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ALL EXTENSIONS BY $C_r^*(\mathbb{F}_n)$ ARE SEMI-INVERTIBLE

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1. Introduction

Extensions of non-nuclear C^*-algebras need not be invertible. The first example of this phenomenon was exhibited by J. Anderson, [A], and the stock of such examples has been slowly and steadily growing since then. Motivated by this fact and the asymptotic homomorphism approach to extensions of Connes and Higson, [CH], Vladimir Manuilov and the author introduced in [MT1] a theory of extensions which basically only differs from the conventional theory, introduced in the work of Brown, Douglas and Fillmore, [BDF], in that the split extensions are replaced by extensions which are only asymptotically split. It was shown in [MT1] that the resulting theory is a version of the E-theory of Connes and Higson when the algebra which plays the role of the quotient in the extensions is a suspended C^*-algebra. Furthermore, it was shown in [MT1] that any extension of a suspended C^*-algebra is semi-invertible in the sense that one can find another extension, namely the one obtained by reversing the orientation in the suspension, such that the addition of the two extensions is asymptotically split. Subsequently it was shown in [MT3] that this nice situation, that all extensions are semi-invertible, does not persist in the general case; certain non-invertible extensions introduced by S. Wasserman were shown not to be semi-invertible. At about the same time U. Haagerup and S. Thorbjørnsen were able to exhibit the first example of a non-invertible extension of the reduced group C^*-algebra $C_r^*(\mathbb{F}_n)$ of a free group, cf. [HT]. Since the example in [MT3] was based on groups with property T, which is a property free groups do not have, it became interesting to decide if there are examples of extensions of $C_r^*(\mathbb{F}_n)$ which are not semi-invertible. Very recently V. Manuilov has shown that the extension constructed by Haagerup and Thorbjørnsen is not only semi-invertible; it can be made asymptotically split by addition of a split extension, [M]. Manuilovs method actually works to obtain the same conclusion for any quasi-diagonal extension of $C_r^*(\mathbb{F}_n)$ and he conjectured that all extensions of $C_r^*(\mathbb{F}_n)$ (by compact operators) are semi-invertible. The purpose of this note is to prove this conjecture. Since asymptotically split extensions are homotopic to 0 it follows that all extensions of $C_r^*(\mathbb{F}_n)$ by the compact operators are invertible up to homotopy; a property which can fail for certain separable C^*-algebras as shown in [MT3].

2. Definitions, results and a proof

Let A be a separable C^*-algebra and let $\mathbb{K}$ denote the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space H. In this note *an extension of A* is a short exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow E \xrightarrow{p} A \longrightarrow 0 \tag{2.1}$$

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of C^*-algebras. The extension is *asymptotically split* when there is an asymptotic $*$-homomorphism $\varphi = (\varphi_t)_{t\in[1,\infty)} : A \to E$, cf. [CH], such that $p \circ \varphi_t = \mathrm{id}_A$ for all t. Let Q denote the Calkin algebra $B(H)/\mathbb{K}$ and let $q : B(H) \to Q$ be the quotient map. If $\psi : A \to Q$ is the Busby invariant of (2.1) the extension (2.1) is split, resp. asymptotically split, exactly when there is a $*$-homomorphism $\pi : A \to B(H)$ such that $q \circ \pi = \psi$, resp. an asymptotic $*$-homomorphism $\pi = (\pi_t)_{t\in[1,\infty)} : A \to B(H)$ such that $\psi = q \circ \pi_t$ for all t. An extension $\pi : A \to Q$ is *invertible*, resp. *semi-invertible*, when there is another extension $\tilde{\pi} : A \to Q$ such that $\pi \oplus \tilde{\pi}$ is split, respectively, asymptotically split. Our results concern the case where A is $C^*_r(\mathbb{F}_n)$, the reduced C^*-algebra of the free group $\mathbb{F}_n$ on n generators.

Theorem 2.1. *Let $\pi : C^*_r(\mathbb{F}_n) \to Q$ be an extension. There is then an invertible extension $\tilde{\pi} : C^*_r(\mathbb{F}_n) \to Q$ such that $\pi \oplus \tilde{\pi}$ is asymptotically split.*

The proof depends on the notion of strong homotopy of extensions, cf. [MT1]. Two extensions $\psi_0, \psi_1 : A \to Q$ are *strongly homotopic* when they are homotopic as $*$-homomorphisms, that is when there is a $*$-homomorphism $\psi : A \to C([0,1], Q)$ such that $\psi_i = \mathrm{ev}_i \circ \psi, i = 0, 1$, where $\mathrm{ev}_t : C([0,1], Q) \to Q$ denotes evaluation at t. We emphasize that strong homotopy is very different from the much weaker and somewhat more natural notion of *homotopy* of extensions which is defined in a similar way, but with $C([0,1], Q)$ replaced by $M(C([0,1],\mathbb{K}))/C([0,1],\mathbb{K})$, where $M(C([0,1],\mathbb{K}))$ is the multiplier algebra of $C([0,1],\mathbb{K})$. It is the latter notion of homotopy which is alluded to in the concluding remarks of the introduction. Strong homotopy comes in here because of the following lemma which is a special case of Lemma 4.3 in [MT2].

Lemma 2.2. *Assume that the extension $\varphi : A \to Q$ is strongly homotopic to a split extension. It follows that φ is asymptotically split.*

Thanks to this lemma Theorem 2.1 is a consequence of the following

Theorem 2.3. *Let $\pi : C^*_r(\mathbb{F}_n) \to Q$ be an extension. There is then an invertible extension $\tilde{\pi} : C^*_r(\mathbb{F}_n) \to Q$ such that $\pi \oplus \tilde{\pi}$ is strongly homotopy to a split extension.*

Proof. As in the work of Manuilov,[M], the main ingredience in the proof is a homotopy of representations of $\mathbb{F}_n$ which was constructed by J. Cuntz based on work of Pimsner and Voiculescu, [PV]. It is described on p. 187 of [C]. It consists of norm-continuous paths, $s^t_i, t \in [0,1], i = 1, 2, \dots, n$, of unitaries on $H = l^2(\mathbb{F}_n)$ such that the corresponding family of representations $\nu_t : \mathbb{F}_n \to B(H)$ has the following properties:

a) $\nu_0 = \lambda$,
b) $\nu_t(g) - \lambda(g) \in \mathbb{K}$ for all $t \in [0,1]$ and all $g \in \mathbb{F}_n$, and
c) ν_1 is unitarily equivalent to $t \oplus \lambda^{(n)}$.

Here λ is the regular representation, t the trivial representation and $\lambda^{(n)}$ is the direct sum of n copies of λ. Let $\mu : C^*(\mathbb{F}_n) \to C^*_r(\mathbb{F}_n)$ be the canonical surjective $*$-homomorphism and $u_i, i = 1, 2, \dots, n$, the canonical unitary generators of $C^*(\mathbb{F}_n)$. Set $v_i = \mu(u_i)$. Let $\pi : C^*_r(\mathbb{F}_n) \to Q$ be an extension of $C^*_r(\mathbb{F}_n)$. Since $\pi(1)$ lifts to a projection in $B(H)$ we can add a split extension to π in order to arrange that the resulting extension is unital. We can therefore assume that $\pi(1) = 1$. By the universal property of $C^*(\mathbb{F}_n)$ there is an extension $\pi_1 : C^*(\mathbb{F}_n) \to Q$ such that $\pi_1(u_i) = \pi(v_i)^*$ for all i. Since $\mu : C^*(\mathbb{F}_n) \to C_r^*(\mathbb{F}_n)$ is invertible in KK-theory by the work of Cuntz,[C], there is a split extension $\varphi : C^*(\mathbb{F}_n) \to Q$ and an invertible extension $\pi^{-1} : C_r^*(\mathbb{F}_n) \to Q$ such that $\pi^{-1} \circ \mu \oplus \varphi$ is unitarily equivalent to $\pi_1 \oplus \varphi$. In particular, $[\pi^{-1}(v_i)] = [\pi^{-1} \circ \mu(u_i)] = [\pi_1(u_i)] = [\pi(v_i)^*]$ in $K_1(Q)$ for all i. Set $\omega = \pi \oplus \pi^{-1} : C_r^*(\mathbb{F}_n) \to Q$. Since $\omega(v_i)$ is homotopic to 1 in the unitary group of Q there is a representation R of $\mathbb{F}_n$ on H such that $\omega \circ \mu = q \circ h_R$, where $h_R : C^*(\mathbb{F}_n) \to B(H)$ is the $*$-homomorphism corresponding to R. Set $\omega_t = q \circ h_{R\otimes\nu_t}, t \in [0,1]$. Since $R \otimes \nu_1$ is equivalent to $R \oplus \left(R \otimes \lambda^{(n)}\right)$ by c), we find that ω_1 is unitarily equivalent to $\omega \circ \mu \oplus q \circ h_{R\otimes\lambda(n)}$. By Fell's absorbtion principle, which was also heavily used in [M], the representation $R \otimes \lambda^{(n)}$ of $\mathbb{F}_n$ is equivalent to a multiple of the regular representation and hence $h_{R\otimes\lambda^{(n)}}$ factors through $C_r^*(\mathbb{F}_n)$. Thus $\omega_1 : C_r^*(\mathbb{F}_n) \to Q$ is (unitarily equivalent to) the direct sum of ω and the split extension $q \circ h_{R\otimes\lambda^{(n)}} : C_r^*(\mathbb{F}_n) \to Q$. Furthermore, thanks to a), $\omega_0 = q \circ h_{R\otimes\lambda}$, where also $h_{R\otimes\lambda}$ factors through $C_r^*(\mathbb{F}_n)$ by Fell's absorbtion principle. That is, $\omega_0 : C_r^*(\mathbb{F}_n) \to Q$ is a split extension. It suffices therefore to show that $\omega_t, t \in [0,1]$, defines a strong homotopy of extensions of $C_r^*(\mathbb{F}_n)$. Since it clearly defines a strong homotopy of extensions of $C^*(\mathbb{F}_n)$, it remains only to verify that this homotopy factors through $C_r^*(\mathbb{F}_n)$. To this end we take an element $x = \sum_j c_j g_j \in \mathbb{C}[\mathbb{F}_n]$, where $c_j \in \mathbb{C}$ and $g_j \in \mathbb{F}_n$. We will complete the proof by showing that

$$\|\omega_t(x)\| \leq 3\|x\|_{C_r^*(\mathbb{F}_n)} \tag{2.2}$$

for any $t \in [0,1]$. To this end write

$$h_{R\otimes\nu_t}(x) = \sum_j c_j R(g_j) \otimes \lambda(g_j) - \sum_j c_j R(g_j) \otimes \Delta(g_j), \tag{2.3}$$

where $\Delta(g_j) = \nu_t(g_j) - \lambda(g_j)$. Note that $\Delta(g_j) \in \mathbb{K}$ by b). It follows from Fell's absorbtion principle that $\|\sum_j c_j R(g_j) \otimes \lambda(g_j)\| \leq \|x\|_{C_r^*(\mathbb{F}_n)}$ and hence

$$\left\| q\Big(\sum_j c_j R(g_j) \otimes \lambda(g_j)\Big)\right\| \leq \|x\|_{C_r^*(\mathbb{F}_n)}. \tag{2.4}$$

To handle the second term in (2.3) we use that $B(H) \otimes \mathbb{K}/\mathbb{K} \otimes \mathbb{K} \simeq Q \otimes \mathbb{K}$ to conclude that

$$\left\| q\Big(\sum_j c_j R(g_j) \otimes \Delta(g_j)\Big)\right\|_Q = \left\|\sum_j c_j \omega(g_j) \otimes \Delta(g_j)\right\|_{Q\otimes\mathbb{K}}.$$

Since $\omega : C_r^*(\mathbb{F}_n) \to Q$ is injective and $\omega \otimes \mathrm{id}_{\mathbb{K}}$ isometric,

$$\left\|\sum_j c_j \omega(g_j) \otimes \Delta(g_j)\right\|_{Q\otimes\mathbb{K}} = \left\|\sum_j c_j \lambda(g_j) \otimes \Delta(g_j)\right\|_{C_r^*(\mathbb{F}_n)\otimes\mathbb{K}}.$$

And

$$\begin{aligned} &\left\|\sum_j c_j \lambda(g_j) \otimes \Delta(g_j)\right\|_{C_r^*(\mathbb{F}_n)\otimes\mathbb{K}} \\ &\qquad = \left\|\sum_j c_j \lambda(g_j) \otimes \nu_t(g_j) - \sum_j c_j \lambda(g_j) \otimes \lambda(g_j)\right\| \leq 2\|x\|, \end{aligned} \tag{2.5}$$

by a final application of Fell's absorbtion principle. In combination with (2.4) this conclusion yields (2.2). □

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