# UNIVERSITY OF AARHUS
## DEPARTMENT OF MATHEMATICS

# LATTICE GAUGE FIELD THEORY AND PRISMATIC SETS

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ABSTRACT. We study prismatic sets analogously to simplicial sets except that realization involves prisms, i.e., products of simplices rather than just simplices. Particular examples are the prismatic subdivision of a simplicial set $S$ and the prismatic star of $S$. Both have the same homotopy type as $S$ and in particular the latter we use to study lattice gauge theory in the sense of Phillips and Stone. Thus for a Lie group $G$ and a set of parallel transport functions defining the transition over faces of the simplices, we define a classifying map from the prismatic star to a prismatic version of the classifying space of $G$. In turn this defines a $G$-bundle over the prismatic star.

## CONTENTS

## 1. INTRODUCTION

In the study of global properties of locally trivial fibre bundles it is a fundamental difficulty that the usual combinatorial methods of algebraic topology depends on the use of simplicial complexes which structure behaves badly with respect to local trivializations. By a theorem of Johnson [8], the base and total space of a locally trivial smooth fibre bundle with projection $\pi : E \to B$ can be triangulated in such a way that $\pi$ is a simplicial map. But obviously even in this case a general fibre is not a simplicial complex in any natural way. However such a fibre has a natural decomposition into prisms, i.e., products of simplices, and the whole triangulated bundle gives the basic example of a prismatic set, analogous to the notion of a simplicial set derived from a simplicial complex. Prismatic sets were introduced and used by the second author and R. Ljungmann in [6] (see also Ljungmann's thesis [10]) in order to construct an explicit fibre integration map in smooth Deligne cohomology, see also [5]. But the important

special case of the prismatic subdivision of a simplicial set was used in Akyar [1] in connection with "Lattice Gauge Theory" in the sense of Phillips and Stone [14], [15], [17] and similar constructions have been used in other connections, see e.g. [13]. One can see [9] for further information about Lattice Gauge Fields.

In this paper we shall give a more systematic treatment of prismatic sets and their properties but we shall concentrate on the applications to lattice gauge theory extending the work of Phillips and Stone to arbitrary simplicial sets and all dimensions. For an arbitrary simplicial set $S$ and a given Lie group $G$ together with a set of parallel transport functions in their sense, we construct a prismatic set $\bar{P}(S)$ of the same homotopy type as $S$ and a classifying map from $\bar{P}(S)$ to a prismatic version of the standard model for $BG$. This is one of our main results (Theorem 8.1). Geometrically, for $S$ a simplicial complex, $\bar{P}(S)$ is closely related to the nerve of the covering by stars of vertices (Theorem 5.1). In turn this gives a principal $G$-bundle with a connection and thus in principle gives rise via the usual Chern-Weil and Chern-Simons theory to explicit formulas for characteristic classes (Corollary 8.2). We shall return to this elsewhere. One can see [2], [3], [4], [7], [18] for further information about Chern-Simons Theory.

The paper is organized as follows:

In chapter 2, prismatic sets are defined and their various geometric realizations are studied.

The third chapter introduces the prismatic triangulation of a simplicial map and in particular of a simplicial set. Furthermore, we comment on the calculation of the homology of the geometric realization of a prismatic set.

In chapter 4 we study prismatic sets associated to stars of simplicial complexes. It turns out that the prismatic set $\bar{P}.(S)$ given in this chapter in the case of a simplicial complex is the nerve of the covering by stars of vertices.

In the fifth chapter, we compare the two star simplicial sets and prove that there is a natural surjective map $\bar{p} : \bar{P}.(S) \to P\operatorname{St} S$. It turns out that this map is an isomorphism for $S = K^s$, where $K$ is a simplicial complex.

In chapter 6, we introduce a prismatic version of the classifying space. This is done by replacing the Lie group $G$ by the singular simplicial set of continuous maps $\operatorname{Map}(\Delta^q, G)$.

In chapter 7, we introduce the notion of "compatible transition functions" similar to the "parallel transport functions" of Phillips-Stone [15] for a simplicial complex $K$. We show how a given bundle on the realization of a simplicial set and socalled "admissible trivializations" give rise to a set of compatible transition functions and vice versa. We end the chapter with a remark on the relation between the compatible transition functions and parallel transport along a piecewise linear path.

Finally in the last chapter we construct the classifying map for a given set of compatible transition functions. For this we construct a prismatic map from $\bar{P}(S)$ to the prismatic model for the classifying space constructed in chapter 6.

## 2. Prismatic Sets

Prismatic sets are similar to simplicial sets but they are realized by using prisms instead of only simplices.

Let $\Delta^p = \{(t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \mid \sum_i t_i = 1, t_i \leq 1\}$ be a standard $p$-simplex given with barycentric coordinates. A prism is a product of simplices, that is, a set of the form $\Delta^{q_0 \cdots q_p} = \Delta^{q_0} \times \cdots \times \Delta^{q_p}$.

The motivating example is triangulated fibre bundles:

**Example 2.1.** Given a smooth fibre bundle $\pi : Y \rightarrow Z$ with $\dim Y = m + n$, $\dim Z = m$ and compact fibres possibly with boundary. By a theorem of Johnson [8], there are smooth triangulations $K$ and $L$ of $Y$ and $Z$, respectively and a simplicial map $\pi' : K \rightarrow L$ in the following commutative diagram

$$\begin{array}{ccc} |K| & \xrightarrow{\approx} & Y \\ \downarrow{\scriptstyle |\pi'|} & & \downarrow{\scriptstyle \pi} \\ |L| & \xrightarrow{\approx} & Z \end{array}$$

and the horizontal maps are homeomorphisms which are smooth on each simplex, here $|K| = \bigcup_{\tau \in K_k} \Delta^k \times \tau / \sim$, $k = 0, \ldots, \dim K$, is the geometric realization.

One can extend a given such triangulation of $\partial Y \rightarrow Z$ to a triangulation of $Y \rightarrow Z$.
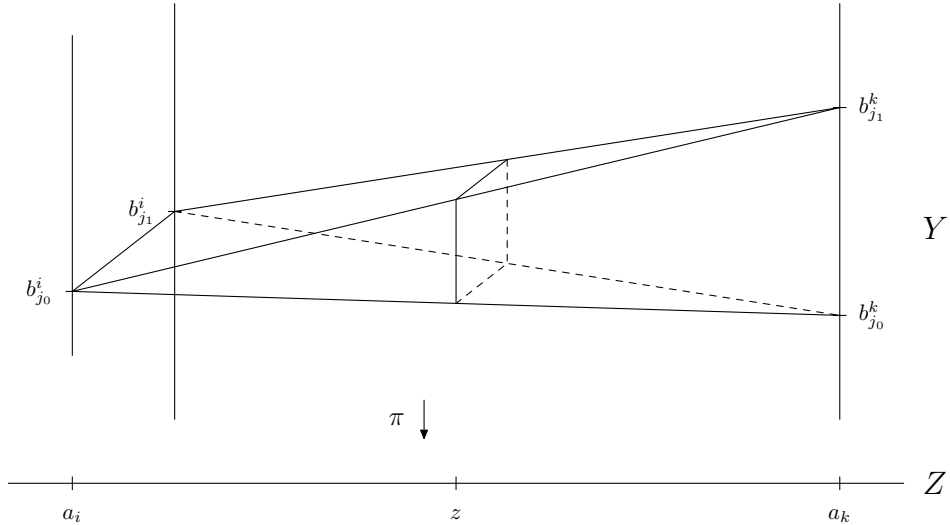


A simplex $\tau$ in $K$ has vertices $\tau = (b_0^0, \ldots, b_{q_0}^0 \mid \ldots \mid b_0^p, \ldots, b_{q_p}^p)$ with $\sigma = (a_0, \ldots, a_p)$ such that $\pi'(b_j^i) = a_i$. Here, we give the set of vertices of the total space, the lexicographical order. So geometrically, for an open simplex $\overset{\circ}{\sigma}$ in $L$, we have

$$\pi^{-1}(|\overset{\circ}{\sigma}|) \approx |\overset{\circ}{\sigma}| \times \bigcup_{\tau \in \pi^{-1}(\sigma)} \Delta^{q_0 \cdots q_p} \times \tau.$$

We collect all these in the formal definition below using simplicial sets. For these we recall the notation but refer otherwise to Mac Lane [11], May [12].

**Definition 2.2.** A simplicial set $S_. = \{S_q\}$ is a sequence of sets with face operators $d_i : S_q \rightarrow S_{q-1}$ and degeneracy operators $s_i : S_q \rightarrow S_{q+1}$, $i = 0, \ldots, q$, satisfying the following identities:

$$d_i d_j = \begin{cases} d_{j-1} d_i & : \quad i < j \\ d_j d_{i+1} & : \quad i \geq j, \end{cases}$$

$$s_i s_j = \begin{cases} s_{j+1} s_i & : \quad i \leq j \\ s_j s_{i-1} & : \quad i > j, \end{cases}$$

and

$$d_i s_j = \begin{cases} s_{j-1} d_i & : \quad i < j \\ \mathrm{id} & : \quad i = j, i = j+1 \\ s_j d_{i-1} & : \quad i > j+1. \end{cases}$$

**Example 2.3.** A simplicial complex $K_.$ gives a simplicial set where

$$K_p = \{(a_{i_0}, \ldots, a_{i_p}) \mid \text{some nondecreasing sequences for a given linear ordering of } K_0\}$$

is the set of $p$-simplices.

**Example 2.4.** Given an open cover $\mathcal{U} = U_i$ of $Z$ we have the nerve $N\mathcal{U} = \{N\mathcal{U}(p)\}$ of the covering, where

$$N\mathcal{U}(p) = \bigsqcup_{i_0, \ldots, i_p} U_{i_0} \cap \cdots \cap U_{i_p},$$

and $(i_0, \ldots, i_p)$ is nondecreasing for a given linear order of the index set.

Let us denote $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0, \ldots, i_p}$. $N\mathcal{U}$ is a simplicial manifold, where the face and degeneracy maps come from the followings

$$d_j : U_{i_0, \ldots, i_p} \rightarrow U_{i_0, \ldots, \hat{i}_j, \ldots, i_p}$$

$$s_j : U_{i_0, \ldots, i_p} \rightarrow U_{i_0, \ldots, i_j, i_j, \ldots, i_p}$$

That is, $N\mathcal{U}(p)$ is a smooth manifold for each $p$ and the face and degeneracy maps are smooth. There is also a corresponding simplicial set $N_d\mathcal{U} = \{N_d\mathcal{U}(p)\}$ called the discrete nerve of the covering. Here $N_d\mathcal{U}(p)$ is simply the set consisting of an element for each non-empty intersection of $p+1$ open sets from $\mathcal{U}$. So there is a natural forgetful map $N\mathcal{U} \rightarrow N_d\mathcal{U}$.

**Note.** If $S_.$ has only face operators, then it is called a $\Delta$-set.

**Definition 2.5.** Given $p \geq 0$, a $(p+1)$-multi-simplicial set is a sequence $\{S_{q_0, \ldots, q_p}\}$ which is a simplicial set in each variable $q_i$, $i = 0, \ldots, p$.

**Definition 2.6.** A prismatic set $P = \{P_{p,.}\}$ is a sequence $P_{p,.} = \{P_{p, q_0, \ldots, q_p}\}$ of $(p+1)$-multi-simplicial sets, i.e., with face and degeneracy operators

$$d_j^i : P_{p, q_0, \ldots, q_p} \rightarrow P_{p, q_0, \ldots, q_i - 1, \ldots, q_p}$$

$$s_j^i : P_{p, q_0, \ldots, q_p} \rightarrow P_{p, q_0, \ldots, q_i + 1, \ldots, q_p}$$

such that $d_j^i$, $s_j^i$ commute with $d_l^k$, $s_l^k$ for $i \neq k$, and such that $d_j^i$, $s_j^i$ for fixed $i$ satisfy the identities in Definition 2.2.

Furthermore there are face operators

$$d_k : P_{p,q_0,\ldots,q_p} \to P_{p-1,q_0,\ldots,\hat{q}_k,\ldots,q_p}$$

commuting with $d_j^i$ and $s_j^i$ (interpreting $d_j^k = s_j^k = \mathrm{id}$ on the right) such that $\{P_{p,.}\}$ is a $\Delta$-set.

**Definition 2.7.** If similarly $P$ has degeneracy operators

$$s_k : P_{p,q_0,\ldots,q_p} \to P_{p+1,q_0,\ldots,q_k,q_k,\ldots,q_p}$$

then $P$ is called a strong prismatic set.

**Remark.** In this case $(P_p, d_k, s_k)$ is a usual simplicial set.

In general, degeneracy operators given in Definition 2.7 do not exist naturally so in this case $(P_{p,.}, d_k)$ is only a $\Delta$-set.

Example 2.1 continued: A triangulated fibre bundle

$$\pi : |K| \to |L|$$

gives a strong prismatic set $P_.(K/L)$ by letting

$$P_p(K/L)_{q_0\ldots q_p} \subseteq K_{p+q_0+\cdots+q_p} \times L_p$$

be the subset of pairs of simplices $(\tau, \sigma)$ so that $q_i + 1$ of the vertices in $\tau$ lies over the $i$-th vertex in $\sigma$. Then we have face and degeneracy operators defined in the obvious way. It is now straight forward to check that this is a strong prismatic set.

**Example 2.8.** For a given simplicial set $S$, consider $E_p S = S_. \times \cdots \times S_.$, $(p+1)$-times. $\pi_i : E_p S \to E_{p-1} S$ is the projection which deletes the $i$-th factor. Similarly, the diagonal map $\delta_i : E_p S \to E_{p+1} S$ repeats the $i$-th factor. This is a strong prismatic set.

Prismatic sets have various geometric realizations.

**Definition 2.9.** First, we have for each $p$ the thin (geometric) realization

$$|P_{p,.}| = \bigsqcup_{q_0,\ldots,q_p} \Delta^{q_0\ldots q_p} \times P_{p,q_0,\ldots,q_p} / \sim \tag{2.10}$$

with equivalence relation "$\sim$" generated by the face and degeneracy maps

$$\varepsilon_j^i : \Delta^{q_0\ldots q_i\ldots q_p} \to \Delta^{q_0\ldots q_i+1\ldots q_p} \quad \text{and}$$
$$\eta_j^i : \Delta^{q_0\ldots q_i\ldots q_p} \to \Delta^{q_0\ldots q_i-1\ldots q_p},$$

respectively. $\{|P_{p,.}|\}$ is a $\Delta$-space hence it gives a fat realization

$$\| \, |P_.| \, \| = \bigsqcup_{p \geq 0} \Delta^p \times |P_{p,.}| / \sim \tag{2.11}$$

by only using face operators $d_k$.

The face and degeneracy operators $d_k$, $s_k$ act on $\Delta^{q_0\ldots q_p}$ as the projection and the diagonal, respectively so they induce a structure of a simplicial set on $|P_p|$. In other words, the projection $\pi_i : \Delta^{q_0\ldots q_p} \to \Delta^{q_0\ldots \hat{q}_i\ldots q_p}$ deletes the $i$-th coordinate and the

diagonal map $\Delta_i : \Delta^{q_0 \dots q_p} \to \Delta^{q_0 \dots q_i q_i \dots q_p}$ repeats the $i$-th factor. Then the further equivalence relation on $|P_\cdot|$ given in (2.11) is generated by

$$(\varepsilon^i t, s, \sigma) \sim (t, \pi_i s, d_i \sigma), \qquad t \in \Delta^{p-1}, \qquad s \in \Delta^{q_0 \dots q_p}, \qquad \sigma \in P_{p, q_0, \dots, q_p}.$$

If $P_\cdot$ is strong then we also have a thin realization

$$|P_\cdot| = \|\|P_\cdot\|\| / \sim$$

given by the above and the further relation

$$(\eta^i t, s, \sigma) \sim (t, \Delta_i s, s_i \sigma), \qquad t \in \Delta^{p+1}, \qquad s \in \Delta^{q_0, \dots, q_p}, \qquad \sigma \in P_{p, q_0, \dots, q_p}.$$

Similarly, we can define for each $p$, the fat realization $\|P_\cdot\|$, that is

$$\|P_p\| = \bigsqcup_{q_0, \dots, q_p} \Delta^{q_0 \dots q_p} \times P_{p, q_0, \dots, q_p} / \sim$$

with equivalence relation given by only the face maps $d_j^i$.

Moreover, we have the very fat realization

$$\| \|P_\cdot\| \| = \bigsqcup_{p \geq 0} \Delta^p \times \|P_{p,\cdot}\| / \sim$$

using only face operators.

For a given simplicial set $S$ and $E_p S$ as in Example 2.8 we have $\| |E_\cdot S| \|$ as the fat realization of the space which maps $p$-th term to $|S_\cdot| \times \dots \times |S_\cdot|$, $(p+1)$-times.

Define for a space $X$, $E_p X = X_\cdot \times \dots \times X_\cdot$, $(p+1)$-times. Let us say $X = |S|$ then $\| |E_\cdot S| \|$ is contractible.

## 3. PRISMATIC TRIANGULATION

Let us return to the case of a triangulated fibre bundle $|K| \to |L|$. In this case the natural map

$$P_p(K/L)_{q_0, \dots q_p} \to K_{q_0 + \dots + q_p + p}$$

induces a homeomorphism

$$
\begin{array}{ccc}
|P_\cdot(K/L)| & \xrightarrow{\approx} & |K| \\
\downarrow{\scriptstyle |\pi'|} & & \downarrow{\scriptstyle \pi} \\
|L| & \xrightarrow{=} & |L|
\end{array}
$$

In this diagram, the top horizontal map we shall call the prismatic triangulation homeomorphism

$$\lambda : |P_\cdot(K/L)| \xrightarrow{\cong} |K|$$

induced by

$$\lambda(t, s^0, \dots, s^p, (\tau, \sigma)) = (t_0 s^0, \dots, t_p s^p, \tau) \in \Delta^{p+q} \times K_{p+q}, \tag{3.1}$$

where $(t, s, \tau, \sigma) \in \Delta^p \times \Delta^{q_0 \dots q_p} \times P_p(K/L)_{q_0 \dots q_p}$ and $q = q_0 + \dots + q_p$.

**Note.** If $\overset{\circ}{\sigma}$ is an open $p$-simplex in $L$ then $\lambda$ provides a natural trivialization of $|K|_\sigma = \pi^{-1}(\overset{\circ}{\sigma})$, that is, a homeomorphism

$$\lambda : \overset{\circ}{\sigma} \times |P_p(K/\sigma)| \overset{\approx}{\to} |K|_\sigma.$$

We can generalize this construction to any simplicial map:

**Example 3.2.** *Prismatic triangulation of a simplicial map.* Let $f : S_. \to \bar{S}_.$ be a simplicial map of simplicial sets and define $P_.(f)$ by

$$P_p(f)_{q_0,\ldots,q_p} = \{(\sigma, \bar{\sigma}) \in S_{q_0+\cdots+q_p+p} \times \bar{S}_p \mid f(\sigma) = \mu_{q_0,\ldots,q_p}(\bar{\sigma})\}$$

where the corresponding map

$$\mu^{q_0,\ldots,q_p} : \Delta^{q_0+\cdots+q_p+p} \to \Delta^p$$

is given by

$$\{0,\ldots,q_0|\ldots|q_0+\cdots+q_{p-1}+p,\ldots,q_0+\cdots+q_p+p\} \to \{0,\ldots,p\}.$$

By this, we mean that the basis vectors $e_0,\ldots,e_{q_0}$ are mapped to $e_0$, and $e_{q_0+1},\ldots,$ $e_{q_0+q_1+1}$ are mapped to $e_1$ and etc. Explicitly

$$\mu_{q_0,\ldots,q_p} = \hat{s}_{q+p} \circ s_{(q_0+\cdots+q_p+p-1)\ldots(q_0+\cdots+q_{p-1}+p)} \circ \cdots \circ \hat{s}_{q_0} \circ s_{(q_0-1)\ldots(0)},$$

where the $\hat{s}_i$ are left out and

$$s_{(q_0+\cdots+q_i+i-1)\ldots(q_0+\cdots+q_{i-1}+i)} = s_{q_0+\cdots+q_i+i-1} \circ \cdots \circ s_{q_0+\cdots+q_{i-1}+i},$$

$i = 0,\ldots,p$. The boundary maps in the fibre direction

$$d_j^i : P_p(f)_{q_0,\ldots,q_p} \to P_p(f)_{q_0,\ldots,q_i-1,\ldots,q_p}$$

are inherited from the face operators defined on $S_{q+p}$. Thus

$$d_j^i(\sigma, \bar{\sigma}) = (d_{q_0+\cdots+q_{i-1}+i+j-1}\sigma, \bar{\sigma}).$$

Similarly the degeneracy maps $s_j^i$ on $P_p(f)_{q_0,\ldots,q_p}$

$$s_j^i : P_p(f)_{q_0,\ldots,q_p} \to P_p(f)_{q_0,\ldots,q_i+1,\ldots,q_p}$$

are inherited from the ones on $S_{q+p}$. That is,

$$s_j^i(\sigma, \bar{\sigma}) = (s_{q_0+\cdots+q_{i-1}+i+j-1}\sigma, \bar{\sigma}).$$

The boundary maps

$$d^i : P_p(f)_{q_0,\ldots,q_p} \to P_{p-1}(f)_{q_0,\ldots,\hat{q}_i,\ldots,q_p}$$

are determined by the boundary maps defined on both $S_{q+p}$ and $\bar{S}_p$. Thus

$$d^i(\sigma, \bar{\sigma}) = (d_{q_0+\cdots+q_{i-1}+i-1} \circ \cdots \circ d_{q_0+\cdots+q_i+i-1}\sigma, d_i\bar{\sigma}),$$

here the composition of the face operators can be shortly written as

$$d_{(q_0+\cdots+q_{i-1}+i-1)\ldots(q_0+\cdots+q_i+i-1)} = d_{q_0+\cdots+q_{i-1}+i-1} \circ \cdots \circ d_{q_0+\cdots+q_i+i-1}.$$

**Note.** $P_.(f)$ is a prismatic set, but in general not a strong one.

**Theorem 3.3.** *There is a pullback diagram*

$$
\begin{array}{ccc}
\| \, |P_\cdot(f)| \, \| & \xrightarrow{\ \lambda\ } & |S_\cdot| \\
\downarrow{\scriptstyle \|f\|} & & \downarrow{\scriptstyle |f|} \\
\|\bar{S}_\cdot\| & \xrightarrow{\ q\ } & |\bar{S}_\cdot|
\end{array}
$$

*In particular $\lambda$ is a homotopy equivalence.*

*Proof.* The map $\lambda : \Delta^p \times \Delta^{q_0 \dots q_p} \times P_p(f)_{q_0 \dots q_p} \to \Delta^{q+p} \times S_{q+p}$ is given by $\lambda(t, s, \sigma, \bar{\sigma}) = (t_0 s^0, \dots, t_p s^p, \sigma)$. The commutativity of the diagram follows from the definition of $P_\cdot(f)$ since

$$
P_p(f)_{q_0, \dots, q_p} \subseteq S_{q+p} \times \bar{S}_p
$$

consists of pairs $(\sigma, \bar{\sigma}) \in S_{q+p} \times \bar{S}_p$ such that $f(\sigma) = \mu_{q_0, \dots, q_p}(\bar{\sigma}) \in \bar{S}_p$.

By the commutativity of the diagram, $\lambda$ factors over the pullback $|S_\cdot| \times_{|\bar{S}_\cdot|} \|\bar{S}_\cdot\|$ in the diagram

$$
\begin{array}{ccc}
|S_\cdot| \times_{|\bar{S}_\cdot|} \|\bar{S}_\cdot\| & \xrightarrow{\ \mathrm{pr}_1\ } & |S_\cdot| \\
\downarrow{\scriptstyle \|f\|} & & \downarrow{\scriptstyle |f|} \\
\|\bar{S}_\cdot\| & \xrightarrow{\ q\ } & |\bar{S}_\cdot|
\end{array}
$$

Here elements in the pullback $|S_\cdot| \times_{|\bar{S}|} \|\bar{S}_\cdot\|$ are represented by pairs $((t, \sigma), (\bar{t}, \bar{\sigma}))$ such that $f(\sigma) = \mu_{q_0, \dots, q_p}(\bar{\sigma})$ and $\bar{t} = \mu^{q_0, \dots, q_p}(t)$, where $\sigma \in S_{q+p}$, $\bar{\sigma} \in \bar{S}_q$. Therefore $\lambda \times \|f\| : \| \, |P_\cdot(f)| \, \| \to |S| \times \|\bar{S}\|$ induces $\Lambda$ in the diagram

$$
\begin{array}{ccccc}
\| \, |P_\cdot(f)| \, \| & \xrightarrow{\ \Lambda\ } & |S_\cdot| \times_{|\bar{S}_\cdot|} \|\bar{S}_\cdot\| & \xrightarrow{\ \mathrm{pr}_1\ } & |S_\cdot| \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle |f|} \\
\|\bar{S}_\cdot\| & \xrightarrow{\ \mathrm{id}\ } & \|\bar{S}_\cdot\| & \xrightarrow{\ \ \ } & |\bar{S}_\cdot|
\end{array}
$$

Now $\Lambda$ is a homotopy equivalence. Indeed, an argument similar to the note following (3.1) gives a homeomorphism of the preimage $\|f\|$ of an open simplex in $\|\bar{S}_\cdot\|$. Hence $\Lambda$ is shown to by a homeomorphism by induction over skeleton of $\|\bar{S}_\cdot\|$. $\qquad \square$

**Example 3.4.** *Prismatic triangulation of a simplicial set.* Let $S_\cdot$ be a simplicial set and $\bar{S}_\cdot = *$ the simplicial set with one element in each degree. Here $P_p(f) = P_p S$ is called the $p$-th prismatic subdivision of $S$ and for each $t \in \Delta^p$ the map $\lambda_p(t, -) : |P_p S| \to |S_\cdot|$ is a homeomorphism. In this case, Theorem 3.3 gives a homeomorphism $\Lambda : \| \, |P_\cdot S| \, \| \xrightarrow{\ \approx\ } \| * \| \times |S_\cdot|$, here $\| * \| = \bigcup_n \Delta^n / \partial \Delta^n$. In particular $\lambda : \| \, |P_\cdot S| \, \| \to |S_\cdot|$ is a homotopy equivalence. We shall call $P_\cdot S$ the prismatic triangulation of $S$.

For later use, let us give the explicit construction of the $p + 1$-prismatic set $P_\cdot S_\cdot$ and its realization:

$$
P_p S_{q_0, \dots, q_p} = S_{q_0 + \dots + q_p + p}.
$$

The face operators

$$
d_j^{(i)} : P_p S_{q_0, \dots, q_i, \dots, q_p} = S_{q+p} \to P_p S_{q_0, \dots, q_i - 1, \dots, q_p} = S_{q+p-1}
$$

are defined by

$$d_j^{(i)} := d_{q_0 + \cdots + q_{i-1} + i + j},$$

$j = 0, \ldots, q_i$. Similarly, the degeneracy operators

$$s_j^{(i)} : P_p S_{q_0, \ldots, q_i, \ldots, q_p} = S_{q+p} \to P_p S_{q_0, \ldots, q_i + 1, \ldots, q_i} = S_{q+p+1}$$

can be defined by

$$s_j^{(i)} := s_{q_0 + \cdots + q_{i-1} + i + j},$$

$j = 0, \ldots, q_i$. The face maps

$$d_{(i)} : P_p S_{q_0, \ldots, q_p} \to P_{p-1} S_{q_0, \ldots, \hat{q}_i, \ldots q_p}$$

are the operators corresponding to

$$\varepsilon^{(i)} : \Delta^{q_0 + \cdots + \hat{q}_i + \cdots + q_p + p - 1} \to \Delta^{q_0 + \cdots + \cdots + q_p + p}$$

take $(e_0, \ldots, e_{q_0 + \cdots + \hat{q}_i + \cdots + q_p + p - 1})$ to $(e_0, \ldots, e_{q_0 + \cdots + q_p + p})$, deleting the elements $q_0 + \cdots + q_{i-1} + i, \ldots, q_0 + \cdots + q_i + i$. It deletes $(q_i + 1)$-elements. In contrary to this, there is no degeneracy operator.

Now we turn to the realizations. For the sequences of spaces $\{|P.S.|\}$, we obtain the fat realization:

$$\| \, |P.S.| \, \| = \bigsqcup_{p \geq 0} \Delta^p \times |P_p S.| /_\sim,$$

where

$$|P_p S.| = \bigsqcup \Delta^{q_0 \ldots q_p} \times S_{q_0 + \cdots + q_p + p} /_\sim$$

and the face operators $\pi_i : |P_p S.| \to |P_{p-1} S.|$ are given by $\pi_i = \text{proj}_i \times d_{(i)}$ with $\text{proj}_i : \Delta^{q_0 \ldots q_p} \to \Delta^{q_0 \ldots \hat{q}_i \ldots q_p}$ beeing the natural projection.

Note that $\lambda_p : \Delta^p \times |P_p S.| \to |S.|$ satisfies

$$\lambda_p \circ (\varepsilon^i \times \text{id}) = \lambda_{p-1} \circ (\text{id} \times \pi_i).$$

Thus $\lambda_p$ induces the map $\lambda$ on the fat realization.

Let $\| \, |P.S.| \, \|^p$ respectively $\| \, |S.| \, \|^p$ denote the subcomplexes generated by $\Delta^p \times |P_p S|$ respectively $\Delta^p \times |S.|$. Then the restriction of $\Lambda$ to $\| \, |P.S.| \, \|^p$ is given by

$$\Lambda_p(t, s, \sigma) = (t, \lambda_p(t, s, \sigma)).$$

**Corollary 3.5.** *The map $\Lambda_p$ induce a homeomorphism*

$$\Lambda : \| \, |P.S.| \, \| \to \| \, |S.| \, \| \approx \| * \| \times |S.|.$$

**Corollary 3.6.** *The composition map $\text{proj}_2 \circ \Lambda = \lambda$*

$$\| \, |P.S.| \, \| \to \| \, |S.| \, \| \to |S.|$$

*is a homotopy equivalence.*

**Remark 1.** We can calculate the homology of the geometric realization of a prismatic set as follows:

A prismatic set $P_{.,.}$ has a double complex $(C_{p,n}(PS), \partial_F, \partial_H)$. Here

$$C_{p,n}(PS) = \bigoplus_{q_0 + \cdots + q_p = n} C_{p,q_0,\ldots,q_p}(PS)$$

is the associated chain complex $C_p(PS)$ generated by $P_{p,q_0,\ldots,q_p}$. The vertical boundary map is defined by using boundary maps in the fibre direction

$$\partial_F{}^i : PC_{p,q_0,\ldots,q_p} \to PC_{p,q_0,\ldots,q_i-1,\ldots,q_p}$$

defined by $\partial_F{}^i = \sum (-1)^j d_j^i$, where, if $q_i = 0$ then $\partial^i{}_F = 0$. The total vertical boundary map is then

$$\partial_V = \partial^0{}_F + (-1)^{q_0+1}\partial^1{}_F + \cdots + (-1)^{q_0+\cdots+q_{p-1}+p}\partial^p{}_F.$$

There is also a horizontal boundary map

$$\partial_H = \partial_0 + (-1)^{q_0+1}\partial_1 + \cdots + (-1)^{q_0+\cdots+q_{p-1}+p}\partial_p,$$

where

$$\partial_k = \begin{cases} 0 & : & \text{if } q_k > 0 \\ d_k & : & \text{if } q_k = 0, \end{cases}$$

so that $\partial = \partial_V + \partial_H$ is a boundary map in the total complex $PC_*$ which is the cellular chain complex for the geometric realization. Hence it calculates the homology. In the case of $P_.(f)$ for $f : S \to \bar{S}$ a simplicial map, the double complex gives rise to a spectral sequence which for a triangulated fibre bundle is the usual Leray-Serre spectral sequence.

**Remark 2.** For each $p$ and each $t \in \overset{\circ}{\Delta}{}^p$, $\lambda_p(t)^{-1} : |S| \to \{t\} \times |P_pS|$ induces a map of cellular chain complexes

$$aw : C_*(S) \to C_{*,*}(PS)$$

given by

$$aw(x) = \sum_{q_0+\cdots+q_p=n} s_{q_0+\cdots+q_{p-1}+p-1} \circ \cdots \circ s_{q_0}(x)_{(q_0,\ldots,q_p)},$$

where $x \in S_n$.

## 4. Prismatic Sets and Stars of Simplicial Complexes

For a simplicial set $S$ and the prismatic triangulation $P_.S$ there is another closely related prismatic set $\bar{P}_pS$, which as we shall see for a simplicial complex is the nerve of the covering by stars of vertices considered as a prismatic set.

**Definition 4.1.** For $S$ a simplicial set let $\bar{P}_.S$ be the prismatic set given by

$$\bar{P}_pS_{q_0,\ldots,q_p} := S_{q_0+\cdots+q_p+2p+1}.$$

where face and degeneracy operators on $\bar{P}_pS_{q_0,\ldots,q_p}$ are inherited from the ones of $S_{q+2p+1}$ as follows:

Let $q = q_0 + \cdots + q_p$, the face operators

$$d_j^{(i)} : S_{q+2p+1} = \bar{P}_pS_{q_0,\ldots,q_p} \to S_{q+2p} = \bar{P}_pS_{q_0,\ldots,q_i-1,\ldots,q_p}$$

are defined by

$$d_j^{(i)} := d_{q_0+\cdots+q_{i-1}+2i+j}, \quad j = 0, \ldots, q_i \quad \text{but} \quad j \neq 2i + 1 + \sum_{k=0}^{i} q_k.$$

So $\bar{P}_p S_{q_0,\ldots,q_p}$ has only $q+p$ face operators, i.e., we skip the following $p+1$ face operators

$$\{d_{q_0+1}, d_{q_0+q_1+3}, \ldots, d_{q+2p+1}\}.$$

Similarly the degeneracy operators

$$s_j^{(i)} : S_{q+2p+1} \to S_{q+2p+2}$$

can be defined by

$$s_j^{(i)} := s_{q_0+\cdots+q_{i-1}+2i+j}, \quad j = 0, \ldots, q_i, \quad \text{but} \quad j \neq 2i + 1 + \sum_{k}^{i} q_k.$$

Furthermore the face operators are

$$d_{(i)} : S_{q+2p+1} = \bar{P}_p S_{q_0,\ldots,q_p} \to S_{q+2p-q_i-1} = \bar{P}_{p-1} S_{q_0,\ldots,\hat{q}_i,\ldots,q_p}$$

corresponding to

$$\varepsilon^{(i)} : \Delta^{q+2p-q_i-1} \to \Delta^{q+2p+1}$$

which take $(e_0, \ldots, e_{q_0+\cdots+\hat{q}_i+\cdots+q_p+2p-1})$ to $(e_0, \ldots, e_{q+2p+1})$, by deleting the vectors with indices $(q_0 + \cdots + q_{i-1} + 2i, \ldots, q_0 + \cdots + q_i + 2i + 1)$. So it deletes $q_i + 2$ elements. That is,

$$d_{(i)} = d_{q_0+\cdots+q_{i-1}+2i} \circ \cdots \circ d_{q_0+\cdots+q_i+2i+1}, \quad i = 0, \ldots, p.$$

**Remark.** As $P.S$, $\bar{P}.S$ is a prismatic set but in general not a strong prismatic set.

**Realization of $\bar{P}.S.$.** Notice that

$$\| |\bar{P}.S.| \| = \bigsqcup_{p \geq 0} \Delta^p \times \Delta^{q_0 \cdots q_p} \times \bar{P}_p S_{q_0,\ldots,q_p} / \sim$$

where the equivalence relation apart from the internal relations in $|\bar{P}_p S|$ using $d_j^{(i)}$ and $s_j^{(i)}$, include the relations

$$(\varepsilon^i t, (s, y)) \sim (t, \pi_i(s, y)),$$

with $\pi_i = (\text{proj}_i) \times d_{(i)}$ the face operators on $|\bar{P}_p S.|$.

The relation of $\bar{P}.S$ with $S.$ and $P.S$ is as follows:

**Proposition 4.2.** *Let* $i : \|S.\| \hookrightarrow \| |\bar{P}_p S.| \|$ *be an inclusion defined for* $(t, x) \in \Delta^p \times S_p$ *by*

$$i(t, x) = (t, 1, s_0 \circ \cdots \circ s_p x) \in \Delta^p \times (\Delta^0)^{p+1} \times S_{2p+1} \subseteq \Delta^p \times |\bar{P}_p S.|,$$

*and* $r : \| |\bar{P}.S.| \| \to \|S.\|$ *be the retraction defined for* $(t, s, y) \in \Delta^p \times \Delta^{q_0 \cdots q_p} \times S_{q+2p+1}$

$$r(t, s, y) = (t, d_{0\ldots q_0} \circ \hat{d}_{q_0+1} \circ \cdots \circ d_{(q_0+\cdots+q_{p-1}+2p)\ldots(q+2p)} \circ \hat{d}_{q+2p+1} y),$$

*where the* $\hat{d}_i$ *are left out and* $d_{(q_0+\cdots+q_{i-1}+2i)\ldots(q_0+\cdots+q_i+2i)} = d_{q_0+\cdots+q_{i-1}+2i} \circ \cdots \circ d_{q_0+\cdots+q_i+2i}$, $i = 0, \ldots, p$.

(1) $i$ is a deformation retract with the retraction $r$.
(2) There is a commutative diagram of homotopy equivalences

$$
\begin{array}{ccc}
||S.|| & \xrightarrow{\ i\ } & ||\,|\bar{P}.S.|\,|| \\
 & & \downarrow{\scriptstyle f} \\
 & & ||\,|P.S.|\,|| \\
 & & \downarrow{\scriptstyle \Lambda} \\
 & & ||\,|S.|\,||
\end{array}
$$

where $f : \Delta^p \times \Delta^{q_0 \dots q_p} \times S_{q+2p+1} \to \Delta^p \times \Delta^{q_0 \dots q_p} \times S_{q+p}$ takes $(t, s^0, \dots, s^p, x)$ to $(t, s^0, \dots, s^p, d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots \circ d_{q+2p+1}x)$, $x \in S_{q+2p+1}$.

The proof is straight forward see [1] for details.

For a simplicial complex $K$ there is another prismatic complex defined using the stars of simplices. That is, let $K_0 = \{a_i \mid i \in I\}$, where $I = \{1, \dots, N\}$, be the set of vertices and let $K_n = \{\sigma = (a_{i_0}, \dots, a_{i_n}) \mid i_0 < \cdots < i_n\}$ be the set of $n$-simplices such that if $\sigma \in K_n$ then any face $\tau = (a_{i_{j_0}}, \dots, a_{i_{j_k}})$ lies in $K_k$. We shall write $\tau \preccurlyeq \sigma$ in this case. Now $K \times K$ is also a simplicial complex with the lexicographical order of the vertices

$$(a_i, b_j) < (a_{i'}, b_{j'}) \Leftrightarrow \text{either } i < i' \text{ or } i = i' \text{ and } j < j',$$

where $\{(a_{i_0}, b_{j_0}), \dots, (a_{i_n}, b_{j_n})\} \in K \times K$.

**Definition 4.3.** Let $K$ be a simplicial complex. The Star of $K$ is defined as

$$\mathrm{St}(K) = \{(\sigma, \tau) \in K \times K \mid \exists\, \sigma' \text{ such that } \sigma \cup \tau \preccurlyeq \sigma'\} \subseteq K \times K.$$

This is equivalent to say that

$$\mathrm{St}(K) = \{\text{faces of } \sigma' \times \sigma' \subseteq K \times K\}.$$

**Remark 3.** For each $\sigma \in K$, $(\{\sigma\} \times K) \cap \mathrm{St}(K)$ is the closure of the usual open star of $\sigma$, i.e., the union of the open simplices having $\sigma$ as a face. Whence the name $\mathrm{St}(K)$. Note that $\mathrm{St}(K) \subseteq K \times K$ is a subcomplex.

Let $K^s$ denote the simplicial set associated to the simplicial complex $K$. That is,

$$K_n{}^s = \{(a_{i_0}, \dots, a_{i_n}) \mid \{a_{i_0}, \dots, a_{i_n}\} \text{ a simplex of } K \text{ (with repetitions) } i_0 \leq \cdots \leq i_n\}.$$

$\mathrm{St}(K)_n^s$ are the following:
Let $(\sigma, \tau) \in K \times K$, where $\sigma = (a_{i_0}, \dots, a_{i_p})$, $\tau = (b_{j_0}, \dots, b_{j_q})$. For $(\sigma, \tau) \in \mathrm{St}(K)$, let $\sigma' = \sigma \cup \tau = (c_{k_0}, \dots, c_{k_n})$. By allowing repetitions in Definition 4.3, i.e., by taking $\sigma' \in K^s$, we can assume $n = p+q$ so that either $c_{k_s} = a_{i_t}$ or $c_{k_s} = b_{j_u}$, where $t = 0, \dots, p$, $u = 0, \dots, q$. Also we can assume $c_{k_n} = a_{i_p}$, and if $a_{i_t} = b_{j_u}$ then $b_{j_u}$ comes before $a_{i_t}$. In other words $(\sigma, \tau)$ is of the form

$$\sigma = d_{\nu_1 \dots \nu_q}\sigma', \quad \tau = d_{\mu_1 \dots \mu_p}\sigma',$$

where $0 \leq \nu_1 < \cdots < \nu_q < n$ and $0 \leq \mu_1 < \cdots < \mu_p \leq n$ and $\mu_i \neq \nu_j$, $\forall i, j$. Therefore we introduce for a general simplicial set $S$ the following.

**Definition 4.4.** Let $\mathrm{St}(S)$ be the simplicial subset of the diagonal $\delta(S \times S)$ containing all simplices of the form

$$(s_{\nu_q \dots \nu_1} \circ d_{\nu_1 \dots \nu_q} \sigma', s_{\mu_p \dots \mu_1} \circ d_{\mu_1 \dots \mu_p} \sigma'),$$

where $0 \le \nu_1 < \dots < \nu_q < n$ and $0 \le \mu_1 < \dots < \mu_p \le n$ with $\mu_i \ne \nu_j$, $\forall i, j$ as above. Here $s_{\nu_q \dots \nu_1} = s_{\nu_q} \circ \dots \circ s_{\nu_1}$, $d_{\nu_1 \dots \nu_q} = d_{\nu_1} \circ \dots \circ d_{\nu_q}$, $s_{\mu_p \dots \mu_1} = s_{\mu_p} \circ \dots \circ s_{\mu_1}$ and $d_{\mu_1 \dots \mu_p} = d_{\mu_1} \circ \dots \circ d_{\mu_p}$.

**Lemma 4.5.** *For $K$ a simplicial complex, there is a map*

$$\mathrm{st} : \mathrm{St}(K)^s \to \mathrm{St}(K^s)$$

*which is an isomorphism.*

*Proof.* By the discussion made before, there is a well-defined map st. Indeed, $\mathrm{St}(K)^s$ is a simplicial set generated by

$$\{(\sigma, \tau) \in K \times K \mid \exists \sigma' \in K^s \text{ such that } \sigma = d_{\nu_1 \dots \nu_q} \sigma', \tau = d_{\mu_1 \dots \mu_p} \sigma'$$
$$\text{and } (s_{\nu_q \dots \nu_1} \sigma, s_{\mu_p \dots \mu_1} \tau) \in \delta(K^s \times K^s)\}.$$

By Definition 4.4, we can put $\mathrm{st}(\sigma, \tau) = (s_{\nu_q \dots \nu_1} \sigma, s_{\mu_p \dots \mu_1} \tau) \in \mathrm{St}(K^s)$. Clearly st is an isomorphism since for $(\sigma, \tau) \in \delta(K^s \times K^s)$ and $\sigma'$ as in Definition 4.4, $(\sigma, \tau) \in K^s \times K^s$ determines an element in $K \times K$ by deleting repetitions and this is unique. $\square$

**Remark 4.** The projection on the first factor $\pi_1 : S \times S \to S$ gives a simplicial map $\pi_1 : \mathrm{St}_{\cdot}(S) \to S$. Hence, we obtain a prismatic set $P_{\cdot} \mathrm{St}(S) = P_{\cdot}(\pi_1)$ as in Example 3.2. Here with $q = q_0 + \dots + q_p$ and $\sigma = s_{\nu_q \dots \nu_1} \circ d_{\nu_1 \dots \nu_q} \sigma' = \mu_{q_0, \dots, q_p} \bar{\sigma}$, $\tau = s_{\mu_p \dots \mu_1} \circ d_{\mu_1 \dots \mu_p} \sigma'$, we have

$$P_p \mathrm{St}(S)_{q_0, \dots, q_p} = \{(\sigma, \tau, \bar{\sigma}) \in \mathrm{St}(S)_{q+p} \times S_p \subset \delta(S \times S)_{q+p} \times S_p \mid \sigma, \tau \text{ given above}\}.$$

That is, $\pi_1(\sigma, \tau) = \mu_{q_0, \dots, q_p}(\bar{\sigma})$, where $\bar{\sigma} = d_{\nu_1 \dots \nu_q} \sigma' \in S_p$. So The elements in $P_p \mathrm{St}(S)_{q_0, \dots, q_p}$ are of the form $(\mu_{q_0, \dots, q_p} \bar{\sigma}, \tau, \bar{\sigma})$, where $\tau \in S_q$. Here explicitly

$$\mu_{q_0, \dots, q_p} = \hat{s}_{q+p} \circ s_{(q+p-1) \dots (q_0 + \dots + q_{p-1} + p)} \dots \hat{s}_{q_0 + q_1 + 1} s_{(q_0 + q_1) \dots (q_0 + 1)} \hat{s}_{q_0} s_{(q_0 - 1) \dots (0)}.$$

## 5. COMPARISON OF THE TWO STAR SIMPLICIAL SETS

We shall now prove that this is closely related to the prismatic set $\bar{P}S$ defined in the previous section.

**Theorem 5.1.**  (1) *There is a natural **(surjective)** map*

$$\bar{p} : \bar{P}_{\cdot} S_{\cdot} \to P \mathrm{St}_{\cdot}(S).$$

(2) *If $S = K^s$, where $K$ is a simplicial complex, then $\bar{p}$ is an isomorphism.*

*Proof.* (1) Take an element $\gamma \in \bar{P}_p S_{q_0, \dots, q_p} = S_{q_0 + \dots + q_p + 2p + 1}$. Then $\gamma$ and $q_0, \dots, q_p$ determine an element $\bar{p}(\gamma)$ in $P_p \mathrm{St}(S)_{q_0, \dots, q_p}$ together with a $(p+1, q+p+1)$-partition

$(i_1, \ldots, i_p, i_{p+1}, j_1, \ldots, j_{q+p+1})$ of $n = q + 2p + 1$, where $q = q_0 + \cdots + q_p$. Here

$$i_1 = q_0 + 1$$
$$i_2 = q_0 + q_1 + 3$$
$$\vdots$$
$$i_p = q_0 + \cdots + q_{p-1} + 2p - 1$$
$$i_{p+1} = q_0 + \cdots + q_p + 2p + 1$$

correspond to the $\mu_i$'s defined in Definition 4.4 and the $j$'s correspond to the complement, that is, $j_1, \ldots, j_{q_0+1}, j_{q_0+2}, \ldots, j_{q_0+q_1+2}, \ldots, j_{q_0+\cdots+q_{p-1}+p}, \ldots, j_{q_0+\cdots+q_p+p+1}$, are $0, \ldots, q_0, q_0 + 2, \ldots, q_0 + q_1 + 2, q_0 + q_1 + 4, \ldots, q_0 + \cdots + q_{p-2} + 2p - 2, \ldots, q_0 + \cdots + q_{p-1} + 2p, q_0 + \cdots + q_{p-1} + 2p, \ldots, q_0 + \cdots + q_p + 2p$, respectively. Then, in terms of Remark 4 at the end of Section 4, we define

$$\bar{p}(\gamma) = (\sigma, \tau, \bar{\sigma}) \in P_p \mathrm{St}(S)_{q_0 \ldots q_p} \subseteq S_{q+p} \times S_{q+p} \times S_p$$

where

$$\bar{\sigma} = d_{0 \ldots q_0} \circ \hat{d}_{q_0+1} \circ \cdots \circ d_{(q_0+\cdots+q_{p-1}+2p)\ldots(q_0+\cdots+q_p+2p)} \circ \hat{d}_{q+2p+1}(\gamma) = d_{j_1 \ldots j_{q+p+1}}(\gamma)$$

$$\tau = d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots \circ d_{q_0+\cdots+q_p+2p+1}(\gamma) = d_{i_1 \ldots i_{p+1}}(\gamma)$$

$$\sigma = \hat{s}_{q_0+\cdots+q_p+p} \circ s_{(q_0+\cdots+q_p+p-1)\ldots(q_0+\cdots+q_{p-1}+p)} \circ \hat{s}_{q_0+\cdots+q_{p-1}+p-1} \circ$$

$$\cdots \circ \hat{s}_{q_0+q_1+1} \circ s_{(q_0+q_1)\ldots(q_0+1)} \circ \hat{s}_{q_0} \circ s_{(q_0-1)\ldots(0)}(\bar{\sigma})$$

$$= s_{(q+p-1)\ldots(q+p-q_p)} \circ \cdots \circ s_{(q_0+q_1)\ldots(q_0+1)} \circ s_{(q_0-1)\ldots(0)}(\bar{\sigma})$$

$$= \mu_{q_0,\ldots,q_p}(\bar{\sigma}).$$

Using the above expression for $\bar{\sigma}$ in terms of $d$'s and $\gamma$, we get

$$\sigma = s_{(q+p-1)\ldots(q+p-q_p)} \circ \cdots \circ s_{(q_0+q_1)\ldots(q_0+1)} \circ s_{(q_0-1)\ldots 0} \circ d_{j_1 \ldots j_{q+p+1}}(\gamma).$$

Now $\mathrm{St}(S)_{q_0+\cdots+q_p+2p+1}$ contains the simplex

$$(s_{j_{q+p+1}\ldots j_1} \circ s_{(q+p-1)\ldots(q+p-q_p)} \circ \cdots \circ s_{(q_0+q_1)\ldots(q_0+1)} \circ s_{(q_0-1)\ldots 0} \circ d_{j_1 \ldots j_{q+p+1}}(\gamma),$$
$$s_{i_{p+1}\ldots i_1} d_{i_1 \ldots i_{p+1}}(\gamma)) = (s_{j_{q+p+1}\ldots j_1}\sigma, s_{i_{p+1}\ldots i_1}\tau).$$

It follows that $(\sigma, \tau) \in \mathrm{St}(S)$ and hence $\bar{p}(\gamma) = (\sigma, \tau, \bar{\sigma}) \in P_p \mathrm{St}(S)_{q_0,\ldots,q_p}$.

Now $\bar{p}$ is a surjective map: Suppose $(\sigma, \tau, \bar{\sigma}) \in P_p \mathrm{St}(S)_{q_0,\ldots,q_p}$ and we shall find $\gamma \in \bar{P}_p S_{q_0,\ldots,q_p}$ such that $\bar{p}(\gamma) = (\sigma, \tau, \bar{\sigma})$. Thus $(\sigma, \tau, \bar{\sigma}) \in P_p \mathrm{St}(S)_{q_0,\ldots,q_p} \subset \delta(S \times S)_{q+p}$ is such that

$$\pi_1(\sigma, \tau) \in \mathrm{Im}\{\mu_{q_0,\ldots,q_p} : S_p \to S_{q+p}\}$$

where $\bar{\sigma} \in S_p$.

Again use the partition $(p + 1, q + p + 1)$ as above, put $\gamma = s_{i_{p+1}\ldots i_1}\sigma \in S_{q+2p+1}$. Indeed since

$$(s_{j_{q+p+1}} \circ s_{j_q-q_p+p-1} \circ \cdots \circ s_{j_{q_0+q_1+2}} \circ s_{j_{q_0+1}}\sigma, s_{i_{p+1}\ldots i_1}\tau)$$

is of the required form as in Definition 4.4 and since

$$(\sigma, \tau) = (d_{k_1 \ldots k_{p+1}} \times d_{i_1 \ldots i_{p+1}})(s_{j_{q+p+1}} \circ s_{j_q-q_p+p-1} \circ \cdots \circ s_{j_{q_0+q_1+3}} \circ s_{j_{q_0+1}}\sigma, s_{i_{p+1}\ldots i_1}\tau)$$

here $d_{k_1 \ldots k_{p+1}} = d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots \circ d_{q_0+\cdots+q_{p-1}+2p-1} \circ d_{q+2p+1}$. So the $d_K$'s and the $d_I$'s are the same, where $d_I = d_{i_1 \ldots i_{p+1}}$.

We have $(\sigma, \tau) \in \mathrm{St}(S)$. Hence $\bar{p}(\gamma) \in P.\mathrm{St}(S).$.

(2) If $S = K^s$, $K$ simplicial complex then

$$P_p\mathrm{St}(K^s)_{q_0,\ldots,q_p} = \{(\sigma,\tau) \in \mathrm{St}(K^s)_{q+p} \subset \delta(K^s \times K^s)_{q+p}$$
$$\mid \pi_1(\sigma,\tau) \in \mathrm{Im}\{\mu_{q_0,\ldots,q_p} : K_p{}^s \to K_{q+p}^s\}\}.$$

The map $\mu_{q_0,\ldots,q_p} : K_p{}^s \to K_{q+p}^s$ takes $(i_0,\ldots,i_p)$ to $(\underbrace{i_0,\ldots,i_0}_{(q_0+1)\text{-times}},\ldots,\underbrace{i_p,\ldots,i_p}_{(q_p+1)\text{-times}})$. Then

$$\sigma = (a_{i_0},\ldots,a_{i_0},\ldots,a_{i_p},\ldots,a_{i_p}) \in K^s{}_{q+p},$$
$$\tau = (b_{j_0},\ldots,b_{j_{q_0}},\ldots,b_{j_{q_0+\cdots+q_{p-1}+p}},\ldots,b_{j_{q+p}}) \in K^s{}_{q+p}.$$

By the definition $\bar{P}_p(K^s)_{q_0,\ldots,q_p} = P_p(K^s)_{q_0+1,\ldots,q_p+1}$. Then $\gamma$ in $K^s{}_{q+2p+1}$ given by $\gamma = (c_0,\ldots,c_{q_0+1}|\ldots|c_{q_0+\cdots+q_{p-1}+2p},\ldots,c_{q+2p+1}) \in K^s_{q+2p+1}$ is uniquely determined by $\sigma$ and $\tau$.

Explicitly the inverse map $\bar{p}^{-1} : P_p\mathrm{St}(K^s)_{q_0,\ldots,q_p} \to \bar{P}_pK^s{}_{q_0,\ldots,q_p}$ is defined by $\bar{p}^{-1}(\sigma,\tau) = \gamma$, where

$$\sigma = (a_{i_0},\ldots,a_{i_p}),$$
$$\tau = (b_{j_0},\ldots,b_{j_{q+p}}) \quad \text{and}$$
$$\gamma = (j_0',\ldots,j_{q_0}',i_0|j_{q_0+1}',\ldots,j_{q_0+q_1+1}',i_1|\ldots|j_{q_0+\cdots+q_{p-1}+p}',\ldots,j_{q+p}',i_p)$$

such that for $\sum_{i=0}^{k-1} q_i + p \leq s \leq \sum_{i=0}^{k} q_i + p$

$$j_s' = \begin{cases} i_{k-1} & : & j_s \leq i_{k-1} \\ j_s & : & i_{k-1} < j_s < i_k \\ i_k & : & i_k \leq j_s, \end{cases}$$

$k = 1,\ldots,p$. Hence $\gamma \in \bar{P}_pK^s{}_{q_0,\ldots,q_p}$ exists and is uniquely determined by $(\sigma,\tau) \in \mathrm{St}(K^s)_{q+2p+1}$.

Therefore $\bar{p} : \bar{P}.K^s \to P.\mathrm{St}(K^s)$ is an isomorphism. $\qquad\square$

**Remark.** Note that $\bar{p}$ is not injective for a simplicial set in general since for constructing the inverse map $P\,\mathrm{St}(S) \to \bar{P}S$, there is no unique choice for the element $\gamma$ in $\bar{P}.S.$. In fact, we do not know which degeneracy operators we will use in order to define $\gamma$, so in general the inverse is not well-defined.

## 6. The Classifying Space and Lattice Gauge Theory

For the definition of a classifying map we need a prismatic version of the standard construction of the classifying space.

Let $G$ be a topological group and the usual classifying space $BG = EG/G$ which is constructed as a simplicial space $EG_p = G. \times \cdots \times G.$, $(p+1)$-times, $BG_p = (G \times \cdots \times G)/G$.

In order to make this simplicial set discrete we can replace $G$ by the singular simplicial set of continuous maps $S_qG = \mathrm{Map}(\Delta^q, G)$ and $E.S.G$ as in Example 2.8 is a prismatic

set. However we shall need another model constructed as follows: For a continuous map $a \in \mathrm{Map}(\Delta^p \times \Delta^{q_0 \cdots q_p}, G^{p+1})$. Then we define

$$a(t, s^0, \ldots, s^p) = (a_0(t, s^0), a_1(t, s^0, s^1), \ldots, a_p(t, s^0, \ldots, s^p)),$$

where $(t, s^0, \ldots, s^p) \in \Delta^p \times \Delta^{q_0 \cdots q_p}$. $S_. G$ acts on this prismatic set and we define

$$P_p EG_{q_0, \ldots, q_p} = \{a : \Delta^p \times \Delta^{q_0 \cdots q_p} \to G^{p+1} \mid a_j(\varepsilon^i t, s) \text{ is independent}$$
$$\text{of } s^i \text{ for all } j \text{ different from } i\}.$$

$P_. BG = P_. EG / S_. G$, that is,

$$P_p BG_{q_0, \ldots, q_p} = P_p EG_{q_0, \ldots, q_p} / S_p G.$$

**Proposition 6.1.** *The evaluation maps give horizontal homotopy equivalences in the diagram*

$$
\begin{array}{ccc}
\| \, |P_. EG_.| \, \| & \xrightarrow{\;ev\;} & EG \\
\Big\downarrow{\scriptstyle \| \, |\gamma| \, \|} & & \Big\downarrow{\scriptstyle \gamma} \\
\| \, |P_. BG_.| \, \| & \xrightarrow{\;ev\;} & EG/G
\end{array}
$$

*Furthermore the top map is equivariant with respect to the homomorphism $ev : |S_. G| \to G$.*

*Proof.* First notice that the evaluation map $ev : |S_. G| \to G$ is a homotopy equivalence. Also the equivariance is obvious by the commutative diagram

$$
\begin{array}{ccc}
\| \, |P_. EG_.| \, \| \times |S_. G| & \xrightarrow{\;ev \times ev\;} & EG \times G \\
\Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} \\
\| \, |P_. EG_.| \, \| & \xrightarrow{\;\;\;\;\;ev\;\;\;\;\;} & EG
\end{array}
$$

Since $\| \, |P_. EG_.| \, \|$ and $EG$ are both contractible, the evaluation map induces a homotopy equivalence on the quotient. $\qquad\square$

## 7. Lattice Gauge Theory, Parallel Transport Function

In Lattice gauge theory in the sense of Phillips and Stone [15] they construct for a given Lie group $G$ and a simplicial complex $K$ a $G$-bundle with connection on $|K|$ associated to a set of $G$-valued continuous functions defined over the faces of a simplex. These they call "parallel transport functions" since they are determined by parallel transport for the connection. In this section we shall introduce similar "compatible transition functions" for $K$ replaced by a simplicial set $S$ and in the following section we shall use these to construct a classifying map on the star complex $\bar{P}_. S_.$. First we consider $G$-bundles over simplicial sets.

**Definition 7.1.** A bundle over $|S|$ is a sequence of bundles over $\Delta^p \times \sigma$ for all $p$, where $\sigma \in S_p$ and with commutative diagrams;

$$
\begin{array}{ccc}
F_{d_j\sigma} & \xrightarrow{\ \bar{\varepsilon}^j\ } & F_\sigma \\
\downarrow & & \downarrow \\
\Delta^{p-1} \times d_j\sigma & \xrightarrow{\ \varepsilon^j\ } & \Delta^p \times \sigma
\end{array}
$$

and

$$
\begin{array}{ccc}
F_{s_j\sigma} & \xrightarrow{\ \bar{\eta}^j\ } & F_\sigma \\
\downarrow & & \downarrow \\
\Delta^{p+1} \times s_j\sigma & \xrightarrow{\ \eta^j\ } & \Delta^p \times \sigma
\end{array}
$$

with the compatibility conditions:

$$
\bar{\varepsilon}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\varepsilon}^{j-1} & : \quad i < j \\ \bar{\varepsilon}^{i+1} \bar{\varepsilon}^j & : \quad i \geq j, \end{cases}
$$

$$
\bar{\eta}^j \bar{\eta}^i = \begin{cases} \bar{\eta}^i \bar{\eta}^{j+1} & : \quad i \leq j \\ \bar{\eta}^{i-1} \bar{\eta}^j & : \quad i > j, \end{cases}
$$

and

$$
\bar{\eta}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\eta}^{j-1} & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ \bar{\varepsilon}^{i-1} \bar{\eta}^i & : \quad i > j+1. \end{cases}
$$

Given a $G$-bundle $F \to |S|$, $G$ a Lie group, since $\Delta^p$ is contractible, we can choose a trivialization $\varphi_\sigma : F_\sigma \to \Delta^p \times \sigma \times G$ for a non-degenerate $\sigma \in S_p$. If $\sigma$ is degenerate, that is, there exists $\tau$ such that $\sigma = s_i\tau$, then the trivialization of $\sigma$ is defined as pullback of the trivialization of $\tau$, that is, $\varphi_\sigma = \eta^{i*}(\varphi_\tau)$.

**Definition 7.2.** ( Admissible Trivializations ) A set of trivializations is called admissible, in case $\varphi_\sigma$ for $\sigma = s_i\tau$ is given by $\varphi_\sigma = s^{i*}(\varphi_\tau)$.

**Lemma 7.3.** *Admissible trivializations always exist.*

Now, let us construct the transition functions for a simplex $\sigma \in S_p$ before giving the following proposition:

**Definition 7.4.** Given a bundle and a set of trivializations, we get for each face $\tau$ of say $\dim \tau = q < p$ in $\sigma$, a transition function $v_{\sigma,\tau} : \Delta^q \to G$. E.g., if $\tau = d_i\sigma$ then the transition function $v_{\sigma,d_i\sigma} : \Delta^{p-1} \to G$ is given by the diagram

$$
\begin{array}{ccc}
\Delta^{p-1} \times (d_i\sigma) \times G & \xrightarrow{\ \Theta\ } & \Delta^p \times (\sigma) \times G \\
\downarrow & & \downarrow \\
\Delta^{p-1} \times d_i\sigma & \xrightarrow{\ \varepsilon^i\ } & \Delta^p \times \sigma
\end{array}
$$

where $d_i\sigma = \tau$ and $\Theta = \varphi_\sigma \circ \bar{\varepsilon}^i \circ \varphi_{d_i\sigma}^{-1}$. So

$$\{v_{\sigma,\tau} \mid \sigma \in S_p \text{ and } \tau \text{ is a face of } \sigma\}$$

are **the transition functions** for the bundle over $|S|$.

**Remark.** The transitions functions are generalized lattice gauge fields. Classically Lattice gauge fields are defined only on 1-skeletons but one can extend them to $p-1$ simplices for all $p$, given rise to transition functions on $\Delta^p$, as above.

We now list a number of propositions stating the properties of these. The proofs are straight forward. For details see Akyar [1].

**Proposition 7.5.** *Given a bundle on a simplicial set and admissible trivializations, the transition function $v_{\sigma,\tau}$, where $\tau$ is a face of $\sigma$, satisfies;*

(i) *$\sigma$ is nondegenerate: if $\gamma = d_j\sigma$ and $\tau = d_i\gamma$ then*

$$v_{\sigma,\tau} = (v_{\sigma,\gamma} \circ \varepsilon^i).v_{\gamma,\tau}.$$

   *This is called the cocycle condition.*

(ii) *$\sigma$ is degenerate: If $\sigma = s_j\sigma'$ and $\tau = d_i\sigma$ then when $i < j$ for $\tau = s_{j-1}\tau'$ one gets $\tau' = d_i\sigma'$ and when $i > j+1$ for $\tau = s_j\tau'$ one gets $\tau' = d_{i-1}\sigma'$. For the other cases, $i = j$ or $i = j+1$, $\tau = \sigma'$. Then the transition functions satisfy:*

$$v_{\sigma,\tau} = \begin{cases} v_{\sigma',\tau'} \circ \eta^{j-1} & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ v_{\sigma',\tau'} \circ \eta^j & : \quad i > j+1. \end{cases}$$

(iii) *If $\tau$ is a composition of face operators of $\sigma$, e.g., $\tau = \tilde{d}^{p-(i-1)}\sigma$, $i = 1, \ldots, p$, where $\tilde{d}^{p-(i-1)} = d_i \circ \cdots \circ d_p$ then*

$$v_{\sigma,\tau} = (v_{\sigma,\tilde{d}^1\sigma} \circ (\varepsilon^i)^{p-i}).(v_{\tilde{d}^1\sigma,\tilde{d}^2\sigma} \circ (\varepsilon^i)^{p-i-1})$$
$$\ldots (v_{\tilde{d}^{p-(i+1)}\sigma,\tilde{d}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{d}^{p-i}\sigma,\tau}.$$

**Proposition 7.6.** *Assume that we have a bundle over $|S|$. Then*

(1) *There exists admissible trivializations such that the transition function is given by*

$$v_{\sigma,d_i\sigma} = 1 \text{ if } i < p.$$

(2) *For $\tau = \tilde{d}^{p-(i-1)}\sigma$, $i = 1, \ldots, p$, we get $v_{\sigma,\tau}$ as product of some transition functions:*

$$v_{\sigma,\tau} = (v_\sigma \circ (\varepsilon^i)^{p-i}).(v_{\tilde{d}^1\sigma} \circ (\varepsilon^i)^{p-i-1}).(v_{\tilde{d}^2\sigma} \circ (\varepsilon^i)^{p-i-2})$$
$$\ldots (v_{\tilde{d}^{p-(i+1)}\sigma} \circ (\varepsilon^i)^1).(v_{\tilde{d}^{p-i}\sigma}).$$

(3) *The transition functions $v_{\sigma,\tau}$ satisfy the compatibility conditions:*

$$v_\sigma \circ \varepsilon^i = \begin{cases} v_{d_i\sigma} & : \quad i < p-1 \\ v_{d_{p-1}\sigma}.v_{d_p\sigma}^{-1} & : \quad i = p-1 \end{cases}$$

(4) *For a degenerate $\sigma$, we have*

$$v_\sigma \circ \eta^j = v_{s_j\sigma}$$

for all $j$.

**Proposition 7.7.** *Given a bundle, one can find admissible trivializations such that the transition functions are determined by functions $v_\sigma : \Delta^{p-1} \to G$ for $\sigma \in S_p$ nondegenerate.*

**Proposition 7.8.** *Suppose given a set of functions*

$$v_\sigma : \Delta^{p-1} \to G$$

for $\sigma \in S_p$ for all $p$, satisfying the compatibility conditions

$$v_\sigma \circ \varepsilon^i = \begin{cases} v_{d_i\sigma} & : \quad i < p-1 \\ v_{d_{p-1}\sigma}.v_{d_p\sigma}{}^{-1} & : \quad i = p-1 \end{cases}$$

and

$$v_{s_j\sigma} = v_\sigma \circ \eta^j.$$

Then one can define for each $\sigma \in S_p$ and each lower dimensional face $\tau$ of $\sigma$, a function $v_{\sigma,\tau}$ such that (i) and (ii) in Proposition 7.5 hold and such that

$$v_{\sigma,\tau} = \begin{cases} v_\sigma & : \quad i = p \\ 1 & : \quad i < p. \end{cases}$$

**Proposition 7.9.** *Given a set of transition functions $v_{\sigma,\tau}$ satisfying (i) and (ii) in Proposition 7.5, there is a bundle $F$ over $|S|$ and trivializations with transition functions $v_{\sigma,\tau}$.*

**Corollary 7.10.** *Given a set of functions $v_\sigma$ satisfying the compatibility conditions in Proposition 7.6, one can construct a bundle $F$ over $|S|$ and the trivializations with the transition functions $v_{\sigma,d_p\sigma} = v_\sigma$ and $v_{\sigma,d_i\sigma} = 1$ when $i < p$ and $v_{s_i\sigma} = v_\sigma \circ \eta^i$ for a degenerate $\sigma$.*

**Definition 7.11.** A set of functions $\{v_\sigma\}_{\sigma \in S}$ as in Proposition 7.8 are called a set of "compatible transition functions".

We end this section by comparing these compatible transition functions with the "parallel transport functions" (p.t.f.) of Phillips and Stone [15]. For $S = K^s$ these consist of a set of maps, $V_\sigma : c_\sigma \to G$ for each $r$-simplex $\sigma$ of $K$, $r \geq 1$, $c_\sigma$ is the $(r-1)$-cube given by $0 \leq s_{a_1} \leq 1, \ldots, 0 \leq s_{a_{r-1}} \leq 1$, where $\sigma = \langle a_0, \ldots, a_r \rangle \in K$ with the compatibility conditions

    1. Cocycle condition

$$V_\sigma(s_{a_1}, \ldots, s_{a_p} = 1, \ldots, s_{a_{r-1}})$$
$$= V_{\langle a_0, \ldots, a_p \rangle}(s_{a_1}, \ldots, s_{a_{p-1}}).V_{\langle a_p, \ldots, a_r \rangle})(s_{a_{p+1}}, \ldots, s_{a_{r-1}}).$$

    2. Compatibility condition

$$V_\sigma(s_{a_1}, \ldots, s_{a_p} = 0, \ldots, s_{a_{r-1}})$$
$$= V_{\langle a_0, \ldots, \hat{a}_p, \ldots, a_{r-1} \rangle}(s_{a_1}, \ldots, \hat{s}_{a_p}, \ldots, s_{a_{r-1}}).$$

Now, suppose we have compatible transition functions $\{v_\sigma\}$ for a principal $G$-bundle $E \to |K|$ with triangulated base. Then for $\sigma = \langle a_0, \ldots, a_r \rangle$, the p.t.f. $V_\sigma : c_\sigma \to G$ is given by the parallel transport $E_{a_0} \to E_{a_r}$ along paths determined as follows:

Let $\sigma = \langle a_0, \ldots, a_r \rangle \in K^s$ and $s = (s_{a_0}, \ldots, s_{a_{r-1}}) \in c_\sigma$.

We pick $r - 1$ points as $P_1, \ldots, P_{r-1}$ so that $P_1$ is on the line segment from $a_0$ to $a_1$, that is,

$$P_1 = (1 - s_{a_1})a_0 + s_{a_1}a_1 = ((1 - s_{a_1}, s_{a_1}), \langle a_0, a_1 \rangle) \in |K|.$$

Similarly, $P_2$ is on the line segment from $P_1$ to $a_2$, $P_2 = (1 - s_{a_2})P_1 + s_{a_2}a_2$. Then

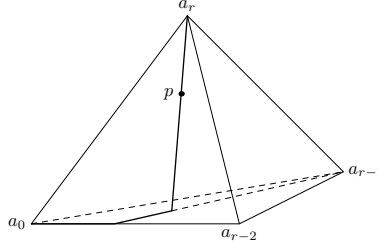$$P_2 = ((1 - s_{a_2})(1 - s_{a_1}), (1 - s_{a_2})s_{a_1}, s_{a_2}, \langle a_0, a_1, a_2 \rangle).$$

By continuing in the same way, we get

$$P_{r-1} = (1 - s_{a_{r-1}})P_{r-2} + s_{a_{r-1}}a_{r-1}.$$

Let $\alpha$ be the piecewise linear path from $a_0$ through $P_1, \ldots, P_{r-1}$ to $a_r$. In other words, $\alpha$ is determined uniquely up to parametrization by $r - 1$ numbers $s_{a_1}, \ldots, s_{a_{r-1}}$. For $P_{r-1} = (t, d_r\sigma) \in \Delta^{r-1} \times K_{r-1}$, $d_r\sigma = \langle a_1, \ldots, a_{r-1} \rangle$, the element

$$V_\sigma(s_1, \ldots, s_{r-1}) = v_\sigma(t) \in G$$

is to be interpreted as the parallel transport along $\alpha$.



## 8. The Classifying Map

**The construction of The Classifying Map**

For a given set of compatible transition functions (c.t.f.) $\{v_\sigma\}$ satisfying Proposition 7.8 we have seen in Proposition 7.9 that there is an associated $G$-bundle $F$ over $|S_\cdot|$. Recall that the composite map $\mathrm{proj} \circ L : \| |\bar{P}_\cdot S_\cdot| \| \to \| |S_\cdot| \| \to |S|$ is a homotopy equivalence, where $L = \Lambda \circ f$ is given as in Proposition 4.2. In this section, we construct a classifying map for the bundle $(\mathrm{proj} \circ L)^* F$ over $\| |\bar{P}_\cdot S_\cdot| \|$.

**Theorem 8.1.**

(1) *For given c.t.f.'s $\{v_\sigma\}$, there is a canonical prismatic map $m : \bar{P}_\cdot S_\cdot \to P_\cdot BG$.*

(2) *The induced map of geometric realizations*

$$ev \circ \| |m| \| = \bar{m} : \| |\bar{P}_\cdot S_\cdot| \| \xrightarrow{\| |m| \|} \| |P_\cdot BG| \| \xrightarrow{ev} BG$$

*is a classifying map for the $G$-bundle $(\mathrm{proj} \circ L)^* F$ over $\| |\bar{P}_\cdot S_\cdot| \|$.*

*Proof.* (1) The map $m : \bar{P}_\cdot S_\cdot \to P_\cdot BG$ is defined as

$$m(\sigma) = [(a_0, a_1, \ldots, a_p)]$$

where $\sigma \in \bar{P}_p S_{q_0 \ldots q_p} = S_{q+2p+1}$, $q = q_0 + \cdots + q_p$ and $a_i : \Delta^p \times \Delta^{q_0 \ldots q_i} \to G$ are given below. In the following, we use for convenience the interior coordinates $(t_1, \ldots, t_p)$ of the standard simplex with barycentric coordinates $(t'_0, \ldots, t'_p)$.

$$t_1 = 1 - t'_0, \ t_2 = 1 - t'_0 - t'_1, \ \ldots, \ t_{r-1} = 1 - t'_0 - \cdots - t'_{r-1}, \ t_r = t'_r$$

such that $0 \le t_i \le 1$, $i = 1, \ldots, p$, $1 \ge t_1 \ge \cdots \ge t_p \ge 0$ and $\sum_{i=0}^p t'_i = 1$, $t'_i \le 1$, $i = 0, \ldots, p$.

In these terms the map $\Lambda$ from Section 3 is induced by the maps $\lambda_p : \Delta^p \times \Delta^{q_0 \ldots q_p} \to \Delta^{q+2p+1}$ given by

$$\begin{aligned}
\lambda_p(t, s^0, 0, \ldots, 0, s^p, 0) = \Big( & s_1^0(1 - t_1) + t_1, \ldots, s_{q_0}^0(1 - t_1) + t_1, t_1, t_1, \\
& s_1^1(t_1 - t_2) + t_2, \ldots, s_{q_1}^1(t_1 - t_2) + t_2, t_2, t_2, \\
& \cdots, \\
& s_1^{p-1}(t_{p-1} - t_p) + t_p, \ldots, s_{q_{p-1}}^{p-1}(t_{p-1} - t_p) + t_p, t_p, t_p, \\
& s_1^p t_p, \ldots, s_{q_p}^p t_p, 0 \Big).
\end{aligned}$$

For convenience, we drop $p$ in $\lambda_p(t)(s)$ and write $\lambda(t)(s)$. Next, let $\rho^{(i)} : \Delta^{q+2p+1} \to \Delta^{q_0 + \cdots + q_{i-1} + 2i-1}$ be the degeneracy map for $i = 1, \ldots, p$ defined by

$$\rho^{(i)} := \eta^{q_0 + \cdots + q_{i-1} + 2i-1} \circ \cdots \circ \eta^{q+2p}$$

deleting the last $q_i + \cdots + q_p + 2(p - i + 1)$ coordinates. So e.g.

$$\begin{aligned}
\rho^{(p)} \lambda(t)(s) = \Big( & s_1^0(1 - t_1) + t_1, \ldots, s_{q_0}^0(1 - t_1) + t_1, t_1, t_1, \\
& s_1^1(t_1 - t_2) + t_2, \ldots, s_{q_1}^1(t_1 - t_2) + t_2, t_2, t_2, \\
& \cdots, \\
& s_1^{p-1}(t_{p-1} - t_p) + t_p, \ldots, s_{q_{p-1}}^{p-1}(t_{p-1} - t_p) + t_p, t_p, t_p \Big),
\end{aligned}$$

where $\rho^{(p)} := \eta^{q - q_p + 2p - 1} \circ \cdots \circ \eta^{q+2p}$ is deleting the last $q_p + 2$ coordinates. With this notation, the maps $a_i : \Delta^p \times \Delta^{q_0 \ldots q_i} \to G$ defining the classifying map $m(\sigma)$ are given by

$$\begin{aligned}
a_p(t, s^0, 0, \ldots, s^p, 0) &= 1 \\
a_{p-1}(t, s^0, 0, \ldots, s^{p-1}, 0) &= v_{\sigma, d_{(p)}\sigma}(\rho^{(p)}(\lambda(t)(s)))^{-1}, \\
a_{p-2}(t, s^0, 0, \ldots, s^{p-2}, 0) &= v_{\sigma, \tilde{d}^{(2)}\sigma}(\rho^{(p-1)}(\lambda(t)(s)))^{-1} \\
&\vdots \\
a_1(t, s^0, 0, s^1, 0) &= v_{\sigma, d_{(2)\ldots(p)}\sigma}(\rho^{(2)}(\lambda(t)(s)))^{-1} \\
a_0(t, s^0, 0) &= v_{\sigma, d_{(1)\ldots(p)}\sigma}(\rho^{(1)}(\lambda(t)(s)))^{-1}.
\end{aligned}$$

Here the boundary operators used above are given as follows:

$$d_{(p)} : S_{q+2p+1} \to S_{q+2p-q_p-1}$$

is defined by $d_{(p)} := d_{q+2p-q_p} \circ \cdots \circ d_{q+2p+1}$, deleting $q_p + 2$ elements. On the other hand, in the formula $\tilde{d}^{(1)} = \hat{d}_{(p)} = d_{(p)}$. Let's denote

$$\tilde{d}^{(p-i)} = \hat{d}_{(i+1)} \circ \cdots \circ \hat{d}_{(p)}$$

$i = 0, \ldots, p-1$, which deletes the elements $(q_0 + \cdots + q_i + 2i - 1, \ldots, q + 2p + 1)$. It deletes $q_{i+1} + \cdots + q_p + 2(p-i) = q - (q_0 + \cdots + q_i) + 2(p-i)$ elements. Here

$$\hat{d}_{(i)} : S_{q+2i+1-\sum_0^{p-i-1} q_{p-j}} \to S_{q+2i-1-\sum_0^{p-i} q_{p-j}}$$

$i = 1, \ldots, p$. By using the equivalence relations on $m$ we can see that $m(d_{(i)}\sigma)$ is independent of $s^i$ for all $j$ different from $i$. Take $t'_0 = 0$ then $v_{\sigma, \tilde{d}^{(p)}\sigma}(1, \ldots, \ldots, 1, 1, 1)$ does not depend on $s^0$ where $j = 1 \neq 0 = i$.

(2) For given c.t.f.'s $v_\sigma$, we now have the map of realizations $\| \, |m| \, \| : \| \, |\bar{P}.S.| \, \| \to \| \, |P.BG| \, \|$ given by

$$\| \, |m| \, \|(t, s, \sigma) = (t, s, [(a_0, \ldots, a_p)]).$$

The associated bundle map is given as follows:

We have a bundle $F$ on $|S|$ by Proposition 7.9 and $|\bar{P}_p S.| \to |S.|$ is an epimorphism, so by pulling back we get a bundle $\bar{F} \to |\bar{P}_p S.|$, i.e.,

$$
\begin{array}{ccc}
\bar{F} & \longrightarrow & F \\
\downarrow & & \downarrow \\
|\bar{P}.S.| & \longrightarrow & |S.|
\end{array}
$$

Transition functions used to define the classifying map $\tilde{m}$ are taken from the bundle $F \to |S|$. Let's take $\sigma \in S_{q+2p+1}$ and there is a fibre at $(\lambda(t, s^0, 0 \ldots, s^p, 0), \sigma)$, by using the trivialization $\varphi_\sigma : F_\sigma \to \Delta^{q+2p+1} \times \sigma \times G$ and the projection on the last factor, we get $F_\sigma \to G$. Let's denote this composition by $\bar{\varphi}_\sigma(\tilde{f})$ where $\tilde{f} := (\lambda(t, s^0, 0, \ldots, s^p, 0), \sigma)$, $\tilde{f}_\sigma \in F_{(\lambda(t, s^0, 0, \ldots, s^p, 0), \sigma)}$, $\sigma \in S_{q+2p+1}$. On the other hand

$$\varphi_{d_{(p)}\sigma} : F_{d_{(p)}\sigma} \to \Delta^{q+2p-q_p-1} \times d_{(p)}\sigma \times G$$

gives us

$$\bar{\varphi}_{d_{(p)}\sigma} : F_{d_{(p)}\sigma} \to G.$$

By the definition,

$$\bar{\varphi}_\sigma(\bar{d}^{(p)}\tilde{f}_\sigma) := v_{\sigma, d_{(p)}\sigma}(\rho^{(p)}\lambda(t, s^0, 0, \ldots, s^p, 0)).\bar{\varphi}_{d_{(p)}\sigma}(\tilde{f})_{d_{(p)}\sigma},$$

where the compatible transition function is

$$v_{\sigma, d_{(p)}\sigma} : \Delta^{q+2p-q_p-1} \to G.$$

The last component in $\| \, |m| \, \|(t, s, \tilde{f}_\sigma)$ is defined via the trivialization $\varphi_\sigma(\tilde{f})$ which is $\bar{\varphi}_\sigma(\tilde{f})$. By using the compatible transition function $v_{\sigma, d_{(p)}\sigma}$ we find the $p$-th component as

$$v_{\sigma, d_{(p)}\sigma}(\rho^{(p)}\lambda(t, s^0, 0, \ldots, s^p, 0))^{-1}.\bar{\varphi}_\sigma(\tilde{f}).$$

We can apply the same method several times to get the other coordinates in $\|m\|(t, s, \tilde{f}_\sigma)$.

By the definition $PEG/SG = PBG$, $PEG = \|N\bar{G}\|$ and $\gamma : N\bar{G} \to NG$ we can identify $PBG = \|NG\|$. Then the required map $\bar{m}$ is

$$\bar{m}(t, s, \sigma) = [(a_0, \ldots, a_p)]. \qquad \square$$

In particular for a simplicial complex $K$ we get the following (c. f. [15])

**Corollary 8.2.** *(Phillips-Stone)*

(1) *A set of compatible transition functions $\{v_\sigma\}$ for $K$ a simplicial complex there is a natural prismatic map*

$$P.\operatorname{St}(K^s) \to PBG.$$

(2) *The induced map on geometric realization gives a classifying map for the bundle $F$ pulled back to $|\operatorname{St}(K)| \subseteq |K| \times |K|$.*

*Proof.* In the second part of Theorem 5.1, we have showed that $\bar{p} : \bar{P}K^s \to P\operatorname{St}K^s$ is an isomorphism. On the other hand in the previous proposition, we have defined the classifying map $m$. This is also valid when $S = K^s$. So the p.t.f. $v_\sigma$ will determine a natural map

$$m : P\operatorname{St}K^s \to PBG. \qquad \square$$

Furthermore $\pi_1 : P\operatorname{St}(K^s) \to K$ is a homotopy equivalence.

**Remark.** The point of the corollary is that there is a connection in the prismatic universal bundle in the simplicial sense (see [6]) which thus pulls back to a connection in the bundle over the star complex. We shall return to this elsewhere.

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