, V_0, χ_0) from x to y such that $U_0 \subseteq U$ and $\chi|_{U_0} = \chi_0$.*

PROOF. Since x does not tend to infinity, there is a finite collection F of edges in $\mathbb{G}$ such that $x_i \in F$ for infinitely many i. Since

$$d_G\left(\sigma^j(x), \sigma^j(y)\right) \leq \frac{1}{2} \min\left\{\frac{1}{h(e)} - \frac{1}{h(e)+1} : \ e \in F\right\}$$

for infinitely many j, we conclude that $x_j = y_j \in F$ for infinitely many j. In particular, y does not tend to infinity. Let F_0 be the set of edges occuring in a maximal collection of pairwise edge-disjoint doublepaths in $\mathbb{G}$. Let $i_0 \in \mathbb{N}$ be so large that $x_{i_0} = y_{i_0}$ and

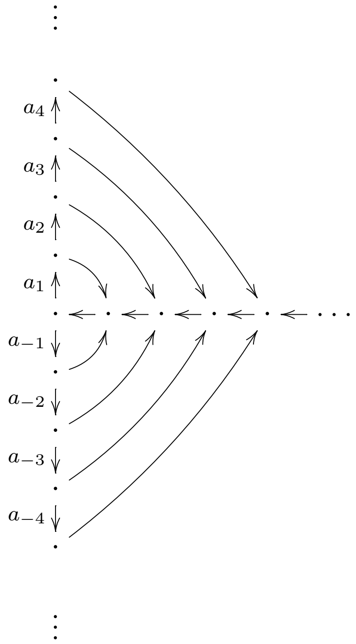
$$d_G\left(\sigma^j(x), \sigma^j(y)\right) \leq \frac{1}{2} \min\left\{\frac{1}{h(e)} - \frac{1}{h(e)+1} : \ e \in F \cup F_0\right\}$$

for all $j \geq i_0$. It follows that $x_i = y_i$ for all $i \geq i_0$ and we can define a normal conjugacy (U_0', V_0', χ_0) from x to y such that $U_0' = \{z \in W_{\mathbb{G}} : \ z_j = x_j, j \leq i_0\}$ and

$$\chi_0(z)_k = \begin{cases} y_k, & k \leq i_0 \\ z_k, & k \geq i_0. \end{cases}$$

By Lemma 1.4 there is a $j_0 \geq i_0$ such that $U_0 = \{z \in W_{\mathbb{G}} : \ z_j = x_j, \ j \leq j_0\} \subseteq U$ and $\chi_0(z) = \chi(z)$ for all $z \in U_0$. Set $V_0 = \chi_0(U_0)$ and note that (U_0, V_0, χ_0) is a normal conjugacy from x to y. □

We are now faced with a new kind of local conjugacies. If, for example, $\mathbb{G}$ is the graph



(7.1)

there is a local conjugacy between any pair $x, y \in W_{\mathbb{G}}$ with $x_i = a_i, i \geq 1$, and $y_i = a_{-i}, i \geq 1$. Such non-normal conjugacies complicate the analysis of the structure of $B_{\mathbb{G}}$ and we have very little to say about the structure of $B_{\mathbb{G}}$ in full generality, but the following observation will be usefull.

LEMMA 7.4. *$B_{\mathbb{G}}$ is stable.*

PROOF. If $\mathbb{G}$ is finite $X_{\mathbb{G}}$ is a Smale space and it follows in this case from Lemma 4.15 that $B_{\mathbb{G}}$ is stable. Assume then that $\mathbb{G}$ is infinite. We use the work of Hjelmborg and Rørdam in the same way as in the proof of Lemma 4.15. Thus it suffices to consider an element $f \in C_c(R_\sigma(X_{\mathbb{G}}, W_{\mathbb{G}}))$ and construct an element $v \in B_{\mathbb{G}}$ such that $vv^*f = f$ and $fv^* = 0$. To this end note that there is a finite set of periodic points $p_1, p_2, \dots, p_N$ in $X_{\mathbb{G}}$ and a finite set of edges $e_1, e_2, \dots, e_M$ in $\mathbb{G}$ such that

$$r(\operatorname{supp} f) \cup s(\operatorname{supp} f) \subseteq \bigcup_{k,j} \left\{ (x_i)_{i \in \mathbb{Z}} \in W^u(p_k) : \ x_0 = e_j \right\}.$$

Let q be a periodic point in $X_{\mathbb{G}}$ whose orbit is disjoint from $\{p_1, p_2, \dots, p_N\}$. Since $\mathbb{G}$ is strongly connected there is for each $(k, j) \in \{1, 2, \dots, N\} \times \{1, 2, \dots, M\}$ a normal conjugacy $(U_{k,j}, V_{k,j}, \chi_{k,j})$ in $W_{\mathbb{G}}$ such that

$$U_{k,j} = \left\{x = (x_i)_{i\in\mathbb{Z}} \in W^u(p_k) : \ x_0 = e_j\right\}$$

and $V_{k,j} \subseteq \bigcup_{l\in\mathbb{N}} W^u\left(\sigma^l(q)\right)$. Let $\{g_{k,j}\}$ be a partition of unity on $r(\operatorname{supp} f)$ subordinate to $\{U_{k,j}\}$. Define $\tilde{g}_{k,j} \in C_c(R_\sigma(X_{\mathbb{G}}, W_{\mathbb{G}}))$ such that

$$\tilde{g}_{k,j}(x,y) = \begin{cases} \sqrt{g_{k,j}(x)} & \text{when } (x,y) \in \{(z, \chi_{k,j}(z)) : \ z \in U_{k,j}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v = \sum_{k,j} \tilde{g}_{k,j} \in C_c(R_\sigma(X_{\mathbb{G}}, W_{\mathbb{G}}))$ does the job. □

To handle local conjugacies that are not normal we make in the following an additional assumption:

iv) There is a finite set, F, of edges in $\mathbb{G}$ such that for every edge $e_0 \in \mathbb{G}$ there is at most one right-infinite ray $r = e_0e_1e_2e_3\dots$ in $\mathbb{G}$ with $e_i \notin F$ for all $i \geq 1$ and at most one left-infinite ray $l = \dots e_{-2}e_{-1}e_0$ in $\mathbb{G}$ with $e_i \notin F$ for all $i \leq -1$.

In the presence of conditions i)–iii) this additional assumption is equivalent to the condition that the canonical extension of the shift to the one-point compactification of $X_{\mathbb{G}}$ is expansive and, therefore, is a compact shift space, cf. [**Sch**], [**F**]. Hence, when condition iv) holds, $(X_{\mathbb{G}}, \sigma)$ is actually expansive with the respect to the Gurevich-metric d_G on the entire space $X_{\mathbb{G}}$, and not only on the doubly transitive points.

We assume now that conditions i), ii), iii) and iv) all hold.

Let $F \subseteq \mathbb{V}$ be a finite subset. An *out-going ray* in $\mathbb{G}$ is a right-infinite path $\gamma = e_0e_1e_2e_3\dots$ in $\mathbb{G}$ such that $t(e_i) \notin F$ for all $i = 0,1,2,\dots$. γ is a *maximal* out-going ray when $i(e_0) \in F$. Likewise an *incoming ray* in $\mathbb{G}$ is a left-infinite path $\dots e_{-3}e_{-2}e_{-1}e_0 = \gamma'$ in $\mathbb{G}$ such that $i(e_i) \notin F$ for all $i = 0,-1,-2,-3,\dots$, and γ' is a *maximal* incoming ray when $t(e_0) \in F$. A finite path $\gamma = e_0e_1e_2\dots e_n$ in $\mathbb{G}$ is said to *avoid* F when $t(e_i) \notin F, i = 0,1,2,\dots,n-1$.

It follows from condition iii) and iv) that there is a finite set $F \subseteq \mathbb{V}$ of vertices such that

$$t(\gamma) = t(\gamma') \ \Rightarrow \ \gamma = \gamma' \tag{7.2}$$

when γ, γ' are maximal incoming rays,

$$i(\gamma) = i(\gamma') \ \Rightarrow \ \gamma = \gamma' \tag{7.3}$$

when γ, γ' are maximal outgoing rays, and

$$i(\gamma) = i(\gamma'), \ t(\gamma) = t(\gamma') \ \Rightarrow \ \gamma = \gamma' \tag{7.4}$$

when γ, γ' are finite paths of the same length that both avoid F.

Note that if $\gamma = e_1e_2e_3\dots$ is an out-going ray and $t(e_i) = t(e_{i+k})$ for some $i, k \in \mathbb{N}$, it follows from condition iv) that $e_j = e_{j+k}$ for all $j \geq i+1$, i.e. γ is eventually k-periodic. If we add to F the vertices of all eventually periodic outgoing rays we still have a finite set of vertices and (7.2), (7.3) and (7.4) hold for the larger set. Thus, by enlarging F if necessary, we can assume that no outgoing ray is pre-periodic, i.e. that

$$i \neq j \ \Rightarrow \ t(e_i) \neq t(e_j) \tag{7.5}$$

when $\gamma = e_1e_2e_3\ldots$ is an outgoing ray. A finite set $F \subseteq \mathbb{V}$ such that (7.2), (7.3), (7.4) and (7.5) all hold will be called *a core* for $\mathbb{G}$. Note that if we add a finite set of vertices to a core the union is still a core.

Let $v \in \mathbb{V}$. In the following *a shortcut from F to v* is a finite path γ which avoids F such that $i(\gamma) \in F$ and $t(\gamma) = v$. Similarly, *a shortcut from v to F* is a finite path γ which avoids F such that $i(\gamma) = v$ and $t(\gamma) \in F$.

The following lemma should be compared with Lemma 3.3 of [**Sch**].

LEMMA 7.5. *Let $F \subseteq \mathbb{V}$ be a core. Let $(x_i)_{i\in\mathbb{Z}} \in X_{\mathbb{G}}$. Then $t(x_j) \in F$ for some $j \in \mathbb{Z}$.*

PROOF. Assume $t(x_j) \notin F$ for all j. Then $x_kx_{k+1}x_{k+2}\ldots$ is the tail of a maximal out-going ray for each k, and hence

$$i \neq j \;\Rightarrow\; x_i \neq x_j, \tag{7.6}$$

because no out-going ray is eventually periodic, cf. (7.5). Since $\mathbb{G}$ is strongly connected there is for each $k \in \mathbb{Z}$ a shortcut γ_k from F to $i(x_k)$. But F is finite so there is a sequence $m_1 > m_2 > m_3 > \ldots$ in $\mathbb{Z}$ and a vertex $v \in F$ such that there is, for each k, a shortcut from F to $i(x_{m_k})$ which starts at v. Choose k such that

$$m_1 - m_k > |\gamma_1|. \tag{7.7}$$

Then $\gamma_kx_{m_k}x_{m_k+1}x_{m_k+2}\ldots$ and $\gamma_1x_{m_1}x_{m_1+1}x_{m_1+2}\ldots$ are out-going rays starting at the same vertex. By condition iv) these rays must be identical, which is impossible by (7.7) and (7.6). □

Let $F \subseteq \mathbb{V}$ be a core for $\mathbb{G}$. We let $F^{\mathbb{E}} = \{e \in \mathbb{E} : \; t(e) \in F\}$ and set $\mathcal{A} = F^{\mathbb{E}} \cup \{\uparrow\}$. Define $X_{\mathbb{G}} \ni x \mapsto \tilde{x} \in \mathcal{A}^{\mathbb{Z}}$ such that

$$\tilde{x}_i = \begin{cases} x_i & \text{when } t(x_i) \in F \\ \uparrow & \text{when } t(x_i) \notin F. \end{cases} \tag{7.8}$$

We define $d'_F : X_{\mathbb{G}} \times X_{\mathbb{G}} \to [0,\infty)$ such that

$$d'_F(x,y) = \sum_{i\in\mathbb{Z}} 2^{-|i|}\delta(\tilde{x}_i,\tilde{y}_i),$$

where

$$\delta(a,b) = \begin{cases} 0 \text{ when } a = b \\ 1 \text{ when } a \neq b. \end{cases}$$

LEMMA 7.6. *d'_F is a metric on $X_{\mathbb{G}}$ equivalent to d_G.*

PROOF. Let $X^+_{\mathbb{G}}$ be the one-point compactification of $X_{\mathbb{G}}$. Even without condition iv) the map $x \mapsto \tilde{x}$ extends to a continuous shift-commuting map $\psi : X^+_{\mathbb{G}} \to \mathcal{A}^{\mathbb{Z}}$ such that $\psi(+) = \uparrow^{\infty}$, cf. Proposition 3.5 of [**Sch**]. The crucial point is that $\psi : X_{\mathbb{G}} \to \mathcal{A}^{\mathbb{Z}} \backslash \{\uparrow^{\infty}\}$ is continuous and proper, which is easy to check. It remains to check that ψ is injective in our case. This follows from (7.2), (7.3), (7.4), (7.5) and Lemma 7.5. □

LEMMA 7.7. *Assume that F is a core. Let $v \in \mathbb{V}$ be a vertex which does not belong to any out-going ray. There is then an $N \in \mathbb{N}$ such that $|\gamma| \leq N$ for every shortcut γ from v to F.*

Proof. If no such N exists it would follow from condition i) that there is an out-going ray $e_0e_1e_2\dots$ in $\mathbb{G}$ such that $i(e_0) = v$, contradicting the assumption on v. $\square$

Lemma 7.8. *Let $e_0e_1e_2\dots$ be an out-going ray in $\mathbb{G}$. For each pair $n, N \in \mathbb{N}$ such that $n < N$ there is an $L \in \mathbb{N}$ with the property that every shortcut γ of length $|\gamma| \geq L$ from $t(e_n)$ to F contains e_N.*

Proof. An edge starting at $t(e_i)$ which is not e_{i+1} can not terminate in a vertex which belongs to an out-going ray. This follows from condition iv). It follows then from condition i) and Lemma 7.7 that for each $i \in \{n, n+1, \dots, N-1\}$ there is an $L_i \in \mathbb{N}$ such that all shortcuts from $t(e_i)$ to F which does not contain e_{i+1} must have length $\leq L_i$. Set

$$L = \max_{n \leq j \leq N-1} (L_j + j).$$

$\square$

Let $\gamma = e_0e_1e_2e_3\dots$ and $\gamma' = f_0f_1f_2f_3\dots$ be out-going rays in $\mathbb{G}$, belonging to different maximal out-going rays. In the following we shall consider the condition that there are numbers $n < N < M$ in $\mathbb{N}$ such that the following holds for each $k \in \mathbb{N}$ and each $v \in F$:

A) If there is a shortcut of length k from $t(e_N)$ to F ending at v, there is also a (necessarily unique) shortcut of length $k + N - n$ from $t(f_n)$ to F ending at v.
B) If there is a shortcut of length k from $t(f_M)$ to F ending at v, there is also a (necessarily unique) shortcut of length $k + M - N$ from $t(e_N)$ to F ending at v.

Let $\gamma = e_1e_2e_3\dots$ be an out-going ray in $\mathbb{G}$. In the following we shall consider the condition that there are $N, k, L \in \mathbb{N}$, $k \neq 0$, such that the following hold for each $v \in F$ and each $l \geq L$:

C) There is a shortcut of length l from $t(e_N)$ to F which terminates at v if and only if there is a shortcut from $t(e_{N+k})$ to F of length l which terminates at v, and
D) every shortcut of length l from $t(e_N)$ to F contains e_{N+k}.

Lemma 7.9. *Let $\mathbb{G}$ be a graph satisfying conditions i) through iv). When $x, y \in W_{\mathbb{G}}$ are conjugate, either*

a) *$x_i = y_i$ for all large i, or*
b) *there are different out-going rays γ and γ' for which A) and B) hold such that $x_i \in \gamma$ and $y_i \in \gamma'$ for all large i, or*
c) *there is a maximal outgoing ray γ for which C) and D) hold, and $x_i, y_i \in \gamma$ for all large i.*

Proof. a) holds if and only if there is a normal conjugacy between x and y. Assume therefore that there is no normal conjugacy from x to y and let (U, V, χ) be a non-normal conjugacy in $W_{\mathbb{G}}$ from x to y. It follows from Lemma 7.3 that x and y both tend to infinity. Furthermore, x and y can not be asymptotically equal. There are out-going rays, $\gamma = e_0e_1e_2\dots$ and $\gamma' = f_0f_1f_2\dots$ in $\mathbb{G}$ and $K \in \mathbb{N}$ such that $x_{[K,\infty)} = \gamma$ and $y_{[K,\infty)} = \gamma'$. Choose $L \geq K$ such that

$$\sup_{z \in U} d_G\left(\sigma^j\left(\chi(z)\right), \sigma^j(z)\right) \leq \frac{1}{2} \min\left\{\frac{1}{h(e)} - \frac{1}{h(e)+1} : \ e \in F\right\} \tag{7.9}$$

and

$$\sup_{z \in V} d_G\left(\sigma^j\left(\chi^{-1}(z)\right), \sigma^j(z)\right) \leq \frac{1}{2} \min\left\{\frac{1}{h(e)} - \frac{1}{h(e)+1} : \ e \in F\right\}$$

for all $j \geq L$. For each $I \in \mathbb{Z}$ and each $u \in W_{\mathbb{G}}$, set

$$U_I^u = \left\{z \in W_{\mathbb{G}} : \ z_i = u_i, \ i \leq I\right\}.$$

There is an $n \geq K$ such that $U_n^y \subseteq V$ and $U_n^x \subseteq U$. There is an $N_1 \in \mathbb{N}$ such that $N_1 > n$ and

$$\chi\left(U_{N_1}^x\right) \subseteq U_{n+L+K}^y. \tag{7.10}$$

Let μ be a shortcut of length k from $t\left(x_{N_1+L}\right)$ to F. There is then a $z \in U_{N_1+L}^x$ such that $z_{[N_1+L+1, N_1+L+|\mu|]} = \mu$. It follows from (7.10) that $\chi(z) \in U_{n+L+K}^y$ and from (7.9) that $z_j \in F \Leftrightarrow \chi(z)_j \in F \ \forall j \geq L$ so we conclude that $\chi(z)_{[n+K+1, N_1+L+K+|\mu|]}$ is a shortcut of length $L + N_1 + |\mu| - n$ from $t\left(y_{n+K}\right)$ to F. Set $N = L + N_1 - K$. Then A) holds. By exchanging the roles of x and y the same argument applies to find $M > N$ such that also B) holds. Hence b) holds, unless γ and γ' are contained in the same maximal outgoing ray. In the case where γ and γ' are contained in the same maximal out-going ray $e_0e_1e_2\ldots$ we observe first that there are $D \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $\gamma = e_De_{D+1}e_{D+2}\ldots$ and $\gamma' = e_{D+k}e_{D+k+1}e_{D+k+2}\ldots$. Note that $k = 0$ is impossible because there is no normal conjugacy from x to y. By exchanging the roles of x and y if needed, we can assume that $k \geq 1$. By combining Lemma 7.8 with A) and B) we find that there are numbers $N, L \in \mathbb{N}$ such that the triple N, k, L satisfy both C) and D). ☐

7.1.1. Grafting. Throughout this section we assume that conditions i), ii), iii) and iv) are all satisfied. Let $F_0 \subseteq \mathbb{V}$ be a core for $\mathbb{G}$.

LEMMA 7.10. *There is then a core $F \subseteq \mathbb{V}$ for $\mathbb{G}$ containing F_0 such that different maximal out-going rays are pair-wise vertex disjoint, i.e.*

$$\left\{i\left(e_k\right) : \ k = 0, 1, 2, \ldots\right\} \cap \left\{i\left(f_k\right) : \ k = 0, 1, 2, \ldots\right\} = \emptyset.$$

when $e_0e_1e_2e_3\ldots$ and $f_0f_1f_2\ldots$ are different outgoing rays in $\mathbb{G}$, and such that F is the vertex set of a finite strongly connected subgraph of $\mathbb{G}$.

PROOF. Let Γ be the finite set of maximal out-going rays from F_0. If two different elements $\gamma = e_1e_2e_3\ldots$ and $\gamma' = f_1f_2f_3\ldots$ of Γ are not vertex disjoint, there are numbers $n, m \in \mathbb{N}$ such that $t\left(e_n\right) = t\left(f_m\right)$ and

$$\left\{i\left(e_1\right), i\left(e_2\right), \ldots, i\left(e_n\right)\right\} \cap \left\{i\left(f_1\right), i\left(f_2\right), \ldots, i\left(f_m\right)\right\} = \emptyset.$$

It follows from condition iv) that $f_{m+k} = e_{n+k}$ for all $k \in \mathbb{N}$. Set

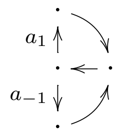
$$\Gamma_{\gamma,\gamma'} = \left\{i\left(e_1\right), i\left(e_2\right), \ldots, i\left(e_{n+1}\right)\right\} \cup \left\{i\left(f_1\right), i\left(f_2\right), \ldots, i\left(f_{m+1}\right)\right\}.$$

Let

$$F' = F_0 \cup \bigcup_{(\gamma,\gamma')} \Gamma_{\gamma,\gamma'}$$

where we take the union over all pairs γ, γ' from Γ that are not vertex disjoint. F' is clearly a finite set, and it is easy to check that F' is a core such that all different maximal out-going rays are pair-wise vertex disjoint. Since $\mathbb{G}$ is strongly connected there is a strongly connected finite subgraph of $\mathbb{G}$ which contains F'. The vertices of such a finite subgraph form a core F with the desired additional properties. $\Box$

Consider then a core $F \subseteq \mathbb{V}$ with the two additional properties described in Lemma 7.10. If $\mathbb{G}$ is the graph (7.1) F could be the vertices of the subgraph



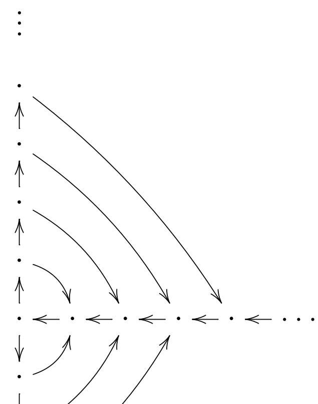
(7.11)

Assume that there are out-going rays γ and γ' such that A) and B) hold. It follows that when γ is a shortcut from $t\,(e_N)$ to F there is a unique shortcut μ_γ of length $|\gamma| + N - n$ from $t\,(f_n)$ to F with $t\,(\gamma) = t\,(\mu_\gamma)$. Let $\mathbb{A}$ be the set of shortcuts from e_N to F such that $f_M \notin \mu_\gamma$.

Let $\mathbb{G}''$ be the subgraph of $\mathbb{G}$ containing the following edges from $\mathbb{G}$:

- all edges $e \in \mathbb{G}$ with $i(e), t(e) \in F$,
- all edges contained in some path γ which starts at a vertex in F, ends in a vertex in F and avoids F and e_{N+1} in between, and
- all edges of the paths $e_0 e_1 \dots e_N \gamma$, where $\gamma \in \mathbb{A}$.

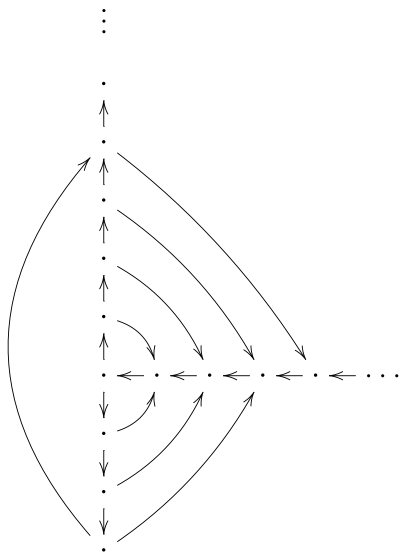
Then $\mathbb{G}''$ satisfies conditions i) through iv) and F remains a core for $\mathbb{G}''$. When $\mathbb{G}$ is the graph (7.1), F the vertices from the subgraph (7.11) and $\gamma = a_{-2}a_{-3}\dots$ we can take $f_n = a_2$ and $e_N = a_{-3}$ and $f_M = a_4$. The resulting graph $\mathbb{G}''$ is then



(7.12)

We add now to $\mathbb{G}''$ a new path, μ_{new}, of length $M - N$ from $t\,(e_N)$ to $t\,(f_M)$. Denote the resulting graph by $\mathbb{G}'$. Then $\mathbb{G}'$ satisfies conditions i) through iv) and F is a core for $\mathbb{G}'$, although different maximal outgoing rays can have common vertices in $\mathbb{G}'$.

In the example (7.12) the graph $\mathbb{G}'$ becomes



(7.13)

In order to relate $X_{\mathbb{G}'}$ to $X_{\mathbb{G}}$ we need some lemmas.

LEMMA 7.11. *Let $e_0e_1e_2\dots$ be a maximal out-going ray in $\mathbb{G}$. Then $t(e_i)$ has in-degree 1 for all i, i.e.*

$$\#\{e \in \mathbb{E} :\ t(e) = t(e_i)\} = 1$$

for all $i = 0, 1, 2, 3, \dots$.

PROOF. This follows because different maximal out-going rays are pair-wise vertex disjoint. ☐

LEMMA 7.12. *Let $v, w \in F$.*

a) *There is a bijection between the set of maximal incoming rays in $\mathbb{G}$ that terminate at v and the set of maximal incoming rays in $\mathbb{G}'$ that terminate at v.*
b) *For each $n \in \mathbb{N}$ there is a bijection between the paths in $\mathbb{G}$ of length n that start at v, end at w and avoid F, and the corresponding set of paths in $\mathbb{G}'$.*
c) *There is a bijection between the set of maximal outgoing rays in $\mathbb{G}$ that start at v and the set of maximal outgoing rays in $\mathbb{G}'$ that start at v.*

PROOF. a) By condition iv) it suffices to show that v is the terminal vertex of a maximal incoming ray in $\mathbb{G}$ if and only if it is in $\mathbb{G}'$. It follows from Lemma 7.11 that an incoming ray in $X_{\mathbb{G}'}$ can not contain μ_{new}. It must therefore be an incoming ray in $\mathbb{G}''$ and hence in $\mathbb{G}$. Let γ be a maximal incoming ray in $\mathbb{G}$ which terminates at v. If every vertex in γ can be reached from F by a finite path which avoids e_{N+1}, γ will be a maximal incoming ray in $\mathbb{G}'$ by construction of $\mathbb{G}'$. Otherwise, v will be the terminal vertex of arbitrarily long shortcuts from $t(e_N)$ to F in $\mathbb{G}$. By condition A) v will then also be the terminal vertex of arbitrarily long shortcuts from $t(f_n)$ to F. It follows from Lemma 7.11 that none of these shortcuts contains e_{N+1}, so they all proceed in $\mathbb{G}''$. If follows that v is also the terminal vertex of a maximal incoming rays in $\mathbb{G}'$.

b) Let γ be a path in $\mathbb{G}$ which avoids F such that $i(\gamma) = v$ and $t(\gamma) = w$. If $e_{N+1} \notin \gamma$, γ is also a path in $\mathbb{G}'$. If $e_{N+1} \in \gamma$, $\gamma = \gamma_1\gamma_2$, where γ_2 is a shortcut from

$t(e_N)$ to F. If $\gamma_2 \in \mathbb{A}$, γ is also a path in $\mathbb{G}'$. If $\gamma_2 \notin \mathbb{A}$, $f_M \in \mu_{\gamma_2}$ which means that $\mu_{\gamma_2} = f_{n+1}f_{n+2}\dots f_M\mu'$ for some shortcut μ' from $t(f_M)$. Then $\gamma_1\mu_{new}\mu'$ is a path in $\mathbb{G}'$ with the same length as γ which avoids F, starts at v and ends at w.

Let then μ be a path in $\mathbb{G}'$ which avoids F, starts at v and ends at w. If $\mu_{new} \nsubseteq \mu$, μ is also a path in $\mathbb{G}$, so assume that $\mu_{new} \subseteq \mu$. It follows that $\mu = \mu_1\mu_{new}\mu_2$, where μ_1 is the unique shortcut from F to $t(e_N)$ and μ_2 is the unique shortcut from $t(f_M)$ to F of length $|\mu| - |\mu_1| - M + N$. By condition B) there is then a unique shortcut ν of length $|\mu| - |\mu_1|$ in $\mathbb{G}$ from $t(e_N)$ to F such that $t(\nu) = w$. It follows that $\mu_1\nu$ is a path in $\mathbb{G}$ with the same length as μ which avoids F, starts at v and ends at w.

c) Since condition iv) is satisfied by both $\mathbb{G}'$ and $\mathbb{G}$ it suffices to show that v is the start vertex of a maximal outgoing ray in $\mathbb{G}$ if and only if it is in $\mathbb{G}'$. So let δ be a maximal outgoing ray in $\mathbb{G}$ with $i(\delta) = v$. If $\gamma \nsubseteq \delta$, δ will also be a maximal outgoing ray in $\mathbb{G}'$. If not $\delta' = \mu_1\mu_{new}f_{M+1}f_{M+2}\dots$ is a maximal outgoing ray in $\mathbb{G}'$ such that $i(\delta') = v$, where μ_1 is the unique shortcut from F to $t(e_N)$. Conversely, let δ' be a maximal outgoing ray in $\mathbb{G}'$ with $i(\delta') = v$. If $\mu_{new} \nsubseteq \delta'$, δ' is also a maximal outgoing ray in $\mathbb{G}$. On the otherhand, if $\mu_{new} \subseteq \delta'$ we find that $i(\delta') = i(\delta)$ where δ is the maximal outgoing ray in $\mathbb{G}$ containing γ. ☐

LEMMA 7.13. *The dynamical systems, $(X_{\mathbb{G}}, \sigma)$ and $(X_{\mathbb{G}'}, \sigma)$ are conjugate.*

PROOF. We define a map $\varphi : X_{\mathbb{G}} \to X_{\mathbb{G}'}$ as follows. Let $x = (x_i)_{i=-\infty}^{\infty} \in X_{\mathbb{G}}$. We will then define $\varphi(x) \in X_{\mathbb{G}'}$ as follows. When $t(x_i) \in F$, set $\varphi(x)_i = x_i$. When $t(x_i) \notin F$, we consider the possibilities

a) $t(x_j) \notin F,\ j \leq i$,
b) $t(x_j) \notin F,\ j \geq i$, and
c) there are $i_1 < i$ and $i < i_2$ such that $t(x_{i_1}), t(x_{i_2}) \in F$.

In case a), set $i_0 = \min\{k > i :\ t(x_k) \in F\}$, cf. Lemma 7.5. Then $\dots x_{i_0-2}x_{i_0-1}x_{i_0}$ is a maximal incoming ray in $\mathbb{G}$ and by a) of Lemma 7.12 there is a unique maximal incoming ray γ in $\mathbb{G}'$ such that $t(\gamma) = t(i_0)$. We set $\varphi(x)_{]-\infty,i_0]} = \gamma$. In particular, this determines $\varphi(x)_i$. In case b) we set $i_0 = \max\{k < i :\ t(x_k) \in F\}$. Then $x_{[i_0+1,\infty[}$ is a maximal out-going ray in $\mathbb{G}$ and by c) of Lemma 7.12 there is a unique maximal outgoing ray δ in $\mathbb{G}'$ such that $i(\delta) = t(x_{i_0})$. We set $\varphi(x)_{[i_0+1,\infty[} = \delta$ in this case. In particular, this determines $\varphi(x)_i$. In case c) we set $i_2 = \min\{k > i :\ t(x_k) \in F\}$ and $i_1 = \max\{k < i :\ t(x_k) \in F\}$. Then $x_{[i_1+1,i_2]}$ is path in $\mathbb{G}$ which avoids F. By b) of Lemma 7.12 there is a unique path μ in $\mathbb{G}'$ of the same length as $x_{[i_1+1,i_2]}$ and with the same initial and terminal vertex. We set $\varphi(x)_{[i_1+1,i_2]} = \mu$. In particular, this determines $\varphi(x)_i$ in case c). Note that φ commutes with the shift. Since φ does not change F-coordinates it follows φ is isometric with respect to the metric d'_F. Hence, by Lemma 7.6 and the bijective correspondances of Lemma 7.12 we conclude that φ is a conjugacy. ☐

COROLLARY 7.14. *There is a $*$-isomorphism $\psi : B_{\mathbb{G}} \to B_{\mathbb{G}'}$ such that $\sigma^\bullet \circ \psi = \psi \circ \sigma^\bullet$.*

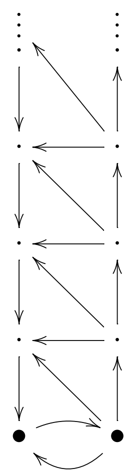
PROOF. It follows from Lemma 7.13 that we can apply Corollary 2.5. ☐

By successive repetition of the process described we obtain from $\mathbb{G}$ a new graph $\mathbb{H}$ – with the same essential properties – such that $(X_{\mathbb{G}}, \sigma)$ and $(X_{\mathbb{H}}, \sigma)$ are conjugate via a shiftcommuting homeomorhism φ with both φ and φ^{-1} uniformly continuous with respect to the Gurevich metrics, and, as a result, $B_{\mathbb{G}}$ and $B_{\mathbb{H}}$ are $*$-isomorphic via a $*$-isomorphism which intertwines the automorphisms induced by the shift. What is true for $\mathbb{H}$ is that A) and B) do not hold for any pair of rays belonging to different maximal outgoing rays. We say that $\mathbb{H}$ is *totally grafted* when this is the case. That is, repeated application of Lemma 7.13 and Corollary 7.14 yields the following

LEMMA 7.15. *Let $\mathbb{G}$ be a graph satisfying i) through iv). There is then a totally grafted graph $\mathbb{H}$ satisfying conditions i) through iv) such that $(X_{\mathbb{G}}, \sigma)$ and $(X_{\mathbb{H}}, \sigma)$ are conjugate.*

7.1.2. Pruning. We still assume that $\mathbb{G}$ satisfies conditions i) through iv) and that F is a core in $\mathbb{G}$ having the additional properties desribed in Lemma 7.10. Assume now that there is an out-going ray $\gamma = e_1e_2e_3\cdots$ in $\mathbb{G}$ such that C) and D) hold.

This is for example the case in the graph



(7.14)

When C) and D) hold for some $N, L, k \in \mathbb{N}$ we call k *an asymptotic period* of the ray γ.

LEMMA 7.16. *Assume that C) and D) hold. It follows that when $j \geq N$, there is an $L_j \in \mathbb{N}$ such that for each $v \in F$ and each $l \geq L_j$,*

- *there is a shortcut of length l from $t(e_j)$ to F which terminates at v if and only if there is a shortcut from $t(e_{j+k})$ to F of length $l-k$ which terminates at v, and*
- *every shortcut of length $l \geq L_j$ from $t(e_j)$ to F contains e_{j+k}.*

PROOF. This follows from Lemma 7.8. □

Let

$$p = \min\{k : \ k \text{ is an asymptotic period for } \gamma\}.$$

We call p *the minimal asymptotic period of* γ.

LEMMA 7.17. *A natural number $m \in \mathbb{N}$ is an asymptotic period of γ if and only if $m \in \mathbb{N}p$.*

PROOF. Left to the reader. □

By increasing L if necessary we may assume that there is in fact a shortcut of length L from $t(e_N)$ to F. We construct first a new graph $\mathbb{G}'''$ as follows:

Let $\mathbb{G}_0$ be the subgraph of $\mathbb{G}$ consisting of

- all edges $e \in \mathbb{G}$ with $i(e), t(e) \in F$,
- all edges contained in some path γ which starts at a vertex in F, ends in a vertex in F and avoids e_1 in between, and
- all edges of the paths $e_1 \dots e_N\gamma$, where γ is a shortcut from $t(e_N)$ to F of length $|\gamma| \geq L$.

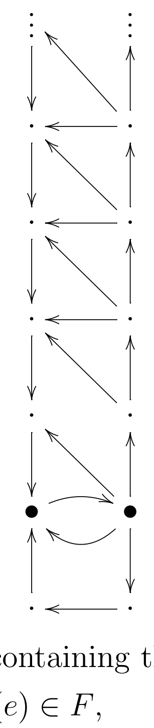
For each shortcut μ from $t(e_1)$ to F which is not of the form $e_2 \dots e_N\gamma$, where γ is a shortcut from $t(e_N)$ to F of length $|\gamma| \geq L$, we add to $\mathbb{G}_0$ a new path μ' of length $|\mu| + 1$ from $s(e_1)$ to $t(\mu)$. All these new paths must be mutually edge disjoint and contain no edges from $\mathbb{G}_0$. The resulting graph $\mathbb{G}'''$ satisfies condition i)–iv) and the set F is a core for $\mathbb{G}'''$ and enjoys the extra properties described in Lemma 7.10. Furthermore, $\mathbb{G}'''$ and $\mathbb{G}$ have exactly the same outgoing and incoming rays with respect to F. Finally C) and D) still hold in $\mathbb{G}_0$. What we have achieved is that in $\mathbb{G}'''$

$$|\gamma| \geq L \text{ when } \gamma \text{ is a shortcut from } t(e_N) \text{ to } F. \tag{7.15}$$

Note now that Lemma 7.12 holds with $\mathbb{G}'$ replaced by $\mathbb{G}'''$ and we can prove the following lemma by the method used to prove Lemma 7.13.

LEMMA 7.18. *The dynamical systems $(X_{\mathbb{G}}, \sigma)$ and $(X_{\mathbb{G}'''}, \sigma)$ are conjugate.*

If $\mathbb{G}$ is the graph (7.14) and we let F be the vertices indicated by fat dots, and $N = 1$, $k = 1$ and $L = 3$, the graph $\mathbb{G}'''$ becomes



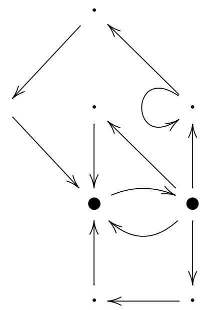
(7.16)

Let $\mathbb{G}''$ be the subgraph of $\mathbb{G}'''$ containing the following edges from $\mathbb{G}$:

- all edges $e \in \mathbb{G}$ with $i(e), t(e) \in F$,
- all edges contained in some path γ which starts at a vertex in F, ends in a vertex in F and avoids F and e_N in between, and
- the edges $e_1, e_2, \dots, e_N$.

Add to $\mathbb{G}''$ a loop ν of length p (the asymptotic period of γ) at $t(e_N)$ which is edge disjoint from $\mathbb{G}''$. For each shortcut μ in $\mathbb{G}'''$ of length $l \in \{L, L+1, \ldots, L+p-1\}$ from $t(e_N)$ to F we add to $\mathbb{G}''$ a path δ_μ of length l from $t(e_N)$ to F such that $t(\delta_\mu) = t(\mu)$. These paths are mutually edge-disjoint and edge-disjoint from $\mathbb{G}''$ and ν. The resulting graph will de denoted by $\mathbb{G}'$.

By applying this recipe in the appropriate way to the graph (7.16) the graph $\mathbb{G}'$ becomes



(7.17)

We have only added finitely many new edges to $\mathbb{G}''$ to obtain $\mathbb{G}'$, and it follows therefore that $\mathbb{G}'$ satisfies condition i) through iv) since $\mathbb{G}'''$ and $\mathbb{G}''$ do. Observe that there is one less maximal out-going ray in $\mathbb{G}'$ than there is in $\mathbb{G}$.

In the following we let ν^∞ denote the set of periodic elements y of $X_{\mathbb{G}'}$ with the property that $y_i \in \nu$ for all $i \in \mathbb{Z}$. Thus ν^∞ is a p-periodic orbit.

Lemma 7.19. *Let $v, w \in F$.*

a) *There is a bijection between the set of maximal incoming rays in $\mathbb{G}'''$ that terminate at v and the set of maximal incoming rays in $\mathbb{G}'$ that terminate at v.*
b) *For each $n \in \mathbb{N}$ there is a bijection between the paths in $\mathbb{G}'''$ of length n that start at v, end at w and avoid F, and the corresponding set of paths in $\mathbb{G}'$.*
c) *There is a bijection between the set of maximal outgoing rays in $\mathbb{G}'''$ that start at v and the set of maximal outgoing rays in $\mathbb{G}'$ that start at v.*

Proof. The proof is quite analogous to the proof of Lemma 7.12. We omit the details. □

Lemma 7.20. *There is a unique shiftcommuting map $\psi : X_{\mathbb{G}'''} \to X_{\mathbb{G}'}$ such that $\psi(x)_i \in F \Leftrightarrow x_i \in F$ for all $i \in \mathbb{Z}$.*

Proof. This follows from Lemma 7.19 in the same way as Lemma 7.13 follows from Lemma 7.12. □

Lemma 7.21. *$F \cup t(e_N)$ is a core for $\mathbb{G}'$.*

Proof. Note that a maximal incoming (outgoing) ray in $\mathbb{G}'$ relative to $F \cup t(e_N)$ is also a maximal incoming (outgoing) ray in $\mathbb{G}'''$ relative to F. Therefore conditions (7.2), (7.3) and (7.5) follow from the corresponding conditions for F in $\mathbb{G}'''$.

Consider then two finite paths γ and γ' of the same length in $\mathbb{G}'$ which both avoid $F \cup t(e_N)$ and have the property that $i(\gamma) = i(\gamma')$, $t(\gamma) = t(\gamma')$. If $t(\gamma) = t(e_N)$ and $i(\gamma) \neq t(e_N)$ we must have that $\gamma = \gamma' = e_1 e_2 \ldots e_N$. If $t(\gamma) = i(\gamma) = t(e_N)$ we have that $\gamma = \gamma' = \nu$. If $i(\gamma) = t(e_N)$ and $t(\gamma) \neq t(e_N)$, we find that $\gamma = \gamma' = \delta_\mu$, where μ is the unique shortcut in $\mathbb{G}'''$ from $t(e_N)$ to F of length $|\gamma|$. All in all we conclude that $F \cup t(e_N)$ also satisfies condition (7.4) in $\mathbb{G}'$. □

Set $F' = F \cup t(e_N)$.

LEMMA 7.22. *The metric d'_F induces the same topology on $X_{\mathbb{G}'}\backslash\nu^\infty$ as $d'_{F'}$.*

PROOF. Since $d'_F \leq d'_{F'}$ it suffices to show that a $d'_{F'}$-open subset of $X_{\mathbb{G}'}\backslash\nu^\infty$ is also d'_F-open. Let therefore U be a non-empty $d'_{F'}$-open subset of $X_{\mathbb{G}'}\backslash\nu^\infty$. To prove that U is also d'_F-open, let $\mathcal{A} = F^{\mathbb{E}} \cup \{\uparrow\}$, where $F^{\mathbb{E}} = \{e \in \mathbb{E} : \ t(e) \in F\}$ and define for each $x \in X_{\mathbb{G}'}$ the element $\tilde{x} \in \mathcal{A}^{\mathbb{Z}}$ as in (7.8). There is an $N \in \mathbb{N}$ with the property that when $x, y \in X_{\mathbb{G}'}, \ i \in \mathbb{Z}, \ k \in \mathbb{N}$, we have the implication

$$x_i = y_i \in F, \ \tilde{x}_{[i-k-N,i+k+N]} = \tilde{y}_{[i-k-N,i+k+N]} \ \Rightarrow \ x_{[i-k,i+k]} = y_{[i-k,i+k]}. \tag{7.18}$$

This follows from Lemma 7.7 and the correponding statement for incoming rays, in combination with Lemma 7.19. Let then $x \in U$. Since U is $d'_{F'}$-open there is an $M \in \mathbb{N}$ such that $y \in U$ when $y_{[-M,M]} = x_{[-M,M]}$. Since $x \notin \nu^\infty$ there is an $i \in \mathbb{Z}$ such that $x_i \in F$. Choose $k > |i| + M$. There is an $\epsilon > 0$ such that $\tilde{x}_{[i-k-N,i+k+N]} = \tilde{y}_{[i-k-N,i+k+N]}$ when $d'_F(x,y) \leq \epsilon$. It follows then from (7.18) that $\{y \in X_{\mathbb{G}'} : \ d'_F(y,x) < \epsilon\} \subseteq U$. $\square$

LEMMA 7.23. *$\psi : (X_{\mathbb{G}'''}, \sigma) \to (X_{\mathbb{G}'}\backslash\nu^\infty, \sigma)$ is a conjugacy.*

PROOF. This follows from Lemma 7.20 and Lemma 7.22. $\square$

LEMMA 7.24. *There is a factor map $\pi : X^+_{\mathbb{G}'} \to (X_{\mathbb{G}'}\backslash\nu^\infty)^+ \simeq X^+_{\mathbb{G}'''}$, and, if $\mathbb{G}'$ is finite, a factor map $\pi : X_{\mathbb{G}'} \to X^+_{\mathbb{G}'''}$.*

PROOF. It follows from Lemma 7.22 that there is an embedding $\mu : (X_{\mathbb{G}'}\backslash\nu^\infty)^+ \to \left(F^{\mathbb{E}} \cup \{\uparrow\}\right)^\infty$ such that $\mu(+) = \uparrow^\infty$. Similarly, by Lemma 7.6, there is an embedding $\mu' : X^+_{\mathbb{G}'} \to \left(F^{\mathbb{E}} \cup \{e_N\} \cup \{\uparrow\}\right)^\infty$ such that $\mu'(+) = \uparrow^\infty$. Let $\pi : \left(F^{\mathbb{E}} \cup \{e_N\} \cup \{\uparrow\}\right)^\infty \to \left(F^{\mathbb{E}} \cup \{\uparrow\}\right)^\infty$ be the one-block factor map sending e_N to $\uparrow$. The diagram

$$\begin{tikzcd}
X_{\mathbb{G}'}\backslash\nu^\infty \arrow[r, hook] \arrow[d, "\mu"] & X_{\mathbb{G}'} \arrow[d, "\mu'"] \\
\left(F^{\mathbb{E}} \cup \{\uparrow\}\right)^\infty & \left(F^{\mathbb{E}} \cup \{e_N\} \cup \{\uparrow\}\right)^\infty \arrow[l, "\pi"]
\end{tikzcd}$$

commutes so we conclude that $\pi \circ \mu'\left(X^+_{\mathbb{G}'}\right) = \mu\left((X_{\mathbb{G}'}\backslash\nu^\infty)^+\right)$. When $\mathbb{G}'$ is finite we find that $\pi \circ \mu'\left(X_{\mathbb{G}'}\right) = \mu\left((X_{\mathbb{G}'}\backslash\nu^\infty)^+\right)$. This proves the lemma. $\square$

LEMMA 7.25. *Assume that $\mathbb{G}$ is totally grafted. There is a $*$-isomorphism $\varphi : B_{\mathbb{G}'''} \to B_{\mathbb{G}'}$ such that $\varphi_* \circ \sigma_* = \sigma_* \circ \varphi_*$ on $K_*\left(B_{\mathbb{G}'''}\right)$.*

PROOF. It follows from Lemma 7.23 and Lemma 4.8 that ψ restricts to a homeomorphism from $W_{\mathbb{G}'''}$ onto $W_{X_{\mathbb{G}'}\backslash\nu^\infty,\sigma}$. Set

$$\tilde{W}_{\mathbb{G}'} = \bigcup_{q \in \operatorname{Per} X_{\mathbb{G}'}\backslash\nu^\infty} W^u(q),$$

which is an open and closed subset of $W_{\mathbb{G}'}$. Note that there is an identification $W_{X_{\mathbb{G}'}\backslash\nu^\infty,\sigma} = \tilde{W}_{\mathbb{G}'}$ which we use in the following. We will check that $\psi : W_{\mathbb{G}'''} \to \tilde{W}_{\mathbb{G}'}$ and $\psi^{-1} : \tilde{W}_{\mathbb{G}'} \to \mathbb{G}'''$ both satisfy conditions 1,2 and 3 of Section 2.1 relative to $(X_{\mathbb{G}'''}, d'_F, W_{\mathbb{G}'''})$ and $\left(X_{\mathbb{G}'}, d'_{F'}, \tilde{W}_{\mathbb{G}'}\right)$. (Recall that $F' = F \cup \{t(e_N)\}$.) Condition 2 is clearly satisfied by both maps. To check the other two note first that

$$d'_{F'}\left(\psi(x), \psi(y)\right) = d'_{F'}\left(x, y\right) \geq d'_F(x,y)$$

for all $x, y \in W_{\mathbb{G}'''}$. It follows that $\left(\psi^{-1}(U'), \psi^{-1}(V'), \psi^{-1} \circ \chi' \circ \psi\right)$ is a local conjugacy in $W_{\mathbb{G}'''}$ from x to y when (U', V', χ') is a conjugacy in $\tilde{W}_{\mathbb{G}'}$ from $\psi(x)$ to $\psi(y)$. In order to have established condition 1 and 3 for both ψ and ψ^{-1} it suffices now to establish the following: When (U, V, χ) is a local conjugacy in $W_{\mathbb{G}'''}$ from x to y, there is an open set U' such that $x \in U' \subseteq U$ and such that $\left(\psi(U'), \psi(\chi(U')), \psi \circ \chi \circ \psi^{-1}\right)$ is a local conjugacy from $\psi(x)$ to $\psi(y)$ in $\tilde{W}_{\mathbb{G}'}$. If there is a normal conjugacy from x to y we have that $x_i = y_i$ for all large i and it follows that $\psi(x)_i = \psi(y)_i$ for all large i. In this case we can use

$$U' = \{z \in W_{\mathbb{G}'''} : \ z_i = x_i, \ i \leq i_0\},$$

provided i_0 is so large that $U' \subseteq U$, $x_{i_0} = y_{i_0}$, $\psi(x)_{i_0} = \psi(y)_{i_0}$ and χ agrees with the normal conjugacy $\chi_0 : U' \to \{z \in W_{\mathbb{G}'''} : \ z_i = y_i, \ i \leq i_0\}$ defined such that $\chi_0(z)_k = z_k, k \geq i_0 + 1$. Assume therefore that there is no normal conjugacy from x to y. Since $\mathbb{G}'''$ is totally grafted we conclude from Lemma 7.9 that x and y must, eventually, stay on the same maximal outgoing ray γ'. If $\gamma' = \gamma$, there are $N, M \in \mathbb{N}$ and a $k \in \mathbb{Z}$ such that $x_i = e_{M+i}, y_i = e_{M+i+k}$ for all $i \geq N$. The assumed conjugacy between x and y then yields the conclusion that k must an asymptotic period of γ and it follows then from Lemma 7.17 that $\psi(x)_i = \psi(y)_i$ for all large enough i. There is then a normal conjugacy (U'', V'', χ'') from $\psi(x)$ to $\psi(y)$ in this case. Since

$$d'_F\left(\sigma^j \circ \psi^{-1} \circ \chi'' \circ \psi(z), \sigma^j(z)\right) \leq d'_{F'}\left(\sigma^j \circ \chi'' \circ \psi(z), \sigma^j \circ \psi(z)\right)$$

tends to zero uniformly on $\psi^{-1}(U'')$ we conclude that $\left(\psi^{-1}(U''), \psi^{-1}(V''), \psi^{-1} \circ \chi'' \circ \psi\right)$ is a conjugacy from x to y in $W_{\mathbb{G}'''}$. The existence of the desired set U' follows then from Lemma 1.4 in this case.

Assume then that $\gamma' \neq \gamma$. There is an $N \in \mathbb{N}$ such that

$$d'_F\left(\sigma^j(\chi(z)), \sigma^j(z)\right) \leq \frac{1}{2} \min\left\{\frac{1}{h(e)} - \frac{1}{h(e)+1} : \ e \in F^{\mathbb{E}}\right\}$$

for all $j \geq N$ and all $z \in U$. Choose $M \geq N$ such that $x_i, y_i \in \gamma'$ for all $i \geq M$. Set

$$U' = U \cap \{z \in W_{\mathbb{G}'''} : \ z_i = x_i, \ i \leq M\} \cap \chi^{-1}\left(\{z \in W_{\mathbb{G}'''} : \ z_i = y_i, \ i \leq M\}\right)$$

and $V' = \chi(U')$. Then

$$\{i \geq M : \ \psi(z)_i \in F'\} = \{i \geq M : \ \psi(\chi(z))_i \in F'\}$$

for all $z \in U'$, and it follows that

$$\lim_{k \to \infty} d'_{F'}\left(\sigma^k(\psi(z)), \sigma^k(\psi(\chi(z)))\right) = 0,$$

uniformly on U'. Then $\left(\psi(U'), \psi(V'), \psi \circ \chi \circ \psi^{-1}\right)$ is a local conjugacy between $\psi(x)$ and $\psi(y)$.

Having established that both ψ and ψ^{-1} satisfy condition 1, 2 and 3 of Section 2.1 it follows from Theorem 2.4 that ψ induces a $*$-isomorphism $\psi^\bullet : A_\sigma\big(X_{\mathbb{G}'}, \tilde{W}_{\mathbb{G}'}\big) \to A_\sigma(X_{\mathbb{G}'''}, W_{\mathbb{G}'''})$ such that $\psi^\bullet \circ \sigma^\bullet = \sigma^\bullet \circ \psi^\bullet$. The existence of the desired $*$-isomorphism $\varphi : B_{\mathbb{G}'''} \to B_{\mathbb{G}'}$ follows now from Corollary 2.14 and Lemma 7.4. On the level of K-theory the isomorphism $\varphi : B_{\mathbb{G}'''} \to B_{\mathbb{G}'}$ induces the same map as the composition

$$A_\sigma(X_{\mathbb{G}'''}, W_{\mathbb{G}'''}) \xrightarrow{(\psi^\bullet)^{-1}} A_\sigma\left(X_{\mathbb{G}'}, \tilde{W}_{\mathbb{G}'}\right) \hookrightarrow A_\sigma(X_{\mathbb{G}'}, W_{\mathbb{G}'}),$$

which is equivariant. Hence $\varphi_* \circ \sigma_* = \sigma_* \circ \varphi_*$. $\square$

7.2. The structure of the heteroclinic algebra

Let $\mathbb{G}$ be a graph satisfying conditions i) through iv). By Lemma 7.15 there is a totally grafted graph $\mathbb{G}''$ which satisfies conditions i) through iv) and an σ-equivariant isomorphism $B_{\mathbb{G}} \simeq B_{\mathbb{G}''}$. We can then use the procedure of Section 7.1.2, called pruning, to remove, one by one, the maximal outgoing rays of $\mathbb{G}''$ for which C) and D) hold. The resulting graph, $\mathbb{G}'$, will still satisfy conditions i) through iv), and it follows from Lemma 7.25 that there is a $*$-isomorphism $B_{\mathbb{G}'} \simeq B_{\mathbb{G}''}$ which is σ_*-equivariant on K-theory. We will say that $\mathbb{G}'$ is obtained from $\mathbb{G}$ by *grafting and pruning.*

THEOREM 7.26. *Assume that $\mathbb{G}$ satisfies conditions i), ii), iii) and iv). It follows that there is a graph $\mathbb{G}'$ such that*

- *$\mathbb{G}'$ satisfies conditions i), ii), iii) and iv).*
- *$\mathbb{G}'$ is obtained from $\mathbb{G}$ by grafting and pruning.*
- *There is a $*$-isomorphism $\varphi : B_{\mathbb{G}} \to B_{\mathbb{G}'}$ such that $\varphi_* \circ \sigma_* = \sigma_* \circ \varphi_*$ on $K_0(B_{\mathbb{G}})$.*
- *The dimension group $K_0(B_{\mathbb{G}})$ is isomorphic, as a partially ordered group, to the dimension group $K_0(X_A)$ of [**Wa**] corresponding to the adjacency matrix A of $\mathbb{G}'$.*

PROOF. At this point only the last assertion requieres a proof. For this note that by Lemma 7.9 every local conjugacy in $W_{\mathbb{G}'}$ is normal, at least after a shrinking of its domain. It is then clear that $B_{\mathbb{G}'}$ is identical with the AF-algebra $B_\sigma(X_{\mathbb{G}'})$ of Section 4.3. As pointed out in Section 4.3, $K_0(B_\sigma(X_{\mathbb{G}'}))$ is isomorphic, as a partially ordered group, to the dimension group $K_0(X_A)$ of [**Wa**]. □

In has been shown by D. Fiebig and U. Fiebig that $X^+_{\mathbb{G}}$ is always a synchronized system when conditions i) through iv) hold. See Lemma 4.5 of [**F**]. The process of grafting and pruning can be used to describe necessary and sufficient conditions for $X^+_{\mathbb{G}}$ to be sofic and of finite type.

THEOREM 7.27. *Let $\mathbb{G}$ be a graph satisfying conditions i) through iv). Then $X^+_{\mathbb{G}}$ is sofic if and only if $\mathbb{G}$ can be made finite by grafting and pruning.*

PROOF. It follows from Lemma 7.15 and Lemma 7.24 that there is a factor map $X^+_{\mathbb{G}'} \to X^+_{\mathbb{G}}$, and if $\mathbb{G}'$ is finite, a factor map $X_{\mathbb{G}'} \to X^+_{\mathbb{G}}$. It follows that $X^+_{\mathbb{G}}$ is sofic in the latter case, cf. Theorem 3.2.1 of [**LM**]. Conversely, assume that $X^+_{\mathbb{G}}$ is sofic. By Lemma 7.15 there is a totally grafted graph $\mathbb{H}$, obtained from $\mathbb{G}$ by grafting, such that $X^+_{\mathbb{H}}$ is conjugate to $X^+_{\mathbb{G}}$. Hence $X^+_{\mathbb{H}}$ is sofic. We claim that all outgoing rays in $\mathbb{H}$ are asymptotically periodic in the sense that conditions C) and D) hold for some N, k, L. To see this we identify $X^+_{\mathbb{H}}$ with a subshift of $\mathcal{A}^{\mathbb{Z}}$ as described in the proof of Lemma 7.6. Let $\gamma = e_0e_1e_2\dots$ be a maximal outgoing ray in $\mathbb{H}$ which is not asymptotically periodic, and choose $e \in \mathbb{E}$ such that $t(e) = i(e_0)$. For each $n \in \mathbb{N}$,

$$w_n = e\underbrace{\uparrow\uparrow\dots\uparrow}_{n}$$

is a word $X^+_{\mathbb{H}}$, and since γ is not asymptotically periodic it follows that w_n and w_m have different follower-sets in $X^+_{\mathbb{H}}$ when $n \neq m$, i.e. the sets

$$\left\{u \in \mathbb{W}\left(X^+_{\mathbb{H}}\right) : \ w_n u \in \mathbb{W}\left(X^+_{\mathbb{H}}\right)\right\}$$

are different for different n. This contradicts that $X^+_{\mathbb{H}}$ is sofic, cf. Theorem 3.2.10 of [**LM**]. Thus all outgoing rays in $\mathbb{H}$ are asymptotically periodic and they can then be pruned off one by one to get a finite graph. $\square$

The necessary and sufficient condition for $X^+_{\mathbb{G}}$ to be sofic is neatly reflected in the structure of the heteroclinic algebra $B_{\mathbb{G}}$:

THEOREM 7.28. *In the setting of Theorem 7.26, $B_{\mathbb{G}}$ is simple if and only if $\mathbb{G}'$ is finite and aperiodic.*

PROOF. When $\mathbb{G}'$ is finite the heteroclinic algebra $B_{\mathbb{G}'}$ is $*$-isomorphic to the stabilized AF-algebra whose Bratteli diagram is stationary with the connecting map given by the adjacency matrix of $\mathbb{G}'$. This is a simple algebra if and only if $\mathbb{G}'$ is aperiodic. By Theorem 7.26 this implies that $B_{\mathbb{G}}$ is simple.

For the converse it suffices to show that $B_{\mathbb{G}'}$ is not simple when $\mathbb{G}'$ is infinite. To see this, note first that when $\mathbb{G}'$ is infinite there has to be an outgoing ray in $\mathbb{G}'$. Let then $\gamma = e_0e_1e_2\dots$ be such a ray. Let $z \in W_{\mathbb{G}'}$ be an element such that $z_0 = e_0$, and $x \in W_{\mathbb{G}'}$ an element such that $x_{[0,\infty)} = e_1e_2e_3\dots$. It is then easy to see that there is no normal conjugacy in $W_{\mathbb{G}'}$ from x to any element of

$$U = \{y \in W_{\mathbb{G}'} :\ y_i = z_i, i \leq 0\}.$$

It follows then from Lemma 7.9 that there is no conjugacy at all from x to an element of U, and then $B_{\mathbb{G}'}$ is not simple by Proposition 4.6 of [**Re1**]. $\square$

For completeness we observe the following consequence of our methods:

THEOREM 7.29. *Let $\mathbb{G}$ be an infinite graph satisfying conditions i) through iv). Then the one-point compactification $X^+_{\mathbb{G}}$ is of finite type if and only if every totally grafted graph $\mathbb{H}$, obtained from $\mathbb{G}$ by grafting, only has one maximal outgoing ray and that ray has asymptotic period 1.*

PROOF. $X^+_{\mathbb{H}}$ is conjugate to $X^+_{\mathbb{G}}$, cf. Lemma 7.13. Let F be a core for $\mathbb{H}$ and represent $X^+_{\mathbb{H}}$ as a subshift of $\left\{F^{\mathbb{E}} \cup \{\uparrow\}\right)^{\infty}$, cf. Lemma 7.6. If $\mathbb{H}$ has two different maximal outgoing rays, conditions A) and B) can not hold for them since $\mathbb{H}$ is totally grafted. It is then not difficult to show that there are infinitely many $n \in \mathbb{N}$ for which $\uparrow^n$ is a word in $X^+_{\mathbb{H}}$ which is not synchronizing. Hence $X^+_{\mathbb{H}}$ is not of finite type when there is more than one maximal outgoing ray in $\mathbb{H}$. Note that the single remaining outgoing ray in $\mathbb{H}$ must be asymptotically periodic, in the sense that conditions B) and C) must hold, if $X^+_{\mathbb{G}}$ is to be sofic or even of finite type by Theorem 7.27. If the asymptotic period is not one it is easy to show that there are also infinitely n for which $\uparrow^n$ is not synchronizing for $X^+_{\mathbb{H}}$ when this shiftspace is realized as a subshift of $\left(F^{\mathbb{E}} \cup \{e_N\} \cup \{\uparrow\}\right)^{\infty}$, in the notation from the proof of Lemma 7.24. In this way we obtain the necessity of the stated condition for $X^+_{\mathbb{G}}$ to be of finite type.

To prove the converse it suffices to prove that the factor map $\pi : X_{\mathbb{G}'} \to X^+_{\mathbb{G}'''}$ of Lemma 7.24 is injective when ν^{∞} only contains one point. We leave this to the reader. $\square$

There are other conclusions one can draw from the methods developed in this chapter. For example that only very special sofic shift spaces can be the one-point compactification of a countable state Markov chain coming from a locallly finite strongly connected countable graph. Specifically, in the terminology of [**Th5**] they must have depth one and the derived shift space must consist entirely of periodic orbits. To reach this conclusion one must also use work by D. Fiebig and M. Schraudner, [**Sch**].

APPENDIX A

Étale equivalence relations from abelian C^*-subalgebras with the extension property

The material in this appendix is to some extend build on ideas from work of Kumjian and Renault from the first half of the 80's. See [**Ku**] and [**Re2**]. The main results, however, are new. This is probably only because we are here interested in the passage from étale equivalence relations to C^*-algebras and back, rather than from C^*-algebras to groupoids and back.

Let A be a C^*-algebra and $D \subseteq A$ an abelian C^*-subalgebra. Let $P(A)$ and $P(D)$ be the pure state spaces of A and D, respectively, considered as topological spaces in their respective weak*-topologies.

Lemma A.1. *Let $\omega \in P(D)$, and let $\tilde{\omega} \in A^*$ be a state extension of ω. Then $\tilde{\omega}(ad) = \tilde{\omega}(a)\omega(d)$ for all $a \in A, d \in D$.*

Proof.

$$\begin{aligned}|\tilde{\omega}(ad) - \tilde{\omega}(a)\omega(d)|^2 &= |\tilde{\omega}\left(a\left(d - \omega(d)\right)\right)|^2 \\ &\leq \tilde{\omega}(aa^*)\omega\left(\left(d - \omega(d)\right)^*\left(d - \omega(d)\right)\right) = 0\end{aligned}$$

since ω is a character on D. □

We assume that D has *the extension property* in A, i.e. that every pure state of D has a unique (pure) state extension to A. Given $\omega \in P(D)$ we let $\tilde{\omega}$ denote the unique pure state $\tilde{\omega} \in P(A)$ extending ω.

Lemma A.2. *The map $P(D) \ni \omega \mapsto \tilde{\omega} \in P(A)$ is continuous.*

Proof. By Corollary 2.7 (c) and Remark 2.6 (iii) of [**ABG**] there is a conditional expectation $\theta : A \to D$ such that $\tilde{\omega}(x) = \omega\left(\theta(x)\right)$ for all $x \in A$. □

Set

$$N(D) = \left\{a \in A : \ aD = Da, \ aa^* \in D, \ a^*a \in D\right\}.$$

Note that when $\omega \in P(D)$ and $v \in N(D)$ we can define a functional on D such that $D \ni d \mapsto \omega\left(vdv^*\right)$ for all $d \in D$. We denote this functional by $\omega\left(v - v^*\right)$ in the following. In fact, we will sligthly extend this notation to similar cases when the meaning is clear.

Definition A.3. Let $\omega, \mu \in P(D)$. A *local A-conjugacy* from ω to μ is a pair (U, v, V) where U and V are neighborhoods in $P(D)$ of ω and μ, respectively, and $v \in N(D)$ is an element such that $\omega\left(v - v^*\right) = \mu$ and $V = \left\{\nu\left(v - v^*\right) : \ \nu \in U\right\}$.

Lemma A.4. *Let (U, v, V) be a local A-conjugacy from ω to μ. Then*

i) $\nu\left(vv^*\right) = 1$ *for all* $\nu \in U$,
ii) $\nu\left(v^*v\right) = 1$ *for all* $\nu \in V$, *and*
iii) (V, v^*, U) *is a local A-conjugacy from μ to ω.*

Proof. i) follow from the fact that $\nu(v - v^*)$ is a state for all $\nu \in U$. iii): Let $\nu' \in V$. Then $\nu' = \nu(v - v^*)$ for some $\nu \in U$ and hence $\nu'(v^* - v) = \nu(vv^* - vv^*) = \nu$ by i) and Lemma A.1.This shows that $\{\nu'(v^* - v) : \ \nu' \in V\} \subseteq U$. Conversely, when $\nu \in U$, $\nu' = \nu(v - v^*) \in V$ and $\nu = \nu'(v^* - v)$. Hence $U \subseteq \{\nu'(v^* - v) : \ \nu' \in V\}$, and iii) follows. ii) follows from i) and iii). $\square$

Lemma A.5. *Let (U, v, V) be a local A-conjugacy from ω to μ and (U', w, V') a local A-conjugacy from μ to ν. Then (U, vw, V') is a local A-conjugacy from ω to ν.*

Proof. Note that $\omega(vw - w^*v^*) = \mu(w - w^*) = \nu$. When $\kappa \in V'$ there is a $\kappa' \in U'$ such that $\kappa = \kappa'(w - w^*)$ and a $\kappa'' \in U$ such that $\kappa''(v - v^*) = \kappa'$. Then $\kappa''(vw - w^*v^*) = \kappa$, proving that $V' \subseteq \{\kappa(vw - w^*v^*) : \ \kappa \in U\}$. The reversed inclusion is also trivial. $\square$

We say that $\omega, \mu \in P(D)$ are *locally conjugate* and write $\omega \sim \mu$ when there is a local A-conjugacy from ω to μ.

Lemma A.6. *Local conjugacy is an equivalence relation on $P(D)$.*

Proof. Symmetry is iii) of Lemma A.4 and transitivity is Lemma A.5. To see that $\omega \in P(D)$ is locally conjugate to itself it suffices to pick an element $d \in D$ such that $\nu(dd^*) = 1$ for all ν in a neighborhood U of ω. Then (U, d, U) is local A-conjugacy from ω to itself. $\square$

Set

$$R(A, D) = \{(\omega, \mu) \in P(D) \times P(D) : \ \omega \sim \mu\}.$$

We go on to make $R(A, D)$ an étale equivalence relation, and for this we proceed in complete analogy with the construction from Section 1.2. The topology of $R(A, D)$ is given by the subbase consisting of the sets of the form

$$\{(\nu, \nu(v - v^*)) : \ \nu \in U\} \tag{A.1}$$

for some local A-conjugacy (U, v, V).

Lemma A.7. *Let (U, v, V) be a local A-conjugacy from ω to μ. Let $U_0 \subseteq U$ be an open subset containing ω. It follows that $V_0 = \{\nu(v - v^*) : \ \nu \in U_0\}$ is open in $P(D)$ and that (U_0, v, V_0) is a local A-conjugacy from ω to μ.*

Proof. Let $\nu' \in V_0$. Then $\nu' = \nu(v - v^*)$ for some $\nu \in U_0$. Note that $\nu = \nu'(v^* - v)$ by Lemma A.4 and Lemma A.1. Furthermore, it follows from Lemma A.2 that $\omega(v^* - v)$ is close to $\nu'(v^* - v) = \nu$ in $P(D)$ when ω is close to ν'. There is therefore an open neighborhood V_0' of ν' such that $V_0' \subseteq V$ and $\omega(v^* - v) \in U_0$ when $\omega \in V_0'$. Since $\omega(v^*v - v^*v) = \omega$ when $\omega \in V$ by Lemma A.4 and Lemma A.1, we conclude that $V_0' \subseteq V_0$, proving that V_0 is open. The rest is trivial. $\square$

For every state τ of A we denote by $(H_\tau, \pi_\tau, \xi_\tau)$ the GNS-representation of A, i.e. H_τ is a Hilbert space, $\pi_\tau : A \to \mathbb{L}_{\mathbb{C}}(H_\tau)$ is a $*$-homomorphism, and ξ_τ is a unit vector in H_τ such that $\tau(a) = \langle \xi_\tau, \pi_\tau(a)\xi_\tau \rangle$ for all $a \in A$, and $\pi_\tau(A)\xi_\tau$ is dense in H_τ.

Lemma A.8. *Let $\omega \in P(D)$.*

i) $\pi_{\tilde{\omega}}(d)\xi_{\tilde{\omega}} = \omega(d)\xi_{\tilde{\omega}}$, $\omega \in P(D)$, $d \in D$.

ii) *For $b \in N(D)$ and $d \in D$, $\pi_{\tilde{\omega}}(d)\pi_{\tilde{\omega}}(b)\xi_{\tilde{\omega}} \in \mathbb{C}\pi_{\tilde{\pi}}(b)\xi_{\tilde{\omega}}$.*

Proof. i) $\|\pi_{\tilde{\omega}}(d)\xi_{\tilde{\omega}} - \omega(d)\xi_{\tilde{\omega}}\|^2 = \omega\left((d-\omega(d))^*\left(d-\omega(d)\right)\right) = 0$.

ii) Since $b \in N(D)$ there is an element $d' \in D$ such that $db = bd'$. Then $\pi_{\tilde{\omega}}(d)\pi_{\tilde{\pi}}(b)\xi_{\tilde{\omega}} = \pi_{\tilde{\pi}}(b)\pi_{\tilde{\omega}}(d')\xi_{\tilde{\omega}} = \omega(d')\pi_{\tilde{\pi}}(b)\xi_{\tilde{\omega}}$ by i). □

Lemma A.9. *Let $\omega \in P(D), a, b \in N(D)$. Assume that $\omega\left(a^*b\right) \neq 0$. It follows that there is a $\lambda \in \mathbb{T}$ such that*

$$\omega\left(a^*a\right)^{-\frac{1}{2}}\pi_{\tilde{\omega}}(a)\xi_{\tilde{\omega}} = \lambda\omega\left(b^*b\right)^{-\frac{1}{2}}\pi_{\tilde{\omega}}(b)\xi_{\tilde{\omega}}.$$

Proof. Since $\left|\omega\left(a^*b\right)\right|^2 \leq \omega\left(a^*a\right)\omega\left(b^*b\right)$, we can set $a' = \omega\left(a^*a\right)^{-\frac{1}{2}}a$ and $b' = \omega\left(b^*b\right)^{-\frac{1}{2}}b$. Then $\pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}}$ and $\pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}}$ unit vectors and

$$\langle \pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}}\rangle \neq 0. \tag{A.2}$$

It follows from ii) of Lemma A.8 that there are characters μ_a, μ_b of D such that $\pi_{\tilde{\omega}}(d)\pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}} = \mu_a(d)\pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}}$ and $\pi_{\tilde{\omega}}(d)\pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}} = \mu_b(d)\pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}}$ for all $d \in D$. It follows from (A.2) that $\mu_a = \mu_b$, and hence that

$$\langle \pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}\left(-\right)\pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}}\rangle = \langle \pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}\left(-\right)\pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}}\rangle$$

on D. It follows from the extension property that this equality holds on A also, and then the irreducibility of $\pi_{\tilde{\omega}}$ implies that $\pi_{\tilde{\omega}}(a')\xi_{\tilde{\omega}} = \lambda\pi_{\tilde{\omega}}(b')\xi_{\tilde{\omega}}$ for some $\lambda \in \mathbb{T}$, cf. e.g. Proposition 3.3.7 of [**Pe**]. □

Lemma A.10. *Let (U, v, V) and (U', w, V') be local A-conjugacies from ω to μ. It follows that there is an open neighborhood $\Omega \subseteq U \cap U'$ of ω such that*

$$\nu\left(vdv^*\right) = \nu\left(wdw^*\right)\ \ \forall d \in D$$

when $\nu \in \Omega$.

Proof. Since $\omega\left(v - v^*\right) = \omega\left(w - w^*\right)$ it follows from the extension property that

$$\langle \pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}(-)\pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}\rangle = \langle \pi_{\tilde{\omega}}\left(w^*\right)\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}(-)\pi_{\tilde{\omega}}\left(w^*\right)\xi_{\tilde{\omega}}\rangle$$

on A. Since $\pi_{\tilde{\omega}}$ is irreducible and $\pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}\left(w^*\right)\xi_{\tilde{\omega}}$ are both unit vectors, we conclude that there is a $\lambda' \in \mathbb{T}$ such that

$$\pi_{\tilde{\omega}}\left(w^*\right)\xi_{\tilde{\omega}} = \lambda'\pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}, \tag{A.3}$$

cf. e.g. Proposition 3.3.7 of [**Pe**]. In particular, $\tilde{\omega}\left(wv^*\right) = \langle \pi_{\tilde{\omega}}\left(w^*\right)\xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}\rangle \neq 0$. It follows then from Lemma A.2 that $\tilde{\nu}\left(w^*v\right) \neq 0$ for all ν in an open neighborhood Ω of ω with $\Omega \subseteq U \cap U'$. By Lemma A.9 this implies that $\pi_{\tilde{\nu}}\left(w^*\right)\xi_{\tilde{\nu}}$ and $\pi_{\tilde{\nu}}\left(v^*\right)\xi_{\tilde{\nu}}$ are proportional when $\nu \in \Omega$. Hence

$$\begin{aligned}\nu\left(v - v^*\right) &= \langle \pi_{\tilde{\nu}}\left(v^*\right)\xi_{\tilde{\nu}}, \pi_{\tilde{\nu}}(-)\pi_{\tilde{\nu}}\left(v^*\right)\xi_{\tilde{\nu}}\rangle \\ &= \langle \pi_{\tilde{\nu}}\left(w^*\right)\xi_{\tilde{\nu}}, \pi_{\tilde{\nu}}(-)\pi_{\tilde{\nu}}\left(w^*\right)\xi_{\tilde{\nu}}\rangle = \nu\left(w - w^*\right)\end{aligned}$$

when $\nu \in \Omega$. □

Corollary A.11. *A base for the topology of $R(A, D)$ is given by the sets of the form (A.1).*

The following generalizes the second proposition on page 437 of [**Re2**].

Theorem A.12. *$R(A, D)$ is an étale equivalence relation.*

Proof. With Lemma A.10 replacing Lemma 1.4 the proof of Theorem 1.7 can be adopted with only the obvious changes. □

Consider now an étale equivalence relation R on the locally compact Hausdorff space X. Let $r : R \to X$ and $s : R \to X$ be the range and source map, respectively. By a *graph* in R we shall mean an open subset $U \subseteq R$ such that $r(U)$ and $s(U)$ are open in X and $r : U \to r(U)$, $s : U \to s(U)$ are homeomorphisms. An element $f \in C_c(R)$ is *localized* when there is a graph $U \subseteq R$ such that $\operatorname{supp} f \subseteq R$. Every element of $C_c(R)$ is a finite linear combination of localized functions. In particular the localized functions span a dense subspace in $C_r^*(R)$.

LEMMA A.13. *$C_0(X)$ has the extension property in $C_r^*(R)$.*

PROOF. Let ω be a pure state of $C_0(X)$, and let $\tilde{\omega}$ be a state extension of ω to $C_r^*(R)$. Let $x_0 \in X$ be the point such that $\omega(h) = h(x_0)$ for all $h \in C_0(X)$. It suffices to show that $\tilde{\omega}(f) = f(x_0, x_0)$ when $f \in C_c(R)$. Since $\operatorname{supp} f \cap X^c$ is a compact subset of R we can find functions k_i, h_i in $C_c(X)$ and $\psi_i \in C_c(R)$, $i = 1, 2, \dots, N$, such that $h_i k_i = 0$ for all i, $\sum_{j=1}^N \psi_j(x,y) = 1$ for all $(x,y) \in \operatorname{supp} f \cap X^c$, and

$$\psi_i(x,y) = k_i(x)\psi_i(x,y)h_i(y) \tag{A.4}$$

for all $(x,y) \in R$ and all $i = 1, 2, \dots, N$. It follows from Lemma A.1 and (A.4) that

$$\tilde{\omega}(\psi_i f) = \tilde{\omega}(k_i \cdot (\psi_i f) \cdot h_i) = k_i(x_0)\, h_i(x_0)\, \tilde{\omega}(\psi_i f).$$

Since $k_i(x_0)\, h_i(x_0) = 0$ we conclude that $\tilde{\omega}(\psi_i f) = 0$ for all i. Since $f - \sum_{i=1}^N \psi_i f$ is supported in $X \subseteq R$, we find that

$$\tilde{\omega}(f) = \omega\Big(f - \sum_{i=1}^N \psi_i f\Big) = \Big(f - \sum_{i=1}^N \psi_i f\Big)(x_0, x_0) = f(x_0, x_0).$$

$\square$

By Lemma A.13 and Theorem A.12 we can consider the étale equivalence relation $R(C_r^*(R), C_0(X))$.

LEMMA A.14. *Let $f \in C_c(R)$ be localized. Then $f \in N(C_0(X))$.*

PROOF. Let $U \subseteq R$ be a graph with $\operatorname{supp} f \subseteq U$. Let $\gamma : s(U) \to r(U)$ be the homeomorphism such that $(\gamma(t), t) \in U$ for all $t \in s(U)$. When $d \in C_0(X)$ there is a $h \in C_c(r(U))$ such that $d \cdot f = (dh) \cdot f = f \cdot ((dh) \circ \gamma)$. This show that $C_0(X) \cdot f = f \cdot C_0(X)$. Since $f \cdot f^*, f^* \cdot f \in C_c(X)$ this completes the proof. $\square$

THEOREM A.15. *There is an isomorphism $R \to R(C_r^*(R), C_0(X))$ of étale equivalence relations given by*

$$R \ni (x,y) \mapsto (\mathrm{ev}_x, \mathrm{ev}_y), \tag{A.5}$$

where $\mathrm{ev}_x \in C_0(X)^*$ is the functional which evaluates functions at x.

PROOF. Set $\Phi(x,y) = (\mathrm{ev}_x, \mathrm{ev}_y)$. When $(x,y) \in R$ there is a localized function $f \in C_c(R)$ such that $f(x', y') = 1$ for all (x', y') in a neighborhood Ω of (x,y) in R. Then $f \in N(C_0(X))$ by Lemma A.14. Let $\mu : r(\Omega) \to s(\Omega)$ be the homeomorphism such that $\Omega = \{(t, \mu(t)) :\ t \in r(\Omega)\}$. Then

$$f \cdot h \cdot f^*(t,t) = \sum_{a,b} f(t,a)h(a,b)\overline{f(b,t)} = h(\mu(t), \mu(t))$$

for all $t \in r(\Omega)$ and all $h \in C_c(R)$. It follows that $\tilde{\mathrm{ev}}_t(f - f^*) = \tilde{\mathrm{ev}}_{\mu(t)}$ for all $t \in r(\Omega)$. This show that (U, f, V) is a local $C_r^*(R)$-conjugacy from ev_x to ev_y. We conclude that $(\mathrm{ev}_x, \mathrm{ev}_y) \in R(C_r^*(R), C_0(X))$, and we have therefore proved that

$\Phi(R) \subseteq R\left(C_r^*(R), C_0(X)\right)$. It remains to show that $\Phi : R \to R\left(C_r^*(R), C_0(X)\right)$ is a homeomorphism.

Let $(\omega, \mu) \in R\left(C_r^*(R), C_0(X)\right)$ and let (U, v, V) be a $C_r^*(R)$-conjugacy from ω to μ. There are points $x, y \in X$ such that $\omega = \mathrm{ev}_x$ and $\mu = \mathrm{ev}_y$. The functional $\tilde{\omega}\left(-v^*\right)$ is non-zero and hence $\tilde{\omega}\left(gv^*\right) \neq 0$ for some localized function $g \in C_c(R)$. Since $g \in N\left(C_0(X)\right)$ it follows from Lemma A.9 that $\pi_{\tilde{\omega}}\left(g^*\right)\xi_{\tilde{\omega}} = \lambda \pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. Then

$$\tilde{\omega}\left(-g^*\right) = \left\langle \xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}(-)\pi_{\tilde{\omega}}\left(g^*\right)\xi_{\tilde{\omega}}\right\rangle = \lambda\left\langle \xi_{\tilde{\omega}}, \pi_{\tilde{\omega}}(-)\pi_{\tilde{\omega}}\left(v^*\right)\xi_{\tilde{\omega}}\right\rangle = \lambda\tilde{\omega}\left(-v^*\right). \tag{A.6}$$

Since $\sum_a g(x,a)\overline{g(a,x)} = \tilde{\omega}\left(g \cdot g^*\right) = \lambda\tilde{\omega}\left(gv^*\right) \neq 0$ there is an element $z \in X$ such that $(x,z) \in R$ and $g\left(x,z\right) \neq 0$. Note that z is unique because g is localized. Assume to obtain a contradiction that $z \neq y$. Choose $h \in C_c(X)$ such that $h(z) = 0$ and $h(y) = 1$. Let $1_x \in l^2[x]$ be the characteristic function at x and note that

$$\omega\left(-\right) = \left\langle 1_x, \kappa_{[x]}(-)1_x\right\rangle$$

on $C_0(X)$. The extension property of $C_0(X)$ in $C_r^*(R)$ and Proposition 3.3.7 of [**Pe**] imply that there is a unitary $U : H_{\tilde{\omega}} \to l^2[x]$ such that $U\xi_{\tilde{\omega}} = 1_x$ and $\kappa_{[x]}(-) = U\pi_{\tilde{\omega}}(-)U^*$. It follows that

$$\tilde{\omega}\left(-g^*\right) = \left\langle 1_x, \kappa_{[x]}(-)\kappa_{[x]}(g^*)1_x\right\rangle = \overline{g(z,x)}\left\langle 1_x, \kappa_{[x]}(-)1_z\right\rangle.$$

Inserting vh we find that

$$\tilde{\omega}\left(vhg^*\right) = \overline{g(z,x)}\left\langle 1_x, \kappa_{[x]}(v)\kappa_{[x]}(h)1_z\right\rangle = 0$$

since $\kappa_{[x]}(h)1_z = h(z)1_z = 0$. In contrast,

$$\tilde{\omega}\left(vhv^*\right) = \mu(h) = h(y) = 1,$$

contradicting (A.6). It follows that $(x,y) \in R$ and that Φ is a bijection.

Let $(x,y) \in R$, and let (U, v, V) be a local $C_r^*(R)$-conjugacy from ev_x to ev_y. There is a localized function $f \in C_c(R)$ such that $f(x',y') = 1$ for all (x',y') in a graph $\Omega \subseteq R$ containing (x,y). Note that

$$\tilde{\mathrm{ev}}_{x'}\left(f - f^*\right) = \left\langle 1_{x'}, \kappa_{[x']}(f)\kappa_{[x']}(-)\kappa_{[x']}(f^*)1_{x'}\right\rangle = \left\langle 1_{y'}, \kappa_{[x']}(-)1_{y'}\right\rangle = \tilde{\mathrm{ev}}_{y'}$$

when $(x',y') \in \Omega$. It follows that (U', f, V') is a local $C_r^*(R)$-conjugacy from ev_x to ev_y, where $U' = \{\mathrm{ev}_z : \ z \in r(\Omega)\}$ and $V' = \{\mathrm{ev}_z : \ z \in s(\Omega)\}$. By Lemma A.10 this implies that $\tilde{\mathrm{ev}}_{x'}\left(f - f^*\right) = \tilde{\mathrm{ev}}_{x'}\left(v - v^*\right)$ for all x' in an open neighborhood U_0 of x such that $U_0 \subseteq U'$. This shows that

$$\Phi\left(\Omega \cap r^{-1}\left(U_0\right)\right) = \left\{\left(\mathrm{ev}_t, \mathrm{ev}_t\left(v - v^*\right)\right) : \ t \in U_0\right\}.$$

It follows that Φ is both open and continuous, and hence a homeomorphism. □

The following theorem is a straightforward consequence of Theorem A.15.

THEOREM A.16. *Two étale equivalence relations, R on X and R' on X', are isomorphic if and only if there is a $*$-isomorphism $\psi : C_r^*(R) \to C_r^*(R')$ such that $\psi\left(C_0(X)\right) = C_0\left(X'\right)$.*

At this point it is natural to ask if we get the pair $D \subseteq A$ back when we construct the reduced groupoid C^*-algebra from $R(A,D)$, and the answer is 'not in general'. This can be seen from the fact that while the set of normalizers $N\left(C_0(X)\right)$ of $C_0(X)$ in $C_r^*(R)$ always span a dense subspace of $C_r^*(R)$ this may not be the case of $N(D)$ in A. See [**Ku**] and [**Re2**] for more on this issue.

APPENDIX B

On certain crossed product C^*-algebras

This appendix contains the results about crossed products which are used in the main body of the text. Most must be known to experts, but I haven't been able to locate the statements in the litterature.

B.1. Translations on tori

In this section we have gathered some technical observations on the structure of crossed products arising from translations in groups, primarily tori.

LEMMA B.1. *Let $\beta : H \to \operatorname{Aut} B$ and $\beta' : H' \to \operatorname{Aut} B'$ be actions of the discrete amenable groups H and H' on the C^*-algebras B and B', respectively. Let $\pi : B \to B'$ be a $*$-homomorphism and $\varphi : H \to H'$ a homomorphism such that*

$$\beta'_{\varphi(h)}\left(\pi(a)\right) = \pi\left(\beta_h(a)\right) \tag{B.1}$$

for all $a \in B$ and all $h \in H$. There is then a $*$-homomorphism $\Pi_{(\pi,\varphi)} : B \rtimes_\beta H \to B' \rtimes_{\beta'} H'$ such that $\Pi_{(\pi,\varphi)}\left(au_h\right) = \pi(a)u_{\varphi(h)}$ for all $a \in B$ and all $h \in H$. $\Pi_{(\pi,\varphi)}$ is injective if π and φ both are.

PROOF. By (B.1) the pair $\left(\pi, u_{\varphi(\cdot)}\right)$ is a covariant representation of (B,β) and as such it gives rise to a $*$-homomorphism between the full crossed products which takes bu_h to $\pi(b)u_{\varphi(h)}$. Since we assume that H is amenable the reduced and the full crossed products agree, cf. [**Pe**]. It follows from the general theory of crossed product that $\Pi_{(\pi,\varphi)}$ is injective when π and φ both are. □

LEMMA B.2. *Let $G_1 \longleftarrow G_2 \longleftarrow G_3 \longleftarrow \dots$ be a sequence of compact abelian groups and continuous surjective group homomorphisms. Let $G = \operatorname{proj\,lim}_k G_k$ be the corresponding inverse limit group, and $H \subseteq G$ a countable subgroup. Let $p^k : G \to G_k$ be the canonical projections and let $H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$ be a sequence of subgroups of H such that $H = \bigcup_j H_j$.*

It follows that $C(G) \rtimes_\tau H$ is $*$-isomorphic to the inductive limit

$$\varinjlim_k \left(C(G_k) \rtimes_{\tau\circ p^k} H_k, \psi_k\right)$$

where each ψ_k is a unital $*$-homomorphism and the action $\tau \circ p^k$ of H_k on $C\left(G_k\right)$ is given by

$$\left(\tau\circ p^k\right)_h(f)(x) = f\left(x - p^k(h)\right), \tag{B.2}$$

$f \in C\left(G_k\right), h \in H_k$.

PROOF. It follows from Lemma B.1 that there is an infinite commuting diagram

$$\begin{tikzcd}
C\left(G_1\right) \rtimes_{\tau\circ p^1} H_1 \arrow[r] \arrow[dr] & C\left(G_2\right) \rtimes_{\tau\circ p^2} H_2 \arrow[r] \arrow[d] & C\left(G_3\right) \rtimes_{\tau\circ p^3} H_3 \arrow[r] \arrow[dl] & \cdots \arrow[dll, "\cdots"] \\
& C(G) \rtimes_\tau H & &
\end{tikzcd} \tag{B.3}$$

of injective $*$-homomorphisms. The actions $\tau \circ p^k$ are here defined such that (B.2) holds. By commutativity of the diagram (B.3) there is a $*$-homomorphism

$$\mu : \varinjlim_k C(G_k) \rtimes_{\tau \circ p^k} H_k \to C(G) \rtimes_\tau H.$$

Note that μ is injective. By using that $C(G)$ is isomorphic to $\varinjlim_j C(G_j)$ it follows that the canonical copy of $C(G)$ is in the range of μ. By construction the same is the case of u_h for each $h \in H$ since $H = \bigcup_j H_j$. Since $\{fu_h : f \in C(G),\ h \in H\}$ generates $C(G) \rtimes_\tau H$, this proves that μ is surjective. $\square$

LEMMA B.3. *Let $\alpha : H \to AutA$ be an action of the countable discrete abelian group H on the separable C^*-algebra A. It follows that $A \rtimes_\alpha H$ is stably isomorphic to $(A \rtimes_\alpha (H/\ker\alpha)) \otimes C\big(\widehat{\ker\alpha}\big)$.*

PROOF. Let $s : H/\ker\alpha \to H$ be a section for the quotient map $H \to H/\ker\alpha$. Then $\omega(x,y) = s(x) + s(y) - s(x+y)$ is a $\ker\alpha$-cocycle which we can consider as a 2-cocycle with values in the unitary group of $C\big(\widehat{\ker\alpha}\big)$. It follows from Theorem 4.1 of [**PR**] that $A \rtimes_\alpha H$ is $*$-isomorphic to the twisted crossed product $A \otimes C\big(\widehat{\ker\alpha}\big) \rtimes_{\alpha\otimes \mathrm{id}_{C(\widehat{\ker\alpha})}, 1_A\otimes\omega} (H/\ker\alpha)$. The 2-cocycle $\omega \otimes 1_{\mathbb{L}}$, which takes values in the unitary group of $\mathbb{L} = \mathbb{L}_{C\left(\widehat{\ker\alpha}\right)}\big(l^2\big(H/\ker\alpha, C\big(\widehat{\ker\alpha}\big)\big)\big)$ is a co-boundary; specifically, $\omega(x,y) \otimes 1_{\mathbb{L}} = v_x v_y v_{x+y}^*$, where

$$v_x\psi(y) = \omega(x,y)\psi(x+y).$$

It follows therefore that

$$\Big(A \otimes C\big(\widehat{\ker\alpha}\big) \rtimes_{\alpha\otimes \mathrm{id}_{C(\widehat{\ker\alpha})}, 1_A\otimes\omega} (H/\ker\alpha)\Big) \otimes \mathbb{K}\left(l^2\left(H/\ker\alpha\right)\right)$$

is $*$-isomorphic to $\big(A \otimes C\big(\widehat{\ker\alpha}\big) \rtimes_{\alpha\otimes \mathrm{id}_{C(\widehat{\ker\alpha})}} (H/\ker\alpha)\big) \otimes \mathbb{K}\left(l^2\left(H/\ker\alpha\right)\right)$, cf. Lemma 3.3 of [**PR**]. This yields the lemma since

$$A \otimes C\big(\widehat{\ker\alpha}\big) \rtimes_{\alpha\otimes \mathrm{id}_{C(\widehat{\ker\alpha})}} (H/\ker\alpha) \simeq (A \rtimes_\alpha (H/\ker\alpha)) \otimes C\big(\widehat{\ker\alpha}\big).$$

$\square$

LEMMA B.4. *Let H be a finitely generated subgroup of the n-torus $\mathbb{T}^n$ and k the rank of the torsion-free part of H. It follows that there is a subgroup $K \subseteq \mathbb{T}^n$ such that $K \simeq \mathbb{Z}^k$ and such that $C(\mathbb{T}^n) \rtimes_\tau H$ is stably isomorphic to $C(\mathbb{T}^n) \rtimes_\tau \mathbb{Z}^k$.*

PROOF. Note that $H \simeq F \oplus \mathbb{Z}^k$ for some finite abelian group F. Then $C(\mathbb{T}^n) \rtimes_\tau H \simeq (C(\mathbb{T}^n) \rtimes_\tau F) \rtimes \mathbb{Z}^k$. Let

$$p = \frac{1}{\#F} \sum_{k\in F} u_k \in C(\mathbb{T}^n) \rtimes_\tau F.$$

A simple calculation shows that p is a projection such that

$$\begin{aligned} p\left(C(\mathbb{T}^n) \rtimes_\tau F\right) p &= \Big\{ \sum_{k\in K} f u_k :\ f \in C(G),\ f(g-k) = f(g),\ \forall g \in G,\ k \in F \Big\} \\ &\simeq C(\mathbb{T}^n/F). \end{aligned}$$

By using that $\mathbb{T}^n/F \simeq \mathbb{T}^n$ we find that

$$p\left(\left(C(\mathbb{T}^n) \rtimes_\tau F\right) \rtimes \mathbb{Z}^k\right) p \simeq C(\mathbb{T}^n) \rtimes_\tau \mathbb{Z}^k.$$

Since the action of H is free, there is bijective correspondance between the ideals of $\left(C\left(\mathbb{T}^n\right) \rtimes_\tau F\right) \rtimes \mathbb{Z}^k$ and the H-invariant ideals of $C\left(\mathbb{T}^n\right)$. It is therefore clear that p is a full projection in $\left(C\left(\mathbb{T}^n\right) \rtimes_\tau F\right) \rtimes \mathbb{Z}^k$ and we conclude from [**Br**] that $\left(C\left(\mathbb{T}^n\right) \rtimes_\tau F\right) \rtimes \mathbb{Z}^k$ is stably isomorphic that $C\left(\mathbb{T}^n\right) \rtimes_\tau \mathbb{Z}^k$. □

The same proof yields also the following, which is a special case of a result of Rieffel, [**Ri1**].

LEMMA B.5. *Let G be a compact group, F a finite normal subgroup of G. It follows that $C\left(G\right) \rtimes_\tau F$ is stably isomorphic to $C\left(G/F\right)$.*

LEMMA B.6. *Let H be finitely generated abelian group and $\varphi : H \to \mathbb{T}^n$ a homomorphism and m the rank of the torsion-free part of H. Let $\tau \circ \varphi : H \to \operatorname{Aut} C\left(\mathbb{T}^n\right)$ be the action given by*

$$\left(\tau \circ \varphi\right)_h (f)(x) = f\left(x - \varphi(h)\right).$$

Then either

i) *there is a natural number $l \in \mathbb{N}$ such that $C\left(\mathbb{T}^n\right) \rtimes_{\tau\circ\varphi} \mathbb{Z}^m$ is stably isomorphic to $\mathbb{C}^l \otimes C\left(\mathbb{T}^{n+m}\right)$, or*

ii) *there are natural numbers l, r, d, k and a non-degenerate special non-commutative torus B of rank (d,k) such that $r+d+k = n+m$ and $C\left(\mathbb{T}^n\right) \rtimes_{\tau\circ\varphi} \mathbb{Z}^m$ is stably isomorphic to $\mathbb{C}^l \otimes C\left(\mathbb{T}^r\right) \otimes B$.*

PROOF. Let l' be the order of the torsion subgroup of $\ker \varphi$. It follows from Lemma B.3 that $C\left(\mathbb{T}^n\right) \rtimes_{\tau\circ\varphi} H$ is stably $*$-isomorphic to

$$\mathbb{C}^{l'} \otimes C\left(\mathbb{T}^a\right) \otimes \left(C\left(\mathbb{T}^n\right) \rtimes_\tau \left(H/\ker\varphi\right)\right)$$

where a is the rank of the torsion-free part of $\ker\varphi$. Write $H/\ker\varphi \simeq F \oplus \mathbb{Z}^k$ where F is a finite group. If $k = 0$ we are in case i) by Lemma B.5. Assume therefore that $k \geq 1$. It follows from Lemma B.4 that $C\left(\mathbb{T}^n\right) \rtimes_\tau \left(H/\ker\varphi\right)$ is stably isomorphic to $C\left(\mathbb{T}^n\right) \rtimes_\tau \mathbb{Z}^k$. Let $\psi : \mathbb{Z}^n \to \mathbb{T}^k$ be the dual of the embedding $\mathbb{Z}^k \subseteq \mathbb{T}^n$ and set $\mu(x) = \psi(x)^{-1}$. Then μ has dense range since ψ has and $C\left(\mathbb{T}^n\right) \rtimes_\tau \mathbb{Z}^k \simeq C\left(\mathbb{T}^k\right) \rtimes_{\tau\circ\mu} \mathbb{Z}^n$. We are now essentially back where we started, but with the crucial difference that μ has dense range. When we repeat the preceding arguments we get therefore that $C\left(\mathbb{T}^k\right) \rtimes_{\tau\circ\mu} \mathbb{Z}^n$ is stably isomorphic to $\mathbb{C}^{l''} \otimes C\left(\mathbb{T}^{a'}\right) \otimes \left(C\left(\mathbb{T}^k\right) \rtimes_\tau \mathbb{Z}^{m'}\right)$, where $\mathbb{Z}^{m'}$ is now a dense subgroup of $\mathbb{T}^k$, l'' is the order of the torsion part of $\ker\psi$ while a' is the rank of the torsion-free part of $\ker\psi$. Then $B = C\left(\mathbb{T}^k\right) \rtimes_\tau \mathbb{Z}^{m'}$ is a non-degenerate noncommutative torus and we have shown that $C\left(\mathbb{T}^n\right) \rtimes_{\tau\circ\varphi} H$ is stably $*$-isomorphic to $\mathbb{C}^{l'l''} \otimes C\left(\mathbb{T}^{a+a'}\right) \otimes B$. Since $a' + a + k + m' = n + m$ this completes the proof. □

PROPOSITION B.7. *Let H be finitely generated abelian group and $\varphi : H \to \mathbb{T}^n$ a homomorphism and m the rank of the torsion-free part of H. Let $\tau \circ \varphi : H \to \operatorname{Aut} C\left(\mathbb{T}^n\right)$ be the action given by*

$$\left(\tau \circ \varphi\right)_h (f)(x) = f\left(x - \varphi(h)\right).$$

There are then a sequence $F_i, i \in \mathbb{N}$, of finite dimensional C^*-algebras, a natural number $k \leq n + m + 1$, projections $p_i \in C\left(\mathbb{T}^k\right) \otimes F_i$ and a sequence

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

of unital C^*-subalgebras of $C(\mathbb{T}^n) \rtimes_{\tau\circ\varphi} H$ such that

$$C(\mathbb{T}^n) \rtimes_{\tau\circ\varphi} H = \overline{\bigcup_{i=k}^{\infty} A_k}$$

and

$$A_i \simeq p_i \left(C\left(\mathbb{T}^k\right) \otimes F_i \right) p_i$$

for all i.

Proof. This follows straightforwardly by combining Lemma B.6 with the main result of [**Ph2**]. □

Lemma B.8. *Let H be a locally compact abelian group, F a finite abelian group and $G \subseteq H \times F$ a countable subgroup. Let $p : H \times F \to F$ the projection. Set $F_0 = p(G)$ and $G_0 = \ker p \cap G$.*

Then $C_0(H \times F) \rtimes_\tau G$ is stably isomorphic to $\oplus_{F/F_0} C_0(H) \rtimes_\tau G_0$.

Proof. It follows from Theorem 4.1 of [**PR**] that there is a twisted action (α, v) on $C_0(H) \rtimes_\tau G_0$ such that $C_0(H \times F) \rtimes_\tau G$ is $*$-isomorphic to the twisted crossed product $((C_0(H) \rtimes_\tau G_0) \otimes C(F)) \rtimes_{\alpha\otimes\tau, v\otimes 1_{C(F)}} F_0$. By using Theorem 3.4 and Lemma 3.3 of [**PR**] we see that there is a genuine action β of F_0 on $\mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0)$ such that $((C_0(H) \rtimes_\tau G_0) \otimes C(F)) \rtimes_{\alpha\otimes\tau, v\otimes 1_{C(F)}} F_0$ is stably $*$-isomorphic to

$$(\mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0) \otimes C(F)) \rtimes_{\beta\otimes\tau} F_0.$$

It is straightforward to see that $(\mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0) \otimes C(F)) \rtimes_{\beta\otimes\tau} F_0$ decomposes as a direct sum

$$\oplus_{F/F_0} (\mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0) \otimes C(F_0)) \rtimes_{\beta\otimes\tau} F_0.$$

But

$$(\mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0) \otimes C(F_0)) \rtimes_{\beta\otimes\tau} F_0 \simeq \mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0) \rtimes_\beta F_0 \rtimes_{\widehat{\beta}} \widehat{F_0},$$

cf. Lemma 7.9.2 of [**Pe**], and $\mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0) \rtimes_\beta F_0 \rtimes_{\widehat{\beta}} \widehat{F_0} \simeq \mathbb{K} \otimes (C_0(H) \rtimes_\tau G_0)$ by Takai-duality, cf. Theorem 7.9.3 of [**Pe**]. □

B.2. On crossed products of abelian C^*-algebras by discrete groups

Let X be a locally compact second countable space. Let H be a countable discrete group and $\beta_h, h \in H$, an action of H by homeomorphisms of X. Let $\tilde{\beta}$ be the corresponding action of H by automorphisms of $C_0(X)$, i.e. $\tilde{\beta}_g(f) = f \circ \beta_{g^{-1}}$, $f \in C_0(X)$.

Theorem B.9. *Assume that for all compact subsets K of X there is a $g \in H$ such that $\beta_g(K) \cap K = \emptyset$.*

It follows that $C_0(X) \rtimes_{\tilde{\beta}} H$ is stable.

Proof. We use Proposition 2.2 and Theorem 2.1 of [**HR**] in the same way as in the proof of Lemma 4.15. Let $F \subseteq H$ be a finite subset, and $f_g \in C_0(X), g \in F$, functions of compact supports. Since elements of the form $\sum_{g\in F} f_g u_g$ are dense in $C_0(X) \rtimes_{\tilde{\beta}} H$ it suffices to find an element $v \in C_0(X) \rtimes_{\tilde{\beta}} H$ such that

$$v^* v \Big(\sum_{g\in F} f_g u_g \Big) = \sum_{g\in F} f_g u_g \tag{B.4}$$

and

$$\Big(\sum_{g\in F} f_g u_g\Big) v = 0. \tag{B.5}$$

To this end, let

$$K = \bigcup_{g\in F^{-1}\cup\{e\}} \beta_g\Big(\bigcup_{h\in F} \operatorname{supp} f_h\Big).$$

Since K is compact there is by assumption an element $k \in G$ such that $\beta_k(K) \cap K = \emptyset$. Let $h \in C_c(X)$ be a non-negative function such that $h(x) = 1, x \in K$, and $\operatorname{supp} h \cap \beta_k(K) = \emptyset$. Set $v = u_{k^{-1}}\sqrt{h}$. It is straightforward to check that (B.4) and (B.5) hold. □

Let τ be a densely defined lower-semicontinuous trace on the crossed product $C_0(X) \rtimes_{\tilde{\beta}} H$. Then τ is finite on the Pedersen ideal of $C_0(X) \rtimes_{\tilde{\beta}} H$, cf. 5.6 of [**Pe**], and since the Pedersen ideal contains all positive elements a of $C_0(X) \rtimes_{\tilde{\beta}} H$ for which there is another positive element b such that $ba = a$, we see that τ restricts to a positive linear functional on $C_c(X) \subseteq C_0(X) \rtimes_{\tilde{\beta}} H$. By the Riesz representation theorem there is therefore a positive measure μ_τ on X such that

$$\tau(f) = \int_X f(t)\ d\mu_\tau(t)$$

for all $f \in C_c(X)$. Note that μ_τ is finite on compact subsets of X and invariant under β in the sense that $\mu_\tau(\beta_g(A)) = \mu_\tau(A)$ for all $g \in H$ and all Borel sets $A \subseteq X$.

THEOREM B.10. *Assume that β is a free action, i.e. that $\beta_g(x) \neq x$ when $x \in X, g \in H\backslash\{e\}$. It follows that the map $\tau \mapsto \mu_\tau$ is a bijection from the densely defined lower-semicontinuous traces on $C_0(X) \rtimes_{\tilde{\beta}} H$ onto the positive β-invariant Borel measures on X that are finite on compact subsets.*

PROOF. Surjectivity: Let ν be a positive β-invariant Borel measure on X, finite on compact subsets. When $a \in C_0(X) \rtimes_{\tilde{\beta}} H$ is positive set

$$\tau(a) = \int_X P(a)(t)\ d\nu(t),$$

where $P : C_0(X) \rtimes_{\tilde{\beta}} H \to C_0(X)$ be the canonical conditional expectation. Since P is continuous it follows from Fatou's lemma that τ is lower semi-continuous. Since ν is finite on compact subsets we have that τ is finite on every positive element of $C_c(X)$ and since this set of elements contains an approximate unit for $C_0(X) \rtimes_{\tilde{\beta}} H$ it follows that τ is densely defined. To see that τ is trace, i.e. that $\tau(uau^*) = \tau(a)$ when u is a unitary from the unitezation $\big(C_0(X) \rtimes_{\tilde{\beta}} H\big)^+$, note that there is a von Neumann algebra $\mathcal{M}$ with a semi-finite faithful trace ψ and a $*$-homomorphism $\pi : C_0(X)^+ \rtimes_{\tilde{\beta}} H \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} C_0(X)^+ \rtimes_{\tilde{\beta}} H \arrow[r, "\pi"] \arrow[d, "\tau"'] & \mathcal{M} \arrow[dl, "\psi"] \\ {[0,\infty]} & \end{tikzcd}$$

commutes on positive elements, cf. pp. 148-149 of [**Di**]. The trace property follows from this because $\big(C_0(X) \rtimes_{\tilde{\beta}} H\big)^+ \subseteq C_0(X)^+ \rtimes_{\tilde{\beta}} H$. Since $\mu_\tau = \nu$ by construction this proves the surjectivity part.

Injectivity: Let τ_1, τ_2 be two densely defined lower-semicontinuous traces on $C_0(X) \rtimes_{\tilde{\beta}} H$. Assume that $\mu_{\tau_1} = \mu_{\tau_2}$, and represent $C_0(X)$ on $L^2(X, \mu_{\tau_1})$ as multiplication operators. Then $C_0(X) \rtimes_{\tilde{\beta}} H$ can be considered as a strongly dense C^*-subalgebra of von Neumann algebra crossed product $L^\infty(X, \mu_{\tau_1}) \rtimes H$. It follows from Proposition 5.6.7 of [**Pe**] that τ_1 and τ_2 both extend to normal semi-finite traces, $\tilde{\tau_1}$ and $\tilde{\tau_2}$, on $L^\infty(X, \mu_{\tau_1}) \rtimes H$. Note that $\tilde{\tau_1}$ and $\tilde{\tau_2}$ agree on $L^\infty(X, \mu_{\tau_1})$ since $\mu_{\tau_1} = \mu_{\tau_2}$. Since β is a free action it follows therefore from Theorem 7.11.13 of [**Pe**] that $\tilde{\tau_1} = \tilde{\tau_2}$. In particular, $\tau_1 = \tau_2$. $\square$

Corollary B.11. *Let G be a locally compact second countable group and $H \subseteq G$ a countable dense subgroup. Assume that G is not compact. It follows that $C_0(G) \rtimes_\tau H$ is stable, simple and and has a densely defined lower semi-continuous trace which is unique up to scalar multiplication.*

Proof. The simplicity of the (reduced) crossed product $C_0(G) \rtimes_\tau H$ follows from [**Z**] (or Proposition 4.6 of [**Re1**]), the stability from Theorem B.9 and the existence and essential uniqueness of the trace from Theorem B.10 by using the essential uniqueness of the Haar-measure. $\square$

B.3. Translations in $\mathbb{R}^k$.

In this section we study the structure of the crossed products arising from vector translations in Euclidian space.

Let $n, k \in \mathbb{N}$, $n \geq k$. Let $v_1, v_2, \dots, v_n$ be a collection of vectors in $\mathbb{R}^k$. Define an action α of $\mathbb{Z}^n$ on $C_0(\mathbb{R}^k)$ such that

$$\alpha_z(f)(x) = f\Big(x + \sum_{i=1}^n z_i v_i\Big).$$

We are interested in the structure of the crossed product C^*-algebra

$$C_0(\mathbb{R}^k) \rtimes_\alpha \mathbb{Z}^n$$

under the assumption that the v_i's span $\mathbb{R}^k$.

Assume that $v_1, v_2, \dots, v_k$ are linearly independent. Let α' be the action of $\mathbb{Z}^k$ on $C_0(\mathbb{R}^k)$ obtained by restricting α to $\mathbb{Z}^k \subseteq \mathbb{Z}^n$, and let α'' be the action of $\mathbb{Z}^{n-k}$ on $C_0(\mathbb{R}^k) \rtimes_{\alpha'} \mathbb{Z}^k$ obtained from α. Then

$$C_0(\mathbb{R}^k) \rtimes_\alpha \mathbb{Z}^n = \left(C_0(\mathbb{R}^k) \rtimes_{\alpha'} \mathbb{Z}^k\right) \rtimes_{\alpha''} \mathbb{Z}^{n-k}.$$

Set

$$L = \Big\{\sum_{i=1}^k z_i v_i : \ (z_1, z_2, \dots, z_k) \in \mathbb{Z}^k\Big\}.$$

For $f \in C_0(\mathbb{R}^k)$ and $x \in \mathbb{R}^k$, define $f^x \in C_0(L)$ such that $f^x(l) = f(x + l)$. Let β be the action of $\mathbb{Z}^k$ on $C_0(L)$ given by

$$\beta_z(g)(l) = g\Big(l + \sum_{i=1}^k z_i v_i\Big).$$

We get in this way a $*$-homomorphism $\Pi : C_0(\mathbb{R}^k) \rtimes_{\alpha'} \mathbb{Z}^k \to C_b\left(\mathbb{R}^k, C_0(L) \rtimes_\beta \mathbb{Z}^k\right)$ defined such that

$$\Pi\Big(\sum_{z \in \mathbb{Z}^k} f(z) u_z\Big)(x) = \sum_{z \in \mathbb{Z}^k} f(z)^x u_z$$

when $f \in C_c\left(\mathbb{Z}^k, C_0\left(\mathbb{R}^k\right)\right)$. For every $l \in L$ define an automorphism γ_l of $C_0(L) \rtimes_\beta \mathbb{Z}^k$ such that

$$\gamma_l\Big(\sum_{z\in\mathbb{Z}^k} g(z)u_z\Big) = \sum_{z\in\mathbb{Z}^k} g(z)^l u_z,$$

where $h^l(l') = h(l+l'),\ h \in C_0(L), l \in L$. Then Π is a $*$-isomorphism mapping $C_0\left(\mathbb{R}^k\right) \rtimes_{\alpha'} \mathbb{Z}^k$ onto

$$A = \left\{g \in C_b\left(\mathbb{R}^k, C_0(L) \rtimes_\beta \mathbb{Z}^k\right) :\ g(x+l) = \gamma_l\left(g(x)\right), x \in \mathbb{R}^k,\ l \in L\right\}.$$

Define an action μ of $\mathbb{Z}^{n-k}$ on A such that

$$\mu_{(z_{k+1},z_{k+2},\ldots,z_n)}(g)(x) = g\Big(x + \sum_{i=k+1}^{n} z_i v_i\Big).$$

Since $\Pi \circ \alpha''_z = \mu_z \circ \Pi$ for all $z \in \mathbb{Z}^{n-k}$ we conclude that

$$C_0\left(\mathbb{R}^k\right) \rtimes_\alpha \mathbb{Z}^n = \left(C_0\left(\mathbb{R}^k\right) \rtimes_{\alpha'} \mathbb{Z}^k\right) \rtimes_{\alpha''} \mathbb{Z}^{n-k} = A \rtimes_\mu \mathbb{Z}^{n-k}.$$

Represent $C_0(L)$ as multiplication operators on $l^2(L)$ in the natural way and define $U_z \in \mathbb{L}\left(l^2(L)\right)$ such that $\left(U_z\psi\right)(l) = \psi\big(l + \sum_{i=1}^k z_i v_i\big)$. Then $C_0(L) \rtimes_\beta \mathbb{Z}^k$ is generated, as a C^*-algebra, by $\left\{fU_z :\ f \in C_0(L), z \in \mathbb{Z}^k\right\}$, and $C_0(L) \rtimes_\beta \mathbb{Z}^k = \mathbb{L}\left(l^2(L)\right) = \mathbb{K}$. The Fourier transformation $W : l^2(L) \to L^2\left([0,1]^k\right)$ is a unitary such that

$$\left(WU_zW^*\psi\right)(t_1, t_2, \ldots, t_k) = e^{2\pi i \sum_{j=1}^k z_j t_j} \psi\left(t_1, t_2, \ldots, t_k\right).$$

For $s \in \mathbb{R}^k$, define $V_s \in \mathbb{L}_{\mathbb{C}}\left(L^2\left([0,1]^k\right)\right)$ such that

$$\left(V_s\psi\right)(t_1, t_2, \ldots, t_k) = e^{2\pi i \sum_{j=1}^k s_j t_j} \psi\left(t_1, t_2, \ldots, t_k\right).$$

Then $U_s = W^*V_sW$ is a strictly continuous representation such that $\operatorname{Ad} U_l = \gamma_l$.

We can then define a $*$-isomorphism Ψ from A onto

$$B = \left\{g \in C_b\left(\mathbb{R}^k, C_0(L) \rtimes_\beta \mathbb{Z}^k\right) :\ g(x+l) = g(x), x \in \mathbb{R}^k,\ l \in L\right\}$$

such that

$$\Psi(f)(x) = U_x^* f(x) U_x.$$

This shows that

$$C_0\left(\mathbb{R}^k\right) \rtimes_\alpha \mathbb{Z}^k \simeq C\left(\mathbb{R}^k/L, C_0(L) \rtimes_\beta \mathbb{Z}^k\right) = C\left(\mathbb{R}^k/L\right) \otimes \mathbb{K}. \tag{B.6}$$

Let $w_{k+1}, w_{k+2}, \ldots, w_n \in \mathbb{R}^k/L$ be the image of $v_{k+1}, v_{k+2}, \ldots, v_n \in \mathbb{R}^k$. Define actions κ and κ' of $\mathbb{Z}^{n-k}$ on $C\left(\mathbb{R}^k/L\right)$ and $\mathbb{K}$ such that

$$\left(\kappa_z f\right)(x) = f\Big(x + \sum_{i=k+1}^{n} z_i w_i\Big),$$

and

$$\kappa'_z = \operatorname{Ad} U_{-z},$$

respectively. It is now clear that the isomorphism (B.6) turns the action α' on $C_0\left(\mathbb{R}^k\right) \rtimes_\alpha \mathbb{Z}^k$ into the diagonal action $\kappa \otimes \kappa'$ on $C\left(\mathbb{R}^k/L\right) \otimes \mathbb{K}$. Thus

$$C_0\left(\mathbb{R}^k\right) \rtimes_\alpha \mathbb{Z}^n \simeq \left(C\left(\mathbb{R}^k/L\right) \otimes \mathbb{K}\right) \rtimes_{\kappa\otimes\kappa'} \mathbb{Z}^{n-k}.$$

Now the formula

$$\sum_{z\in\mathbb{Z}^{n-k}} f(z)u_z \mapsto \sum_{z\in\mathbb{Z}^{n-k}} f(z)\left(1\otimes U_z\right)u_z$$

gives rise to a $*$-isomorphism

$$\begin{aligned}\left(C\left(\mathbb{R}^k/L\right)\otimes\mathbb{K}\right)\rtimes_{\kappa\otimes\kappa'}\mathbb{Z}^{n-k} &\simeq \left(C\left(\mathbb{R}^k/L\right)\otimes\mathbb{K}\right)\rtimes_{\kappa\otimes\mathrm{id}}\mathbb{Z}^{n-k}\\ &= \left(C\left(\mathbb{R}^k/L\right)\rtimes_{\kappa}\mathbb{Z}^{n-k}\right)\otimes\mathbb{K},\end{aligned}$$

and we can summarize the result as follows:

Let $e_i, i=1,2,\dots,k$, be the standard basis in $\mathbb{R}^k$. Let $V\in M_k(\mathbb{R})$ be the matrix such that $Ve_i=v_i$. Set $\alpha_j=\rho\left(V^{-1}v_{j+k}\right), j=1,2,\dots,n-k$, where $\rho:\mathbb{R}^k\to\mathbb{T}^k=\mathbb{R}^k/\mathbb{Z}^k$ is the quotient map. Let β be the action of $\mathbb{Z}^{n-k}$ on $C\left(\mathbb{T}^k\right)$ obtained from rotation by the α_j's, i.e.

$$\beta_z(f)(x)=f\left(\alpha_1^{z_1}\alpha_2^{z_2}\dots\alpha_{n-k}^{z_{n-k}}x\right).$$

THEOREM B.12. *Assume that the v_i's span $\mathbb{R}^k$. Then*

$$C_0\left(\mathbb{R}^k\right)\rtimes_\alpha\mathbb{Z}^n\simeq\mathbb{K}\otimes\left(C\left(\mathbb{T}^k\right)\rtimes_\beta\mathbb{Z}^{n-k}\right).$$

COROLLARY B.13. *Assume that $\left\{\sum_{j=1}^n z_jv_j:\ (z_1,z_2,\dots,z_n)\in\mathbb{Z}^n\right\}$ is dense in $\mathbb{R}^k$. It follows that $C_0(\mathbb{R}^k)\rtimes_\alpha\mathbb{Z}^n$ is a stable AT-algebra of real rank zero with a lower semi-continuous densely defined trace which is unique up to scalar multiplication.*

PROOF. Under the present assumption $C_0(\mathbb{R}^k)\rtimes_\alpha\mathbb{Z}^n$ is simple and hence the crossed product $C\left(\mathbb{T}^k\right)\rtimes_\beta\mathbb{Z}^{n-k}$ is a (special) nondegenerate non-commutative torus. Therefore the corollary follows by combining Theorem B.12 with the main result of [**Ph2**]. $\square$

APPENDIX C

On an example of Bratteli, Jorgensen, Kim and Roush

The purpose of this Appendix is to supply the details of the argument which shows that the stationary dimension groups corresponding to the matrix

$$\begin{pmatrix} 65 & 7 \\ 24 & 67 \end{pmatrix} \tag{C.1}$$

and its transpose are not isomorphic, even when we ignore the orderings. Compare with Example 3.6 of [**BJKR**]. I am grateful to Kim and Roush for explaining me the part of the argument which is missing in [**BJKR**].

Let A be an 2×2 matrices with entries from $\mathbb{N}$. The inductive limit group of the sequences

$$\mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \cdots \tag{C.2}$$

will be denoted by G_A. Since the entries of A are non-negative, G_A is a partially ordered group in a natural way, cf. e.g. [**BJKR**]. Assume that A is invertible in $M_2(\mathbb{Q})$, i.e. assume that $\operatorname{Det} A \neq 0$. Then the commuting diagram

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "A"] \arrow[dr, "\mathrm{id}"'] & \mathbb{Z}^2 \arrow[r, "A"] \arrow[d, "A^{-1}"'] & \mathbb{Z}^2 \arrow[r, "A"] \arrow[dl, "A^{-2}"] & \mathbb{Z}^2 \arrow[r, "A"] \arrow[dll, "A^{-3}"] & \cdots \\
 & \mathbb{Q}^2 & \cdots & &
\end{tikzcd}$$

gives rise to an embedding $G_A \subseteq \mathbb{Q}^2$ which identifies G_A with

$$\left\{ z \in \mathbb{Q}^2 : \ A^k z \in \mathbb{Z}^2 \text{ for some } k \in \mathbb{N} \right\}.$$

LEMMA C.1. *Assume that the eigenvalues, n and m, of A are positive natural numbers, different and relatively prime. Choose $v \in \mathbb{Z}^2 \backslash \{(0,0)\}$ such that $Av = nv$. It follows that*

$$\left\{ z \in G_A : \ \frac{1}{n^k} z \in G_A \text{ for all } k \in \mathbb{N} \right\} = \mathbb{Q}v \cap G_A. \tag{C.3}$$

If, furthermore, $v = (a,b)$ is chosen such that a and b are mutually prime, we have that

$$\mathbb{Q}v \cap G_A = \left\{ \frac{k}{l} v : \ k, l \in \mathbb{Z}, \ l | n^i \text{ for some } i \in \mathbb{N} \right\}. \tag{C.4}$$

PROOF. Let $w \in \mathbb{Q}v \cap G_A$ and consider some $k \in \mathbb{N}$. Then $A^{r+k}\left(\frac{1}{n^k} w\right) = A^r w \in \mathbb{Z}^2$ for some $r \in \mathbb{N}$ since $w \in G_A$. It follows that $\frac{1}{n^k} w \in G_A$.

Conversely, assume that $z \in G_A$ and that $\frac{1}{n^k} z \in G_A$ for all $k \in \mathbb{N}$. Let $u \in \mathbb{N}^2$ be an eigenvector for A corresponding to the eigenvalue m. Then $z = qv + q'u$ for some $q, q' \in \mathbb{Q}$, and we aim to show that $q' = 0$. Let $N \in \mathbb{N}$ be so large that $p = Nq$

and $p' = Nq'$ are both integers. Then $Nz = pv + p'u$. If $q' \neq 0$ we can choose $k \in \mathbb{N}$ so large that

$$\frac{1}{n^k} p' u \notin \mathbb{Z}^2. \tag{C.5}$$

Since $\frac{1}{n^k} Nz \in G_A$ there must be an $r \in \mathbb{N},\ r \geq k$, such that

$$\mathbb{Z}^2 \ni A^r \left(\frac{1}{n^k} Nz \right) = pn^{r-k} v + p' \frac{m^r}{n^k} u.$$

Since $pn^{r-k}v \in \mathbb{Z}^2$ this implies that $p' \frac{m^r}{n^k} u \in \mathbb{Z}^2$. This contradicts (C.5) since n and m are mutually prime. Hence $q' = 0$ and we conclude that $z \in \mathbb{Q}v \cap G_A$. (C.3) is established.

To prove (C.4) note first that one inclusion, namely $\supseteq$, is trivial. Let $\frac{p}{q} v \in G_A$, where $p, q \in \mathbb{Z}$ are relatively prime. Then $A^i \left(\frac{p}{q} v\right) = \frac{p}{q} n^i v \in \mathbb{Z}^2$ for some $i \in \mathbb{N}$ and it follows then that $q | n^i$ since (p, q) and (a, b) are relatively prime. $\square$

For the following lemma recall that a non-negative matrix is *primitive* when some power of it has all entries strictly positive.

LEMMA C.2. *Let A and B be primitive 2×2 matrices with entries from $\mathbb{N}$. Assume that A and B have the same distinct and mutually prime eigenvalues from $\mathbb{N} \setminus \{0\}$. Assume that G_A and G_B are isomorphic as groups.*

It follows that A and B are shift equivalent, i.e. there are matrices $U, V \in M_2(\mathbb{Z})$ with non-negative entries such that $UA = BU,\ VB = AV,\ UV = B^l$ and $VU = A^l$ for some $l \in \mathbb{N}$.

PROOF. Let n, m be the two common eigenvalues for A and B. Choose v, u, v', $u' \in \mathbb{Z}^2$ be non-zero vectors such that $Av = nv$, $Au = mu$, $Bv' = nv'$ and $Bu' = mu'$. Let $\varphi : G_A \to G_B$ be an isomorphism. It follows from Lemma C.1 that $\varphi(v) \in \mathbb{Q}v'$ and $\varphi(u) \in \mathbb{Q}u'$, which in turn implies that

$$\varphi \circ A = B \circ \varphi. \tag{C.6}$$

Let $X \in M_2(\mathbb{Q})$ be the matrix determined by the condition that $Xu = \varphi(u)$ and $Xv = \varphi(v)$. It follows from (C.6) that

$$XA = BX. \tag{C.7}$$

Similarly, we construct a matrix $Y \in M_2(\mathbb{Q})$ such that $Yu' = \varphi^{-1}(u')$, $Yv' = \varphi^{-1}(v')$ and

$$YB = AY. \tag{C.8}$$

Let $e_1 = (1, 0), e_2 = (0, 1)$. There is then a natural number k such that

$$B^k \varphi(e_1),\ B^k \varphi(e_2),\ A^k \varphi^{-1}(e_1),\ A^k \varphi^{-1}(e_2)$$

are all in $\mathbb{Z}^2$. Define $U, V \in M_2(\mathbb{Z})$ such that $Ue_1 = B^k \varphi(e_1)$, $Ue_2 = B^k \varphi(e_2)$, $Ve_1 = A^k \varphi^{-1}(e_1)$ and $Ve_2 = A^k \varphi^{-1}(e_2)$. Since there are integers k_i, l_i, n_i such that $k_i e_i = l_i v + n_i u, i = 1, 2$, it is easy to check that

$$B^{-k} U = X \tag{C.9}$$

and

$$A^{-k} V = Y. \tag{C.10}$$

It follows from (C.7) through (C.10) that $UA = BU$ and $VB = AV$. Finally, since $XY = YX = 1$, we find that $UV = B^k X A^k Y = B^{2k}$ and $VU = A^k Y B^k X = A^{2k}$.

U and V have the required properties, except that they may contain negative entries. Since both A and B are primitive, this can be remedied by Perron-Frobenius theory, cf. Lemma 2.2.8 of [**K**]. $\square$

Since the matrix (C.1) is primitive, we can now complete our task by appealing to the reasoning from Example 3.6 of [**BJKR**]. For completeness we include the argument here. So let now A be the matrix (C.1). The common eigenvalues for A and A^t are the primes $n = 53$ and $m = 79$. Choose vectors

$$\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right), \ \left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix}\right), \ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \ \left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)$$

in $\mathbb{Z}^2$ such that

$$A\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = n\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right),$$
$$A\left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix}\right) = m\left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix}\right),$$
$$A^t\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = n\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right),$$

and

$$A^t\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = m\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right).$$

We 'normalize' these vectors such that the pairs (α, β), (α', β'), (x, y) and (x', y') are all relatively prime. Assume to get a contradiction that G_A and G_{A^t} are isomorphic as groups. By Lemma C.2 there are then matrices $U, V \in M_2(\mathbb{Z})$ such that $UA = A^tU$, $VA^t = AV$, $UV = (A^t)^l$ and $VU = A^l$ for some $l \in \mathbb{N}$. Then $UG_A = G_{A^t}$ and it follows from the Lemma C.1 that

$$U\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = \frac{k}{l}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \ U\left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix}\right) = \frac{k'}{l'}\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)$$

for some $k, k', l, l' \in \mathbb{Z}$ such that $l|n^i$ and $l'|m^i$ for some $i \in \mathbb{N}$. Then

$$UA^i\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = d\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \ UA^i\left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix}\right) = d'\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)$$

where $d = \frac{k}{l}n^i \in \mathbb{Z}$ and $d' = \frac{k'}{l'}m^i \in \mathbb{Z}$. Since $UA^i : G_A \to G_{A^t}$ is an isomorphism of groups, it follows from Lemma C.1 that there must be some $k, l \in \mathbb{Z}$ such that $l|n^r$ for some $r \in \mathbb{N}$ and

$$UA^i\frac{k}{l}\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right).$$

This implies that $d\frac{k}{l} = 1$ and hence that $d|n^r$ since $l|n^r$. By using the same argument with $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ replaced by $\left(\begin{smallmatrix} \alpha' \\ \beta' \end{smallmatrix}\right)$ and by increasing r, if necessary, we conclude that $d|n^r$ and $d'|m^r$. Since

$$UA^i\left(\begin{smallmatrix} \alpha & \alpha' \\ \beta & \beta' \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right)\left(\begin{smallmatrix} d & 0 \\ 0 & d' \end{smallmatrix}\right),$$

we conclude that

$$\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right)\left(\begin{smallmatrix} d & 0 \\ 0 & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & \alpha' \\ \beta & \beta' \end{smallmatrix}\right)^{-1} = UA^i \in M_2(\mathbb{Z}). \tag{C.11}$$

Now, it is elementary to check that

$$\left(\begin{smallmatrix} \alpha & \alpha' \\ \beta & \beta' \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -7 & 1 \\ 12 & 2 \end{smallmatrix}\right)^{-1} = \begin{pmatrix} -\frac{1}{13} & \frac{1}{26} \\ \frac{6}{13} & \frac{7}{26} \end{pmatrix},$$

while

$$\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right) = \left(\begin{smallmatrix} -2 & 12 \\ 1 & 7 \end{smallmatrix}\right).$$

The $(1,1)$ entry in the product $\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right)\left(\begin{smallmatrix} d & 0 \\ 0 & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & \alpha' \\ \beta & \beta' \end{smallmatrix}\right)^{-1}$ is

$$\frac{2d}{13} + \frac{72d'}{13}.$$

As noted above $d|n^r$ and $d'|m^r$ for some $r \in \mathbb{N}$. Since n and m are primes and both congruent to -1 modulo 13, we conclude that d and d' are both congruent to ± 1 modulo 13. That is, we can write $d = \epsilon_1 + n_1 \cdot 13$ and $d' = \epsilon_2 + n_2 \cdot 13$ for some $\epsilon_i \in \{1, -1\}$ and some $n_i \in \mathbb{Z},\ i = 1, 2$. It follows that

$$\frac{2d}{13} + \frac{72d'}{13} = \frac{2\epsilon_1}{13} - \frac{6\epsilon_2}{13}$$

modulo 1. This is not an integer, contradicting (C.11).

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