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THE HOMOCLINIC AND HETEROCLINIC C^*-ALGEBRAS OF A GENERALIZED ONE-DIMENSIONAL SOLENOID

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1. Introduction

The homoclinic and heteroclinic structure in dynamical systems was first used to produce C^*-algebras in the way breaking work of Cuntz and Krieger in [Kr1], [Kr2] and [CuK]. This work has been generalized in many directions where the relation to dynamical systems is either absent or appears very implicit, but Ian Putnam described in [Pu1] a natural way to extend the constructions of Cuntz and Krieger to higher dimensions such that the point of departure is the heteroclinic structure in a Smale space, just as the work of Cuntz and Krieger departed from the heteroclinic structure in a shift of finite type, which is a zero-dimensional Smale space. Putnam builds his approach on the work of D. Ruelle, [Ru1], [Ru2], who introduced the notion of a Smale space in [Ru1] and constructed the so-called asymptotic algebra from the homoclinic equivalence relation in [Ru2].

The work of Putnam and Ruelle was further generalized by the author in [Th] where it was shown that Ruelle's approach can be adopted as soon as there is enough expansiveness in the underlying dynamical system; the local product structure in a Smale space is not crucial for the construction. Furthermore, in [Th] the alternative approach was used to obtain inductive limit decompositions for the algebras of Putnam arising from particular classes of Smale spaces, e.g. expansive group automorphisms and one-dimensional generalized solenoids in the sense of R.F. Williams, [Wi], and I. Yi, [Y1]. For expansive group automorphisms it was shown that the C^*-algebras are all AT-algebras of real rank zero, and hence are classified by their K-theory groups, thanks to the work of G. Elliott, [Ell1]. For one-dimensional generalized solenoids the exact nature of the inductive limit decomposition was not determined and the homoclinic algebra was not examined. In particular, it was not decided if the C^*-algebras are classified by K-theory. The main purpose of the present paper is tie up this loose end by showing that they are, although they turn out to be more general AH-algebras and exhibit more complicated K-theory than the algebras arising from expansive group automorphisms, at least in the sense that torsion appears. We obtain this conclusion by combining a thorough study of the inductive limit decomposition of the heteroclinic algebra with results from the classification program for C^*-algebras, the study of minimal homeomorphisms of the Cantor set and a tensor product decomposition for the homoclinic algebra of a Smale space obtained by Putnam in [Pu1].

One of the main motivations for the work on the C^*-algebras of homoclinic and heteroclinic structure in dynamical systems is to use C^*-algebras and their associated invariants to detect properties of dynamical systems which are invariant under conjugacy or other appropriate notions of equivalence for dynamical systems. The present work opens the way to do this for one-dimensional generalized solenoids, and in a final section we describe the significance of two of the ingredients in the classifying invariant of the homoclinic and heteroclinic algebra, the unique trace state of the homoclinic algebra and the K_1-group of the heteroclinic algebra. In particular the latter study has an interesting outcome: The K_1-group is infinite cyclic if the underlying solenoid admits an orientable presentation and is the cyclic group with two elements when it does not. This collapse of the K_1-group in the non-orientable case can be used to show that the corresponding Ruelle algebras R_u and R_s, as defined in [PS], can not be isomorphic or even KK-equivalent. This provides apparently the first examples of this sort and gives an answer to a couple of questions raised by Ian Putnam in [Pu2].

1.1. **The homoclinic and heteroclinic algebras of an expansive homeomorphism.** Let (X, d) be a compact metric space and $\psi : X \to X$ a homeomorphism. Recall that ψ is *expansive* when there is a $\delta > 0$ such that

$$x \neq y \ \Rightarrow \ \sup_{k \in \mathbb{Z}} d\left(\psi^k(x), \psi^k(y)\right) \geq \delta.$$

We say that two points $x, y \in X$ are *locally conjugate* when there are open neighborhoods U and V of x and y, respectively, and a homeomorphism $\chi : U \to V$ such that $\chi(x) = y$ and $\lim_{k \to \pm\infty} \sup_{z \in U} d\left(\psi^k(z), \psi^k(\chi(z))\right) = 0$. This is an equivalence relation on X and its graph

$$R_\psi(X) = \left\{(x, y) \in X^2 : \ x \text{ is locally conjugate to } y\right\}$$

is a locally compact Hausdorff space in the topology for which every local conjugacy (U, V, χ) defines an element of a canonical base:

$$\left\{(z, \chi(z)) : \ z \in U\right\}.$$

This topology is typically different from the topology which $R_\psi(X)$ inherits from $X \times X$; it has more open sets. The crucial fact is that $R_\psi(X)$ is what is nowadays known as an étale equivalence relation, cf. [Ph1], [GPS1], [Th], so that the reduced groupoid C^*-algebra $C_r^*\left(R_\psi(X)\right)$ of Renault, [Re], can be defined. We call this *the homoclinic algebra* of (X, ψ) and denote it by $A_\psi(X)$. When (X, ψ) is a Smale space, as defined by Ruelle in [Ru1], the homoclinic algebra $A_\psi(X)$ is the asymptotic algebra of Ruelle and Putnam, [Ru2],[Pu1]. When X is a sub shift the homoclinic algebra is the AF-algebra whose dimension group was defined by Krieger in [Kr1] and Section 2 of [Kr2].

Let $\operatorname{Per} \psi$ denote the set of ψ-periodic points. We assume that $\operatorname{Per} \psi \neq \emptyset$. For $p \in \operatorname{Per} \psi$, set

$$W^u(p) = \left\{x \in X : \ \lim_{k \to -\infty} d\left(\psi^k(x), \psi^k(p)\right)\right\} = 0.$$

Since ψ is expansive each $W^u(p)$ is a locally compact Hausdorff space in a topology with base

$$\left\{y \in X : d\left(\psi^j(y), \psi^j(x)\right) < \epsilon, j \leq k\right\},$$

where $x \in W^u(p), \ k \in \mathbb{Z}, \ \epsilon \in \,]0, \epsilon_p[$ are arbitrary, and $\epsilon_p > 0$ only depends on p. See Lemma 4.6 of [Th]. The set of *post-periodic points* of (X, ψ) is

$$W_{X,\psi} \ = \bigcup_{p \in \operatorname{Per} \psi} W^u(p),$$

and it is a locally compact Hausdorff space in the topology, which we call *the Wagoner topology,* defined such that each $W^u(p)$ is open in $W_{X,\psi}$ and has the topology we have just described above.

Define the equivalence relation $\sim$ on $W_{X,\psi}$ such that $x \sim y$ if and only if there are open neighborhoods, U of x and V of y in $W_{X,\psi}$, and a homeomorphism $\gamma : U \to V$, again called *a local conjugacy*, such that $\gamma(x) = y$ and

$$\lim_{k\to\infty} \sup_{z\in U} d\left(\psi^k(z), \psi^k(\gamma(z))\right) = 0.$$

This is an étale equivalence relation $R_\psi\left(X, W_{X,\psi}\right)$, cf. [Th], and the corresponding (reduced) groupoid C^*-algebra $C^*_r\left(R_{X,\psi}, W_{X,\psi}\right)$ is *the heteroclinic algebra* of (X, ψ). As in [Th] we denote it by $B_\psi\left(X\right)$. When (X, ψ) is a mixing Smale space the heteroclinic algebra is $*$-isomorphic to the stabilization of Putnams 'stable algebra', cf. Theorem 4.17 of [Th]. In particular, it is a higher dimensional analogue of the 'AF-core' in the Cuntz-Krieger construction, cf. [CuK].

1.2. **Generalized one-dimensional solenoids.** Let Γ be a finite (unoriented) graph with vertexes $\mathbb{V}$ and edges $\mathbb{E}$. Consider a continuous map $h : \Gamma \to \Gamma$ such that the following conditions are satisfied for some metric d for the topology of Γ:

α) (Expansion) There are constants $C > 0$ and $\lambda > 1$ such that

$$d\left(h^n(x), h^n(y)\right) \geq C\lambda^n d(x, y)$$

for every $n \in \mathbb{N}$ when $x, y \in e \in \mathbb{E}$ and there is an edge $e' \in \mathbb{E}$ with $h^n\left([x, y]\right) \subseteq e'$.

β) (Non folding) h^n is locally injective on e for each $e \in \mathbb{E}$ and each $n \in \mathbb{N}$.

γ) (Markov) $h\left(\mathbb{V}\right) \subseteq \mathbb{V}$.

δ) (Mixing) For every edge $e \in \mathbb{E}$ there is an $m \in \mathbb{N}$ such that $\Gamma \subseteq h^m(e)$.

ϵ) (Flattening) There is a $d \in \mathbb{N}$ such that for all $x \in \Gamma$ there is a neighborhood U_x of x with $h^d\left(U_x\right)$ homeomorphic to $]-1, 1[$.

When all conditions hold we are in a setting first introduced by Williams in [Wi] and later studied by I.Yi in [Y1],[Y2],[Y3],[Y4],[Y5]. We say then that h is a *pre-solenoid*. Throughout this paper $h : \Gamma \to \Gamma$ is a pre-solenoid.

Set

$$\overline{\Gamma} = \left\{(x_i)_{i=0}^\infty \in \Gamma^{\mathbb{N}} :\ h\left(x_{i+1}\right) = x_i,\ i = 0, 1, 2, \dots\right\}.$$

We consider $\overline{\Gamma}$ as a compact metric space with the metric

$$D\big((x_i)_{i=0}^\infty, (y_i)_{i=0}^\infty\big) = \sum_{i=0}^\infty 2^{-i} d\left(x_i, y_i\right).$$

Define $\overline{h} : \overline{\Gamma} \to \overline{\Gamma}$ such that $\overline{h}(x)_i = h\left(x_i\right)$ for all $i \in \mathbb{N}$. $\overline{h}$ is a homeomorphism with inverse

$$\overline{h}^{-1}\left(z_0, z_1, z_2, \dots\right) = \left(z_1, z_2, z_3, \dots\right).$$

Following Williams and Yi, [Wi], [Y1], we call $\left(\overline{\Gamma}, \overline{h}\right)$ a *generalized one-dimensional solenoid* or just a *1-solenoid*.

2. The heteroclinic algebra of a 1-solenoid

In this section we study the structure of the heteroclinic algebra of a 1-solenoid. The point of departure is the inductive limit decomposition of the heteroclinic algebra obtained in Theorem 5.13 of [Th].

2.1. **The building blocks.** Let Γ be a finite graph and $h : [-1,1] \to \Gamma$ a locally injective continuous map. We define an equivalence relation $\sim$ on $]-1,1[$ such that $t \sim s$ if and only if $h(t) = h(s)$ and there are open neighborhoods U_s and U_t of s and t in $]-1,1[$, respectively, such that $h(U_s) = h(U_t) \simeq]-1,1[$. Set

$$R_h = \left\{(s,t) \in]-1,1[^2 : \ s \sim t\right\}.$$

Give R_h the topology inherited from $]-1,1[^2$.

Lemma 2.1. *R_h is an étale equivalence relation.*

Proof. This is Lemma 5.10 in [Th]. □

We are going to use the étale equivalence relations of Lemma 2.1 in the special case where $g(-1), g(1) \in \mathbb{V}$ and $g(]-1,1[) = \Gamma$. When this holds we say that R_h is an *open interval-graph relation*.

To give a manageable description of $C_r^*(R_h)$, let $m,n \in \mathbb{N}$. Consider some $a = (a(1), a(2), \dots, a(m)) \in \mathbb{N}^m$, $b = (b(1), b(2), \dots, b(n)) \in \mathbb{N}^n$ and two $n \times m$-matrices, I, U, with $\{0,1\}$-entries. Assume that

$$\sum_{k=1}^{m} I_{ik} a(k) \leq b(i) \tag{2.1}$$

and

$$\sum_{k=1}^{m} U_{ik} a(k) \leq b(i) \tag{2.2}$$

for all $i = 1, 2, \dots, n$. Set $F_a = M_{a(1)} \oplus M_{a(2)} \oplus \cdots \oplus M_{a(m)}$ and $F_b = M_{b(1)} \oplus M_{b(2)} \oplus \cdots \oplus M_{b(n)}$, where M_k denotes the C^*-algebra of complex $k \times k$ matrices. Let $\varphi^I, \varphi^U : F_a \to F_b$ be $*$-homomorphisms with partial maps $\varphi^I_{ij}, \varphi^U_{ij} : M_{a(j)} \to M_{b(i)}$ of multiplicity I_{ij} and U_{ij}, respectively. Set

$$A(a,b,I,U) = \left\{(x,f) \in F_a \oplus C([0,1], F_b) : \ \varphi^I(x) = f(0), \ \varphi^U(x) = f(1)\right\}.$$

In the following we shall only be interested in algebras of this type when

$$\sum_{i=1}^{n} I_{ik} + \sum_{i=1}^{n} U_{ik} = 2 \tag{2.3}$$

for all $k = 1, 2, \dots, m$. We will refer to such an algebra as a *building block*. Note that $A(a,b,I,U)$ is unital if and only if

$$\sum_{k=1}^{m} U_{ik} a(k) = \sum_{k=1}^{m} I_{ik} a(k) = b(i)$$

for all i.

For the explicit identification of $C_r^*(R_h)$ with a building block we need to decide for each edge which of its endpoints corresponds to $0 \in [0,1]$ and which to $1 \in [0,1]$. For this purpose we give each edge of Γ an (arbitrary) orientation so that they may be considered as directed arrows instead of undirected edges. The start vertex of the arrow e is then denoted by $s(e)$ and the terminal vertex of e by $t(e)$. For each edge e of Γ choose a homeomorphism $\psi_e : e \to [0,1]$ such that $\psi_e(s(e)) = 0$, $\psi_e(t(e)) = 1$.

Let $-1 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be the elements of $h^{-1}(\mathbb{V})$. *A passage in* Γ is a closed subset $J \subseteq \Gamma$ containing exactly one vertex v such that there is a homeomorphism $\varphi : J \to [-1,1]$ with $\varphi^{-1}(0) = v$. We will identify two passages J and J' in Γ when $J \cap J'$ is also a passage in Γ. A *h-passage* is a passage in Γ which contains $h([x_i-\epsilon, x_i+\epsilon])$ for some $i \in \{1,2,\dots,N-1\}$ and all sufficiently small $\epsilon > 0$. An arrow e is then an *entry-arrow* in the h-passage p when $s(e) \in p$ and an *exit-arrow* when $t(e) \in p$.

Note that for every element $x \in h^{-1}(\mathbb{V})$ there is a unique h-passage p_x such that $p_x = h([x-\epsilon, x+\epsilon])$ for all small $\epsilon > 0$. Let $\mathbb{A}_h$ denote the finite-dimensional C^*-algebra generated by the matrix-units $e_{x,y}, x, y \in h^{-1}(\mathbb{V})$ such that $p_x = p_y$. Similarly, let $\mathcal{I}_h$ denote the set of connected components of $J \backslash h^{-1}(\mathbb{V})$ and let $\mathbb{B}_h$ denote the finite-dimensional C^*-algebra generated by the matrix units $e_{I,J}$, where $I, J \in \mathcal{I}_h$ and $h\left(\overline{I}\right) = h\left(\overline{J}\right)$. Define $\pi^I : \mathbb{A}_h \to \mathbb{B}_h$ such that $\pi^I(e_{x,y}) = \sum_{I,J} e_{I,J}$ where we sum over the pairs $I, J \in \mathcal{I}_h$ with the property that $x \in \overline{I}, y \in \overline{J}, h\left(\overline{I}\right) = h\left(\overline{J}\right)$ and $h\left(\overline{I}\right)$ is an entry-arrow in both p_x and p_y. There are at most two such pairs - when there is none we set $\pi^I(e_{x,y}) = 0$. Define $\pi^U : \mathbb{A}_h \to \mathbb{B}_h$ such that $\pi^U(e_{x,y}) = \sum_{I,J} e_{I,J}$ where we sum over the pairs $I, J \in \mathcal{I}_h$ with the property that $x \in \overline{I}, y \in \overline{J}, h(\overline{I}) = h(\overline{J})$ and $h(\overline{I})$ is an exit-arrow in both p_x and p_y. There are at most two such pairs - when there is none we set $\pi^U(e_{x,y}) = 0$. Set

$$\mathbb{D}_h = \left\{(x,f) \in \mathbb{A}_h \oplus C([0,1], \mathbb{B}_h) : \ f(0) = \pi^I(x), \ f(1) = \pi^U(x)\right\}.$$

We can then define a $*$-isomorphism $\Phi_h : C^*_r(R_h) \to \mathbb{D}_h$ such that

$$\Phi_h(f) = \left(\sum_{x,y \in h^{-1}(\mathbb{V})} f(x,y) e_{x,y}, \sum_{I,J \in \mathcal{I}_h} f\left((h|_{\overline{I}})^{-1} \circ \psi^{-1}_{h\left(\overline{I}\right)}(\cdot), (h|_{\overline{J}})^{-1} \circ \psi^{-1}_{h\left(\overline{J}\right)}(\cdot) \right) e_{I,J} \right)$$

when $f \in C_c(R_h)$. This is essentially the $*$-isomorphism from Lemma 5.12 of [Th], revised to avoid the assumption about the absence of loops in Γ which was imposed there.

Let $p_i, i = 1,2,\dots,m$, be a numbering of the different h-passages and $e_i, i = 1,2,\dots,n$, a numbering of the edges in Γ. Set

$$a(i) = \ \#\left\{j \in \{1,2\dots,N-1\} : \ h([x_j-\epsilon, x_j+\epsilon]) = p_i \text{ for all small } \epsilon\right\},$$

$i = 1,2,\dots,m$, and

$$b(i) = \ \#\left\{j \in \{0,1,\dots,N-1\} : \ h([x_j, x_{j+1}]) = e_i\right\},$$

$i = 1,2,\dots,n$. Set $a = (a(1), a(2), \dots, a(m)) \in \mathbb{N}^m, \ b = (b(1), b(2), \dots, b(n)) \in \mathbb{N}^n$. Then there are obvious $*$-isomorphisms $\kappa : \mathbb{A}_h \to F_a$ and $\kappa' : \mathbb{B}_h \to F_b$. Let

$$I_{ik} = \begin{cases} 1, & \text{when } \ e_i \ \text{ is an entry-edge in } p_k \\ 0, & \text{otherwise,} \end{cases}$$

and

$$U_{ik} = \begin{cases} 1, & \text{when } \ e_i \ \text{ is an exit-edge in } p_k \\ 0, & \text{otherwise.} \end{cases}$$

Note that (2.3) holds, i.e. $A(a,b,I,U)$ is a building block. Furthermore, $\varphi^I = \kappa' \circ \pi^I \circ \kappa^{-1}$ and $\varphi^U = \kappa' \circ \pi^U \circ \kappa^{-1}$, at least up to unitary equivalences which we can safely ignore. We get therefore a $*$-isomorphism $\Psi_h : \mathbb{D}_h \to A(a,b,I,U)$ defined such that

$$\Psi_h(x,f) = \left(\kappa(x), \left(\mathrm{id}_{C[0,1]} \otimes \kappa'\right)(f)\right).$$

In this way we get

Lemma 2.2. *(cf. Lemma 5.12 of* [Th]*)* $\Psi_h \circ \Phi_h : C^*_r(R_h) \ \to \ A(a,b,I,U)$ *is a $*$-isomorphism.*

Note that the building block $A(a,b,I,U)$ of Lemma 2.2 satisfies condition (2.4) of the following lemma by construction.

Lemma 2.3. *Let $A(a,b,I,U)$ be a building block corresponding to the vectors $a = (a(1), a(2), \dots, a(m)) \in \mathbb{N}^m$ and $b = (b(1), b(2), \dots, b(n)) \in \mathbb{N}^n$. Assume that*

$$\sum_{j=1}^{m} \varphi^U_{k,j} \geq 1, \quad \sum_{j=1}^{m} \varphi^I_{k,j} \geq 1 \tag{2.4}$$

for all $k = 1, 2, \dots, n$, and $a(i) \geq 2n+1$ for all $i = 1, 2, \dots, m$. Then $A(a,b,I,U)$ contains a non-zero projection.

Proof. We define a labeled graph $\mathcal{G}$ as follows: The vertexes of $\mathcal{G}$ consist of the tuples $(i, \rightarrow), i = 1, 2, \dots, n$, and the tuples $(i, \leftarrow), i = 1, 2, \dots, n$. The arrows $\rightsquigarrow$ in $\mathcal{G}$ are labeled by the elements of $\{1, 2, \dots, m\}$ and there is a labeled arrow $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$ when $\varphi^U_{i,j} \neq 0$ and $\varphi^I_{i',j} \neq 0$, there is a labeled arrow $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)$ when $i \neq i'$, $\varphi^U_{i,j} \neq 0$ and $\varphi^U_{i',j} \neq 0$, there is a labeled arrow $(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)$ when $\varphi^I_{i,j} \neq 0$ and $\varphi^U_{i',j} \neq 0$ and there is a labeled arrow $(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$ when $i \neq i'$, $\varphi^I_{i,j} \neq 0$ and $\varphi^I_{i',j} \neq 0$. If there is an arrow in $\mathcal{G}$ of the form

$$(i, \rightarrow) \overset{j}{\rightsquigarrow} (i, \rightarrow) \tag{2.5}$$

we choose rank 1 projections $p \in C\left([0,1], M_{b(i)}\right)$ and $q \in M_{a(j)}$ such that $p(1) = \varphi^U_{i,j}(q)$ and $\varphi^I_{i,j}(q) = p(0)$. Then (q,p) is a non-zero projection in $A(a,b,I,U)$. A similar construction gives a non-zero projection when there is an arrow of the form

$$(i, \leftarrow) \overset{j}{\rightsquigarrow} (i, \leftarrow). \tag{2.6}$$

We will therefore assume in the following that there are no arrows in $\mathcal{G}$ of the form (2.5) or (2.6).

Given an arrow $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$ in $\mathcal{G}$ we write $p \overset{q}{\rightsquigarrow} p'$ when $p \in C\left([0,1], M_{b(i)}\right)$, $p' \in C\left([0,1], M_{b(i')}\right)$ and $q \in M_{a(j)}$ are rank 1 projections such that $p(1) = \varphi^U_{i,j}(q)$ and $\varphi^I_{i',j}(q) = p'(0)$. We say that $p \overset{q}{\rightsquigarrow} p'$ *realizes* $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$. In the following all projections, with a single obvious exception, will be rank 1 projections. We say that $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \rightarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)$ when $p(1) = \varphi^U_{i,j}(q)$ and $\varphi^U_{i',j}(q) = p'(1)$, that $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \leftarrow)$ when $p(0) = \varphi^I_{i,j}(q)$ and $\varphi^U_{i',j}(q) = p'(1)$ and finally that $p \overset{q}{\rightsquigarrow} p'$ realizes $(i, \leftarrow) \overset{j}{\rightsquigarrow} (i', \rightarrow)$ when $p(0) = \varphi^I_{i,j}(q)$ and $\varphi^I_{i,j}(q) = p'(1)$. Given a finite path

$$c = (i_1, *_1) \overset{j_1}{\rightsquigarrow} (i_2, *_2) \overset{j_2}{\rightsquigarrow} \cdots \overset{j_{d-1}}{\rightsquigarrow} (i_d, *_d)$$

in $\mathcal{G}$, where $*_k \in \{\leftarrow, \rightarrow\}$, we say that c is realized by $p_1 \overset{q_1}{\rightsquigarrow} p_2 \overset{q_2}{\rightsquigarrow} p_3 \overset{q_3}{\rightsquigarrow} \cdots \overset{q_{d-1}}{\rightsquigarrow} p_d$ when $p_k \overset{q_k}{\rightsquigarrow} p_{k+1}$ realizes $(i_k, *_k) \overset{j_k}{\rightsquigarrow} (i_{k+1}, *_{k+1})$ for all $k = 1, 2, \dots, d-1$, and, in addition, the projections $p_1, p_2, \dots, p_d$ are mutually orthogonal in $C([0,1], F_b)$, and $q_1, q_2, \dots, q_{d-1}$ mutually orthogonal in F_a.

We claim that every path in $\mathcal{G}$ of length $\leq k$ can be realized when $\min_i a(i) \geq k$. To prove this by induction in k, assume that it is true for k and consider a path

$$c = (i_1, *_1) \overset{j_1}{\rightsquigarrow} (i_2, *_2) \overset{j_2}{\rightsquigarrow} \cdots \overset{k_k}{\rightsquigarrow} (i_{k+1}, *_{k+1})$$

of length $k+1$. By induction hypothesis there are projections such that $p_1 \overset{q_1}{\rightsquigarrow} p_2 \overset{q_2}{\rightsquigarrow} p_3 \overset{q_3}{\rightsquigarrow} \cdots \overset{q_{d-1}}{\rightsquigarrow} p_k$ realizes $(i_1, *_1) \overset{j_1}{\rightsquigarrow} (i_2, *_2) \overset{j_2}{\rightsquigarrow} \cdots \overset{j_{k-1}}{\rightsquigarrow} (i_k, *_k)$. Consider first the case where $*_k = \rightarrow$ and $*_{k+1} = \rightarrow$. Since $a(j_k) \geq k+1$ there is a projection $q_k \in M_{a(j_k)}$ which is orthogonal in F_a to each $q_i, i \leq k-1$. It follows that $\varphi^U_{i_k,j_k}(q_k)$ is orthogonal in F_b to $p_i(1)$ for all $i \leq k-1$. There is then a rank 1 projection

$p_k' \in C\left([0,1], M_{b(k)}\right)$ such that $p_k'(t) = p_k(t), t \leq \frac{1}{2}$, $p_k' \perp p_i$ in $C([0,1], F_b)$ for all $i \leq k-1$, and $p_k'(1) = \varphi^U_{i_k,j_k}(q_k)$. Note that $\varphi^I_{i_{k+1},j_k}(q_k)$ is orthogonal to $p_i(0)$ in F_b for $i \leq k$. Using that $b(i_{k+1}) \geq \min_i a(i) \geq k+1$ we can find a projection p_{k+1} in $C\left([0,1], M_{b(i_{k+1})}\right)$ which is orthogonal in $C([0,1], F_b)$ to $p_i, i \leq k-1$, and to p_k', and satisfies that $p_{k+1}(0) = \varphi^I_{i_{k+1},j_k}(q_k)$. Then $p_1 \overset{q_1}{\rightsquigarrow} p_2 \overset{q_2}{\rightsquigarrow} p_3 \overset{q_3}{\rightsquigarrow} \cdots \overset{q_{k-1}}{\rightsquigarrow} p_k' \overset{q_k}{\rightsquigarrow} p_{k+1}$ realizes c. Since exactly the same proof works in the other three case, $*_k = \rightarrow$ and $*_{k+1} = \leftarrow$, $*_k = \leftarrow$ and $*_{k+1} = \rightarrow$, and finally $*_k = \leftarrow$ and $*_{k+1} = \leftarrow$, we have completed the induction step. The case $k = 1$ follows from the observation that any arrow $(i,*) \overset{j}{\rightsquigarrow} (i',*)$ in $\mathcal{G}$ can be realized because we have abandoned the cases where $i = i'$.

Note that it follows from conditions (2.3) and (2.4) that $\mathcal{G}$ has no sinks or sources. There is therefore a loop

$$c = (i_1, *_1) \overset{j_1}{\rightsquigarrow} (i_2, *_2) \overset{j_2}{\rightsquigarrow} \cdots \overset{j_{d-1}}{\rightsquigarrow} (i_d, *_d) \overset{j_d}{\rightsquigarrow} (i_1, *_1),$$

in $\mathcal{G}$ of length not exceeding $2n+1$. Set $(i_{d+1}, *_{d+1}) = (i_1, *_1)$. Since we assume that $a(i) \geq 2n+1$ for all i, it follows from the preceding that there are projections such that $p_1 \overset{q_1}{\rightsquigarrow} p_2 \overset{q_2}{\rightsquigarrow} p_3 \overset{q_3}{\rightsquigarrow} \cdots \overset{q_d}{\rightsquigarrow} p_{d+1}$ realizes c. Assume $*_1 = \leftarrow$. Change p_1 to p_1' such that $p_1'(t) = p_1(t), t \leq \frac{1}{2}$, and $p_1'(1) = \varphi^U_{i_1,j_d}(q_d)$. Since $\varphi^U_{i_1,j_d}(q_d)$ is orthogonal to $p_j(1), 2 \leq j \leq d$, in F_b and p_1 to $p_j, 2 \leq j \leq d$, in $C([0,1], F_b)$, we can do this such that the elements of $\{p_1', p_2, \ldots, p_d\}$ are mutually orthogonal in $C([0,1], F_b)$. Then $\left(\sum_{j=1}^d q_j, p_1' + \sum_{j=2}^d p_j\right)$ is a non-zero projection in $A(a,b,I,U)$. This completes the proof because the case $*_1 = \rightarrow$ is completely analogous. □

It is an easy exercise in the K-theory of C^*-algebras to calculate the K-theory groups of a building block $A(a,b,I,U)$.

Lemma 2.4. *$K_0(A(a,b,I,U))$ is isomorphic, as a partially ordered group, to*

$$\{z = (z_1, z_2, \ldots, z_m) \in \mathbb{Z}^m : \ (I-U)z = 0\},$$

and

$$K_1(A(a,b,I,U)) \simeq \textit{coker}\,(I-U) = \mathbb{Z}^n/(I-U)(\mathbb{Z}^m).$$

2.2. **The inductive limit decomposition.** In the notation of Theorem 5.13 in [Th], there is an open interval $J \subseteq \Gamma$, contained in some edge of Γ, with endpoints in $h^{-1}(\mathbb{V})$, such that $h^d(J) = \Gamma$ for some $d \in \mathbb{N}$, and such that $B_{\overline{h}}\left(\overline{\Gamma}\right)$ is $*$-isomorphic to $A \otimes \mathbb{K}$, where A is the inductive limit of the sequence

$$C_r^*\left(R^d\right) \subseteq C_r^*\left(R^{d+1}\right) \subseteq C_r^*\left(R^{d+2}\right) \subseteq \cdots$$

Here $R^i = R_{h^i}$ is the open interval-graph relation defined by $h^i : J \to \Gamma$ and the inclusions $C_r^*(R^i) \subseteq C_r^*(R^{i+1})$ are induced by considering R^i as an open sub relation of R^{i+1}. To give a more detailed description of the sequence (2.2) we need some terminology which we now introduce. A $*$-homomorphism $\varphi : C([0,1], M_n) \to C([0,1], M_m)$ is *regular* when there are continuous functions $g_1, g_2, \ldots, g_k : [0,1] \to [0,1]$ such that

$$\varphi(f) = \operatorname{diag}(f \circ g_1, f \circ g_2, \ldots, f \circ g_k, 0, \ldots, 0).$$

Of course, this requires that $kn \leq m$. φ is unital when $kn = m$. The functions $g_1, \ldots, g_k$ will be called *the characteristic functions* of φ. A $*$-homomorphism $\chi : \oplus_{i=1}^N C\left([0,1], M_{n(i)}\right) \to \oplus_{i=1}^M C\left([0,1], M_{m(i)}\right)$ is *regular* when the $*$-homomorphisms $\chi_{ij} : C\left([0,1], M_{n(i)}\right) \to C\left([0,1], M_{m(j)}\right), i = 1, 2, \ldots, N, j = 1, 2, \ldots, M$, which it defines are all regular. The union of the characteristic functions of the χ_{ij}'s will be called *the characteristic functions* of χ. A $*$-homomorphism $\chi : \oplus_{j=1}^{N} M_{n(j)} \to \oplus_{j=1}^{M} M_{m(j)}$ will be said to be *non-increasing in rank* when $\operatorname{Rank}\chi_{ij}(1) \leq n(j)$ for all i, j.

Let $A(a,b,I,U)$ and $A(a',b',I',U')$ be building blocks. A $*$-homomorphism

$$\psi : A(a,b,I,U) \to A(a',b',I',U') \tag{2.7}$$

is *regular* when there are $*$-homomorphisms $\mu : C([0,1],F_b) \to F_{b'}$, $\chi : F_a \to F_{a'}$ and a regular $*$-homomorphism $\varphi : C([0,1],F_b) \to C([0,1],F_{b'})$ such that

1) $\mu(f)\chi(x) = \chi(x)\mu(f) = 0,\ f \in C([0,1],F_b), x \in F_a$,
2) $\psi(x,f) = (\chi(x) + \mu(f), \varphi(f)),\ (x,f) \in A(a,b,I,U)$, and
3) χ is non-increasing in rank.

We will refer to the $*$-homomorphism μ as *the skew map* of ψ.

Let F be a finite-dimensional C^*-algebra. We define the *spectral variation* $\operatorname{var} f$ of an element $f \in C([0,1],F)$ to be the number

$$\operatorname{var} f = \inf_U \sup_{s,t\in[0,1]} \|U(s)f(s)U(s)^* - f(t)\|,$$

where we take the infimum over all unitaries U in $C([0,1],F)$. The *spectral variation* $\operatorname{var} z$ of an element $z = (x,f) \in A(a,b,I,U)$ of a building block is then defined to be the number $\operatorname{var} z = \operatorname{var} f$.

Lemma 2.5. *The heteroclinic algebra $B_{\overline{h}}\left(\overline{\Gamma}\right)$ is $*$-isomorphic to the inductive limit of a sequence of building blocks*

$$A_1' \xrightarrow{\pi_1} A_2' \xrightarrow{\pi_2} A_3' \xrightarrow{\pi_3} \cdots \tag{2.8}$$

and injective regular $*$-homomorphisms with the property that

$$\lim_{k\to\infty} \operatorname{var}\left[\pi_k \circ \pi_{k-1} \circ \pi_{k-2} \circ \cdots \circ \pi_i(a)\right] = 0 \tag{2.9}$$

for all $a \in A_i'$ and all $i \in \mathbb{N}$.

Proof. For each $i \geq d$ we let A_i' be the building block obtained from $C_r^*(R_{h^i})$ by use of Lemma 2.2. Define $\chi : \mathbb{A}_{h^i} \to \mathbb{A}_{h^{i+1}}$ such that $\chi(e_{x,y}) = e_{x,y}$ when $x, y \in h^{-i}(\mathbb{V})$ and $p_x = p_y$. This is well-defined because h maps h^i-passages to h^{i+1}-passages by the Markov condition γ) and the non folding condition β). Define $\mu : C([0,1],\mathbb{B}_{h^i}) \to \mathbb{A}_{h^{i+1}}$ such that

$$\mu(f \otimes e_{I,J}) = \sum_{(x,y)\in N_{I,J}} f\left(\psi_{h^i(\overline{I})} \circ h^i(x)\right) e_{x,y}$$

when $f \in C[0,1]$ and $I, J \in \mathcal{I}_{h^i}$ are such that $h^i(I) = h^i(J)$, and we sum over the set

$$N_{I,J} = \left\{(x,y) \in h^{-i-1}(\mathbb{V})^2 :\ x \in I, y \in J,\ h^i(x) = h^i(y)\right\}.$$

Define a $*$-homomorphism $\varphi : C([0,1],\mathbb{B}_{h^i}) \to C([0,1],\mathbb{B}_{h^{i+1}})$ such that

$$\varphi(f \otimes e_{I,J}) = \sum_{(I_1,J_1)\in M_{I,J}} f \circ \psi_{h^i(\overline{I})} \circ \left(h|_{h^i(\overline{I})\cap h^{-1}\left(h^{i+1}(\overline{I_1})\right)}\right)^{-1} \circ \psi^{-1}_{h^{i+1}(\overline{I_1})} \otimes e_{I_1,J_1},$$

for $f \in C[0,1]$ and $I, J \in \mathcal{I}_{h^i}$ with $h^i\left(\overline{I}\right) = h^i\left(\overline{J}\right)$, where we sum over the set

$$M_{I,J} = \left\{(I_1,J_1) \in \mathcal{I}^2_{h^{i+1}} :\ I_1 \subseteq I,\ J_1 \subseteq J,\ h^{i+1}\left(\overline{I_1}\right) = h^{i+1}\left(\overline{J_1}\right)\right\}.$$

Then χ, μ, φ define a $*$-homomorphism $\psi : \mathbb{D}_{h^i} \to \mathbb{D}_{h^{i+1}}$ such that $\psi(x, f) = (\chi(x) + \mu(f), \varphi(f))$. Note that the diagram

$$\begin{tikzcd} \mathbb{D}_{h^i} \arrow[r, "\psi"] & \mathbb{D}_{h^{i+1}} \\ C_r^*(R_{h^i}) \arrow[u, "\Phi_{h^i}"] \arrow[r, phantom, "\subseteq"] & C_r^*(R_{h^{i+1}}) \arrow[u, "\Phi_{h^{i+1}}"'] \end{tikzcd}$$

commutes. Set $\pi_i = \Psi_{h^{i+1}} \circ \psi \circ \Psi_{h^i}^{-1}$. Then π_i is a regular $*$-homomorphism and the diagram

$$\begin{tikzcd} A_i' \arrow[r, "\pi_i"] & A_{i+1}' \\ C_r^*(R_{h^i}) \arrow[u, "\Psi_{h^i}\circ\Phi_{h^i}"] \arrow[r, phantom, "\subseteq"] & C_r^*(R_{h^{i+1}}) \arrow[u, "\Psi_{h^{i+1}}\circ\Phi_{h^{i+1}}"'] \end{tikzcd}$$

commutes. The characteristic functions of π_i are of the form

$$\psi_e \circ \left(h|_{\overline{I_j}}\right)^{-1} \circ \psi_{e'}^{-1}$$

where e, e' are edges of Γ and $I_1, I_2, \ldots, I_k$ are the intervals in $e \cap h^{-1}(e')$ such that $h : \overline{I_j} \to e'$ are homeomorphisms. It is then clear that the expansion condition $\alpha)$ on h ensures that $\lim_{k\to\infty} \operatorname{var}[\pi_k \circ \pi_{k-1} \circ \pi_{k-2} \circ \cdots \circ \pi_i(a)] = 0$ for all $a \in A_i'$ and all i. $\square$

As seen in the proof of Lemma 2.5 the sequence (2.8) is stationary in the sense that the characteristic functions of the connecting $*$-homomorphisms, the π_i's, are the same. Let H_{ij} be the number of times the edge e_i is covered by the edge e_j under h, i.e. for arbitrary $x \in \operatorname{int} e_i$,

$$H_{ij} = \#h^{-1}(x) \cap e_j.$$

The characteristic functions of each π_i in (2.8) are given by continuous functions $\chi_{ji}^k : [0,1] \to [0,1], i, j = 1, 2, \ldots, n, k = 1, 2, \ldots, H_{ij}$, where n is the number of edges in Γ. The characteristic functions are injective and have the following properties:

i) $\bigcup_{j,k} \chi_{ij}^k([0,1]) = [0,1]$ for all $i = 1, 2, \ldots, n$.

ii) When $(j,k) \neq (j',k')$ the intersection $\chi_{ij}^k([0,1]) \cap \chi_{ij'}^{k'}([0,1])$ consists of at most one point.

iii) For each i, j the points $\chi_{ij}^k(0), k = 1, 2, \ldots, H_{ij}$, are distinct.

For convenience we shall substitute h with a power h^m of h in order to ensure further convenient properties of the connecting maps of (2.8). Such a substitution is justified by the following lemma.

Lemma 2.6. *For every $m \in \mathbb{N}$ the map $h^m : \Gamma \to \Gamma$ is a pre-solenoid, $B_{\overline{h^m}}(\overline{\Gamma}) \simeq B_{\overline{h}}(\overline{\Gamma})$ and $B_{\overline{h^m}^{-1}}(\overline{\Gamma}) \simeq B_{\overline{h}^{-1}}(\overline{\Gamma})$.*

Proof. The first statement is straightforward to check, and the others follow from the observation that $(\overline{\Gamma}, \overline{h^m})$ is conjugate to $(\overline{\Gamma}, \overline{h}^m)$, combined with the observation that the heteroclinic algebra of a homeomorphism is $*$-isomorphic to the heteroclinic algebra of any of its positive powers. $\square$

It follows from the mixing condition $\delta)$ that there is a an $m \in \mathbb{N}$ such that $h^m(e) = \Gamma$ for every edge e in Γ. Hence, for a study of the heteroclinic algebra, we may assume, by Lemma 2.6, that

$$h(e) = \Gamma, \ e \in \mathbb{E}. \tag{2.10}$$

In fact we are going to move to higher powers of h in order to obtain other properties. Note that this will not violate (2.10). Let $\mathcal{P}_i$ denote the h^i-passages. It follows from (2.10) that $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$. Since there are only finitely many passages in Γ it follows that there is an m so big that $\mathcal{P}_{i+m} = \mathcal{P}_m$ for all $i \geq 0$. A similar argument shows that if m is big enough the set of passages contained in $h^i(\operatorname{int} e)$ is the same for each $i \geq m$ and each $e \in \mathbb{E}$. Then (2.10) shows that this 'stabilized' set of passages is independent of e. Hence by substituting h with h^m for some sufficiently large m we can arrange that there is a set Q of passages in Γ such that

$$Q \text{ is the set of } h^i\text{-passages for all } i \in \mathbb{N}, \tag{2.11}$$

and

$$Q \text{ is the set of passages contained in } h(\operatorname{int} e) \text{ for all } e \in \mathbb{E}. \tag{2.12}$$

By using the properties (2.10), (2.11) and (2.12) in the proof of Lemma 2.5 we can regularize the resulting sequence of C^*-algebras further. To describe the property of the connecting maps which we obtain from (2.12), consider a regular $*$-homomorphism as in (2.7) with skew map μ. We say that ψ is *full* when the composition

$$C_0(0,1) \otimes M_{b(i)} \hookrightarrow C([0,1], F_b) \xrightarrow{\mu} F_{a'} \rightarrow M_{a'(j)}$$

is non-zero for all i, j. By using the properties (2.10), (2.11) and (2.12) in the proof of Lemma 2.5, we obtain the following

Lemma 2.7. *There are natural numbers $n, m \in \mathbb{N}$ and $n \times m$ $\{0,1\}$-matrices I, U and a set $\chi^k_{ji} : [0,1] \rightarrow [0,1]$, $k = 1, 2, \ldots, H_{ij}, i, j = 1, 2, \ldots, n$, of continuous functions such that $H_{ij} \geq 1$ for all i, j, and sequences $\{a_i\} \subseteq \mathbb{N}^m$, $\{b_i\} \subseteq \mathbb{N}^n$, such that $B_{\overline{h}}\left(\overline{\Gamma}\right)$ is the inductive limit of a sequence*

$$A(a_1, b_1, I, U) \xrightarrow{\pi_1} A(a_2, b_2, I, U) \xrightarrow{\pi_2} A(a_3, b_3, I, U) \xrightarrow{\pi_3} \cdots \tag{2.13}$$

where each π_i is a full regular $*$-homomorphism with $\left\{\chi^k_{ji}\right\}$ as characteristic functions such that (2.9) *holds.*

Lemma 2.8. *Let $\varphi : A \rightarrow B$ be a regular $*$-homomorphism between building blocks A and B. Let $e \in A$ be a projection. There are unital building blocks, A' and B', a unital regular $*$-homomorphism $\varphi' : A' \rightarrow B'$ and $*$-isomorphisms $eAe \rightarrow A'$ and $\varphi(e)B\varphi(e) \rightarrow B'$ such that*

$$\begin{array}{ccc} eAe & \longrightarrow & A' \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ \varphi(e)B\varphi(e) & \longrightarrow & B' \end{array} \tag{2.14}$$

commutes.

Proof. Left to the reader. □

In the following we let $\mathbb{K}$ denote the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space.

Proposition 2.9. *There are natural numbers $n, m \in \mathbb{N}$ and $n \times m$ $\{0,1\}$-matrices I, U and a set $\chi^k_{ji} : [0,1] \rightarrow [0,1]$, $k = 1, 2, \ldots, H_{ij}, i, j = 1, 2, \ldots, n$, of continuous functions such that $H_{ij} \geq 1$ for all i, j, and sequences $\{a_i\} \subseteq \mathbb{N}^m$, $\{b_i\} \subseteq \mathbb{N}^n$, such that $B_{\overline{h}}\left(\overline{\Gamma}\right)$ is $*$-isomorphic to $A \otimes \mathbb{K}$ where A is the inductive limit of a sequence*

$$A(a_1, b_1, I, U) \xrightarrow{\pi_1} A(a_2, b_2, I, U) \xrightarrow{\pi_2} A(a_3, b_3, I, U) \xrightarrow{\pi_3} \cdots \tag{2.15}$$

of unital building blocks, where each π_i is a unital full regular $*$-homomorphism with $\left\{\chi^k_{ji}\right\}$ as characteristic functions such that (2.9) holds.

Proof. Let n, m, U, I, H and $\left\{\chi^k_{ij}\right\}$ be as in Lemma 2.7. The mixing condition combined with the fullness of the connecting maps in (2.13) guarantees that

$$\lim_{i\to\infty}\min_k a_i(k) = \infty$$

and it follows therefore from Lemma 2.3 that the building blocks in (2.13) contain projections, at least from a certain stage. The fullness of the connecting maps ensure that these projections become full projections. $B_{\overline{h}}(\overline{\Gamma})$ is simple and stable by Theorem 5.13 of [Th] so it follows from [Br] that $B_{\overline{h}}(\overline{\Gamma}) \simeq pB_{\overline{h}}(\overline{\Gamma})p \otimes \mathbb{K}$ for any non-zero projection $p \in B_{\overline{h}}(\overline{\Gamma})$. As we have just argued we can assume that p is a full projection in the first building block occurring in (2.13). The proof is then completed by use of Lemma 2.8. $\square$

Remark 2.10. It follows from the mixing condition that the $n \times n$ matrix H of Proposition 2.9 is mixing, i.e. all entries of H^k are positive for all k large enough. By telescoping the sequence (2.15) we can therefore achieve that the characteristic functions $\left\{\chi^k_{ij}\right\}$ are either all increasing or else that for all pairs i, j there are both increasing and decreasing functions in $\left\{\chi^k_{ij} : \ k = 1, 2, \dots, H_{ij}\right\}$. $\square$

2.3. **Real rank zero and the consequences.**

Lemma 2.11. *The C^*-algebra A of Proposition 2.9 has a unique trace state.*

Proof. Consider the sequence (2.15), and set $A_i = A(a_i, b_i, I, U)$. There is a $*$-homomorphism $\iota_i : A_i \to C([0,1], F_{b_i}) = B_i$ such that $\iota_i(x, f) = f$. Since the connecting maps are regular we get an infinite commuting diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\pi_1} & A_2 & \xrightarrow{\pi_2} & A_3 & \xrightarrow{\pi_3} & \cdots \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} & & \\ B_1 & \xrightarrow{\varphi_1} & B_2 & \xrightarrow{\varphi_2} & B_3 & \xrightarrow{\varphi_3} & \cdots \end{array} \tag{2.16}$$

where the φ_i's are regular $*$-homomorphisms between interval algebras. It follows then from condition (2.9) that the inductive limit C^*-algebra $B = \varinjlim(B_i, \varphi_i)$ is AF. Furthermore, the sequence $K_0(B_i) \xrightarrow{\varphi_{i*}} K_0(B_{i+1})$ is stationary: $K_0(B_i) \simeq \mathbb{Z}^n$ where n is the number of edges in Γ and φ_{i*} is given by the matrix $H = (H_{ij})$ of Proposition 2.9. As observed in the proof of Proposition 2.9 H is mixing and it follows that B is a unital simple AF-algebra with a unique trace state, cf. Theorem 6.1 of [Ef].

To complete the proof we need some notation. When D is a unital C^*-algebra we denote by $T(D)$ its tracial state space and by $\operatorname{Aff} T(D)$ the order unit space of real-valued affine continuous functions on $T(D)$. When $\nu : D \to C$ is a $*$-homomorphism between unital C^*-algebras we let $\widehat{\nu} : D_{sa} \to \operatorname{Aff} T(C)$ be the bounded linear map from the self adjoint part of D to $\operatorname{Aff} T(C)$ given by

$$\widehat{\nu}(d)(\omega) = \omega(\nu(d)), \ \omega \in T(C).$$

Note that

$$\left\|\widehat{\nu' \circ \nu}\right\| \leq \left\|\widehat{\nu'}\right\| \left\|\widehat{\nu}\right\| \tag{2.17}$$

when $\nu' : C \to E$ is another $*$-homomorphism between unital C^*-algebras. Let $\pi_{k,i}(x, f) = (\chi_{k,i}(x) + \mu_{k,i}(f), \varphi_{k,i}(f))$ be the decomposition of $\pi_{k,i} = \pi_{k-1} \circ \pi_{k-2} \circ \cdots \circ \pi_i : A_i \to A_k$ arising from the fact that the π_j's, and hence also $\pi_{k,i}$, are regular

$*$-homomorphisms. Let $z = (x, f) \in A_i$ be a self-adjoint element in the unit ball of A_i and let $\epsilon > 0$. We will show is that there is a constant $\lambda \in [-1, 1]$ such that

$$\|\widehat{\pi_{\infty,i}}(z) - \lambda 1\| \leq 2\epsilon \tag{2.18}$$

in $\operatorname{Aff} T(A)$, where $\pi_{\infty,i} : A_i \to A$ is the canonical embedding going with the inductive limit construction. The desired conclusion follows easily from (2.18) since i, z and $\epsilon > 0$ are arbitrary. Observe first that since B has a unique trace state there is a constant $\lambda \in [-1, 1]$ such that $\lim_{k\to\infty} \|\widehat{\varphi_{k,i}}(f) - \lambda 1\| = 0$. Hence $\|\widehat{\varphi_{j,i}}(f) - \lambda 1\| \leq \epsilon$ for all large j. Write $\pi_{j,i}(z) = (y, g)$ where $g = \varphi_{j,i}(f)$. Since $\varphi_{j,i}$ is unital it follows from (2.17) that

$$\|\widehat{\mu_{k,j}}(g) - \lambda\widehat{\mu_{k,j}}(1)\| = \left\|\widehat{\mu_{k,j} \circ \varphi_{j,i}}(f - \lambda 1)\right\| \leq \|\widehat{\varphi_{j,i}}(f) - \lambda 1\| \leq \epsilon \tag{2.19}$$

for all $k \geq j$. Since $\chi_{k,i}$ does not increase rank by condition 3) on a regular $*$-homomorphism, we find that

$$\lim_{k\to\infty} \|\widehat{\chi_{k,j}}(y - \lambda 1)\| = 0. \tag{2.20}$$

By combining (2.19) and (2.20) we find that

$$\|\widehat{\chi_{k,j}}(y) + \widehat{\mu_{k,j}}(g) - \lambda 1\| = \|\widehat{\chi_{k,j}}(y - \lambda 1) + \widehat{\mu_{k,j}}(g) - \lambda\widehat{\mu_{k,j}}(1)\| \leq 2\epsilon$$

for all large k. It follows that (2.18) holds. □

Lemma 2.12. *$B_{\overline{h}}\left(\overline{\Gamma}\right)$ has real rank zero.*

Proof. By [BP] we must show that the C^*-algebra A of Proposition 2.9 has real rank zero. Since A is a simple unital recursively subhomogenous C^*-algebra in the sense of [Ph3] and since A only has one trace state by Lemma 2.11 it suffices by Theorem 4.2 of [Ph3] to show that there are non-zero projections in A of arbitrary small trace. For this purpose observe that the projection p constructed in the proof of Lemma 2.3 satisfies the inequality

$$\tau(p) \leq \frac{2n + 1}{\min_k a(k)}$$

for all $\tau \in T\left(A(a, b, I, U)\right)$. As we have already used above, the mixing condition and the fullness of the connecting $*$-homomorphisms in (2.15) imply that the number $\min_k a(k)$ becomes arbitrarily large in the building blocks of (2.15). It follows that A has non-zero projections of arbitrarily small trace. □

Remark 2.13. My original proof of Lemma 2.12 was more complicated and more direct. I am grateful to N.C. Phillips for his suggestion to use the result from [Ph3] to shorten the proof. □

We know now that $B_{\overline{h}}\left(\overline{\Gamma}\right) \simeq A \otimes \mathbb{K}$ where A

- is the inductive limit of a sequence of unital building blocks with unital connecting maps,
- has real rank zero, and
- has a unique trace state.

This allows us to use the work of H. Lin on simple C^*-algebras of tracial rank 0.

Lemma 2.14. *A has tracial rank 0 in the sense of H. Lin, cf.* [L].

Proof. By Proposition 5.4 of [L] $T(A)$ is approximately AC. It is easily shown that the invertibles are dense in a unital building block and it follows that the same is true in A, i.e. A has stable rank one. Furthermore, it follows from Lemma 2.4 that $K_0(A)$ is unperforated. Hence Theorem 4.15 of [L] shows that A has tracial rank zero. □

By an *AH-algebra with no dimension-growth* we mean in the following the inductive limit of a sequence of C^*-algebras of the form

$$p_1 C(X_1, M_{k_1}) p_1 \to p_2 C(X_2, M_{k_2}) p_2 \to p_3 C(X_3, M_{k_3}) p_3 \to \cdots , \tag{2.21}$$

where each X_i is a compact metric space, p_i is a projection in $C(X_i, M_{k_i})$ and $\sup_i \dim X_i < \infty$.

Theorem 2.15. *Let $B_{\overline{h}}(\overline{\Gamma})$ be the heteroclinic algebra of the 1-solenoid $(\overline{\Gamma}, \overline{h})$. Then $B_{\overline{h}}(\overline{\Gamma}) \simeq A \otimes \mathbb{K}$ where A is a unital simple AH-algebra with real rank zero, no dimension growth and a unique trace.*

Proof. As pointed out by N.C. Phillips in Proposition 4.7 of [Ph2] this follows from the work of Elliott, Gong and Lin, cf. [L] and [EG]. □

In particular, it follows now, thanks to the work of Elliott and Gong, [EG], or of Elliott, Gong and Li, [EGL], that $B_{\overline{h}}(\overline{\Gamma})$ is classified by K-theory. In fact, it follows from the concluding remark in [EGL] that $B_{\overline{h}}(\overline{\Gamma})$ can also be realized by the inductive limit of a sequence of direct sums of circle algebras and matrix algebras over so-called dimension drop algebras, i.e. algebras of the form

$$\{f \in C([0,1], M_k) : \ f(0) \in \mathbb{C}, f(1) \in \mathbb{C}\}.$$

Thus $B_{\overline{h}}(\overline{\Gamma})$ is actually in the class of real rank zero algebras classified in Elliotts breakthrough paper, [Ell2].

Since $B_{\overline{h}}(\overline{\Gamma})$ is stable the projections in (2.21) are in fact redundant here. This, however, need not be the case for the AH-algebras of Section 4 below.

3. The heteroclinic algebra of the inverse of a 1-solenoid

We seek here to clarify the structure of the heteroclinic algebra of the inverse of $\overline{h}$, i.e. we seek to describe $B_{\overline{h}^{-1}}(\overline{\Gamma})$. This is the same algebra as the stabilization of the 'unstable algebra' of Putnam, cf. [Pu1] and Section 4.4 of [Th].

Thanks to Lemma 2.6, in order to study the structure of $B_{\overline{h}^{-1}}(\overline{\Gamma})$ we may substitute h with any positive power of h. As in [Y4] we can use this together with a result of Williams, 5.2 of [Wi], to restrict our attention to the case where Γ is the wedge of n circles, each of which is an edge in Γ, such that the common point of the circles is the only vertex in Γ.

Let $e_i, i = 1, 2, \dots, n$, be the edges of Γ and let v be the unique vertex in Γ. By the flattening condition ϵ) and Lemma 2.6 we can also assume that there is an open neighborhood U of v in Γ such that $h(U) \simeq \,]-1,1[$ and that $h(e_i) = \Gamma$ for all i. There are numbers $a_{ij} \in \mathbb{N}$ and disjoint intervals $I^l_{ij}, l = 1, 2, \dots, a_{ij}$, such that

$$\operatorname{int}(e_i) \cap h^{-1}(\operatorname{int}(e_j)) = \bigcup_{l=1}^{a_{ij}} I^l_{ij}. \tag{3.1}$$

Since $h(e_i) = \Gamma$ we have that $a_{ij} \geq 1$ for all i, j. It follows from the mixing condition that the h-periodic points are dense in Γ. For each i, j, k, choose a point $b^k_{ij} \in I^k_{ij}$ which is periodic under h. Let $\overline{b^k_{ij}} \in \overline{\Gamma}$ be the $\overline{h}$-periodic point with $\overline{b^k_{ij}}_0 = b^k_{ij}$.

Lemma 3.1. *For each i, j, k, $\left\{y \in \overline{\Gamma} : \ y_0 = b^k_{ij}\right\}$ is an open and compact subset of $W^u\left(\overline{b^k_{ij}}\right)$ in the Wagoner topology of $W_{\overline{\Gamma}, \overline{h}^{-1}}$.*

Proof. By Lemma 2.9 of [Y1] there is a $\kappa > 0$ and an $l \in \mathbb{N}$ such that

$$d\left(h^k(a), h^k(b)\right) < \kappa \ \forall k \in \mathbb{N} \ \Rightarrow \ h^l(a) = h^l(b). \tag{3.2}$$

Choose $L \geq l$ such that $\sum_{j \geq L} 2^{-j} \sup_{a,b \in \Gamma} d(a,b) < \kappa$. For each $a \in h^{-L}\left(b^k_{ij}\right)$ choose an element $\overline{a} \in \overline{\Gamma}$ such that $\overline{a}_L = a$. Then

$$\left\{y \in \overline{\Gamma} : \ y_0 = b^k_{ij}\right\} = \bigcup_{a \in h^{-L}\left(b^k_{ij}\right)} \left\{y \in \overline{\Gamma} : \ D\big(\overline{h}^j(y), \overline{h}^j(\overline{a})\big) < \kappa, \ j \geq -L\right\}.$$

Thus $\left\{y \in \overline{\Gamma} : \ y_0 = b^k_{ij}\right\}$ is open in $W_{\overline{\Gamma},\overline{h}^{-1}}$ by Lemma 4.6 of [Th]. Furthermore,

$$\overline{h}^m\left(\left\{y \in \overline{\Gamma} : \ y_0 = b^k_{ij}\right\}\right) \subseteq W^u\Big(\overline{b^k_{ij}}, 0, \eta_{\overline{b^k_{ij}}}\Big)$$

in the notation of Section 4 in [Th] for some sufficiently large multiple m of the period of b^k_{ij}. It follows that $\left\{y \in \overline{\Gamma} : \ y_0 = b^k_{ij}\right\}$ is compact in $W^u(\overline{b^k_{ij}})$ since it is in $\overline{\Gamma}$. □

Set $W_0 = \bigcup_{i,j,k} \left\{y \in \overline{\Gamma} : \ y_0 = b^k_{ij}\right\}$. By Lemma 3.1 W_0 is an open and compact subset of $W_{\overline{\Gamma},\overline{h}^{-1}}$. Furthermore, $\left(\overline{\Gamma}, \overline{h}^{-1}\right)$ is a mixing Smale space and hence every element of $W_{\overline{\Gamma},\overline{h}^{-1}}$ is conjugate to an element of W_0, cf. the first part of the proof of Lemma 4.6 in [Th]. It follows then from Corollary 2.14 of [Th] that $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ is stably $*$-isomorphic to the Ruelle-algebra $A_{\overline{h}^{-1}}\left(\overline{\Gamma}, W_0\right)$.

Let $\mathbb{B}$ be the stationary Bratteli diagram with n vertices at each level and with a_{ij} arrows, labeled by $I^l_{ij}, l = 1, 2, \dots, a_{ij}$, from vertex j at each level to vertex i at the next level. Then every element $z = (z_0, z_1, z_2, \dots) \in W_0$ defines an infinite path $\xi(z) = \xi^z = \left(I^{l_0}_{i_0 j_0}, I^{l_1}_{i_1 j_1}, I^{l_2}_{i_2 j_2}, \dots\right)$ in $\mathbb{B}$ by the requirement that $z_k \in I^{l_k}_{i_k j_k}$. Let $X_{\mathbb{B}}$ be the space of infinite paths in $\mathbb{B}$ equipped with the relative topology inherited from the infinite product space $\left\{I^l_{ij} : \ l = 1, 2, \dots, a_{ij}, \ i, j = 1, 2, \dots, n\right\}^{\mathbb{N}}$.

Lemma 3.2. *$X_{\mathbb{B}}$ is homeomorphic to the Cantor set and the map $z \mapsto \xi(z)$ is a homeomorphism from W_0, equipped with the Wagoner-topology, to $X_{\mathbb{B}}$.*

Proof. It is straightforward to check that $X_{\mathbb{B}}$ is metrizable, compact, totally disconnected and without isolated points. It is therefore a copy a of the Cantor set. That ξ is surjective follows because we chose an element b^k_{ij} from each of the intervals I^k_{ij} plus the observation that $h\left(I^k_{ij}\right) = \operatorname{int}(e_j)$. That ξ is injective follows from the expansion condition α) and it remains only to prove the continuity of ξ. To this end note that since h is finite-to-one there is for any given $k \in \mathbb{N}$ an $\epsilon > 0$ so small that $x, y \in W_0, \ D\big(\overline{h}^{-j}(x), \overline{h}^{-j}(y)\big) < \epsilon, j \leq k \ \Rightarrow \ x_i = y_i \ \Rightarrow \ \xi^x_i = \xi^y_i, \ i = 0, 1, 2, \dots, k.$ This proves the continuity of ξ by Lemma 4.6 of [Th]. □

Lemma 3.3. *For every $\delta_0 > 0$ there is a $\delta > 0$ such that the following holds when $x, y \in W_0$ and $d(x_k, y_k) < \delta$ for all $k \in \mathbb{N}$:*

$$i \geq 1, \ \xi^x_i \neq \xi^y_i \ \Rightarrow \ \max\left\{d\left(x_{i-1}, v\right), d\left(y_{i-1}, v\right)\right\} < \delta_0,$$

and

$$i \geq 0, \ \xi^x_i = \xi^y_i \ \Rightarrow \ \xi^x_{i+1} = \xi^y_{i+1} \ .$$

Proof. The first property is easy to arrange. The second is also elementary and uses conditions γ) and ϵ). We leave the details to the reader. □

Since we assume that $h(U) \simeq]-1,1[$ for all small neighborhoods U of v we can renumber the edges of Γ to arrange that for some $j \in \{1, n\}$ it holds that

$$h(U) \subseteq \left(e_1 \cap h^{-1}(e_1)\right) \cup \left(e_j \cap h^{-1}(e_j)\right) \tag{3.3}$$

or

$$h(U) \subseteq \left(e_1 \cap h^{-1}(e_j)\right) \cup \left(e_j \cap h^{-1}(e_1)\right).$$

If the latter occurs (and $j \neq 1$) we substitute h with h^2, as we may by Lemma 2.6, to ensure that (3.3) holds. By an appropriate numbering of the intervals in $\operatorname{int} e_i \cap h^{-1}(\operatorname{int} e_j)$ we can in this way arrange that

$$h(U) \subseteq \overline{I_{11}^1} \cup \overline{I_{jj}^{a_{jj}}},$$

also when $j = 1$.

Lemma 3.4. *Let $x, y \in W_0$. Then x and y are locally conjugate in $W_{\overline{\Gamma},\overline{h}^{-1}}$ if and only if there is a number $N_1 \in \mathbb{N}$ such that $\xi_i^x = \xi_i^y,\ i \geq N_1$, or a number $N_2 \in \mathbb{N}$ such that $\{\xi_i^x, \xi_i^y\} = \left\{I_{11}^1, I_{jj}^{a_{jj}}\right\}$ and $\xi_i^x = \xi_{i+1}^x,\ \xi_i^y = \xi_{i+1}^y$ for all $i \geq N_2$.*

Proof. $\left(\overline{\Gamma}, \overline{h}\right)$ is a Smale space by Theorem 5.3 of [Th] and hence $\left(\overline{\Gamma}, \overline{h}^{-1}\right)$ is also a Smale space. It follows therefore from Lemma 4.14 of [Th] that x and y are locally conjugate in $W_{\overline{\Gamma},\overline{h}^{-1}}$ if and only if $\lim_{k\to\infty} d(x_k, y_k) = 0$. Thus, if x and y are locally conjugate, it follows from Lemma 3.3 that ξ^x and ξ^y must behave as described. Conversely, if ξ^x and ξ^y behave this way it follows from the expansion condition α) that there are constants $C' > 0$ and $\lambda > 1$ such that $d(x_k, y_k) \leq C'\lambda^{-k}$ for all large k. In particular, $\lim_{k\to\infty} d(x_k, y_k) = 0$. □

Consider a vertex w in $\mathbb{B}$, say the vertex i at some level of the Bratteli diagram. Then the arrows in $\mathbb{B}$ terminating at w are labeled by $I_{il}^k, k = 1, 2, \ldots, a_{il}^k, l = 1, 2, \ldots, n$. We equip these arrows with a total ordering such that I_{i1}^1 is the minimal element and $I_{ij}^{a_{ij}}$ the maximal. Add a zeroth level with one vertex with n emitting arrows, one to each of the vertices at the first level. With these additional features $\mathbb{B}$ becomes a simple ordered Bratteli diagram in the sense of [GPS1] and by construction there is a unique minimal and a unique maximal path in $X_{\mathbb{B}}$. Hence the Bratteli-Vershik map $\psi : X_{\mathbb{B}} \to X_{\mathbb{B}}$ is defined and minimal. It follows from Lemma 3.4 that two elements $x, y \in W_0$ are locally conjugate if and only if ξ^x and ξ^y lie in the same ψ-orbit, cf. Remarks (ii) on page 72 of [GPS1]. Thus the homeomorphism ξ of Lemma 3.2 takes the equivalence relation given by local conjugacy under $\overline{h}^{-1}$ to orbit equivalence under ψ. We leave the reader to check that this identification of equivalence relations is also topological, i.e. that ξ is an isomorphism between $R_{\overline{h}^{-1}}\left(\overline{\Gamma}, W_0\right)$ and the étale equivalence relation R_ψ given by orbit equivalence under ψ. See [GPS2]; in particular Example 2.7 (i) and Definition 2.4 of [GPS2].

Theorem 3.5. *Let $\overline{h} : \overline{\Gamma} \to \overline{\Gamma}$ be a 1-solenoid. There is a minimal homeomorphism ψ of the Cantor set K, which is either an odometer or a primitive substitution shift, such that the heteroclinic algebra $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ of $\overline{h}^{-1}$ is $*$-isomorphic to*

$$\mathbb{K} \otimes \left(C(K) \rtimes_\psi \mathbb{Z}\right).$$

Hence $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ is an AT-algebra of real rank zero, $K_1\left(B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)\right) \simeq \mathbb{Z}$ and $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ has a densely defined lower-semicontinous trace which is unique up scalar multiplication.

Proof. It follows from the preceding that $A_{\overline{h}^{-1}}\left(\overline{\Gamma}, W_0\right) \simeq C_r^*\left(R_\psi\right)$. Since ψ is minimal, and hence in particular free, $C_r^*\left(R_\psi\right) \simeq C(K) \rtimes_\psi \mathbb{Z}$. As we have already used, the proof of Lemma 4.6 of [Th] shows that every element of W is locally conjugate to an element of W_0. It follows therefore from Corollary 2.14 of [Th] that $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right) \simeq A_{\overline{h}^{-1}}\left(\overline{\Gamma}, W_0\right) \otimes \mathbb{K} \simeq \mathbb{K} \otimes (C(K) \rtimes_\psi \mathbb{Z})$. It is apparent that the Bratteli-Vershik model of the minimal homeomorphism ψ is given by a stationary diagram. It follows then from [DHS] that ψ is either an odometer system or a primitive substitution shift. Since both odometers and substitution shifts are uniquely ergodic there is only one trace state on $C(K) \rtimes_\psi \mathbb{Z}$. Hence $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ has a densely defined lower-semicontinuous trace which is unique up scalar multiplication. The remaining parts of the statements follow from [HPS]. □

Theorem 3.5 was obtained by I. Yi in [Y4] for orientable 1-solenoids.

Remark 3.6. Note that it follows from the preceding and [HPS] that the K_0-group of $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ is the inductive limit of the stationary system

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots$$

where $A = (a_{ij})$ is the matrix defined by (3.1), the transpose of the matrix H considered in Section 2.2. □

4. The homoclinic algebra of a 1-solenoid

Theorem 4.1. *The homoclinic C^*-algebra $A_{\overline{h}}\left(\overline{\Gamma}\right)$ of the 1-solenoid $\left(\overline{\Gamma}, \overline{h}\right)$ is a simple unital AH-algebra of real rank zero with no dimension growth and a unique trace state.*

Proof. By Theorem 3.1 of [Pu1] $A_{\overline{h}}\left(\overline{\Gamma}\right)$ is stably isomorphic to $B_{\overline{h}}\left(\overline{\Gamma}\right) \otimes B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$. It follows then from Theorem 3.5 and Theorem 2.15 that $A_{\overline{h}}\left(\overline{\Gamma}\right)$ is stably isomorphic to a tensorproduct $D_1 \otimes D_2$ where D_1 is a unital AT-algebra of real rank zero with a unique trace state, and D_1 is a unital simple AH-algebra with real rank zero, no dimension growth and a unique trace state. It follows immediately that $A_{\overline{h}}\left(\overline{\Gamma}\right)$ is a simple unital AH-algebra with no dimension growth. Note that D_1 is approximately divisible (as defined in [BKR]) by a result of Elliott, [Ell1]. Hence $D_1 \otimes D_2$ is also approximately divisible. Furthermore, since D_1 and D_2 both have a unique trace state, it follows that so does $D_1 \otimes D_2$. It follows therefore from Theorem 1.4 of [BKR] that $D_1 \otimes D_2$ has real rank zero. (We have here used that quasi-traces on AH-algebras are traces.) □

The topological spaces of the AH-algebras needed to realize $A_{\overline{h}}\left(\overline{\Gamma}\right)$ can be taken as CW-complexes of dimension at most 3, cf. [EG]. But there are examples of (unorientable) 1-solenoids where both the K_0-group and the K_1-group of $A_{\overline{h}}\left(\overline{\Gamma}\right)$ have 2-torsion. It follows that it is not possible to restrict the dimensions of the AH-algebras of Theorem 4.1 further in general.

5. On the invariants

5.1. **The trace.** Theorem 4.1 singles out a unique trace state τ on the homoclinic algebra $A_{\overline{h}}\left(\overline{\Gamma}\right)$. Since $A_{\overline{h}}\left(\overline{\Gamma}\right)$ is the reduced groupoid C^*-algebra of the étale equivalence relation $R_{\overline{h}}\left(\overline{\Gamma}\right)$ there is a bijective correspondence between trace states of $A_{\overline{h}}\left(\overline{\Gamma}\right)$ and the Borel probability measures μ on $\overline{\Gamma}$ which are invariant under local conjugacy in the sense that

$$\mu\left(\gamma(U)\right) = \mu(U), \tag{5.1}$$

when $\gamma : U \to \gamma(U)$ is a local conjugacy. This invariance property can be described in other ways, see [Re], [Ph1] and [GPS2]. The trace state τ_μ of $A_{\overline{h}}\left(\overline{\Gamma}\right) = C_r^*\left(R_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ corresponding to such an invariant measure μ is given by the requirement that

$$\tau_\mu(f) = \int_{\overline{\Gamma}} f(x,x)\ d\mu(x)$$

for all $f \in C_c\left(R_{\overline{h}}\left(\overline{\Gamma}\right)\right)$. The Borel probability measure on $\overline{\Gamma}$ which corresponds to the trace state of $A_{\overline{h}}\left(\overline{\Gamma}\right)$ is the Williams measure μ_w of [Wi] and [Y4]. To see this it suffices to observe that the invariance property (5.1) for the Williams measure μ_w follows readily from the the description of the local product structure in the Smale space $\left(\overline{\Gamma}, \overline{h}\right)$ and the corresponding local conjugacies given in [Th]. Since $\left(\overline{\Gamma}, \overline{h}\right)$ is a Smale space by Theorem 5.3 of [Th], local conjugacy is the same as homoclinicity in the ordinary sense. In this way we obtain the following conclusion.

Proposition 5.1. *Let $\left(\overline{\Gamma}, \overline{h}\right)$ be a 1-solenoid. The Williams measure μ_w is the only Borel probability measure on $\overline{\Gamma}$ which is invariant under homoclinicity.*

5.2. **The K-theory.** With the detailed information on the connecting maps at hand it is easy to obtain the following description of the K-theory of $B_{\overline{h}}\left(\overline{\Gamma}\right)$ from the 'stationary' sequence of Lemma 2.9.

Proposition 5.2. *The partially ordered group $K_0\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ is the inductive limit of the sequence*

$$H \xrightarrow{\chi_* + \mu_* \circ I} H \xrightarrow{\chi_* + \mu_* \circ I} H \xrightarrow{\chi_* + \mu_* \circ I} \cdots,$$

where $H = \{x \in \mathbb{Z}^m : \ Ix = Ux\}$, and

$$K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right) \simeq \operatorname{coker}(I - U) = \mathbb{Z}^n/(I-U)\left(\mathbb{Z}^m\right).$$

Note that $K_0\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ is a stationary partially ordered group, but in a slightly more general sense than the usual because H is not (obviously) a simplicial group. For this reason we must resort to Theorem 2.15 in order to conclude that $K_0\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ is a dimension group. In the following we focus on $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)$.

Let A be an $n \times m$ matrix. Let G_A be the oriented graph with vertex set $\{1, 2, \ldots, n\}$ and an edge between i and j if and only if $i \neq j$ and there is a $k \in \{1, 2, \ldots, m\}$ such that $a_{ik} \neq 0$ and $a_{jk} \neq 0$. We say that A is *irreducible* when G_A is connected.

Lemma 5.3. *Let $n, m \in \mathbb{N}, n \geq 2$. Let A be an irreducible $n \times m$ integer matrix with the property that each non-zero column has exactly two non-zero entries from the set $\{1, -1\}$ or exactly one non-zero entry from the set $\{2, -2\}$.*

a) If A contains an entry from $\{2, -2\}$ it follows that the cokernel of $A : \mathbb{Z}^m \to \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}_2$.

b) If each non-zero column of A has exactly two non-zero entries, 1 and -1, it follows that the cokernel of $A : \mathbb{Z}^m \to \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}$.

Proof. a) We will prove by induction in n that there is a matrix $H \in \mathrm{Gl}_n(\mathbb{Z})$ such that $HA\left(\mathbb{Z}^m\right) = 2\mathbb{Z} \oplus \mathbb{Z}^{n-1}$. It is easy to see that this holds when $n = 2$. Consider then the case $n > 2$. By permuting the rows and columns we can arrange that $a_{11} \neq 0$ and $a_{21} \neq 0$. If a_{11} and a_{21} have the same sign we subtract the first row from the second; otherwise we add the first row to the second. Both operations correspond to the multiplication by an element of $\mathrm{Gl}_n(\mathbb{Z})$. The resulting $n \times m$-matrix A' has the form

$$A' = \begin{pmatrix} \pm 1 & * \\ 0 & B \end{pmatrix}, \tag{5.2}$$

where $*$ is an $1\times(m-1)$ matrix with entries from $\{-1,0,1,2\}$, the 0 is $(n-1)\times 1$ and B is an irreducible $(n-1)\times(m-1)$ integer matrix for which we can use the induction hypothesis. In particular B contains an entry from $\{2,-2\}$. Let $H_0 \in \mathrm{Gl}_{n-1}(\mathbb{Z})$ be such that $H_0B\left(\mathbb{Z}^{m-1}\right) = 2\mathbb{Z}\oplus\mathbb{Z}^{n-2}$. Then

$$H = \begin{pmatrix} 1 & 0 \\ 0 & H_0 \end{pmatrix} \in \mathrm{Gl}_n(\mathbb{Z})$$

has the property that $HA'\left(\mathbb{Z}^m\right) = \mathbb{Z}\oplus 2\mathbb{Z}\oplus\mathbb{Z}^{n-2}$. This completes the induction step and hence the proof.

b) is proved in the same way: Use induction to show that there is a matrix $H \in \mathrm{Gl}_n(\mathbb{Z})$ such that $HA\left(\mathbb{Z}^m\right) = 0\oplus\mathbb{Z}^{n-1}$. □

Lemma 5.4. *Let $n,m\in\mathbb{N}, n\geq 2$. Let A be an irreducible $n\times m$ integer matrix with the property that each non-zero column has exactly two non-zero entries from the set $\{1,-1\}$. It follows that the cokernel of $A:\mathbb{Z}^m\to\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_2$.*

Proof. After appropriate permutations of rows and columns and a single permissible row operation we obtain a matrix A' as in (5.2) such that the matrix B is covered by a) or b) in Lemma 5.3. □

We say that the pre-solenoid $h:\Gamma\to\Gamma$ is *orientable* when the edges of Γ can be given an orientation such that $h: e\cap h^{-1}\left(e'\right)\to e'$ is orientation preserving for any pair of edges $e,e'\in\Gamma$.

Theorem 5.5. *Let $B_{\overline{h}}\left(\overline{\Gamma}\right)$ be the heteroclinic algebra of the 1-solenoid $\left(\overline{\Gamma},\overline{h}\right)$. Then $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)\simeq\mathbb{Z}$ when $h^2:\Gamma\to\Gamma$ is orientable and $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)\simeq\mathbb{Z}_2$ when it is not.*

Proof. If Γ is given an orientation of the edges which makes h^2 orientation preserving the matrix $U-I$ will have the property described in b) of Lemma 5.3 and hence $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)\simeq\mathbb{Z}$.

Assume that $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)\simeq\mathbb{Z}$. By Lemma 5.4 it suffices now to show that h^2 is orientable. Let Q be the passages of (2.12). Let e,f be two edges of Γ. The passages from Q which involve both e and f are of the form

$$\begin{aligned} + \quad &: \quad \xrightarrow{e}\xrightarrow{f} \quad \text{or} \quad \xrightarrow{f}\xrightarrow{e}, \\ - \quad &: \quad \xrightarrow{e}\xleftarrow{f} \quad \text{or} \quad \xleftarrow{e}\xrightarrow{f}\ . \end{aligned}$$

We use these parities to define a labeled unoriented graph H whose vertexes are the edges of Γ. Let $e,f\in\mathbb{E}$. If all passages involving e and f are of parity $+$ draw an edge between e and f with label $+$. If not, and there is in fact at least one passage involving both e and f, draw an edge between e and f with label $-$. Note that H is connected. A change in the orientation of the edges in Γ will not change the graph H - only its labels. Let $\mathbb{G}$ denote the collection of connected subgraphs H' of H with the property that there is an orientation of the edges in Γ such that all edges of H' are labeled $+$. Then $\mathbb{G}$ is not empty and is partially ordered by inclusion. Let H_{max} be a maximal element. It is easy to see that if $H_{max}\neq H$ we can obtain a matrix B from $U-I$ by permutations, and addition and/or subtraction of rows such that

$$B = \begin{pmatrix} 1 & * & * & * & \dots & * \\ 0 & 1 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & B_0 \end{pmatrix}$$

where B_0 is an irreducible matrix for which a) Lemma 5.3 applies. It follows that $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right) \simeq \mathbb{Z}_2$ which contradicts the assumption, proving that $H_{max} = H$.

Give the edges of Γ an orientation such that $H_{max} = H$. It follows that for all large enough i, $h^i : J \to \Gamma$ will either be orientation preserving or orientation reversing. If $h^i : J \to \Gamma$ and $h^{i+1} : J \to \Gamma$ are either both orientation preserving or both orientation reversing, $h : \Gamma \to \Gamma$ will be orientation preserving. If $h^i : J \to \Gamma$ is orientation preserving and $h^{i+1} : J \to \Gamma$ orientation reversing, or vice versa, $h : \Gamma \to \Gamma$ will be orientation reserving. In both cases h^2 will be orientation preserving. □

Note that it follows from Theorem 5.5 that a 1-solenoid can be presented by an orientation preserving pre-solenoid (is orientable in the sense of [Y4]) if and only if the given pre-solenoid $h : \Gamma \to \Gamma$ has the property that h^2 is orientable.

Theorem 5.6. *Assume that the pre-solenoid $h : \Gamma \to \Gamma$ is orientable. It follows that the homoclinic algebra $A_{\overline{h}}\left(\overline{\Gamma}\right)$ and the heteroclinic algebra $B_{\overline{h}}\left(\overline{\Gamma}\right)$ are AT-algebras.*

Proof. It follows from Proposition 5.2 and Lemma 5.4 that the K-theory of $B_{\overline{h}}\left(\overline{\Gamma}\right)$ is torsion-free in this case. The same is true for $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ by [Y4] (or Theorem 3.5). Hence the Künneth theorem combined with Theorem 3.1 of [Pu1] show that also the K-theory of $A_{\overline{h}}\left(\overline{\Gamma}\right)$ is torsion-free. The conclusion follows from Theorem 2.15 and Theorem 4.1 in combination with Corollary 6.7 of [Go]. □

Remark 5.7. As pointed out in [Th] the algebras $A_{\overline{h}}\left(\overline{\Gamma}\right)$, $B_{\overline{h}}\left(\overline{\Gamma}\right)$ and $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ all come equipped with a canonical automorphism $\alpha_{\overline{h}}$ which is given, in all cases, by the formula

$$\alpha_{\overline{h}}(f)(x,y) = f\left(\overline{h}^{-1}(x), \overline{h}^{-1}(y)\right)$$

when f is a continuous compactly supported function on the respective étale equivalence relations. The crossed products $B_{\overline{h}}\left(\overline{\Gamma}\right) \rtimes_{\alpha_{\overline{h}}} \mathbb{Z}$ and $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right) \rtimes_{\alpha_{\overline{h}}} \mathbb{Z}$ are stably isomorphic to the Ruelle algebras R_s and R_u of [PS]. (This does not follow directly from the statement of Theorem 4.17 in [Th], but it can be deduced in the same way.) If we consider a case where $\overline{h} : \overline{\Gamma} \to \overline{\Gamma}$ is not orientable we find from the preceding that $K_0\left(B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)\right)$, $K_1\left(B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)\right)$ and $K_0\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ are all torsion-free while $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right) \simeq \mathbb{Z}_2$. It is then not difficult to conclude from the Pimsner-Voiculescu exact sequence that $K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right) \rtimes_{\alpha_{\overline{h}}} \mathbb{Z}\right)$ contains $\mathbb{Z}_2$ while $K_1\left(B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right) \rtimes_{\alpha_{\overline{h}}} \mathbb{Z}\right)$ is torsion free. It follows in this way that R_u and R_s are not KK-equivalent, answering in the negative two questions raised by Putnam in [Pu2].

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