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HYPERSURFACES WITH ISOLATED SINGULARITIES WITH SYMMETRY

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Hypersurfaces with isolated singularities with symmetry

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Abstract

Those hypersurfaces admitting a 1-parameter symmetry group are characterised by failure of versality of a certain unfolding of their set of singularities, which in the simplest cases (sextic curves, quartic surfaces and cubic 4-folds) is the unfolding by hypersurfaces of the same degree. We give a classification of these hypersurfaces, and calculate their total Milnor and Tjurina numbers. The maximal Tjurina number occurs if and only if the equation (of degree d) is annihilated by a vector field of degree $(d-2)$ independent of that given by the action; in these cases the enumeration is more explicit, and when also $d = 3$ we have a 2-parameter group. It is conjectured, and proved in low dimensions, that any hypersurface with maximal Tjurina number admits a 1-parameter symmetry group.

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Introduction

We will assume throughout that $V \subset P^n(\mathbb{C})$ is given by a homogeneous equation $f(x_0, \ldots, x_n) = 0$ of degree d, and that V has isolated singularities and is not a cone (e.g. we exclude equations with f independent of x_0). Write μ (resp. τ) for the sum of the Milnor numbers (resp. Tjurina numbers) of all singularities of V. We say that V is r-symmetric if there is an r-parameter subgroup $G \subset PGL_{n+1}$ leaving V invariant. In this article, which collects results of several earlier papers of the authors, we study the cases when V is 1-symmetric.

By way of introduction we state (without proofs) results on characterisation, enumeration, calculation of μ and τ, moduli, and higher symmetry for the case $n = 2$ when V is a curve: thus V is reduced, i.e. has no repeated component. Nearly all the proofs can be found in [6].

In the remaining sections we show (with outlines of most proofs) how far these generalise to the general hypersurface case. The second section contains the main theoretical results, culled from various papers. In particular, we give outline proofs for the characterisation of 1-symmetry by versality, that in most case topological versality holds anyway, for bounds on τ, and for the equivalence of τ attaining its maximum (for 1-symmetric hypersurfaces) with annihilation of f by a vector field, independent of the first, of degree $d-2$: in these cases we call V *oversymmetric*.

We then discuss in detail geometry and enumeration and show how to calculate μ and τ; in the next section in the semi-simple case, mostly following [8]; and in the following section in the unipotent case, following a manuscript, work on which is still in progress. We conclude with detailed lists of 1-symmetric cubic 3-folds, of oversymmetric hypersurfaces, and of all 2-symmetric cases.

1 The curve case

Characterisation: The following are equivalent:

- V is 1-symmetric;
- there is a 1-parameter subgroup $\tilde{G} \subset GL_3$ leaving f invariant;
- there is a vector field $\xi = \sum_0^2 a_i \partial/\partial x_i$ with the a_i linear and $\xi(f) = 0$;
- (if the line $x_0 = 0$ passes through no singular point and is transverse to V) the unfolding of $f(1, x_1, x_2)$ by all functions of degree $\leq 2d-6$ does NOT induce a simultaneous versal deformation of all singularities of V.
- If V is 1-symmetric, $\tau \geq d^2 - 3d + 2$. If $\tau > d^2 - 4d + 7$, V is 1-symmetric. Thus if $d \geq 6$, V is 1-symmetric if and only if $\tau \geq d^2 - 3d + 2$.

Enumeration: If G (and hence $\tilde{G}$) is isomorphic to the multiplicative group $\mathbb{C}^*$ (semi-simple case), it is conjugate to a diagonal subgroup $t \to \mathrm{diag}(t^{w_0}, t^{w_1}, t^{w_2})$. Thus if the coefficient of $x_0^{r_0} x_1^{r_1} x_2^{r_2}$ in f is non-zero, the exponents lie on the line segment $\sum r_i = d$, $\sum w_i r_i = 0$, $r_i \geq 0$. Since x_i^2 is not a factor of f, there is a solution with r_i equal to 0 or 1. Thus one end of the line segment must have two of the $r_i \leq 1$. Permuting the x_i, we may take one end of the segment as one of $A = (0, d, 0)$, $B = (1, d-1, 0)$, $C = (1, d-2, 1)$ and the other as $\lambda_r = (r, 0, d-r)$ $(1 \leq r \leq d-1)$ or $\nu_r = (r, 1, d-r-1)$ $(1 \leq r \leq d-2)$. Using the symmetry interchanging x_0 and x_2 we may suppose $r \leq d/2$ for the A, C cases.

If G (and hence $\tilde{G}$) is isomorphic to the additive group $\mathbb{C}^+$ (unipotent case), we can take $\xi = x_0 \partial/\partial x_1 + x_1 \partial/\partial x_2$. The ring of invariants is then generated by x_0 and $B := 2x_0x_2 - x_1^2$. Thus f has the form $\prod_0^k (B + a_i x_0^2)$ $(d = 2k)$ or $x_0 \prod_0^k (B + a_i x_0^2)$ $(d = 2k+1)$, with the a_i distinct.

Calculation of $\boldsymbol{\mu}$ and $\boldsymbol{\tau}$: In the semi-simple case, $\mu = \tau = d^2 - 3d + 3$ in the B, C cases and $\mu = \tau = d^2 - 3d + 2$ in the A cases.

In the unipotent case, $\tau = d^2 - 3d + 3$ and $\mu = \lfloor \frac{1}{2}(2d^2 - 5d + 3) \rfloor$.

Moduli: The $\mathcal{K}$-class of the singularities determines the 1-symmetric curve V up to projective equivalence, except if f depends only on x_1 and x_0x_2, and the coefficient of x_1^d is non-zero, when we have a 1-parameter family of curves.

Higher symmetry: There are no 3-symmetric curves. If V is 2-symmetric, then $d = 3$. If $d = 3$, then V is 2-symmetric if and only if $\tau = 3$. There are two cases:

(A) we can take $f = x_0x_1x_2$, singularities $3A_1$, vector fields

$$\xi = x_0\partial/\partial x_0 - x_1\partial/\partial x_1, \quad \eta = x_0\partial/\partial x_0 - x_2\partial/\partial x_2;$$

(C) we can take $f = x_0(x_1^2 - 2x_0x_2)$, singularities A_3, vector fields

$$\xi = -2x_0\partial/\partial x_0 + x_1\partial/\partial x_1 + 4x_2\partial/\partial x_2, \quad \eta = x_0\partial/\partial x_1 + x_1\partial/\partial x_2.$$

2 Vector fields and Koszul complexes

We choose co-ordinates so that the intersection V_0 of V with the hyperplane $x_0 = 0$ is non-singular: thus $f_0(x_1, \ldots, x_n) := f(0, x_1, \ldots, x_n)$ has an isolated singularity at the origin, with Milnor number $\tau_0 = (d-1)^n$. Set $R := \mathbb{C}[x_0, \ldots, x_n]$ and $\overline{R} := R/\langle x_0\rangle = \mathbb{C}[x_1, \ldots, x_n]$. Denote by Jf_0 the Jacobian ideal of f_0. Choose monomials $\{\phi_i \,|\, 0 \le i < \tau_0\}$ (with $\phi_0 = 1$) mapping to a basis of $\overline{R}/Jf_0$, and write F for the versal unfolding $F(x,u) = (f(x,u), u)$, where $f(x,u) = f_0(x) + \sum_1^{\tau_0-1} u_i\phi_i(x)$: this is a stable map. Denote the target coordinates y and v_i; we will also write $v_0 = -y$.

There exist relations

$$f_0(x,u)\phi_i(x) = \sum_j a_{ij}(u)\phi_j(x) + \sum_k b_{ik}(x,u)\partial f(x,u)/\partial x_k.$$

Thus the vector field $\eta_i = \sum_j (a_{ij}(v) - y\delta_{ij})\partial/\partial v_j$ on the target of F lifts to the vector field $\xi_i = \sum_k b_{ik}(x,u)\partial/\partial x_k + \sum_j (a_{ij}(u) - f(x,u)\delta_{ij})\partial/\partial u_j$ on the source; indeed, the η_i form a free basis over $\mathbb{C}\{v_0, \ldots, v_{\tau_0-1}\}$ of the module of liftable vector fields. We call the matrix $(a_{ij}(v) - y\delta_{ij})$ the *discriminant matrix*.

The unfolding F consists of the functions $f_v(x) := f(x,v) + v_0$, among which is $f(1, x_1, \ldots, x_n)$. By inspection, a linear relation $\sum c_i(a_{ij} - y\delta_{ij}) = 0$ between the columns of the discriminant matrix holds at a point v of the target if and only if $g = \sum_i c_i\phi_i$ satisfies $gf_v \in Jf_v$, i.e. $g \in (Jf_v : f_v)$, where for I an ideal in a ring R and $S \subset R$, $(I : S)$ denotes $\{r \in R \,|\, rS \subseteq I\}$.

The following properties of the discriminant (i.e. locus of critical values of F) appear in Looijenga [11]; a detailed exposition was also given in [4]. Let us say that two points v, v' of the target are equivalent if the germs of F at $\Sigma F \cap F^{-1}(v)$ and $\Sigma F \cap F^{-1}(v')$ are $\mathcal{K}-$, (hence $\mathcal{A}-$) equivalent; we will call the equivalence classes *leaves.*

(i) A vector field is liftable if and only if it is tangent to the discriminant.

(ii) Each leaf is smooth; the tangent space at v is spanned by values at v of liftable vector fields.

(iii) The codimension of the leaf through v is equal to $\sum_{x \in (\Sigma F \cap F^{-1}(v))} \tau_x(f_v)$.

Since $\sum_i c_i\eta_i$ vanishes at v if and only if $g = \sum_i c_i\phi_i$ satisfies $gf_v \in Jf_v$, the codimension of the leaf is equal to the codimension in the space Φ of linear combinations $g = \sum_i c_i\phi_i$ of those $g \in (Jf_v : f_v)$. Since by definition $\Phi + Jf_0 = \overline{R}$, and hence $\Phi + Jf_v = \overline{R}$, this is equal to $\dim(\overline{R}/(Jf_v : f_v))$. The equality of this with $\tau_0 - \tau(f_v) = (d-1)^n - \tau(f_v)$ is the basis of all our calculations of τ.

For our account of the necessary algebra we follow the version of [10], which evolved through several earlier of our joint papers.

It follows from our standing hypothesis that $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ is a regular sequence, generating the ideal $J_0 f$, say; to use this we introduce the Koszul complex. Write Λ for the exterior algebra over R on n generators $\theta_1, \ldots, \theta_n$. There is a unique differential on Λ such that $d\theta_i = \partial f/\partial x_i$; denote this differential algebra Λf. This is bi-graded: if we define θ_i to have grade 1 and degree 0, the differential lowers grade by 1 and raises degree by $d-1$.

There is an exact sequence $0 \to R \xrightarrow{x_0} R \to \overline{R} \to 0$; tensoring with Λf gives an exact sequence $0 \to \Lambda f \xrightarrow{x_0} \Lambda f \to \overline{\Lambda f} \to 0$ of chain complexes. Since the $\partial f/\partial x_i$ form a regular sequence, these chain complexes are acyclic. Thus $H_0(\Lambda f) \cong R/J_0 f$ and since multiplying by x_0 is injective, this is a free module over $\mathbb{C}[x_0]$, of rank τ_0.

If we add a further generator θ_0, with $d\theta_0 = \partial f/\partial x_0$, we obtain a Koszul complex $\Lambda^+ f$, say, an exact sequence $0 \to \Lambda f \to \Lambda^+ f \xrightarrow{[-1]} \Lambda f \to 0$ of chain complexes, and hence an exact sequence of homology, whose only non-zero terms are $0 \to H_1(\Lambda^+ f) \to R/J_0 f \xrightarrow{\partial f/\partial x_0} R/J_0 f \to H_0(\Lambda^+ f) \to 0$. Again all except the last term are free over $\mathbb{C}[x_0]$. It follows that $H_1(\Lambda^+ f) \cong (J_0 f : \partial f/\partial x_0)/J_0 f$, and the image of the middle map is $\cong R/(J_0 f : \partial f/\partial x_0)$.

It is not hard to show (see [10, Lemma 3.3]) that $(J_0 f : \partial f/\partial x_0) = (Jf : f)$, so this image has rank $\tau_0 - \tau$; it follows that $H_1(\Lambda^+ f)$ is a free $\mathbb{C}[x_0]-$module of rank τ. The simplest argument for this is to go via the affine case, factoring out $x_0 - 1$.

We can interpret the abstract symbols θ_i as differential operators $\partial/\partial x_i$. Then the terms of grade 1 in $\Lambda^+ f$ are the first order differential operators, i.e. vector fields ξ, and the differential in $\Lambda^+ f$ is given by $d\xi = \xi f$, so its kernel consists of the $R-$module $\operatorname{Ann}(f)$ of vector fields annihilating f. The terms of grade 2 have basis $\partial/\partial x_i \wedge \partial/\partial x_j$, and the differential operator maps this to the basic Hamiltonian vector field $\eta_{ij} = \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$. Thus $H_1(\Lambda^+ f)$ is isomorphic to the quotient of $\operatorname{Ann}(f)$ by the $R-$submodule $\operatorname{Ham}(f)$ of Hamiltonian vector fields. More precisely, if $\xi = \sum_i \alpha_i \partial/\partial x_i \in \operatorname{Ann}(f)$, then $\alpha_0 \in (Jf : \partial f/\partial x_0)$, while in this case, $\xi \in \operatorname{Ham}(f) \Leftrightarrow \alpha_0 \in Jf$.

The quotient $\operatorname{Ann}(f)/\operatorname{Ham}(f)$ is infinite dimensional but is graded; the dimension of a subspace of fixed large enough degree is τ; and this statement is independent of co-ordinates.

The singularities of V can be simultaneously versally deformed by perturbing the equation f by the addition of all homogeneous functions of sufficiently high degree r; since F is versal, $r \geq n(d-2)$ suffices. Conversely, if the addition of homogeneous functions of degree $n(d-2)-1-a,\ a \geq 0$, *fails* to simultaneously versally deform the singularities of V, then we will say that V or f is *a-non-versal.*

Theorem 2.1. *The hypersurface V is a-non-versal if and only if there is a vector field of degree a in* $\operatorname{Ann}(f) \setminus \operatorname{Ham}(f)$.

The idea of the proof comes from [3]; the proof appears in [2]; the argument below is closer to the version in [10].

The unfolding of f_0 or f_v by all monomials of degree $\leq k$ is a trivial unfolding of the unfolding F^k by those ϕ_i of degree $\leq k$. These ϕ_i are the $\tau_0 - c(k)$ (say) unfolding monomials of least degree. We seek to characterise the set of points where stability of F^k fails. Now for F itself the vanishing of $\theta(F)/tF(\theta_n)+\omega F(\theta_p)$ is guaranteed by the fact that the images of the vector fields η_i span $\theta F/tF\theta_n$ as a module over the ring of functions on the target. The map F^k is obtained from F by setting the last $c(k)$ source and target variables equal to 0. We thus lose the last $c(k)$ of the vectors η_i, i.e. the last $c(k)$ rows of the discriminant matrix; stability fails at the points where the rank of the matrix drops below $\tau_0 - c(k)$.

By a result of Mond and Pellikaan [12] we may suppose that the discriminant matrix is symmetric. Thus the last $c(k)$ rows correspond to the last $c(k)$ columns; we can interpret 'last' as 'of least degree'. Stability fails at v if and only if there is a linear combination of the $\tau_0 - c(k)$ rows of least degree which vanishes at v, i.e. if there is a linear combination $g=\sum_i c_i\phi_i$ of the ϕ_i of degree $\leq k$ with $g\in(Jf_v : f_v)$. Now stability of F^k means the same as versality of the corresponding unfolding of f_v. In view of the above identifications $(Jf : f) = (J_0 f : \partial f/\partial x_0)$, $(J_0 f : \partial f/\partial x_0)/J_0 f \cong H_1(\Lambda^+ f) \cong \operatorname{Ann}(f)/\operatorname{Ham}(f)$, the result follows.

Taking $a=1$ we deduce that V is 1-symmetric if and only if it is 1-non-versal. This result is particularly attractive if $d = n(d-2)-2$, so the unfolding is by all monomials of degree d and we get a direct tie-up between deformations of hypersurfaces of degree d and those of the set of singularities. This occurs in just three cases: $d=3$, $n=5$ (cubic 4-folds), $d=4$, $n=3$ (quartic surfaces), and $d=6$, $n=2$ (sextic curves).

Although the 1-symmetric case is characterised by a certain failure of transversality, topological transversality usually holds even here. First we have

Theorem 2.2. *[2] Suppose the vector field ξ generates the degree k part of $\operatorname{Ann}(f)/\operatorname{Ham}(f)$. If there exists a non-simple singular point P of X at which ξ does not vanish, then V is topologically k-versal.*

The idea of the proof is that versality only fails by a single dimension, and that for a weighted homogeneous non-simple singularity, the deformation omitting only the unfolding monomial of highest weight from the versal deformation is topologically versal.

Eliminating the cases when only simple singularities appear, we obtain

Theorem 2.3. *[8, Theorem 3.8] Every 1-symmetric hypersurface V of degree d in P^n is topologically 1-versal provided either $d=3$, $n\geq 5$ and V is not 2-symmetric; $d\geq 4$, $n\geq 3$; or $d\geq 6$, $n=2$.*

We turn to estimates of τ. Write r for the least degree of a non-zero homogeneous element of $(Jf : f)$, or equivalently, of a vector field ξ with $\xi f=0$. As V is not a cone, $r>0$; we can show $r\leq d-1$. The first estimate (see [5, Theorem 3.2] for the curve case and [10, Theorem 4.4] in general) is

$$r(d-1)^{n-1} \geq \tau_0 - \tau \geq r(d-1-r)(d-1)^{n-2}. \tag{1}$$

In the case $n=2$ we succeeded in [5] in improving the right hand side from $\tau \leq (d-1)(d-r-1)+r^2$, when $2r+1>d$, to

$$\tau \leq (d-1)(d-r-1)+r^2-\tfrac{1}{2}(2r+1-d)(2r+2-d),$$

and it is this result which leads to the algebraic characterisation cited in §1. In general, the estimates we have been able to obtain are not so good.

Conjecture 2.4. *We have* $\tau_0 - \tau \leq (d-1)^{n-1}$ *if and only if* $r = 1$.

This would imply a characterisation of 1-symmetry. We have succeeded in proving the conjecture if $r < d-2$ or if $n \leq 4$. We abstain here from further details of such estimates: see [7] and [10].

The case of equality in the second part of (1) is of particular interest.

Theorem 2.5. *[10, Theorems 4.7, 4.9] (i) If* $\xi \in \mathrm{Ann}(f)$ *has degree* r*, and* $\tau_0 - \tau = r(d-1-r)(d-1)^{n-2}$*, there exists* $\xi' \in \mathrm{Ann}(f)$ *of degree* $d-r-1$ *with* α, α' *coprime.*

(ii) Let $\xi, \xi' \in \mathrm{Ann}(f)$ *have degrees* r, r'*. If* $\xi \wedge \xi' \neq 0$*, e.g. if* α_0 *and* α_0' *are coprime,* $r + r' \geq d-1$*.*

(iii) If (ii) applies and $r + r' = d-1$*, then* α *and* α' *generate* $(Jf : f)/Jf$*;* $\mathrm{Ann}(f) = \mathrm{Ham}(f) + R\xi + R\xi'$*, and* $\tau_0 - \tau = rr'(d-1)^{n-2}$*.*

This is the content of [10, Theorems 4.7, 4.9] when expressed in geometrical terms. (Or see [9, Lemma 4.3, Theorem 4.5].)

The idea of the proof of (1) is to start with the relation $\sum_{i=0}^{n} a_i \partial f/\partial x_i = 0$ and adjust co-ordinates till $\{a_0, \partial f/\partial x_2, \ldots, \partial f/\partial x_n\}$ is a regular sequence; then work on estimates.

As to 2.5, for (i) we deduce from the hypothesis and further care about co-ordinates that $\partial f/\partial x_2 \in \langle a_0, a_1, \partial f/\partial x_3, \ldots, \partial f/\partial x_n \rangle$, so there is a relation $0 = \lambda_0 a_0 + \lambda_1 a_1 + \sum_3^n c_i \partial f/\partial x_i$ with $c_2 = 1$, and then show that we can take $\alpha = a_0, \alpha' = \lambda_1$.

For (ii) it now suffices to note that $\xi \wedge \xi'$ is a cycle, hence a boundary in the Koszul complex, and that any non-zero boundary has degree at least $3d-3$ (or we can argue that $\alpha'\xi - \alpha\xi'$ is a boundary, since the coefficient of $\partial/\partial x_0$ vanishes and Λf is acyclic). For (iii) we take a further $\xi'' \in \mathrm{Ann}(f)$ and form $\eta'' = \alpha\xi' - \alpha'\xi$ and similarly for other permutations of ξ, ξ', ξ''. Since the coefficients of $\partial/\partial x_0$ vanish, these are cycles in $\Lambda^+ f$, say $\eta = d\omega$ etc. By inspection, $\alpha\omega + \alpha'\omega' + \alpha''\omega''$ is again a cycle, hence a boundary. Here we can pick out one coefficient which must be a non-zero constant; eventually this shows that ξ'' is a linear combination of ξ and ξ'.

Taking $r = 1$ we obtain

Corollary 2.6. *Suppose* V *quasi-smooth and 1-symmetric of degree* d *with* $\xi(f) = 0$*. Then* $\tau(V) \leq (d-1)^{n-2}(d^2 - 3d + 3)$*, and equality holds if and only if* V *is oversymmetric, with a second vector field* η*. When this holds, any vector field annihilating* f *is a linear combination of* ξ*,* η *and Hamiltonian vector fields.*

This gives the maximal value of τ for 1-symmetric, and conjecturally for all quasi-smooth hypersurfaces.

Corollary 2.7. *The hypersurface* V *cannot be 3-symmetric; it is 2-symmetric if and only if it is oversymmetric and* $d = 3$*.*

For by [10, Lemma 5.2], if f is annihilated by vector fields ξ, ξ' with $\xi \wedge \xi' \neq 0$, of degrees r, r' we must have $r + r' \geq d-1$. If V is 2-symmetric, we have $r = r' = 1$, hence $d = 3$ and V is oversymmetric; the converse is immediate. It follows from the theorem that now any vector field annihilating f is a linear combination of ξ, ξ' and Hamiltonian vector fields; hence if linear, is a linear combination of ξ and ξ'.

* * *

For V a hypersurface, write $\Sigma(V)$ for the set of singularities of V, $O(V)$ for the orbit of V under $\mathrm{PGL}_{n+1}(\mathbb{C})$. Given a finite list Σ of singularities, write $V^{an}(\Sigma)$, $V^{es}(\Sigma)$ for the set of hypersurfaces W of degree d in $P^n(\mathbb{C})$ such that $\Sigma(W)$ is analytically ($\mathcal{K}-$) equivalent, resp. equisingular (μ-equivalent) to Σ.

As our estimates relate mainly to τ, but equisingularity to μ, we can only expect general results about the former. Indeed, it is not in general the case that if $V^{an}(\Sigma)$ contains a 1-symmetric manifold, then all its hypersurfaces are 1-symmetric (we abstain here from giving examples about these $V^{es}(\Sigma)$ in low dimensions), but it is possible to say something.

Lemma 2.8. *[10, Proposition 5.7] (or [9, Proposition 5.3]) Suppose V is 1-symmetric and $d \geq 3$: then* $\dim V^{an}(\Sigma(V)) - \dim O(V) \leq \binom{n}{2}$*; if $d = 3$, we can replace the bound by $1+\binom{n-1}{2}$.*

If moreover V is oversymmetric, the dimensions are equal.

Thus in the oversymmetric case, the isomorphism class of the singularities determines V up to projective equivalence (up to a possible finite ambiguity, which is eliminated by a glance at the enumerations to follow).

3 The semi-simple case

As in the curve case, very different patterns present themselves in the semi-simple and unipotent cases. In this section we discuss the semi-simple case.

We diagonalise the group G, say $t.(x_0, \ldots, x_n) = (t^{w_0}x_0, \ldots, t^{w_n}x_n)$. If $f = 0$ is invariant, then $f(t.x) = t^D f(x)$ for some D, so for each monomial $\prod_i x_i^{a_i}$ appearing in f we must have $\sum_i a_i w_i = D$, as well as $\sum_i a_i = d$. In particular, $\sum_i a_i(dw_i - D) = 0$: this gives a 1-parameter group (denoted $\widetilde{G}$ in §1) leaving f invariant, and we consider such a group from now on. Also, we see that the $G-$invariant functions f are linear combinations of a fixed set $R(W)$ of monomials. For enumeration we have to decide for which actions we can find a linear combination of $R(W)$ defining a V with isolated singularities.

Let $\mathbb{C}^I$ the vector space with coordinates x_i corresponding to $i \in I$ (I an index set); $\mathbf{x}$ denotes the vector $\{x_i\}$. For $A \subset I$, we set $\mathbb{C}^A := \{\mathbf{x} \mid i \notin A \Rightarrow x_i = 0\}$. Let R be a set of monomials $m_r = \mathbf{x}^{\alpha(r)}$ in the x_i, and u_r coordinates on $\mathbb{C}^R$. Linear combinations of the monomials in R, or R-functions, are sections f_u of the evaluation map $F : \mathbb{C}^R \times \mathbb{C}^I \to \mathbb{C}$ defined by $F(\mathbf{u}, \mathbf{x}) = f_u(\mathbf{x}) = \sum_{r \in R} u_r \mathbf{x}^{\alpha(r)}$.

Write $V := F^{-1}(0)$ for the zero locus and $\Sigma V = V \cap \Sigma F$ for its singular set. For $A \subset I$, set

$$M_A := \{i \in I - A \mid \exists r \in R \text{ such that } m_r = x_i m \text{ with } m \mid \mathbb{C}^A \not\equiv 0\}.$$

Theorem 3.1. *[14, 5-7], [8, Theorem 4.1] The following conditions on R are equivalent:*
(a) there exists an R-function with isolated singularities on $\mathbb{C}^I$;
(b) a generic R-function has isolated singularities on $\mathbb{C}^I$;
(e) $\dim \Sigma V \leq \#R$*;*
(f) for all non-empty $A \in \mathcal{A}$, either for some $r \in R$, $m_r \mid \mathbb{C}^A \not\equiv 0$, or $\#M_A \geq \#A$.

This is not difficult: the main idea is that if (e) holds, the genericity of regular values of a smooth map implies (b). Translating this to the projective setting yields

Theorem 3.2. *[8, Theorem 4.2] A generic linear combination of monomials in R defines a hypersurface with isolated singularities in $P(\mathbb{C}^I)$ if and only if, for each $A \subseteq I$ with $\#A \geq 2$, we have*
C_A: either for some $r \in R$, $m_r|\mathbb{C}^A \not\equiv 0$, or $\#M_A \geq \#A - 1$.

Taking R to be the set $R(W)$ of monomials $\prod_0^n x_i^{a_i}$ such that $\sum a_i = d$ and $\sum w_i a_i = 0$ now gives the desired criterion. Unfortunately, there is no simple condition on the weights equivalent to this.

We can deduce, however, that if W is a system of weights with $R(W)$ satisfying 3.2, the same holds for the system W^* obtained by adding a zero weight to W (alternatively, if f admits W and has isolated singularities, so has $f(x_0, \ldots, x_n) + x_{n+1}^d$). In some cases this implication can be reversed.

Lemma 3.3. *[9, Prop 7.4] Suppose W is a system of weights containing 0 at least once; form W^* by adding a further zero weight. If $R(W^*)$ satisfies the conditions of 3.2, so does $R(W)$.*

Proof. Suppose not: then C_A fails for some set A of coordinates. Choose such an A with the minimum number $\#A$ of elements. Since C_A fails, there is no monomial of degree d and weight 0 formed from the variables in A.

Since C_A holds in the system W^*, there are at least $\#A - 1$ variables $x_i \notin A$ such that for some monomial m_i in the variables in A, $x_i m_i$ has degree d and weight 0. Now x_* must appear among these variables, otherwise C_A would hold. Thus A cannot include a variable of weight 0, since if x_0 were one such, $x_0 m_*$ would contradict the conclusion of the preceding paragraph. By hypothesis there is at least one other variable, y say, of weight 0, and as $y m_*$ has weight 0, $y \in N_A$.

Partition $A = A^+ \cup A^-$ according to the signs of the weights of the variables. Neither can be empty since if, for example, $A = A^+$ then $x_* m_*$ would have positive weight, a contradiction. So by inductive hypothesis, C_{A^+} and C_{A^-} both hold. But now N_A contains N_{A^+}, with at least $\#A^+ - 1$ elements (all of negative weight), and N_{A^-}, with at least $\#A^- - 1$ elements (all of positive weight), as well as at least one variable y of weight zero. It thus has at least $\#A - 1$ elements, and C_A holds, a contradiction. □

If f, homogeneous of degree d, is invariant under the action with weights w_i $(0 \leq i \leq n)$, then substituting $x_0 = 1$ gives a function $\tilde{f}$ on affine space, homogeneous of weight $-dw_0$ if x_i is assigned weight $w_i - w_0$. In general, several of these weights are negative. This is not inconsistent with having an isolated singularity and occurs, for example, for $\sum_1^r x_i y_i + g(z_1, \ldots, z_s)$ if g has an isolated singularity and the weights of x_i and y_i can be arbitrary provided they sum to $-dw_0$. It is not hard to show that this is the general picture.

Lemma 3.4. *[8, Lemma 2.2] Let $\mathbb{C}^\times$ act as a group of symmetries of a function f with an isolated singularity. Then*
(i) if the action on the target has weight $v > 0$, those weights on $\mathbb{C}^n$ which do not satisfy $0 < w < v$ fall into pairs $(w, v - w)$

(ii) if the action on the target has weight 0, all non-zero weights on $\mathbb{C}^n$ fall into pairs $(w, -w)$.

We call a system of weights as in (ii) 0-*symmetric.*

Next we analyse the singularities of V. Write F_λ for the eigenspace corresponding to the weight λ and for its image in projective space. The fixed point set F of the action of $\mathbb{C}^*$ on projective space is the union of the F_λ; as V has isolated singular points, each of these lies in F. Also, if $\lambda \neq 0$, then $F_\lambda \subset V$.

Lemma 3.5. *[8, Lemma 3.4] If the zero weight space F_0 contains a singular point, the set of weights is* 0*-symmetric. The type of each singular point of V lying in F_0 is the suspension of its type as singular point of $V \cap F_0$. The hypersurface $V \cap F_0$ of degree d in F_0 may be chosen arbitrarily (subject to having isolated singularities).*

The first two assertions are immediate consequences of lemma 3.4. The proof of the third uses Bertini's theorem.

If $\lambda \neq 0$, we have a fairly complete description of the singularities of V in V_λ: see [8, Proposition 3.3]. Denote by m_λ the multiplicity with which λ appears as a weight. It follows from Theorem 3.1 that if $m_\lambda > 1$, then $m_{(1-d)\lambda} \geq m_\lambda - 1$. If $P \in F_\lambda$ is a singular point of V, then in local coordinates at P, V is invariant by an action of $\mathbb{C}^*$ with weights $w_i - \lambda$ and degree $-d\lambda \neq 0$. It follows by lemma 3.4 that the weights not between 0 and $-d\lambda$ fall into pairs $(-w, w - d\lambda)$. Thus the weights equal to λ (with one removed) correspond to weights 0, and are paired with weights $-d\lambda$ and hence weights $(1-d)\lambda$, so $m_{(1-d)\lambda} = m_\lambda - 1$. Thus if $m_{(1-d)\lambda} > m_\lambda - 1$ there are no singular points in F_λ.

If $m_{(1-d)\lambda} = m_\lambda - 1$, further analysis shows that there are $(d-1)^{m_\lambda - 1}$ singular points of f on F_λ, each one weighted homogeneous, and a suspension of a singularity with weights $w_i - \lambda$, where the above $2m_\lambda - 1$ values of i are removed.

* * *

Next we calculate $\mu(V)$ and $\tau(V)$, following [8, Proposition 3.5]. Write $s_n(d) := \{(d-1)^{n+1} - (-1)^{n+1}\}/d$. Then for any hypersurface V of degree d in P^n with isolated singularities,

$$\mu(V) - s_n(d) = (-1)^n(\chi(V) - (n+1)), \tag{2}$$

where $\chi(V)$ denotes the Euler characteristic. Thus it suffices to calculate $\chi(V)$.

Proposition 3.6. *If 0 is not a weight, $\mu(V) = s_n(d)$. In all cases,*

$$-s_{m_0-1}(d) \leq (-1)^{n-m_0-1}(\mu(V) - s_n(d)) \leq s_{m_0-2}(d),$$

where the left hand equality holds if V_0 is smooth, e.g. if the set of weights is not 0*-symmetric; the right hand equality holds if V_0 is a cone. The same assertions hold for $\tau(V)$.*

In the semi-simple case, each non-trivial orbit has $\chi = 0$. Hence $\chi(V) = \chi(V \cap F)$. We have $F = \bigcup F_\lambda$, and $F_\lambda \subset V$ if $\lambda \neq 0$: set $V_0 := V \cap F_0$. Now as λ has multiplicity m_λ, F_λ is a projective space of dimension $m_\lambda - 1$, so $\chi(F_\lambda) = m_\lambda$. Thus if 0 does not occur as a weight, $\chi(V) = n+1$, so $\mu(V) = s_n(d)$; in general, $\chi(V) - \chi(V_0) =$

$n+1-m_0$. Since each singularity outside V_0 is weighted homogeneous, $\tau(V)-\tau(V_0) = \mu(V) - \mu(V_0)$.

Now $\mu(V_0) \geq 0$, with equality only if V_0 is smooth, e.g. if the set of weights is not 0-symmetric. Since V_0 has isolated singularities, its Milnor number is greatest when it is a cone over a non-singular variety, and then $\tau(V_0) = \mu(V_0) = (d-1)^{m_0-1}$. The result follows.

Corollary 3.7. *For fixed n, m_0 and d, μ (and hence τ) is greatest when either $m_0 = n-1$, V_0 is a cone, and so the non-zero weights are ± 1 or $m_0 = n-2$ and V_0 is smooth. In both cases we have $\tau(V) = (d^2-3d+3)(d-1)^{n-2}$, so V is oversymmetric.*

For if $n - m_0$ is odd, the right hand inequality must be an equality: V_0 is a cone and $\tau(V) = s_n(d) + s_{m_0-2}(d)$ is greatest when $m_0 = n-1$. Thus there are two non-zero weights; as the zero weight space contains a singularity, these are ± 1.

If $n - m_0$ is even, we need equality on the left hand: V_0 is smooth and $\tau(V) = s_n(d) + s_{m_0-1}(d)$ is greatest when $m_0 = n-2$. Thus the set of weights is obtained from a set of 4 weights, which includes one zero, by adding further zeros.

* * *

We thus next enumerate the 1-symmetric surfaces in $P^3(\mathbb{C})$. Condition $C_{0,1,2}$ states that R must contain some $x_0^{a_0}x_1^{a_1}x_2^{a_2}$, and $C_{0,1}$ that there must be a monomial of the form $x_0^{a_0}x_1^{a_1}$, $x_0^{a_0}x_1^{a_1}x_2$ or $x_0^{a_0}x_1^{a_1}x_3$ in R.

First suppose two weights are equal: say $w_2 = w_3$. It is then sufficient to consider monomials in x_0, x_1, x_2 only; we may consider them as functions on the plane. Then $C_{2,3}$ states that we have a term x_2^d or $x_2^{d-1}x_0$ or $x_2^{d-1}x_1$. This already implies $C_{0,2} = C_{0,3}$ and $C_{1,2} = C_{1,3}$. $C_{0,1}$ gives an invariant monomial of the form $x_0^r x_1^{d-r}$ or $x_0^r x_1^{d-r-1}x_2$. Now conditions $C_{0,1,2}$ and $C_{0,1,3}$ are automatic, and $C_{1,2,3}, C_{0,2,3}$ demand monomials not involving x_0 and x_1 respectively. Taking all cases, and rearranging the list of weights in increasing order gives $[w_0, w_1, w_2, w_3]$ equal to one of the following: $[r-d, 0, 0, r]$, $[r+1-d, 0, 0, r]$, $[-1, -1, 0, d-1]$ or $[1-d, 1-d, 1, (d-1)^2]$.

If $w_0 < w_1 < w_2 < w_3$ then since by $C_{0,1,2}$ some $a_0w_0 + a_1w_1 + a_2w_2 = 0$ we have $w_0 \leq 0$ (so by symmetry $w_3 \geq 0$); and by $C_{2,3}$ some $w_0 + a_2w_2 + a_3w_3$ or $w_1 + a_2w_2 + a_3w_3$ or $a_2w_2 + a_3w_3$ is 0, so $w_0 + (d-1)w_2 \leq 0$ (hence by symmetry $(d-1)w_1 + w_3 \geq 0$). Now setting $x_1 = 1$ and applying Lemma 3.4 gives a list of possibilities from which we see that one of the following must be the case:

$$\begin{array}{llll} (\alpha) & w_1 = 0 & (\beta) & (d-1)w_1 + w_2 = 0 \\ (\gamma) & (d-1)w_1 + w_3 = 0 & (\delta) & w_0 + (d-2)w_1 + w_3 = 0. \end{array}$$

By symmetry, one of the following must also hold:

$$\begin{array}{llll} (\alpha') & w_2 = 0 & (\beta') & w_1 + (d-1)w_2 = 0 \\ (\gamma') & w_0 + (d-1)w_2 = 0 & (\delta') & w_0 + (d-2)w_2 + w_3 = 0. \end{array}$$

These imply all conditions $C_{i,j}$ except $C_{0,3}$, which implies that one of the following holds:

$$\begin{array}{lll} (\lambda_r) & rw_0 + (d-r)w_3 = 0 & (0 \leq r \leq d) \\ (\mu_r) & rw_0 + w_1 + (d-r-1)w_3 = 0 & (0 \leq r \leq d-1) \\ (\nu_r) & rw_0 + w_2 + (d-r-1)w_3 = 0 & (0 \leq r \leq d-1). \end{array}$$

Starting from this point we gave in [8, §5] a complete enumeration.

We saw in Lemma 3.3 that to list oversymmetric cases we need systems with one weight zero. We may suppose $w_1 = 0$ (case α); as we have already considered cases with two equal weights, $w_2 \neq 0$, so we do not have (α') or (β'). Each of the remaining combinations determines the ratios of all w_i. If we change the names of γ', δ' to B, C, we recognise the cases $B\lambda_r$, $B\nu_r$, $C\lambda_r$, $C\nu_r$ as obtained from the enumeration of curves by adjoining a zero weight, while $A\lambda_r$, $A\nu_r$ give the cases with two equal (zero) weights. The cases $B\mu_r$, $C\mu_r$ are new. So if just one weight is non-zero, we have $B\lambda_r$, $B\mu_r$, $B\nu_r$, $C\lambda_r$, $C\mu_r$, $C\nu_r$. In these cases, $\partial f/\partial x_2$ and $\partial f/\partial x_3$ are both divisible by x_0 and we can take the second vector field as $x_0^{-1}\eta_{2,3}(f)$.

4 The unipotent case

The unipotent case is perhaps more interesting. We first focus on enumeration.

Begin with a vector field $\sum_{ij} a_{ij}x_i\partial/\partial x_j$ whose matrix (a_{ij}) is nilpotent. We put this matrix in Jordan normal form. This is determined by the sizes of the Jordan blocks, which we denote by $r_1+1 \geq r_2+1 \geq \ldots$. Our standard notation for a vector field with a single block is $\xi = \sum_{i=0}^{r} x_{i-1}\partial/\partial x_i$ (when we need a second block we use y_i). Write R for the sequence of r_i, where zeros are omitted.

Theorem 4.1. *We have one of the following four cases: $R = (2)$, $d \geq 3$; $R = (3)$, $d = 4$; $R = (4)$, $d = 3$; $R = (2,1)$, $d = 3$.*

The proof involves several steps. First we can extend the action of any nilpotent endomorphism on a finite dimensional vector space K to an action of the Lie group sl_2 (with basis e, f, h and $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$) where the given endomorphism is f. The eigenvalues of h are integers, and hence define a grading. For each r, f^r gives an isomorphism of K_r on K_{-r}.

The action of sl_2 extends to an action on the symmetric algebra on K; also if we have actions on K, K' we get one on $S^rK \otimes S^{r'}K'$.

Lemma 4.2. *Let f be a polynomial of degree d and weight W in variables x_i, with weights w_i (in increasing order) which defines an affine variety V with* $\dim(\operatorname{Sing} V) \leq k$. *Then $(d-1)w_i \leq W - w_{n+1-i-k}$ for $1 \leq i \leq n-k$.*

For if all $\partial f/\partial x_j$ vanish on $\{x_1 = \ldots = x_{n-k-1} = 0\}$ then this subspace is singular on V, and $\dim(\operatorname{Sing}\ V) \geq k+1$. Hence some $\partial f/\partial x_j$ depends only on $x_{n-k}, \ldots, x_n$, so $(d-1)w_{n-k} \leq W - w_j \leq W - w_1$. Now repeat the argument.

In the case of interest here, $k = 1$, $W = 0$ and $w_{n+1-i} = -w_i$. Thus $(d-1)w_i \leq w_{i+1}$ for $1 \leq i \leq n-1$.

In the situation of Theorem 4.1, for each i we have basis elements of weights $-r_i, 2-r_i, \ldots, r_i-2, r_i$; and f contains only monomials of weight ≤ 0.

If $r_2 = r_1$, we have $w_{n-1} = w_n = r_1$, so $r_1 \geq r_1(d-1)$, a contradiction.

If $r_2 = r_1 - 1$, we have $w_n = r_1, w_{n-1} = r_1 - 1$, so $r_1 \geq (r_1-1)(d-1)$ and $r_1 \leq \frac{d-1}{d-2}$. If $d > 3$ this implies $r_1 = 1$, which implies that V is a cone, so is excluded. If $d = 3$ we may also have $r_1 = 2$. If now $r_3 = 1$, we have $w_{n-2} = 1$, contradicting $(d-1)w_{n-2} \leq w_{n-1}$. Thus $r_3 = 0$, and we have Case 21 of the Theorem.

Otherwise $w_n = r_1$ and $w_{n-1} = r_1 - 2$, whence $r_1 \geq (r_1-2)(d-1)$ and $r_1 \leq \frac{2(d-1)}{d-2}$. This gives $r_1 \leq 4$ if $d = 3$, $r_1 \leq 3$ if $d = 4$ and $r_1 \leq 2$ if $d > 4$.

If $r_1 = 2$, we have Case 2 of the theorem. If $d = 4$, the remaining possibility is $r_1 = 3$, so that $w_n = 3, w_{n-1} = 1$. Since $3w_{n-2} \leq w_{n-1}$, we have $w_{n-2} = 0$, and Case 3 of the Theorem. The cases $d = 3$ and r_1 equal to 3 or 4 are dealt with by the same methods, but require a little more care.

To find equations we need the (known: see e.g. [15]) calculation of invariant rings.

Lemma 4.3. *We have rings of invariants*

Case 2:	$\mathbb{C}[W, X, B]$,
Case 3:	$\mathbb{C}[W, X, B, C, \Delta / X^2\Delta + C^2 + B^3 = 0]$,
Case 4:	$\mathbb{C}[W, X, B, U, C, E / X^3 E = 3X^2 BU - B^3 - C^2]$,
Case 21:	$\mathbb{C}[W, X, Y, T, B, S / XS = Y^2 B + T^2]$,

where $W := w$, $X := x_0$, $Y := y_0$, *and*

$$
\begin{aligned}
B &:= T^2_{x,x} = 2x_0x_2 - x_1^2, \\
C &:= 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3, \\
\Delta &:= -9x_0^2x_3^2 + 18x_0x_1x_2x_3 - 8x_0x_2^3 + 3x_1^2x_2^2 - 6x_1^3x_3, \\
U &:= T^4_{x,x} = 2x_0x_4 - 2x_1x_3 + x_2^2, \\
E &:= 12x_0x_2x_4 - 9x_0x_3^2 + 6x_1x_2x_3 - 2x_2^3 - 6x_1^2x_4, \\
T &:= T^1_{x,y} = x_0y_1 - x_1y_0, \\
S &:= x_0y_1^2 - 2x_1y_0y_1 + 2x_2y_0^2.
\end{aligned}
$$

Although the latter three rings are not polynomial, we can simplify as follows. Define a projection $\pi : K \to L$ by the invariant functions (Case 2) (W, X, B), (Case 3) (W, X, B, Δ), (Case 4) (W, X, U, E), (Case 21) (W, X, Y, B, S).

Lemma 4.4. *Suppose f, invariant under the group, defines a hypersurface V with isolated singularities. Then there is an allowable change of co-ordinates which puts f in the form $f = \phi \circ \pi$, where the polynomial ϕ is*

Case 3	$\Delta + a_0B^2 + a_2B + a_4$,
Case 4	$E + 3a_1U + a_3$,
Case 21	$S + a_1B + a_3$.

In each case, a_i is homogeneous of degree i in the invariant co-ordinates W, X and (for a_3 in Case 21) Y.

For it follows from the proof of the previous result that the coefficient in f of (Case 3) $x_2^3x_0$, (Case 4) $x_3^2x_0$, (Case 21) $y_1^2x_0$ is non-zero, and hence that the expression of f by the invariants contains Δ, E or S respectively. It is now easy to write down co-ordinate changes which eliminate C, B and C, or T respectively.

In Case 2 with $d = 3$, we can write $\phi = a_1B + a_3$. Here, and in Cases 4 and 21, if the coefficient of any W_i in a_1 is non-zero, we can take new co-ordinates with $a_1 = W_1$. Otherwise, a_1 is a multiple of X which in Cases 2 and 21 cannot be identically 0, so we can take $a_1 = X$.

In Case 21 there is a further linear vector field that annihilates f, viz. $\xi_1 = y_0\partial/\partial x_1 + y_1\partial/\partial x_2 - a_1\partial/\partial y_1$. If $a_1 = X$, the vector fields $\xi_0 \pm \xi_1$ both fall into Case 2: we will now call this Case 21_0. We rename the case $a_1 = W_1$ as Case 5, and also rename w_1 as z_0. Then $f = J + a_3(w, x_0, y_0, z_0)$ where

$$J := 2x_2(y_0^2 - x_0z_0) + x_1^2z_0 - 2x_1y_1y_0 + y_1^2x_0.$$

We next consider criteria for isolated singularities. In each case, the main singularity occurs at the point P where all co-ordinates except the last vanish. We define a number of auxiliary hypersurfaces, mostly sections of $\phi = 0$. Our notation is chosen so that each V_r (also V_r^*, etc.) has dimension $m + 1 - r$, where m is the number of variables w_i; the variable Z below has weight 1 and arises as BX^{-1}.

(Case 2) V_1 is defined by $\phi = 0$, and V_2 and V_3 are its intersections with $X = 0$ and with $X = B = 0$ respectively.

(Case 3) V_0 is $\phi = 0$, V_1 its intersection with $X = 0$, and V_3 its intersection with $X = B = \Delta = 0$. Write $\psi(W, X, Z) := -XZ^3 + a_0X^2Z^2 + a_2XZ + a_4 = 0$, V_1' for the hypersurface $\psi = 0$, and V_2' for its intersection with $X = 0$.

(Case 4) Write $a_3^* := a_1^3 + a_3$. We define V_2 by $a_3(W, X) = 0$, V_2^* by $a_3^*(W, X) = 0$, and their respective intersections with $X = 0$ by V_3, V_3^*.

Lemma 4.5. *V has isolated singular points if and only if*

(Case 2) V_1 has no singular points and V_3 has isolated singular points;

(Case 3) V_1', V_3 are both non-singular;

(Case 4) V_2^* and V_3^* are non-singular and V_3 has isolated singular points.

The singular points of f are then P and (in Cases 2, 4) points P_i corresponding to the singular points Q_i of V_3 which (Case 2) are not also singular on V_2. Moreover, the singularity at P_i is a suspension of that of V_3 at Q_i.

Proof. We discuss Case 2 only; the other arguments are similar but more involved.

At a critical point of f, the following vanish:

$$\frac{\partial f}{\partial w_i} = \frac{\partial \phi}{\partial W_i}, \ \frac{\partial f}{\partial x_0} = \frac{\partial \phi}{\partial X} - 2x_2\frac{\partial \phi}{\partial B}, \ \frac{\partial f}{\partial x_1} = 2x_1\frac{\partial \phi}{\partial B}, \ \frac{\partial f}{\partial x_2} = -2x_0\frac{\partial \phi}{\partial B}.$$

If $\partial\phi/\partial B = 0$, we have a critical point of ϕ. If $W = X = B = 0$, the only corresponding point in $P(K)$ is P. Otherwise we have a singular point of V_1. Conversely, if we have a singular point of V_1, all the points in its pre-image are singular on V, so are non-isolated singular points of V.

If $\partial\phi/\partial B \neq 0$, then $x_0 = x_1 = 0$, hence $B = 0$. We thus have a critical point of the restriction of ϕ to $X = B = 0$. If $W = 0$, we again have the point P. Otherwise we have a singular point of V_3. Conversely, if we have such a singular point with $\partial\phi/\partial B(W, 0, 0) \neq 0$, there is a unique corresponding value of x_2 giving a critical point of f, hence a unique corresponding singular point of V. However, if we have a singular point of V_3 at which $\partial\phi/\partial B(W, 0, 0) = 0$, then as above we have a singular point of V_1.

The final assertion follows from a sharp version of the splitting theorem. □

The structure in Case 5 is somewhat different. We have two hypersurfaces in (w, x_0, y_0, z_0) space: the cone $C : y_0^2 - x_0z_0 = 0$ and the variety V_0 defined by $a_3 = 0$.

Write $V_1 := V_0 \cap C$, and V_3 for the variety $a_3(w, 0, 0, 0) = 0$. Also define V_1^* as the variety $\phi(w, t, u) := a_3(w, t^2, tu, u^2) = 0$ in weighted projective space $P(2^m 1^2)$.

Proposition 4.6. *The variety V has isolated singularities if and only if*

(i) for no (w, x_0, y_0, z_0) in C with $(x_0, y_0, z_0) \neq (0, 0, 0)$ do we have $\partial a_3/\partial w_i = 0$ for each i, and the matrix A of rank 1, where

$$A := \begin{pmatrix} \partial f/\partial x_0 & \partial f/\partial y_0 & \partial f/\partial z_0 \\ z_0 & -2y_0 & x_0 \end{pmatrix}, \quad \textit{and}$$

(ii) for any singular point of V_3 we have $(\partial a_3/\partial y_0)^2 \neq 4(\partial a_3/\partial x_0)(\partial a_3/\partial z_0)$. In particular, singular points of V_3 are isolated.

When this holds, P is the only singular point of V.

Theorem 4.7. *The values of μ are given by*

Case	μ
2	$\frac{1}{2}(d-2)(2d-1)(d-1)^{n-2} + \mu(V_2) + \mu(V_3)$
3	22.3^{n-3}
4	$11.2^{n-4} + \mu(V_3)$
5	$25.2^{n-5} + \mu(V_3)$

We find μ by decomposing V into pieces whose Euler characteristics we can determine directly, and infer μ from (2). We illustrate with Case 2. The quotient map $\pi : K \to L$ induces a map $\overline{\pi} : P(K) \to P(L)$ of (weighted) projective spaces. We stratify these by

$$\begin{array}{llll} \mathcal{S}_0 : & x_0 \neq 0; & \overline{\mathcal{S}}_0 : & X \neq 0 \\ \mathcal{S}_1 : & x_0 = 0,\ x_1 \neq 0; & \overline{\mathcal{S}}_1 : & X = 0,\ B \neq 0 \\ \mathcal{S}_2 : & x_0 = x_1 = 0; & \overline{\mathcal{S}}_2 : & X = B = 0. \end{array}$$

By direct calculation of pre-images under π in the affine case and taking some care we infer the Euler characteristics of pre-images under $\overline{\pi}$ in the projective case: note that $\overline{\pi}$ also has an exceptional set $\mathcal{E}$ which is a point.

The preimage $\overline{\pi}^{-1}(W, X, B)$ is as follows:
($\overline{\mathcal{S}}_0$) one orbit, $\chi = 1$,
($\overline{\mathcal{S}}_1$) if $W \neq 0$, two orbits, $\chi = 2$; if $W = 0$, one orbit, $\chi = 1$,
($\overline{\mathcal{S}}_2$) infinitely many point orbits, $\chi = 1$.

Note that V_1 meets the respective strata in $V_1 \setminus V_2$, $V_2 \setminus V_3$ and V_3.

First suppose d odd. Then V_1 necessarily contains the point Q where $W = X = 0$, $B \neq 0$, so we have

$$\chi(V) = \chi(\mathcal{E}) + \chi(V_1 \setminus V_2) + 2\chi(V_2 \setminus V_3) - 1 + \chi(V_3) = \chi(V_1) + \chi(V_2) - \chi(V_3).$$

Now V_1 is non-singular, and V_3 has isolated singularities. Thus we can apply (2) to V_3. A formula due to Steenbrink (see [1], [13]) gives the value of χ for a non-singular hypersurface V^* of degree d in weighted projective space with weights w_i. Set $p(t) = \prod_{i=0}^{n+1}(1 - t^{d-w_i})/(1 - t^{w_i})$: note that this is a polynomial. Then

$$\chi(V^*) = n + 1 + (-1)^n \tfrac{1}{d} \textstyle\sum_{\epsilon^d=1} \epsilon^W p(\epsilon).$$

In particular,

Lemma 4.8. *A non-singular hypersurface V^* of dimension n and degree d with respect to weights $w_0 = 2$ and $w_i = 1$ for $1 \le i \le n+1$ has*

$$\chi(V^*) = n + 2 + \tfrac{(-1)^n}{d}\{\tfrac{d-2}{2}(d-1)^{n+1} + (-1)^{n+1}\}$$

if d is odd, and is $\frac{1}{2}$ less than this if d is even. In particular, if $d = 3$ we have $n + 2 + \frac{(-1)^n}{3}\{2^n - (-1)^n\}$.

We can apply this with $n = m$ to V_1, and with $n = m-1$ to V_2. If V_2 is singular, its singularities must be isolated, and we can show that (as for hypersurfaces in ordinary projective space) the above formula must be adjusted by $\mu(V_2)$. Substituting in, and performing cancellations, this gives the result stated.

In the case d even, the point $Q \notin V_1$; otherwise it would be a singular point. Thus in the above formula for $\chi(V)$ we must add 1 (corresponding to this point) and subtract $\frac{1}{2}$ for each of $\chi(V_1)$ and $\chi(V_2)$. Hence the formula in terms of d is the same as before.

When Theorem 4.7 gives a precise value for the Milnor number of the singularity at P, these are all in the same $\mu-$constant stratum as the parameters vary. Adding an extra variable w_i has the effect of adding a term w_i^d to the local equation; this is why the value of μ for the singularity at P is multiplied by $d-1$. We can then easily identify the stratum by taking n small. Thus provided in Case 2 that V_2 non-singular and in Case 5 that V_3 is, the $\mu-$constant stratum at P is that of a semi-quasi-homogeneous singularity with degree and weights given by the table (3).

In fact, in each case a suitable substitution shows that the singularity is itself semi-quasi-homogeneous. We illustrate with Case 2. Take local affine co-ordinates at P with $x_2 = 1$, and substitute $x_0 := \frac{1}{2}(y + x_1^2)$. Now assign weights 1 to x_1, 4 to y and 2 to the w_i. All terms have weight at least $2d$: the sum of those of exactly this weight is $\phi(w, y, \frac{1}{2}x_1^2)$. Using the fact that V_1 is non-singular, it follows easily that this has an isolated singular point.

Case	Degree	Weights
2	$2d$	1, 4, 2 (m times)
3	12	1, 4, 6, 3 (m times)
4	12	1, 6, 6, 6, 4 (m times)
5	6	1, 1, 3, 3, 3, 2 (m times)

(3)

We turn to calculation of τ. For Case 21, we have already seen that there is a second linear vector field annihilating f, so by Theorem 2.5 we have $\tau = 3.2^{n-2}$.

For Case 3, we set $\eta' := x_1^2\partial/\partial x_1 + (3x_1x_2 - 3x_0x_3)\partial/\partial x_2 + (4x_2^2 - 3x_1x_3)\partial/\partial x_3$. Then $\eta'\Delta = 0$ and $\partial\Delta/\partial x_3 = 3\eta' B$. Hence f is annihilated by $\eta' - \frac{1}{3}(2a_0B + a_2)\partial/\partial x_3$, of degree 2 and independent of ξ. Thus again V is oversymmetric; by Theorem 2.5 we have $\tau = 7.3^{n-2}$.

In Case 4, since $\tau \le \mu = 11.2^{n-4} < 3.2^{n-2}$, V cannot be oversymmetric. In fact this bound for τ is achieved in some cases. Suppose $f = E + a_0x_0U + a_1x_0^2 + a_3$ with a_3 independent of x_0. Then we can construct a vector field $\zeta = \sum_0^4 \phi_i\partial/\partial x_i$ with $\zeta(f) = 0$ and ϕ_4 non-vanishing at the singular point P. Thus by Saito's criterion the singularity is weighted homogeneous, and $\tau_P = \mu_P$.

Case 2 has numerous subcases, and we do not know the generic value of τ. The maximal value is characterised by

Theorem 4.9. *A function in Case 2 is oversymmetric if and only if either: (a) V_3 is a cone, or (b) after change of co-ordinates if necessary, $\partial\phi/\partial B$ and $\partial\phi/\partial X$ both vanish along $X = B = 0$.*

Proof. Since V_1 is non-singular, the sequence $\{\partial\phi/\partial W_1, \ldots, \partial\phi/\partial W_m, \partial\phi/\partial X, \partial\phi/\partial B\}$ is regular, and any vector field annihilating ϕ is a linear combination of the Hamiltonian fields $\partial(\phi,*)/\partial(W_i,W_j)$, $\partial(\phi,*)/\partial(W_i,X)$, $\partial(\phi,*)/\partial(W_i,B)$ and $\partial(\phi,*)/\partial(X,B)$. We seek a vector field η which is a lift of a linear combination of these. We are only interested in η modulo Hamiltonian fields: removing the corresponding linear combination of the $\partial(f,*)/\partial(w_i,w_j)$ and $\partial(f,*)/\partial(w_i,x_0)$, we can take $\sum C_i\partial(\phi,*)/\partial(W_i,B) + D\partial(\phi,*)/\partial(X,B)$. Since we seek η of degree $d-2$, we want the C_i and D to be constants. We now have

$$\eta = \textstyle\sum_1^m p_i\partial/\partial w_i + \sum_0^2 q_j\partial/\partial x_j,$$

where $p_i = -C_i\partial\phi/\partial B$, $q_0 = -D\partial\phi/\partial B$ and

$$2(x_2q_0 - x_1q_1 + x_0q_2) = \textstyle\sum C_i\partial\phi/\partial W_i + D\partial\phi/\partial X.$$

Thus

$$2(x_0q_2 - x_1q_1) = \sum C_i\partial\phi/\partial W_i + D\partial\phi/\partial X + 2x_2D\partial\phi/\partial B.$$

The right hand side of this equation must vanish identically along $X = B = 0$.

If $D = 0$, changing co-ordinates, we may suppose $\eta = \partial(\phi,*)/\partial(W_1,B)$. If we set $a_d(W) := \phi(W,0,0)$, we need $\partial a_d/\partial W_1 \equiv 0$, i.e. a_d independent of W_1. This holds if and only if V_3 is a cone.

If $D \neq 0$, a suitable substitution $W_i' := W_i + \lambda_i X$, $X' := X$ reduces the C_i to zero, so it suffices to consider $\eta = \partial(\phi,*)/\partial(X,B)$. Here the condition is that both $\partial\phi/\partial B$ and $\partial\phi/\partial X$ vanish along $X = B = 0$. □

In practice, (a) breaks up into subcases according as the vertex of the cone V_3 (a1) is not, or (a2) is also singular on V_2.

The method of proof can be applied to obtain calculations of τ in some other cases: for the case $n = 3$ see [8].

5 Lists

To illustrate the enumerations, we offer three new lists: 1-symmetric cubic 3-folds (surfaces with $d \leq 5$ were listed in [8]), oversymmetric hypersurfaces and 2-symmetric hypersurfaces.

1-symmetric cubic 3-folds

In the semi-simple case, the weights may be enumerated by the methods of [8, §5]; the singularities are determined by the rules in [8, §3], except in the 0-symmetric cases (denoted *), when there may be additional singularities: A_1 for [-2,-1,0,1,2] and A_1, A_2, $2A_1$ A_3, $3A_1$ or D_4 for $[-1,0,0,0,1]$. In particular, $\mu(V)$ is 11, 10, 12 or 8 according as the number of zero weights is 0, 1, 2 or 3.

μ	Weights	Singularities	μ	Weights	Singularities
11	$[-8,-2,1,4,16]$	S_{11}	11	$[-10,-4,2,5,8]$	A_7A_4
11	$[-8,-2,1,4,4]$	D_52A_3	11	$[-8,-2,1,4,7]$	D_8A_3
11	$[-5,-2,1,1,4]$	$D_6A_32A_1$	11	$[-2,-2,1,1,4]$	$\tilde{E}_72A_1$
10	$[-4,-1,0,2,8]$	Q_{10}	10	$[-4,-1,0,2,5]$	E_8A_2
10	$[-4,-1,0,2,4]$	D_7A_3	10	$[-2,-2,0,1,4]$	E_62A_2
10	$[-2,-1,0,1,3]$	$E_7A_2A_1$	10$*$	$[-2,-1,0,1,2]$	$2A_5$
12	$[-2,0,0,1,4]$	U_{12}	12	$[-2,0,0,1,2]$	$\tilde{E}_8A_2$
8	$[-1,0,0,0,2]$	$\tilde{E}_6$	8$*$	$[-1,0,0,0,1]$	$2D_4$

The unipotent cases are enumerated in §4: only Case 4 and Case 2 of that enumeration appear with $d=3$ and $n=4$.

$f = 12x_0x_2x_4 - 9x_0x_3^2 + 6x_1x_2x_3 - 2x_2^3 - 6x_1^2x_4 + 3ax_0(2x_0x_4 - 2x_1x_3 + x_2^2) + bx_0^3$, with $a^3+b \neq 0$. A_{11} singularity.

$f = x_0(2x_0x_2 - x_1^2) + a_3(x_0,x_3,x_4)$, where $a_3(0,x_3,x_4)$ has distinct roots; U_{12} singularity; $\tau = 12$ if $\partial a_3/\partial x_0 \in \langle x_0, \partial a_3/\partial x_3, \ldots, \partial a_3/\partial x_4\rangle$, $\tau = 11$ otherwise.

$f = x_3(2x_0x_2 - x_1^2) + a_3(x_0,x_3,x_4)$, where $a_3(x_0,0,x_4)$ has distinct roots and $a_3(0,x_3,x_4) \not\equiv 0$. The singularity at P has type $T_{2,3+r,6}$ if the highest power of x_3 dividing $a_3(0,x_3,x_4)$ is x_3^r; there is an additional singularity A_s where $s=1$ if $a_3(0,x_3,x_4)$ has a repeated factor (so $r=0$ or 1), $s=2$ if a 3-fold factor, distinct from x_3 (so $r=0$). Thus $\mu(V) = 10+r+s$, $\tau(V) = 10+s$ if $r=0$, $\tau(V) = 9+r+s$ if $r>0$.

Oversymmetric hypersurfaces

In the semi-simple case, the weights are given by adding zeros to one of $[-1,0,1]$, $B\lambda_r$, $B\mu_r$, $B\nu_r$, $C\lambda_r$, $C\mu_r$ or $C\nu_r$.

In the unipotent case we have all hypersurfaces in Cases 3 and 5 and those in Case 2 given by Theorem 4.9.

2-symmetric hypersurfaces

By Theorem 2.5, higher symmetry occurs only if $d=3$, and then a 1-symmetric V is 2-symmetric if and only if it is oversymmetric. Thus in the semi-simple case, the set of weights is obtained from $[-1,0,0,1]$, $[-2,0,1,2]$, or $[-2,0,1,4]$ by adding zeros. In the unipotent case, we have the V in Case 2 given by Theorem 4.9 with $d=3$, and those in Case 5. However as several cases occur twice in this list in virtue of the two vector fields, it is better to give the list explicitly. In each case we begin with a convenient normal form and conclude with an example of a singularity equisingular (ES) to that presented.

(A) $f = x_0x_1x_2 + a_3(x_3,\ldots,x_n)$, where $a_3=0$ is non-singular. This admits the action $(\lambda,\mu).(x_0,x_1,x_2,\ldots,x_n) = (\lambda^{-1}x_0, \mu^{-1}x_1, \lambda\mu x_2, \ldots, x_n)$: the weights $w_i=0$ for $i>2$ but we can take any w_0,w_1,w_2 with $w_0+w_1+w_2=0$. There are 3 singular points, mutually isomorphic, each ES to $x_1x_2 + \sum_3^n x_i^3$.

(B) $f = x_0x_1^2 + x_0x_2x_3 + a_3(x_3, \dots, x_n)$, where $a_3 = 0$ is non-singular, and has non-singular intersection with $x_3 = 0$. This admits weights $[-2, 1, 2, 0, \dots]$ and is also annihilated by $x_3\partial/\partial x_1 - 2x_1\partial/\partial x_2$, which is nilpotent in Case 2; conversely, a Case 2 cubic $w_1(2x_0x_2 - x_1^2) + a_3(x_0, w)$ can be put in this form if a_3 is independent of w_1 (Case (a1) of Theorem 4.9). Two singularities, ES to $x_1x_2 + x_3^2 + \sum_4^n x_i^3$ and $x_1x_2 + x_3^6 + \sum_4^n x_i^3$.

(C) $f = x_0(2x_0x_2 - x_1^2) + a_3(x_3, \dots, x_n)$, where $a_3 = 0$ is non-singular. This admits weights $[-2, 1, 4, 0, \dots]$ and the vector field $x_0\partial/\partial x_1 + x_1/pd/\partial x_2$ in Case 2: the special case of our normal form where $a_1 = x_0$ and a_3 is independent of x_0 (Case (b) of Theorem 4.9). Singularity ES to $x_1^2 + x_2^4 + \sum_3^n x_i^3$.

(D) $f = x_3(2x_0x_2 - x_1^2) + x_0x_4^2 + a_3(x_0, x_3, x_5, \dots, x_n)$ $(n \geq 4)$, with $a_3 = 0$ non-singular. We have vector fields $x_0\partial/\partial x_1 + x_1\partial/\partial x_2$ in Case (a2) of Theorem 4.9 and $x_4\partial/\partial x_2 - x_3\partial/\partial x_4$ in Case 21_0. The singularity has $\mu = 13.2^{n-4}$, and is ES to $x_1x_2 + x_3^6 + x_3^2x_4^2 + x_4^6 + \sum_5^n x_i^3$ (it is *not* semi-quasi-homogeneous.)

(E) $f = 2x_2(y_0^2 - x_0z_0) + x_1^2z_0 - 2x_1y_1y_0 + y_1^2x_0 + a_3(x_0, y_0, z_0, w_1, \dots, w_m)$ $(n \geq 5)$, satisfying the conditions of Proposition 4.6. This is invariant by $x_0\partial/\partial x_1 + x_1\partial/\partial x_2 + y_0\partial/\partial y_1$ and $y_0\partial/\partial x_1 + y_1\partial/\partial x_2 + a_1\partial/\partial y_1$; any non-zero linear combination of these is in Case 5. If V_3 is non-singular, the singularity is ES to $x_1x_2 + x_3^2 + x_4^6 + x_5^6 + \sum_6^n x_i^3$; otherwise, μ is larger.

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