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A GENERALIZATION OF ABEL'S THEOREM AND THE ABEL–JACOBI MAP[1]

JOHAN L. DUPONT AND FRANZ W. KAMBER

Abstract. We generalize Abel's classical theorem on linear equivalence of divisors on a Riemann surface. For every closed submanifold $M^d \subset X^n$ in a compact oriented Riemannian n–manifold, or more generally for any d–cycle Z relative to a triangulation of X, we define a (simplicial) $(n-d-1)$–gerbe Λ_Z, the Abel gerbe determined by Z, whose vanishing as a Deligne cohomology class generalizes the notion of 'linear equivalence to zero'. In this setting, Abel's theorem remains valid. Moreover we generalize the classical Inversion Theorem for the Abel–Jacobi map, thereby proving that the moduli space of Abel gerbes is isomorphic to the harmonic Deligne cohomology; that is, gerbes with harmonic curvature.

Contents

1. Introduction

In this paper we shall expand on some beautiful ideas of Hitchin [15] and Chatterjee [2], generalizing the classical notion of linear equivalence of divisors and Abel's theorem about the existence of meromorphic functions with prescribed zeroes and poles on a compact Riemann surface (see Section 2). As is well-known, this problem is equivalent to the existence of a parallel section, for some complex connection,

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in the holomorphic line bundle of the divisor. In general, for a closed oriented Riemannian manifold X of dimension n, we replace the divisor by a cycle Z of arbitrary dimension $d,\ d = 0, \dots, n-1$ for a smooth triangulation of X.

In Section 4, we associate to Z an abelian *gerbe* Λ_Z which we call the *Abel gerbe* for Z, whose class $[\Lambda_Z]$ in the smooth Deligne cohomology $H^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$ only depends on Z. Two cycles are then defined to be *linearly equivalent*, if their Abel gerbes represent the same class in Deligne cohomology. This definition is in agreement with the definition in the classical situation. At this level of generality we prove in Section 5 Abel's theorem 5.3, characterizing linear equivalence of Abel gerbes in terms of period integrals (cf. Chatterjee [2], Theorem 6.4.2 for 2–gerbes associated to submanifolds of codimension 3).

Abel's theorem. *Let $Z = \partial\Gamma$, $\Gamma \in C_{d+1}(K)$. Then Z is linearly equivalent to zero, that is $[\Lambda_Z] = 0 \in H^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$, if and only if*

$$\int_\Gamma \theta \in \mathbb{Z}\ ,$$

for all harmonic $\theta \in \Omega^{d+1}(X)$ with integral periods.

Other well-known results from the theory of Riemann surfaces make sense in higher dimensions as well. Thus in Section 6, we introduce the *Picard torus* of Deligne classes represented by *topologically trivial flat* gerbes and the *Jacobi torus* which is the recipient of the period map. The former is analogous to the Picard variety of holomorphic line bundles of degree zero on a Riemann surface, in which case every holomorphic line bundle is associated to a divisor. The Jacobi torus is analogous to the Jacobi variety of a Riemann surface. We determine the moduli space $\mathcal{M}_d(X)$ of Abel gerbes in full generality, as well as the moduli space $\mathcal{M}^\circ_d(X)$ of topologically trivial Abel gerbes.

Prior to stating and proving our main theorem 6.14, we illustrate our method by a number of examples (Examples 6.7 to 6.13). Below, we quote Theorem 6.14.

Moduli theorem. *Let X be a compact connected oriented Riemannian manifold X of dimension $n \geq 2$ and let $d = 0, \dots, n-1$. Then*

(1) *The Picard map $\overline{\alpha}\colon \mathcal{M}^\circ_d(X) \to \mathrm{Pic}^{n-d-1}(X)$ is an isomorphism.*

(2) *The Abel–Jacobi map $\overline{J}\colon \mathcal{M}^\circ_d(X) \to \mathrm{Jac}^{d+1}(X)$ is an isomorphism.*

(3) *The mapping $\overline{\Lambda}\colon \mathcal{M}_d(X) \to \mathcal{H}^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$ is an isomorphism.*

(4) *Every equivalence class of $(n-d-1)$–gerbes in the harmonic Deligne cohomology $\mathcal{H}^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$, given by classes in $H^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$ whose curvature is harmonic, can be realized by a unique (up to linear equivalence) Abel gerbe.*

In the final Sections 7 and 8, we shall investigate the Abel gerbe associated to the fundamental cycle of an embedded closed submanifold $M \subset X$. In particular, we shall compare the restriction of this gerbe to M with the *characteristic gerbe* ([10]) called the *Euler gerbe*, which respresents the *Cheeger–Chern–Simons class* for the normal bundle with the Riemannian connection and is defined in terms of the Pfaffian polynomial. We prove in Theorem 7.1 that these two gerbes differ by a third canonical gerbe, called the *difference gerbe*. This is a topologically trivial gerbe whose curvature is the difference between the harmonic form representing the Poincaré dual of $[M] \in H_d(X)$ and a specific choice for the form representing the Thom class of the normal bundle.

For the construction of these gerbes it is important to use the representation of Deligne cohomology and gerbes by *simplicial differential forms* as developed in our previous paper [10]. For completeness, we recall in Section 2 the basic definitions and properties of these topics.

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2. Abel's Theorem on linear equivalence of divisors on a Riemann surface

For motivation, let us recall the classical Abel theorem. Let X be a compact Riemann surface and $\mathfrak{d} = \sum_{i=1}^{k} a_i p_i$, $a_i \in \mathbb{Z}$, $p_i \in X$ a divisor. A first necessary condition for finding a *meromorphic* function with *zeros* and *poles* exactly in $\{p_i\}$ of order a_i, is that the degree $\operatorname{Deg}\mathfrak{d} = \sum_i a_i = 0 \in \mathbb{Z}$; that is, there is chain Γ with $\partial\Gamma = \mathfrak{d}$.

Abel's theorem. *Suppose that* $\operatorname{Deg}(\mathfrak{d}) = 0$ *and* $\mathfrak{d} = \partial\Gamma$*, where* Γ *is a (smooth)* 1*–chain on* X*. Then* $\mathfrak{d}$ *admits a global meromorphic function, that is* $\mathfrak{d} \sim 0$*, if and only if*

$$\int_\Gamma \theta \in \mathbb{Z} \ ,$$

for every harmonic 1*–form* $\theta \in \mathcal{H}^1(X, \mathbb{Z})$ *with integral periods.*

The relationship with smooth connections in the holomorphic line bundle $\mathcal{L}(\mathfrak{d})$ for the divisor $\mathfrak{d}$ is given by the following Lemma.

Lemma 2.1. $\mathcal{L}(\mathfrak{d})$ *admits a non–vanishing holomorphic section; that is,* $\mathfrak{d} \sim 0$*, if and only if* $\mathcal{L}(\mathfrak{d})$ *admits a non–vanishing* C^∞*–section, which is parallel with respect to a suitable complex connection in* $\mathcal{L}(\mathfrak{d})$*.*

Proof. Recall that $\mathcal{L}(\mathfrak{d})$ is given by first choosing an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ and local solutions f_i on U_i. Then

$$g_{ij} = f_i/f_j \colon U_i \cap U_j \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

is a *Čech cocycle* defining a *holomorphic* line bundle $\mathcal{L}(\mathfrak{d})$. If $h_i \colon U_i \to \mathbb{C}^*$ defines a *holomorphic* section of $\mathcal{L}(\mathfrak{d})$ i.e. if $g_{ij}h_j = h_i$ on $U_i \cap U_j$ $\forall i, j \in I$, then

$$f_i/h_i = f_i/(g_{ij}h_j) = f_j/h_j$$

defines a *global meromorphic solution.* For finding $\{h_i\}$ we define a smooth connection in $\mathcal{L}(\mathfrak{d})$, i.e. a family $\omega_i \in \Omega^1(U_i)$, such that $g_{ij}^{-1}dg_{ij} = \omega_i - \omega_j$ on $U_i \cap U_j$ and we can arrange that $\omega_i = f_i^{-1}df_i$ away from small neighborhoods of $\{p_i\}$.

Now suppose $\mathcal{L}(\mathfrak{d})$ has a non–vanishing *parallel* C^∞–section $\{k_i\}$, i.e. a section satisfying

$$k_i^{-1}dk_i = \omega_i \quad \text{in } U_i,$$

then away from $\{p_i\}$ we have

$$d\log k_i = \omega_i = d\log f_i \ .$$

Hence $\log k_i$ is holomorphic away from p_i. But $\log k_i$ is smooth all through U_i so the singularity of $\log k_i$ is removable. Hence we can redefine k_i throughout U_i to give a holomorphic section. □

Our goal is to generalize these classical results to submanifolds $M^d \subset X^n$ of compact oriented Riemannian manifolds X^n, and more generally to cycles $Z \subset X$, by using the notion of the Abel gerbe.

3. Review of 'gerbes with connection' and simplicial gerbes

3.1. **Gerbes with connections.** Let X be a smooth manifold and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering. We assume that the covering $\mathcal{U}$ is good, i.e. all

$$U_{i_0\dots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$$

are contractible. We identify the circle group $U(1)$ with $\mathbb{R}/\mathbb{Z}$ via the exponential map; that is

$$U(1) = \text{circle group} \cong \mathbb{R}/\mathbb{Z}$$
$$\exp(2\pi i t) \leftrightarrow t$$

A *Hermitian ℓ–gerbe* is given by a cocycle in the Čech complex

$$\theta \in \check{C}^\ell(\mathcal{U}, \underline{\mathbb{R}/\mathbb{Z}});$$

that is, $\theta_{i_0\dots i_\ell}\colon U_{i_0\dots i_\ell} \to \mathbb{R}/\mathbb{Z}$ satisfying

$$0 \equiv \check{\delta}\theta_{i_0\dots i_\ell} = \sum_\nu (-1)^\nu \theta_{i_0\dots\check{i}_\nu\dots i_{\ell+1}} .$$

For $\ell = 1$, θ defines a line bundle.

We consider the modified Čech-deRham bi–complex:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\check{C}^2(\mathcal{U},\mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{C}^2(\mathcal{U},\underline{\mathbb{R}/\mathbb{Z}}) & \xrightarrow{d} & \check{C}^2(\mathcal{U},\underline{\Omega}^1) & \xrightarrow{d} & \check{C}^2(\mathcal{U},\underline{\Omega}^2) & \xrightarrow{d} & \cdots \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
\check{C}^1(\mathcal{U},\mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{C}^1(\mathcal{U},\underline{\mathbb{R}/\mathbb{Z}}) & \xrightarrow{d} & \check{C}^1(\mathcal{U},\underline{\Omega}^1) & \xrightarrow{d} & \check{C}^1(\mathcal{U},\underline{\Omega}^2) & \xrightarrow{d} & \cdots \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
\check{C}^0(\mathcal{U},\mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{C}^0(\mathcal{U},\underline{\mathbb{R}/\mathbb{Z}}) & \xrightarrow{d} & \check{C}^0(\mathcal{U},\underline{\Omega}^1) & \xrightarrow{d} & \check{C}^0(\mathcal{U},\underline{\Omega}^2) & \xrightarrow{d} & \cdots \\
 & & \uparrow\varepsilon^* & & \uparrow\varepsilon^* & & \uparrow\varepsilon^* & & \\
 & & \mathrm{Map}(X,\underline{\mathbb{R}/\mathbb{Z}}) & \xrightarrow{d} & \Omega^1(X) & \xrightarrow{d} & \Omega^2(X) & \xrightarrow{d} & \cdots
\end{array}$$

where the dotted lines indicate the total complex with differential $D = \check{\delta} + (-1)^p d$ on $\check{C}^p(\mathcal{U}, \Omega^*)$.

A *connection* in an ℓ-gerbe θ is a sequence $\omega = (\omega^0, \dots, \omega^\ell)$ in the Čech–deRham bi-complex

$$\omega^\nu \in \check{C}^\nu(\mathcal{U}, \Omega^{\ell-\nu}), \quad \nu = 0, \dots, \ell,$$

satisfying

$$\omega^\ell \equiv -\theta \bmod \mathbb{Z}, \quad \check\delta\omega^{\nu-1} + (-1)^\nu d\omega^\nu = 0, \quad \nu = 1, \dots, \ell \ .$$

In particular, we have $\check\delta(d\omega^0) = 0$, so that $d\omega^0$ is given by a global form F_ω, the *curvature* of (θ, ω); that is, we set

$$d\omega^0 \in \mathrm{Im}\big\{\varepsilon^* \colon \Omega^{\ell+1}(X) \to \check C^0(\mathcal{U}, \Omega^{\ell+1})\big\},$$

where $\varepsilon\colon \sqcup_i U_i \to X$ is the natural map and

$$F_\omega := (\varepsilon^*)^{-1}(d\omega^0) \in \Omega^{\ell+1}(X).$$

Definition 3.1. (1) Two gerbes with connection are *equivalent*, $(\theta_1, \omega_1) \sim (\theta_2, \omega_2)$, if $\omega_1 - \omega_2$ is a coboundary in

$$\big(\check C^*(\mathcal{U}, \Omega^*)/\check C^*(\mathcal{U}, \mathbb{Z}), D\big).$$

(2) $H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z})$, the set of equivalence classes $[\theta, \omega]$ of ℓ–gerbes with connection, is the smooth Deligne cohomology of X.
(3) $H^\ell(X, \mathbb{R}/\mathbb{Z})$ is the set of equivalence classes of ℓ--gerbes with *flat* connection; that is $F_\omega = 0$. Hence we have the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^\ell(X, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z}) & \xrightarrow{d_*} & \Omega^{\ell+1}_{\mathrm{cl}}(X, \mathbb{Z}) & \longrightarrow 0 \\ & & & [\theta, \omega] & \longmapsto & F_\omega & \end{array} \tag{3.1}$$

where $\Omega^{\ell+1}_{\mathrm{cl}}(X, \mathbb{Z})$ denotes the closed $(\ell+1)$–forms with *integral periods*.

Let us introduce the notation

$$H^{\ell+1}_{\mathcal{D}}(X) = \Omega^\ell(X)/d\Omega^{\ell-1}(X) \tag{3.2}$$

The elements $[\omega] \in H^{\ell+1}_{\mathcal{D}}(X)$ can be interpreted as equivalence classes of connections on the trivial ℓ–gerbe $\theta = 0$ by setting

$$\omega^0 = \varepsilon^*\omega \ , \qquad F_\omega = d\omega \ , \qquad \check\delta\omega^0 = 0 \ , \qquad \omega^1 = \dots = \omega^\ell = 0. \tag{3.3}$$

Thus $\iota(\omega) = (0; \varepsilon^*\omega, 0, \dots, 0)$ induces a well–defined mapping

$$\iota_* \colon H^{\ell+1}_{\mathcal{D}}(X) \to H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z}),$$

since $\iota(d\alpha) = D(0; \varepsilon^*\alpha, 0, \dots, 0)$. Clearly the connection is flat if and only if $F_\omega = d\omega = 0$, that is $[\omega] \in H^\ell(X, \mathbb{R})$.

We then have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & j_*H^\ell(X,\mathbb{Z}) & \xrightarrow{\cong} & \Omega^\ell_{\rm cl}(X,\mathbb{Z})/d\Omega^{\ell-1}(X) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^\ell(X,\mathbb{R}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X) & \longrightarrow & \Omega^\ell(X)/\Omega^\ell_{\rm cl}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & H^\ell(X,\mathbb{R}/\mathbb{Z}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) & \xrightarrow{d_*} & \Omega^{\ell+1}_{\rm cl}(X,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_*} & \overset{c}{\swarrow} & \downarrow & & \downarrow & & \\
 & & H^{\ell+1}(X,\mathbb{Z}) & \underset{\cong}{\xleftarrow{\check\delta_*}} & H^\ell(X,\underline{\mathbb{R}/\mathbb{Z}}) & & j_*H^{\ell+1}(X,\mathbb{Z}) & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle j_*} & & \downarrow & \swarrow & \downarrow & & \\
 & & H^{\ell+1}(X,\mathbb{R}) & & 0 & & 0 & &
\end{array}
\tag{3.4}
$$

Remarks 3.2. The diagram (3.4) incorporates many properties of our construction:

(1) The second exact row follows from the definition (3.2).

(2) The third exact row is (3.1), with d_* being the curvature.

(3) The exact column on the left is the Bockstein sequence for

$$0 \to \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{\rho} \mathbb{R}/\mathbb{Z} \to 0.$$

Note that the image of β_* is the (finite) torsion subgroup of $H^{\ell+1}(X,\mathbb{Z})$; that is, β_* induces an isomorphism

$$H^\ell(X,\mathbb{R}/\mathbb{Z})/\overline{\rho}_*\left(H^\ell(X,\mathbb{R})/j_*H^\ell(X,\mathbb{Z})\right) \cong \mathrm{Tor}_{\mathbb{Z}}(H^{\ell+1}(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}) \subseteq H^{\ell+1}(X,\mathbb{Z}).$$

(4) The map c is the characteristic class $\check\delta_*[\theta] = -[\check\delta\omega^l]$ of the gerbe $[\theta,\omega]$ (the Douady–Dixmier invariant); it is equivalent to the last map $[\theta,\omega] \mapsto [\theta]$ in the middle exact column, which simply forgets the connection. These maps are surjective, since every (naked) gerbe $[\theta] \in H^\ell(X,\underline{\mathbb{R}/\mathbb{Z}})$ admits a connection.

(5) The image of ι_* ; that is, the equivalence classes of trivial gerbes with connection, is given exactly by the the kernel of the characteristic class c, so we may call these gerbes *topologically trivial.*

(6) It follows that the Deligne cohomology is given by an exact sequence (i.e. the middle exact column in (3.4))

$$0 \to \Omega^\ell(X)/\Omega^\ell_{\rm cl}(X,\mathbb{Z}) \underset{\subseteq}{\xrightarrow{\overline{\iota}_*}} H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) \xrightarrow{c} H^{\ell+1}(X,\mathbb{Z}) \to 0\ . \tag{3.5}$$

(7) The commutativity of the diagram involving the slanted arrows expresses the fact that the characteristic class of a gerbe determines the deRham class of the curvature in $H^{\ell+1}(X,\mathbb{R})$; that is, $j_*c([\theta,\omega]) = [F_\omega]$.

3.2. **Simplicial forms and gerbes.** In this section, we recall the reformulation of Deligne cohomology in terms of simplicial deRham theory [10]. For simplicial deRham theory we refer to [6], [7].

Consider the standard simplex $\Delta^p \subseteq \mathbb{R}^{p+1}$

$$\Delta^p = \{(t_0, \dots, t_p) \mid \textstyle\sum_i t_i = 1,\ t_i \geq 0\}$$

with face maps $\varepsilon^i \colon \Delta^{p-1} \to \Delta^p$, $i = 0, \dots, p$, given by

$$\varepsilon^i(t_0, \dots, t_{p-1}) = (t_0, \dots, 0, \dots, t_{p-1}),\ (t_0, \dots, t_{p-1}) \in \Delta^{p-1}.$$

The open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X determines a *simplicial manifold* $N\mathcal{U}$

$$N\mathcal{U}(p) = \bigsqcup_{(j_0,\dots,j_p)} U_{j_0\dots j_p}, \quad p = 0, 1, \dots$$

with *face operators* $\varepsilon_j \colon N\mathcal{U}(p) \to N\mathcal{U}(p-1)$, $i = 0, \dots, p$, given by

$$U_{j_0\dots j_p} \hookrightarrow U_{j_0\dots\check{j}_i\dots j_p}$$

The *fat realisation* is

$$\|N\mathcal{U}\| = \bigsqcup_p \Delta^p \times N\mathcal{U}(p)/\sim,$$

with identifications $(t, \varepsilon_i x) \sim (\varepsilon^i t, x)$, $t \in \Delta^{p-1}$, $x \in N\mathcal{U}(p)$.

Definition 3.3. A *simplicial* k–form ω on $N\mathcal{U}$ is a sequence $\omega^{(p)} \in \Omega^k\big(\Delta^p \times N\mathcal{U}(p)\big)$ satisfying

$$(\varepsilon^i \times \mathrm{id})^*\omega^{(p)} = (\mathrm{id} \times \varepsilon_i)^*\omega^{(p-1)}, \quad i = 0, \dots, p,\ \forall p\ ,$$

and we denote by $\Omega^k(\|N\mathcal{U}\|)$ the set of simplicial k-forms.

Theorem 3.4 (deRham)**.** [6], [7] *There are quasi–isomorphisms (inducing isomorphisms in cohomology)*

$$\mathcal{I}_\Delta \colon \Omega^*(\|N\mathcal{U}\|) \to \check{C}(\mathcal{U}, \Omega^*),$$

given by

$$\mathcal{I}_\Delta(\omega) = (\omega^\nu), \qquad \omega^\nu = \int_{\Delta^\nu} \omega^{(\nu)};$$

and

$$\varepsilon^* \colon \Omega^*(X) \to \Omega^*(\|N\mathcal{U}\|),$$

induced by the natural map

$$\varepsilon \colon \Delta^p \times N\mathcal{U}(p) \to N\mathcal{U}(p) \to X.$$

We also need the following

Definition 3.5. $\omega \in \Omega^k(\|N\mathcal{U}\|)$ is *integral* if

(1) $\omega^{(p)} = \sum \alpha_{i_0,\dots,i_k}(t) dt_{i_1} \wedge \dots \wedge dt_{i_k}$,
(2) $\mathcal{I}_\Delta(\omega) \in \check{C}^*(\mathcal{U}, \mathbb{Z}) \subseteq \check{C}^*(\mathcal{U}, \Omega^0)$.

We denote by $\Omega^*_{\mathbb{Z}}(\|N\mathcal{U}\|) \subseteq \Omega^*(\|N\mathcal{U}\|)$ the subcomplex of integral forms.

Remark 3.6. Note that we now have that $\mathcal{I}_\Delta \colon \Omega^*_{\mathbb{Z}}(\|N\mathcal{U}\|) \to \check{C}^*(\mathcal{U}, \mathbb{Z})$ is also a quasi–isomorphism.

Theorem 3.7. [10] *Every ℓ-gerbe with connection is up to equivalence determined by a simplicial form $\Lambda \in \Omega^\ell(\|N\mathcal{U}\|)$ satisfying*

$$d\Lambda = \varepsilon^*\alpha - \beta, \quad \text{with } \alpha \in \Omega^{\ell+1}(X),\ \beta \in \Omega^{\ell+1}_{\mathbb{Z}}(\|N\mathcal{U}\|). \tag{3.6}$$

In fact

$$\omega^\nu = \int_{\Delta^\nu} \Lambda^\nu, \quad \nu = 0, \dots, \ell, \quad -\theta = \omega^\ell,$$

and α is the curvature.

Equivalently, we have

Theorem 3.8. *Every element in $H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z})$ is represented by a unique class $[\Lambda]$ in*

$$\Omega^\ell(\|N\mathcal{U}\|)/\big(\Omega^\ell_{\mathbb{Z}}(\|N\mathcal{U}\|) + d\Omega^{\ell-1}(\|N\mathcal{U}\|)\big),$$

satisfying (3.6) *above.*

Proof. Let $H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z})'$ be the subgroup of such classes $[\Lambda]$ satisfying (3.6). Then there is a diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^\ell\big(\Omega^*(\|N\mathcal{U}\|)/\Omega^*_{\mathbb{Z}}(\|N\mathcal{U}\|)\big) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z})' & \longrightarrow & \Omega^{\ell+1}_{\text{cl}}(X, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \cong \downarrow \mathcal{I}_\Delta & & \downarrow \mathcal{I}_\Delta & & \big| \text{id} & & \\
0 & \longrightarrow & H^\ell(X, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z}) & \longrightarrow & \Omega^{\ell+1}_{\text{cl}}(X, \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

The vertical map on the left is an isomorphism by deRham's Theorem. Hence Theorem 3.8 follows from the 5-lemma. □

4. Abel gerbes associated to submanifolds and cycles

Classically on a Riemann surface two divisors $\mathfrak{d}_1, \mathfrak{d}_2$ are called *linearly equivalent* if $\mathfrak{d}_1 - \mathfrak{d}_2$ is the divisor of a meromorphic function. We have seen that this is equivalent to finding a parallel section for a suitable connection in the line bundle $\mathcal{L}(\mathfrak{d}_1 - \mathfrak{d}_2)$. Using gerbes we can generalise that to higher dimensions as follows.

Let $X = X^n$ be a *compact connected oriented* manifold, $\partial X = \emptyset$, with Riemannian metric. Choose a *smooth triangulation*, i.e. a homeomorphism to a finite simplicial complex $X \approx |K|$, such that the homeomorphism is a diffeomorphism on each simplex. Let

$$Z \in C_d(K)$$

be a *cycle*, and let $|Z| \subseteq |K|$ be the subcomplex consisting of all simplices of Z and their faces. Also choose a good covering $\mathcal{U}$ of X, finer than the covering by *open stars* of K. Let

$$\eta_Z \in \mathcal{H}^{n-d}(X, \mathbb{Z}) \subset \Omega^{n-d}(X) \quad , \quad \beta_Z \in \Omega^{n-d}_{\mathbb{Z}}(\|N\mathcal{U}\|).$$

both represent the *Poincaré dual* of $[Z] \in H_d(X)$; η_Z is a *harmonic* form, and β_Z is an *integral* form with $\operatorname{supp} \beta_Z \subseteq \|N\mathcal{U}_Z\|$, $\mathcal{U}_Z = \{U_i \mid U_i \cap Z \neq \emptyset\}$. Here $\mathcal{H}^\ell(X, \mathbb{Z}) \subset \mathcal{H}^\ell(X)$ denote the harmonic forms, respectively the integral lattice of harmonic forms.

Following Hitchin [15] , we can solve the distributional Poisson equation (in $\Omega^d(X)'$):

$$\Delta H_Z = \eta_Z - \delta_Z, \tag{4.1}$$

where Δ is the Laplace operator, η_Z the harmonic form dual to $[Z]$ and δ_Z the Dirac measure associated to Z; that is,

$$\eta_Z(\psi) = \int_X \eta_Z \wedge \psi \quad , \quad \delta_Z(\psi) = \int_Z \psi, \ \psi \in \Omega^d(X).$$

H_Z is uniquely defined up to a global harmonic $(n-d)$–form, and is smooth outside $|Z|$. Since η_Z and δ_Z represent the same cohomology class, we get from the deRham–Hodge decomposition

$$\Delta H_Z = d * d * H_Z,$$

where $*$ is the Hodge $*$–operator. Setting $F_Z = * \; d * H_Z$, it follows that F_Z is uniquely defined by Z and we have

$$\Delta H_Z = dF_Z = \eta_Z - \delta_Z. \tag{4.2}$$

In particular, $F = F_Z|_{X-|Z|} = * \; d * H_Z|_{X-|Z|}$ is smooth and

$$dF = \eta_Z|_{X-|Z|}. \tag{4.3}$$

Theorem 4.1. *There is a canonical Deligne class $[\Lambda_Z] \in H^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$, such that $\Lambda_Z \in \Omega^{n-d-1}(\|N\mathcal{U}\|)$ satisfies:*

(1) *$d\Lambda_Z = \varepsilon^*\eta_Z - \beta_Z$. Thus the curvature of Λ_Z is the integral harmonic form $\eta_Z \in \mathcal{H}^{n-d}(X, \mathbb{Z})$ and the characteristic class of Λ_Z is the integral class $[\beta_Z] \in H^{n-d}(X, \mathbb{Z})$.*

(2) *$F = F_Z|_{X-|Z|} \in \Omega^{n-d-1}(X - Z)$ is smooth, satisfying $dF = \eta_Z|_{X-|Z|}$.*

(3) *$\Lambda_{Z|W} = \varepsilon^* F$, where $W = \|N\mathcal{U}\| - \|N\mathcal{U}_Z\|$.*

Proof. Let

$$K_0 = \{a_0, \ldots, a_m, \ldots, a_N\}$$
$$Z_0 = \{a_0, \ldots, a_m\}$$

be the vertices of K and the subcomplex $|Z|$ respectively. Then the coverings of $|K|$ and $|Z|$ respectively are $\mathcal{U} = \{U_i \mid i = 0, \ldots, N\}$ and $\mathcal{U}_Z = \{U_i \mid i = 0, \ldots, m\}$, where $U_i = \mathrm{Star}(a_i)$. Let $V = \bigcup_{i=0}^m U_i$, which is a regular neighborhood of $|Z|$. Then by Lefschetz and Poincaré duality we have a commutative diagram

$$\begin{array}{ccccc} H^{n-d}(\overline{V}, \partial\overline{V}) & \xrightarrow{\cong} & H^{n-d}(X, X-|Z|) & \xrightarrow{\cong} & H_d(|Z|) \\ \downarrow\cong & & \downarrow & & \downarrow \\ H^{n-d}(X, X-V) & \longrightarrow & H^{n-d}(X) & \xrightarrow{\cong} & H_d(X). \end{array}$$

It follows, as claimed above, that the Poincaré dual of $[Z] \in H_d(X)$ is represented in $H^{n-d}(X) \cong H^{n-d}(\|\mathcal{U}\|)$ by an integral simplicial form $\beta_Z \in \Omega^{n-d}_{\mathbb{Z}}(\|N\mathcal{U}\|)$ with $\mathrm{supp}(\beta_Z) \subseteq \|N\mathcal{U}_Z\|$.

For the construction of Λ_Z, we first define the following simplicial forms

$$\eta_0, \ \eta_1, \ \eta_2 \in \Omega^{n-d}(\|N\mathcal{U}\|)$$

and $F_1 \in \Omega^{n-d-1})(\|N\mathcal{U}\|)$. They are given on $\Delta^p \times U_{i_0} \cap \cdots \cap U_{i_p}$ respectively by the forms:

$$(\eta_0)_{i_0\ldots i_p} = \sum_{i_s \le m} t_{i_s}\ \eta_Z \quad , \quad (\eta_1)_{i_0\ldots i_p} = \sum_{i_s > m} t_{i_s}\ \eta_Z$$
$$(\eta_2)_{i_0\ldots i_p} = \sum_{i_s > m} dt_{i_s} \wedge F = -\sum_{i_s \le m} dt_{i_s} \wedge F,$$
$$(F_1)_{i_0\ldots i_p} = \sum_{i_s > m} t_{i_s} \wedge F.$$

Notice that both η_2 and F_1 vanish on $\Delta^p \times U_{i_0\ldots i_p} \cap |Z|$, since $U_{i_0\ldots i_p} \cap |Z| \neq \emptyset$ only if all $i_s \le m$. From these formulas, we clearly have

$$dF_1 = \eta_1 + \eta_2 \quad , \quad \eta_Z = \eta_0 + \eta_1,$$
$$\eta_Z = (\eta_0 - \eta_2) + dF_1,$$

The second equation implies that $d(\eta_0 - \eta_2) = 0$. Furthermore, by construction

$$\mathrm{supp}(\eta_0)\ ,\ \mathrm{supp}(\eta_2) \subset \|N\mathcal{U}_Z\|.$$

It follows that both β_Z and $\eta_0 - \eta_2$ lie in $\Omega^{n-d}(\|N\mathcal{U}\|)$, both have support in $\|N\mathcal{U}_Z\|$ and both represent the Lefschetz dual of $[Z] \in H_d(|Z|)$ in $H^{n-d}(\|N\mathcal{U}\|, \|N\mathcal{U}\| - \|N\mathcal{U}_Z\|) \cong H^{n-d}(X, X - V)$. Hence there is a simplicial form $\gamma \in \Omega^{n-d-1}(\|N\mathcal{U}\|)$, also with $\mathrm{supp}(\gamma) \subseteq \|N\mathcal{U}_Z\|$, such that $\eta_0 - \eta_2 = \beta_Z + d\gamma$. Now we define $\Lambda_Z = \gamma + F_1$ so that we have

$$\begin{aligned} \Lambda_Z &= \gamma + F_1 \in \Omega^{n-d-1}(\|N\mathcal{U}\|), \\ d\Lambda_Z &= d\gamma + dF_1 = \eta_Z - \beta_Z. \end{aligned} \tag{4.4}$$

We now must show that the class of Λ_Z in Deligne cohomology $H^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$ depends only on Z. Recalling that F and hence F_1 are uniquely defined by the Poisson equation (4.2), let β' be another integral form representing the Poincaré dual of $[Z]$ and suppose γ' satisfies the same properties as γ relative to β'; in particular $\eta_0 - \eta_2 = \beta' + d\gamma'$. Then $d(\gamma - \gamma') = (\beta' - \beta_Z) = d\kappa$; that is $d(\gamma - \gamma' - \kappa) = 0$, with $\kappa \in \Omega^{n-d-1}_{\mathbb{Z}}(\|N\mathcal{U}\|)$ integral and all forms γ, γ', κ having support in $\|N\mathcal{U}_Z\|$. But since $H^{n-d-1}(X, X - |Z|) \cong H_{d+1}(|Z|) = 0$, we have $\gamma - \gamma' - \kappa = d\tau$, for $\tau \in \Omega^{n-d-2}(\|N\mathcal{U}\|)$ also with support in $\|N\mathcal{U}_Z\|$. Thus we have from (4.4)

$$\gamma' + F_1 = \gamma + F_1 - (\kappa + d\tau)$$
$$\Lambda'_Z = \Lambda_Z - (\kappa + d\tau).$$

By Theorem 3.8, this shows that the equivalence class $[\Lambda_Z]$ is well–defined and we get

$$[\Lambda'_Z] = [\Lambda_Z] \in H^{n-d}_{\mathcal{D}}(X, \mathbb{Z}). \tag{4.5}$$

The properties of Λ_Z stated in (1) to (3) are clear from the construction. □

Note that from the preceding proof, we have

$$\Lambda_Z|_{\|N\mathcal{U}_Z\|} = \gamma|_{\|N\mathcal{U}_Z\|}. \tag{4.6}$$

Definition 4.2. The Deligne cohomology class

$$[\Lambda_Z] \in H^{n-d}_{\mathcal{D}}(X, \mathbb{Z}) \tag{4.7}$$

we shall call the *Abel gerbe* associated to the cycle Z.

Remark 4.3. In particular, if $M = M^d \subset X$ is a closed oriented submanifold, we choose a triangulation K of X, such that $M^d \approx |L|$, where L is a subcomplex of K. Then there is a canonical simplicial cycle $Z_M \in C_d(L) \subseteq C_d(K)$, such that $|Z_M| = |L| \subseteq |K|$ and Z_M represents the fundamental class $[M] \in H_d(M) \cong \mathbb{Z}$. Viewed as a cycle on $X \approx |K|$, $Z_M \in C_d(K)$ represents the image of $[M]$ under the homomorphism $H_d(M) \cong \mathbb{Z} \to H_d(X)$, also denoted by $[M]$; that is, we have $[Z_M] = [M] \in H_d(X)$. Then we put $\Lambda_M = \Lambda_{Z_M}$, so that

$$[\Lambda_M] \in H^{n-d}_{\mathcal{D}}(X, \mathbb{Z}) \tag{4.8}$$

is well–defined, with $d\Lambda_M = \varepsilon^* \eta_M - \beta_M$ having the obvious meaning, namely $\eta_M = \eta_{Z_M}$ and $\beta_M = \beta_{Z_M}$.

5. Linear equivalence of cycles and Abel's Theorem

For X a Riemann surface and $Z = \mathfrak{d}$ a divisor as in Section 2, the Abel gerbe is the associated holomorphic line bundle $\mathcal{L}(\mathfrak{d})$ with the complex connection given by the holomorphic structure. In this case, by Lemma 2.1 Z has a meromorphic solution, i.e. it is linearly equivalent to zero, if and only if $[\Lambda_Z] = 0$ in $H^2_{\mathcal{D}}(X, \mathbb{Z})$. Motivated by this, we introduce the following definition of linear equivalence for cycles.

Definition 5.1. Two cycles $Z_1, Z_2 \in C_d(K)$ are called *linearly equivalent* if

$$[\Lambda_{Z_1 - Z_2}] = [\Lambda_{Z_1}] - [\Lambda_{Z_2}] = 0 \in H^{n-d}_{\mathcal{D}}(X, \mathbb{Z}). \tag{5.1}$$

Remark 5.2. If $[\Lambda_Z] = 0$ then in particular $\eta_Z = 0$ and $[\beta_Z] = 0$ in $H^{n-d}(X, \mathbb{Z})$, that is, Z is homologous to zero.

Theorem 5.3 (Abel's Theorem). *Let $Z = \partial\Gamma$, $\Gamma \in C_{d+1}(K)$. Then Z is linearly equivalent to zero, if and only if*

$$\int_\Gamma \theta \in \mathbb{Z} \ ,$$

for all harmonic $\theta \in \mathcal{H}^{d+1}(X, \mathbb{Z})$ with integral periods.

For the proof, we again solve the distributional equation (4.1) with $\eta_Z = 0$:

$$\Delta H_Z = -\delta_Z = -d\delta_\Gamma, \tag{5.2}$$

where $\delta_\Gamma(\psi) = \int_\Gamma \psi, \psi \in \Omega^{d+1}(X)$. Hence for $F_{\partial\Gamma} = F_Z = * \, d * H_Z$ as before, we get

$$\Delta H_Z = dF_{\partial\Gamma} = -d\delta_\Gamma,$$

and

$$d(F_{\partial\Gamma} + \delta_\Gamma) = 0,$$

so that by Hodge theory for currents

$$F_{\partial\Gamma} + \delta_\Gamma = \alpha_\Gamma + dT, \tag{5.3}$$

for a *harmonic* form $\alpha_\Gamma \in \mathcal{H}^{n-d-1}(X)$ and a $(n-d-2)$–current $T \in \Omega^{d+2}(X)'$. Note that α_Γ is smooth by elliptic regularity.

We shall first prove the following theorem.

Theorem 5.4. *For $Z = \partial\Gamma$, the simplicial gerbe Λ_Z and the harmonic form α_Γ have the following properties:*

(1) *As simplicial forms, we have*

$$\Lambda_Z = \Lambda_{\partial\Gamma} \equiv \varepsilon^*\alpha_\Gamma \mod \left(\Omega^{n-d-1}_{\mathbb{Z}}(\|N\mathcal{U}\|) + d\Omega^{n-d-2}(\|N\mathcal{U}\|)\right); \tag{5.4}$$

that is, the simplicial form Λ_Z is given by the global harmonic form α_Γ.

(2) *There exists an integral form $\kappa \in \Omega^{n-d-1}_{\mathbb{Z}}(\|N\mathcal{U}\|)$ with support in a regular neighborhood V_Γ of $|\Gamma|$, such that for all harmonic $(d+1)$–forms with integral periods $\theta \in \mathcal{H}^{d+1}(X,\mathbb{Z})$, we have*

$$\int_{[X]} (\Lambda_Z + \kappa) \wedge \varepsilon^*\theta \equiv \int_\Gamma \theta \mod \mathbb{Z}. \tag{5.5}$$

(3) *If $Z = \partial\Gamma = \partial\Gamma'$, then $\zeta = \alpha_{\Gamma'} - \alpha_\Gamma \in \mathcal{H}^{n-d-1}(X,\mathbb{Z})$; that is, ζ is a harmonic form with integral periods. Hence, $[\alpha_\Gamma]$ is well-defined in the Picard torus* $\mathrm{Pic}^{n-d-1}(X) = \mathcal{H}^{n-d-1}(X)/\mathcal{H}^{n-d-1}(X,\mathbb{Z})$ *in* (6.9).

Proof. First notice that since $F_Z = * \, d * H_Z$ and θ is harmonic, we get from (5.3)

$$\int_X \alpha_\Gamma \wedge \theta = \langle F_Z + \delta_\Gamma, \theta\rangle = \langle \delta_\Gamma, \theta\rangle = \int_\Gamma \theta. \tag{5.6}$$

This shows that (1) and (2) are equivalent.

For the proof of (2), we let $V_\Gamma = \bigcup\{U_i \in \mathcal{U}_\Gamma\}$, where $\mathcal{U}_\Gamma$ is the set of open sets $U_i \in \mathcal{U}$ intersecting Γ, so that V_Γ is a regular neighborhood of $|\Gamma|$. Since formula (5.5) is additive in Γ, we can without loss of generality assume that Γ consists of a single simplex and that V_Γ is contractible. Therefore we can assume that $\theta|_{V_\Gamma} = d\nu$ for some $\nu \in \Omega^d(V_\Gamma)$. From the formulas for integration of simplicial forms (cf. Dupont–Kamber [10] and Dupont–Ljungmann [11]), together with the construction of Λ_Z in the proof of Theorem 4.1, we now get

$$\begin{aligned}
\langle F_Z + \delta_\Gamma, \theta\rangle &= \int_{[X-V_\Gamma]} F_Z \ \wedge\theta + \langle (F_Z|_{\overline{V}_\Gamma} + \delta_\Gamma), d\nu\rangle \\
&= \int_{[X-V_\Gamma]} \Lambda_Z \wedge \varepsilon^*\theta - \int_{[\partial\overline{V}_\Gamma]} \Lambda_Z \wedge \varepsilon^*\nu \\
&= \int_{[X-V_\Gamma]} \Lambda_Z \wedge \varepsilon^*\theta + \int_{[\overline{V}_\Gamma]} d\kappa \wedge \varepsilon^*\nu + \int_{[\overline{V}_\Gamma]} \Lambda_Z \wedge \varepsilon^*\theta \\
&= \int_{[X]} \Lambda_Z \wedge \varepsilon^*\theta + \int_{[\overline{V}_\Gamma]} \kappa \wedge \varepsilon^*\theta + \int_{[\partial\overline{V}_\Gamma]} \kappa \wedge \varepsilon^*\nu \\
&= \int_{[X]} (\Lambda_Z + \kappa) \wedge \varepsilon^*\theta.
\end{aligned} \tag{5.7}$$

Here we have used that, since $Z = \partial\Gamma \sim 0$, we have $\eta_Z = 0$ and $\beta_Z = -d\kappa$ and hence $d\Lambda_Z = d\kappa$ for some integral simplicial form κ with support in V_Γ. We used also the simplicial Stokes' theorem [11] to see that $\int_{[\overline{V}_\Gamma]} d(\kappa \wedge \varepsilon^*\nu) = \int_{[\partial\overline{V}_\Gamma]} \kappa \wedge \varepsilon^*\nu = 0$, since κ vanishes on $\partial\overline{V}_\Gamma$. Equations (5.6) and (5.7) now prove (2).

For the proof of (3), let $Z = \partial\Gamma = \partial\Gamma'$. Then $\partial(\Gamma' - \Gamma) = 0$ and $Z' = \Gamma' - \Gamma$ is an integral $(d+1)$–cycle. Equation (5.3) implies $\delta_{Z'} = \delta_{\Gamma'} - \delta_\Gamma = (\alpha_{\Gamma'} - \alpha_\Gamma) + d(T' - T)$.

So $\zeta = \alpha_{\Gamma'} - \alpha_\Gamma$ satifies $\delta_{Z'} = \zeta + d(T' - T)$. Since Z' is an integral cycle, ζ must be an integral harmonic form $\zeta \in \mathcal{H}^{n-d-1}(X, \mathbb{Z})$. □

Abel's Theorem 5.3 is now a consequence of the following Corollary to Theorem 5.4.

Corollary 5.5. *For $Z = \partial\Gamma$ as above, the following statements are equivalent:*
(1) $[\Lambda_Z] = 0$ *in* $H^{n-d}_{\mathcal{D}}(X, \mathbb{Z})$*;*
(2) *For all harmonic $(d+1)$–forms θ with integral periods; that is, $\theta \in \mathcal{H}^{d+1}(X, \mathbb{Z})$, we have*

$$\int_\Gamma \theta \in \mathbb{Z}. \tag{5.8}$$

(3) *There exists Γ_0 with $\partial\Gamma_0 = Z$, such that*

$$F_{\partial\Gamma_0} + \delta_{\Gamma_0} = dT_0,$$

where $F_Z = F_{\partial\Gamma_0}$ is given as before. By (5.3)*, we have $\alpha_{\Gamma_0} = 0$.*

Proof. By Theorem 5.4 $[\Lambda_Z]$ is represented in $H^{n-d-1}(X, \mathbb{R})$ by the harmonic form α_Γ. Hence (1) and (2) are equivalent to $\alpha_\Gamma \equiv 0 \mod H^{n-d-1}(X, \mathbb{Z})$. From Theorem (5.4) (3), we know that $[\alpha_\Gamma] = 0 \in \mathcal{H}^{n-d-1}(X)/\mathcal{H}^{n-d-1}(X, \mathbb{Z})$. By changing Γ by a cycle, we can make $\alpha_\Gamma = 0$. This proves that (3) is equivalent to (1) and (2). □

Remark 5.6. Notice that $F_{\partial\Gamma}|_{X-|Z|}$ is harmonic by (4.3) and $F_{\partial\Gamma}$ is thus analogous to a meromorphic solution in the classical Abel Theorem.

6. Moduli spaces

In this section, we need to enlarge the chain complex $C_*(K)$ with respect to a smooth triangulation K of X; that is $X \approx |K|$, which was introduced at the beginning of Section 4. Therefore we look at the limit complex

$$C_*(X) = \varinjlim_K C_*(K) \subset \mathcal{S}_*(X), \tag{6.1}$$

taking into account the inclusions of chain complexes $C_*(K) \subseteq C_*(K')$ where K' is a subtriangulation of K, e.g. barycentric subtriangulations. Obviously, we can view $C_*(X)$ as a subcomplex of the singular complex $\mathcal{S}_*(X)$ of X. Then $C_*(K) \subseteq C_*(K') \subset \mathcal{S}_*(X)$ induces isomorphisms in homology, so that we have canonical isomorphisms

$$H_*(C_*(X)) \cong \varinjlim_K H_*(K) \cong H_*(X).$$

As the construction of the Abel gerbe in Section 4 involves deRham–Hodge theory on the compact oriented Riemannian manifold X, we need now to better understand the terms in diagram (3.4) for the Deligne cohomology in view of the deRham–Hodge decomposition of forms on X:

$$\Omega^\ell(X) \cong \mathcal{H}^\ell(X) \oplus d\Omega^{\ell-1}(X) \oplus d^*\Omega^{\ell+1}(X). \tag{6.2}$$

We recall that $\mathcal{H}^\ell(X, \mathbb{Z}) \subset \mathcal{H}^\ell(X)$ denotes the harmonic forms, respectively the integral lattice of harmonic forms. Further, the sum decompositions in (6.2) and the following formulas are orthogonal. Thus the deRham–Hodge decomposition (6.2) implies that

$$H^{\ell+1}_{\mathcal{D}}(X) \cong \mathcal{H}^\ell(X) \oplus d^*\Omega^{\ell+1}(X) \qquad , \qquad \Omega^\ell(X)/\Omega^\ell_{\mathrm{cl}}(X) \cong d^*\Omega^{\ell+1}(X) \tag{6.3}$$

and also

$$\Omega^\ell_{\rm cl}(X,\mathbb{Z}) \cong \mathcal{H}^\ell(X,\mathbb{Z}) \oplus dd^*\Omega^\ell(X). \tag{6.4}$$

This implies

$$j_*H^\ell(X,\mathbb{Z}) \cong \Omega^\ell_{\rm cl}(X,\mathbb{Z})/dd^*\Omega^\ell(X) \cong \mathcal{H}^\ell(X,\mathbb{Z}), \tag{6.5}$$

as well as

$$\Omega^\ell(X)/\Omega^\ell_{\rm cl}(X,\mathbb{Z}) \cong \mathcal{H}^\ell(X)/\mathcal{H}^\ell(X,\mathbb{Z}) \oplus d^*\Omega^{\ell+1}(X). \tag{6.6}$$

Remarks 6.1. This has the following consequences for the diagram (3.4):

(1) By (6.3), the right arrow in the second exact row is of the form

$$H^{\ell+1}_{\mathcal{D}}(X) \cong \mathcal{H}^\ell(X) \oplus d^*\Omega^{\ell+1}(X) \to \Omega^\ell(X)/\Omega^\ell_{\rm cl}(X) \cong d^*\Omega^{\ell+1}(X), \tag{6.7}$$

and is given by orthogonal projection to the second summand. Here the infinite dimensional part $d^*\Omega^{\ell+1}(X)$ consists of topologically trivial gerbes of the form $\omega_0 = d^*\alpha$ whose curvature $d\omega_0 = dd^*\alpha$ uniquely determines $\omega_0 = d^*\alpha$.

(2) Using (6.3), (6.5), the kernel of ι_* are the harmonic forms $\mathcal{H}^\ell(X,\mathbb{Z})$ with integral periods. Thus the image of ι_* contains the torus

$$\begin{array}{ccc} H^\ell(X,\mathbb{R})/j_*H^\ell(X,\mathbb{Z}) & \xrightarrow{\cong} & \mathcal{H}^\ell(X)/\mathcal{H}^\ell(X,\mathbb{Z}) \\ {\scriptstyle\bar\rho_*}\big\downarrow{\scriptstyle\subseteq} & & {\scriptstyle\bar\iota_*}\big\downarrow{\scriptstyle\subseteq} \\ H^\ell(X,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\subseteq} & H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) \end{array} \tag{6.8}$$

of topologically trivial *flat* ℓ–gerbes. In our motivating situation in Section 2, where $\ell = 1$, this torus corresponds to the Picard variety of topologically trivial holomorphic line bundles. We will refer to it as the *Picard torus* and write

$$\mathrm{Pic}^\ell(X) = \mathcal{H}^\ell(X)/\mathcal{H}^\ell(X,\mathbb{Z}). \tag{6.9}$$

Note that from (3.4) and Remark 3.2 (3), the Picard torus in (6.8) differs from the moduli space $H^\ell(X,\mathbb{R}/\mathbb{Z})$ of flat ℓ–gerbes by the torsion subgroup of $H^{\ell+1}(X,\mathbb{Z})$. This is encoded in diagram (3.4) by the left exact column; that is, the Bockstein exact sequence. In fact, the torus on the left side of (6.8) is exactly the kernel of the Bockstein boundary map β_* and the image of β_* is the torsion subgroup of $H^{\ell+1}(X,\mathbb{Z})$.

(3) It follows from (3.5) and (6.6) that the Deligne cohomology is given by an exact sequence (i.e. the middle exact column in (3.4))

$$0 \to \mathrm{Pic}^\ell(X) \oplus d^*\Omega^{\ell+1}(X) \xrightarrow[\subseteq]{\bar\iota_*} H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) \xrightarrow{c} H^{\ell+1}(X,\mathbb{Z}) \to 0\ , \tag{6.10}$$

since $\bar\iota_*$ is injective on $d^*\Omega^{\ell+1}(X)$ by exactness of the third column of (3.4).

(4) Harmonic Deligne cohomology: If we pull back the exact sequence (3.1) along the inclusion $\mathcal{H}^{\ell+1}(X,\mathbb{Z}) \subset \Omega^{\ell+1}_{\rm cl}(X,\mathbb{Z})$, we obtain the *harmonic* Deligne cohomology $\mathcal{H}^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z})$ of ℓ–gerbes with *harmonic curvature*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^\ell(X,\mathbb{R}/\mathbb{Z}) & \longrightarrow & \mathcal{H}^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) & \xrightarrow{d_*} & \mathcal{H}^{\ell+1}(X,\mathbb{Z}) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\rm id} & & \big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\subset} & & \\ 0 & \longrightarrow & H^\ell(X,\mathbb{R}/\mathbb{Z}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) & \xrightarrow{d_*} & \Omega^{\ell+1}_{\rm cl}(X,\mathbb{Z}) & \longrightarrow & 0. \end{array} \tag{6.11}$$

Then the exact sequence (6.10) becomes

$$0 \to \mathrm{Pic}^{\ell}(X) \underset{\subseteq}{\xrightarrow{\bar{\imath}_*}} \mathcal{H}_{\mathcal{D}}^{\ell+1}(X,\mathbb{Z}) \xrightarrow{c} H^{\ell+1}(X,\mathbb{Z}) \to 0\ , \tag{6.12}$$

For these reasons, we call these gerbes *harmonic* gerbes and $\mathcal{H}_{\mathcal{D}}^{\ell+1}(X,\mathbb{Z})$ the *harmonic* Deligne cohomology.

6.1. **The Picard torus and the Picard map.** From the construction of the Abel gerbe in Theorem 4.1 and the definition of linear equivalence in Definition 5.1, we have an injection of abelian groups

$$\overline{\Lambda}\colon \mathcal{M}_d(X)\colon = Z_d(X)/\{\text{lin. equiv.}\} \subseteq \mathcal{H}_{\mathcal{D}}^{n-d}(X,\mathbb{Z}), \tag{6.13}$$

where the inclusion $\overline{\Lambda}$ is induced by $Z \mapsto [\Lambda_Z]$.

For the boundaries $B_d(X) \subset Z_d(X)$, we have the following inclusion from Theorem 5.4 (1), (3):

$$\overline{\alpha}\colon \mathcal{M}_d^{\circ}(X)\colon = B_d(X)/\{\text{lin. equiv.}\} \subseteq \mathrm{Pic}^{n-d-1}(X). \tag{6.14}$$

where the inclusion is given by $Z = \partial\Gamma \mapsto \overline{\alpha}_\Gamma = [\alpha_\Gamma]$. Thus $\mathcal{M}_d(X)$, respectively $\mathcal{M}_d^{\circ}(X)$, is the *moduli space of Abel gerbes*, respectively the *moduli space of topologically trivial Abel gerbes*. From Theorem 5.4 (1) and (6.8) we have the following Cartesian diagram; that is, a pull–back diagram:

$$\begin{array}{ccc} \mathcal{M}_d(X) & \overset{\overline{\Lambda}}{\underset{\subseteq}{\rightarrowtail}} & \mathcal{H}_{\mathcal{D}}^{n-d}(X,\mathbb{Z}) \\ \subseteq\uparrow & & \subseteq\uparrow \bar{\imath}_* \\ \mathcal{M}_d^{\circ}(X) & \overset{\overline{\alpha}}{\underset{\subseteq}{\rightarrowtail}} & \mathrm{Pic}^{n-d-1}(X) \end{array} \tag{6.15}$$

Recall that by construction the image of $\mathcal{M}_d(X)$ in $H_{\mathcal{D}}^{n-d}(X,\mathbb{Z})$ is contained in the group of gerbes whose curvature is harmonic with integral periods; that is, in the harmonic Deligne cohomology $\mathcal{H}_{\mathcal{D}}^{n-d}(X,\mathbb{Z})$ (cf. (6.11)). In contrast, $\mathcal{M}_d^{\circ}(X)$ is exactly the part of $\mathcal{M}_d(X)$ which maps into the Picard torus (6.8), (6.9), namely $\mathrm{Pic}^{n-d-1}(X)$; that is, it consists of flat, topologically trivial gerbes. We call $\overline{\alpha}$ the *Picard* map.

From (6.12) and the fact that the characteristic class of the Abel gerbe $[\Lambda_Z]$ is the Poincaré dual $[\beta_Z]$ of $[Z]$, it follows that we have canonical isomorphisms

$$\mathcal{M}_d(X)/\mathcal{M}_d^{\circ}(X) \cong H_d(X,\mathbb{Z}) \underset{\cong}{\xrightarrow{PD}} H^{n-d}(X,\mathbb{Z}), \tag{6.16}$$

the second being Poincaré duality, induced by the characteristic class. We will see in Proposition 6.6 that $\mathcal{M}_d^{\circ}(X)$ is connected in $\mathrm{Pic}^{n-d-1}(X)$. So if $\mathcal{M}_d^{\circ}(X) \cong \mathrm{Pic}^{n-d-1}(X)$, then $\mathcal{M}_d(X) \cong \mathcal{H}_{\mathcal{D}}^{n-d}(X,\mathbb{Z})$, the harmonic Deligne cohomology of classes with harmonic curvature.

Thus, we need to understand the image of $\mathcal{M}_d^{\circ}$ in the Picard torus $\mathrm{Pic}^{n-d-1}(X)$ of topologically trivial flat gerbes.

Remark 6.2. Torsion classes (cf. Remark 6.1 (2)): Suppose that the Abel gerbe Λ_Z is flat; that is $\eta_Z = 0$, so that $[\Lambda_Z] \in H^{n-d-1}(X,\mathbb{R}/\mathbb{Z})$. By diagram (3.4), the characteristic class $[\beta_Z] \in H^{n-d}(X,\mathbb{Z})$ is given by $[\beta_Z] = \beta_*[\Lambda_Z]$, where β_* is the Bockstein homomorphism. Thus β_Z is a torsion class, say $m \cdot [\beta_Z] = 0$ for some $m \in \mathbb{N}^+$. By Poincaré duality, we have also $m \cdot Z = \partial\Gamma$ and so $m \cdot Z$ determines an element in $\mathcal{M}_d^\circ(X)$. Finally the Bockstein formula implies $m \cdot \beta_*[\Lambda_Z] = 0$; that is, $m \cdot [\Lambda_Z] = [\alpha_\Gamma]$ takes value in the Picard torus $\mathrm{Pic}^{n-d-1}(X)$.

6.2. **The Jacobi torus and the Abel–Jacobi map.** First, we observe that $\alpha \mapsto \int_X \alpha \wedge$ induces by Poincaré duality a canonical isomorphism $\varphi\colon \mathcal{H}^{n-d-1}(X) \cong \mathcal{H}^{d+1}(X)^*$. It further induces an isomorphism of abelian tori of (real) dimension $\dim H^{d+1}(X.\mathbb{R})$:

$$\overline{\varphi}\colon \mathcal{H}^{n-d-1}(X)/\mathcal{H}^{n-d-1}(X,\mathbb{Z}) \cong \mathrm{Hom}(\mathcal{H}^{d+1}(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}). \tag{6.17}$$

This is valid for $d = 0,\dots,n-1$. The torus on the right hand side of (6.17) corresponds classically to the Jacobi variety of a Riemann surface, where $n = 2$, $d = 0$. We shall call it the *Jacobi torus* and denote it by $\mathrm{Jac}^{d+1}(X)$. We now recall formula (5.6); that is,

$$\int_X \alpha_\Gamma \wedge \theta = \int_\Gamma \theta,\ \ \theta \in \mathcal{H}^{d+1}(X).$$

Combining (5.6) with (6.17), we obtain a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Pic}^{n-d-1}(X) \\ & \overset{\overline{\alpha}}{\nearrow}\ \subseteq & \\ \mathcal{M}_d^\circ(X) & & \cong \Big\downarrow \overline{\varphi} \\ & \underset{\subseteq}{\searrow}\ \overline{J} & \\ & & \mathrm{Jac}^{d+1}(X), \end{array} \tag{6.18}$$

where $\overline{J}$ is induced by the functional

$$Z = \partial\Gamma \mapsto J_{\partial\Gamma}(\theta) = \int_\Gamma \theta,\ \ \theta \in \mathcal{H}^{d+1}(X,\mathbb{Z}). \tag{6.19}$$

Note that $\overline{J}$ is well-defined and injective by Abel's Theorem 5.3, plus the fact that $J_{\partial\Gamma}(\theta)$ has integral values if Γ is a cycle; that is $\partial\Gamma = 0$. We shall call $\overline{J}$ the *Abel–Jacobi map*. Thus we may just as well use the map $\overline{J}$ to investigate the image of $\mathcal{M}_d^\circ(X)$. The Abel–Jacobi map $\overline{J}$ is given in terms of period integrals and therefore is more explicit than the Picard map $\overline{\alpha}$, which is determined by the solution of a Laplace–Poisson equation. Therefore it is in general easier to deal with and more effective in explicit calculations, as we shall see.

Remarks 6.3. Intermediate Jacobians:

(1) For a Kähler manifold, our definition of the Jacobians agrees with the tori underlying the complex *intermediate* Jacobians in odd degrees, which are related to *holomorphic* Deligne cohomology (cf. Griffiths–Harris [13], Ch. 2.6, Dupont–Hain-Zucker [8] and also Clemens [5]). For divisors on algebraic manifolds of complex dimension greater than one, it is not clear how our version of Abel's theorem is related to the version by Griffiths [12].

(2) We also remark that for $\dim X = n = 4k+2,\ d = 2k,\ k \geq 0$, the Picard and the Jacobi tori in degree $n-d-1 = d+1 = 2k+1$ carry a canonical *complex* and *symplectic* structure, compatible with the isomorphism

$$\mathrm{Pic}^{2k+1}(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^{2k+1}(X).$$

The former is induced by the Hodge $*$–operator on $\mathcal{H}^{2k+1}(X)$ and the latter is defined by the pairing $\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta$ on $\mathcal{H}^{2k+1}(X)$.

6.3. **Deformations.** We now consider 'deformations' of Abel gerbes as follows:

Definition 6.4.

(1) A *regular* $(d+1)$*–simplex* in X is a smooth embedding $\Gamma\colon \Delta^{d+1} \to X$ of the standard simplex Δ^{d+1} (or rather an open neighborhood in the hyperplane $\sum_{i=0}^{d+1} t_i = 1$). Note that any simplex in a triangulation K of X can be parametrized as a regular simplex.

(2) A *deformation* of a cycle $Z = \partial\Gamma \in B_d(X)$ for a triangulation K of X is a family of cycles $Z_r = \partial\Gamma_r \in B_d(X),\ r \in [0,1]$, for some subdivisions K_r of K, such that $\Gamma_1 = \Gamma,$ each simplex of Γ_r is regular and $r \mapsto \overline{J}(Z_r) = J_{\partial\Gamma_r} = \int_{\Gamma_r}(\cdot) \in \mathrm{Jac}^{d+1}(X)$ is a smooth curve.

The following deformation techniques are used repeatedly in what follows and we state them in a separate Lemma.

Lemma 6.5. *Let $Z = \partial\Gamma$, for Γ any $(d+1)$–chain in $C_{d+1}(K)$ consisting of regular simplices in $C_{d+1}(K)$, where K is an arbitrary triangulation of X. Then there is a deformation $Z_r = \partial\Gamma_r,\ r \in [0,1]$ of Z satisfying:*

(1) $\Gamma_r \in C_{d+1}(K_r),\ |K_r|$ *a subdivision of* $|K|$.

(2) *For $\alpha_r = \alpha_{\Gamma_r}$, the map $]0,1] \to \mathrm{Pic}^{n-d-1}(X)$ given by $r \mapsto [\alpha_r]$ is smooth.*

(3) *For $J_r = J_{\partial\Gamma_r}$, the map $]0,1] \to \mathrm{Jac}^{d+1}(X)$ given by $r \mapsto \overline{J}_r$ is smooth.*

(4) $[\alpha_r] \to 0$ *in* $\mathrm{Pic}^{n-d-1}(X)$ *for* $r \downarrow 0$.

(5) $J_r(\theta) = \int_{\Gamma_r} \theta \to 0,\ r \downarrow 0$, *for all* $\theta \in \mathcal{H}^{d+1}(X)$.

Proof. It is clearly enough to take Γ to be a regular $(d+1)$–simplex $\Gamma\colon \Delta^{d+1} \to X$ of K. Then we simply define $\Gamma_r = \Gamma \circ \phi_r$, with $\phi_r(t_0, \ldots, t_{d-1}, t_d, t_{d+1}) = (t_0, \ldots, t_{d-1}, t_d + (1-r)\ t_{d+1}, r\ t_{d+1}),\ t \in \Delta^{d+1}$. Then $\Gamma_r,\ r \in [0,1]$ clearly satisfies (1) and (2). (2) and (3) are equivalent by formula (5.6). Furthermore by Theorem 5.4 and formula (5.6), conditions (4) and (5) are equivalent and are fulfilled, since $\int_{\Gamma_r} \theta \to 0, r \downarrow 0$, for all $\theta \in \mathcal{H}^{d+1}(X)$ by construction of Γ_r. □

Proposition 6.6. (1) *$\mathcal{M}_d^\circ(X)$ is connected in the Picard torus, respectively the Jacobi torus.*

(2) *The closure $\overline{\mathcal{M}_d^\circ}(X)$ in the induced topology is a subtorus of $\mathrm{Pic}^{n-d-1}(X)$.*

(3) *For $\overline{\alpha}$, respectively $\overline{J}$ to be surjective, it is necessary and sufficient that their image contain an open neighborhood of the origin (or an open neighborhood of any point in their image).*

Proof. Again let $\partial\Gamma$ for Γ any $(d+1)$–chain in $C_{d+1}(K)$, where K is an arbitrary triangulation of X. To prove (1), we again take Γ to be a regular $(d+1)$–simplex and we define as before $\Gamma_r(t) = \Gamma(t_0, \ldots, t_{d-1}, t_d + (1-r)t_{d+1}, r\ t_{d+1}),\ t \in \Delta^{d+1}$. Then (1) follows from Lemma 6.5 and (2) clearly follows from (1). To prove (3), we have only to observe that $\overline{\alpha}$, respectively $\overline{J}$ are homomorphisms of abelian groups. The statement follows from the fact that any open neighborhood of the origin in either torus generates the entire torus. Observe that, except for (1), the above deformations can take place in the interior of the fundamental domain of $\mathcal{H}^{n-d-1}(X)$ relative to the integral lattice $\mathcal{H}^{n-d-1}(X, \mathbb{Z})$. □

6.4. **The moduli theorem.** In this Section we determine the moduli space of Abel gerbes by establishing an inversion theorem for the Abel–Jacobi map. Before stating and proving the main Theorem 6.14, we will illustrate the technique involved in some important examples.

Example 6.7. The case $n \geq 2,\ d = n-1$:
The Jacobi map $\overline{J}\colon \mathcal{M}^\circ_{n-1}(X) \to \mathrm{Jac}^n(X)$ is an isomorphism. Therefore, so is the Picard map $\overline{\alpha}\colon \mathcal{M}^\circ_{n-1}(X) \to \mathrm{Pic}^0(X)$. This is the easiest case, since the Picard and Jacobi tori are now in degree 0, respectively n. Thus

$$\mathrm{Pic}^0(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^n(X) \cong \mathrm{Hom}(\mathcal{H}^n(X,\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z},$$

with the generator of the integral lattice given by $\theta_0 = \mathrm{Vol}$, assuming that the volume is normalized. Taking a deformation $\Gamma_r,\ r \in [0,1]$ of a regular n–simplex as in Lemma 6.5, we get $J_{\Gamma_r}(\theta_0) = \int_{\Gamma_r} \mathrm{Vol} > 0$, respectively $J_{\Gamma_r}(\theta_0) = \int_{\Gamma_r} \mathrm{Vol} < 0$, if the orientation of the regular simplex Δ^n is reversed. Further, we have $J_{\Gamma_r}(\theta_0) = \int_{\Gamma_r} \mathrm{Vol} \to 0,\ r \downarrow 0$. Thus the image contains an interval around the origin and so the Jacobi map must be an isomorphism $\mathcal{M}^\circ_{n-1}(X) \cong \mathrm{Jac}^n(X)$.

In this case, the Deligne cohomology $H^1_{\mathcal{D}}(X,\mathbb{Z})$ consists of 0–gerbes, which are given by $f^0 \in \check{C}^0(\mathcal{U}, \underline{\mathbb{R}})$, such that $\check{\delta} f^0 \equiv 0 \mod \mathbb{Z}$, so that f^0 defines a global smooth function $\theta\colon X \to \mathbb{R}/\mathbb{Z} \cong U(1)$, modulo global functions $\Omega^0(X)$. Since $\check{\delta} df^0 = d\check{\delta} f^0 = 0$, the curvature F_f is a closed 1–form with integral periods, determined by $\varepsilon^*(F_f) = df^0$. The characteristic class of $[\theta, f]$ is given by $c[\theta, f] = [\check{\delta} f^0] \in H^1(X,\mathbb{Z})$; that is, the obstruction to lift θ to a global function $f \in \Omega^0(X)$.

Example 6.8. The case $n > 2,\ d = n-2$:
Here we look at Abel 1–gerbes associated to submanifolds $M^{n-2} \subset X^n$ of codimension 2 or more generally to cycles $Z \in C_{n-2}(X)$. In this case, we have

$$\mathrm{Pic}^1(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^{n-1}(X) \cong \mathrm{Hom}(\mathcal{H}^{n-1}(X,\mathbb{Z}), \mathbb{R}/\mathbb{Z}),$$

and the Abel–Jacobi map $\overline{J}\colon \mathcal{M}^\circ_{n-2}(X) \cong \mathrm{Jac}^{n-1}(X)$ is an isomorphism. The moduli space $\mathcal{M}_{n-2}(X)$ of Abel 1–gerbes is generated by cycles $Z \in C_{n-2}(X)$ and is given by $\mathcal{M}_{n-2}(X) \cong \mathcal{H}^2_{\mathcal{D}}(X,\mathbb{Z})$; that is, the moduli space of complex line bundles with unitary connection and harmonic curvature.

Example 6.9. $X = \mathbb{T}^n, n \geq 2,\ d = 0, \ldots, n-1$:
The Jacobi map $\overline{J}\colon \mathcal{M}^\circ_d(X) \to \mathrm{Jac}^{d+1}(X)$ is an isomorphism; so is the Picard map $\overline{\alpha}\colon \mathcal{M}^\circ_d(X) \to \mathrm{Pic}^{n-d-1}(X)$. Here, we take $X = \mathbb{T}^n$, an n–dimensional torus with the flat (invariant) Riemannian metric. In this case, the dimension of the Picard–, respectively the Jacobi torus is $\dim H^{d+1}(X,\mathbb{R}) = \binom{n}{d+1},\ d = 0, \ldots, n-1$. There is an orthonormal basis $\{\theta_1, \ldots, \theta_n\}$ of integral, harmonic, invariant 1–forms which form a framing of the cotangent bundle $T^*(X)$ and determine an orthonormal basis of $\mathcal{H}^{d+1}(X,\mathbb{Z})$ by $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_{d+1}}$, where $I = (i_1 < i_2 < \cdots < i_{d+1})$. The dual basis $\{e_1, \ldots, e_n\}$ determines $(d+1)$–subspaces $e_I = e_{i_1} \wedge \cdots \wedge e_{i_{d+1}}$ of $\mathbb{R}^n$, respectively basis elements of $\Lambda^{d+1}(\mathbb{R}^n)$ for all multi–indices I as above. By deforming small $(d+1)$–parallelepipeds P_I in the direction of e_I by the method of Lemma 6.5 and considering the families of Jacobi integrals $J_{P_I}(\theta_I) = \int_{P_I} \theta_I$ or their linear combinations, one generates (small) open sets in the range of the Jacobi map.

Example 6.10. The case $n \geq 2,\ d = 0$:
This is similar to the classical case of divisors on a Riemann surface. In this case, we have

$$\mathrm{Pic}^{n-1}(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^1(X) \cong \mathrm{Hom}(\mathcal{H}^1(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}).$$

Theorem 6.14 asserts that the Jacobi map $\overline{J}\colon \mathcal{M}_0^\circ(X) \to \mathrm{Jac}^1(X)$ is an isomorphism. Therefore, so is the Picard map $\overline{\alpha}\colon \mathcal{M}_0^\circ(X) \to \mathrm{Pic}^{n-1}(X)$. The moduli space $\mathcal{M}_0(X)$ of Abel $(n-1)$–gerbes defined by points $\{p\} \subset X$, whose curvature is the normalized harmonic volume form Vol, satisfies $\mathcal{M}_0(X) \cong \mathcal{H}^n_{\mathcal{D}}(X,\mathbb{Z})$; that is, the space of $(n-1)$–gerbes with harmonic curvature. In this case, we have $\mathcal{M}_0(X)/\mathcal{M}_0^\circ(X) \cong H_0(X,\mathbb{Z})$. If X is connected with basepoint p_0, the Abel–Jacobi map $\overline{J}\colon B_0(X) \to \mathrm{Jac}^1(X)$ defines a smooth mapping $j\colon X \to \mathrm{Jac}^1(X)$ by $j(p)\ (\theta) = \int_{p_0}^p \theta \mod \mathbb{Z}$, for $\theta \in \mathcal{H}^1(X,\mathbb{Z})$. In turn, the mapping j determines the Abel–Jacobi map $\overline{J}$ completely. To see this, choose regular 1–simplices Γ_i, such that $\partial\Gamma_i = \{p_i\} - \{q_i\},\ i = 1,\ldots,m,\ m \geq 1$. Then for $\Gamma = \sum_i \Gamma_i$, we have $J_{\partial\Gamma}\ (\theta) = \sum_i \int_{\Gamma_i} \theta \equiv \sum_i\ (j(p_i)\ (\theta) - j(q_i)\ (\theta)) \mod \mathbb{Z}$. Thus Abel's Theorem implies that $\sum_i\ j(p_i) = \sum_i\ j(q_i)$, if and only if the 0–chains $\sum_i\ \{p_i\}$ and $\sum_i\ \{q_i\}$ are linearly equivalent. Note that this argument does not prove surjectivity of $\overline{J}$.

Example 6.11. Riemann surfaces X of genus $g \geq 1,\ n = 2,\ d = 0$:
In this case, we have

$$\mathrm{Pic}^1(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^1(X) \cong \mathrm{Hom}(\mathcal{H}^1(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}),$$

and $\overline{\alpha}\colon \mathcal{M}_0^\circ(X) \to \mathrm{Pic}^1(X)$, respectively $\overline{J}\colon \mathcal{M}_0^\circ(X) \to \mathrm{Jac}^1(X)$, correspond to the Picard, respectively the Abel–Jacobi map of the Riemann surface. The Jacobi integral involves path integrals over 1–chains Γ of the form $J_\Gamma(\theta) = \int_\Gamma \theta$. The Deligne cohomology $H^2_{\mathcal{D}}(X,\mathbb{Z})$ is the moduli space of 1–gerbes; that is *complex line bundles with unitary connection.*

On a Riemann surface the first cohomology group $H^1(X,\mathcal{M}_X^*)$ vanishes (cf. [14], Ch.7, Theorem 12). This is a non–trivial consequence of the Riemann–Roch theorem and Serre duality. From the exact cohomology sequence

$$0 \to H^0(X,\mathcal{O}_X^*) \to H^0(X,\mathcal{M}_X^*) \xrightarrow{D} H^0(X,\mathcal{D}_X) \xrightarrow{\delta_*} H^1(X,\mathcal{O}_X^*) \to H^1(X,\mathcal{M}_X^*)$$

of the divisor sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \xrightarrow{D} \mathcal{D}_{\mathcal{X}} \to 0,$$

it follows that $H^0(X,\mathcal{D}_X) \xrightarrow{\delta_*} H^1(X,\mathcal{O}_X^*)$ is surjective and every holomorphic line bundle is the line bundle of a divisor. In particular, the divisors of degree zero are mapped onto the Picard variety

$$\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \subseteq H^1(X,\mathcal{O}_X^*).$$

$\mathrm{Pic}(X)$ is a complex torus of $\dim_{\mathbb{C}} \mathrm{Pic}(X) = g$, the variety of holomorphic line bundles $\mathcal{L}$ with $\mathrm{Deg}(\mathcal{L}) = c_1(\mathcal{L}) = 0$; that is, topologically trivial holomorphic line bundles. In our context, the Picard torus $\mathrm{Pic}^1(X)$ is a real torus of dimension $2g$ and the above shows that $\mathcal{M}_0^\circ(X) \cong \mathrm{Pic}^1(X)$. Our proof of $\overline{J}\colon \mathcal{M}_0^\circ(X) \cong \mathrm{Jac}^1(X)$ in Theorem 6.14 is much more elementary and closer to the direct generation of all holomorphic line bundles via divisors in [14], Ch.7 (c) and the Inversion Theorem in [13], Ch. 2.2. Thus, we choose suitable 1–simplices Γ_i on cycles Y_i representing a basis $[Y_i] \in H_1(X,\mathbb{Z}),\ i = 1,\dots,2g$ and deform their endpoints $p_i = \Gamma_i(0,1)$ along the curves Γ_i by $\Gamma_i(r)(t_0,t_1) = \Gamma_i(t_0 + (1-r)t_1, rt_1), r \in [0,1]$ to the fixed initial points $p_{0,i} = \Gamma_i(1,0)$. In this way one generates an open set in the image of the Abel–Jacobi map by the functionals $J_{\partial\Gamma(r_1,\dots,r_{2g})} = \int_{\Gamma(r_1,\dots,r_{2g})}$, where $\Gamma(r_1,\dots,r_{2g}) = \sum_{i=1}^{2g}\Gamma_i(r_i)$. The same procedure applies also to Example 6.10 for the generation of $(n-1)$–gerbes defined by points in X.

Example 6.12. The case $n = 3,\ d = 0$:
This is a special case of Example 6.10; so, we have

$$\mathrm{Pic}^2(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^1(X) \cong \mathrm{Hom}(\mathcal{H}^1(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}).$$

Theorem 6.14 asserts that the Abel–Jacobi map $\overline{J}\colon \mathcal{M}_0^\circ(X) \cong \mathrm{Jac}^1(X)$ is an isomorphism. The moduli space $\mathcal{M}_0(X)$ of Abel gerbes defined by points $\{p\} \subset X^3$ and whose curvature is the normalized harmonic volume form Vol is given by $\mathcal{M}_0(X) \cong \mathcal{H}^3_{\mathcal{D}}(X,\mathbb{Z})$; that is, the moduli space of 2–gerbes with harmonic curvature. In this situation, the Abel–Jacobi map was introduced and Abel's theorem proved by Hitchin [15], Ch. 3.2 and Chatterjee [2] in the context of 2–gerbes. This was one of our motivating examples.

Example 6.13. The case $n > 3,\ d = n-3$:
Here we look at Abel 2–gerbes associated to submanifolds $M^{n-3} \subset X^n$ of codimension 3 or more generally to cycles $Z \in C_{n-3}(X)$. In this case, we have

$$\mathrm{Pic}^2(X) \xrightarrow[\cong]{\overline{\varphi}} \mathrm{Jac}^{n-2}(X) \cong \mathrm{Hom}(\mathcal{H}^{n-2}(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}),$$

Theorem 6.14 asserts that the Abel–Jacobi map $\overline{J}\colon \mathcal{M}_{n-3}^\circ(X) \to \mathrm{Jac}^{n-2}(X)$ is an isomorphism. Therefore, so is the Picard map $\overline{\alpha}\colon \mathcal{M}_{n-3}^\circ(X) \to \mathrm{Pic}^2(X)$. The moduli space $\mathcal{M}_{n-3}(X)$ of Abel 2–gerbes is generated by cycles $Z \in C_{n-3}(X)$ and is given by $\mathcal{M}_{n-3}(X) \cong \mathcal{H}^3_{\mathcal{D}}(X,\mathbb{Z})$; that is, the moduli space of 2–gerbes with harmonic curvature. However, except possibly for the case $n = 4,\ d = 1$, it is not clear whether it would be sufficient to consider only codimension three submanifolds $M \subset X$; at any rate, in our proof of Theorem 6.14 we need to consider cycles $Z \in C_{n-3}(X)$ (cf. Bohr–Hanke–Kotschick [1]).

For codimension 3 submanifolds $M^{n-3} \subset X^n$, the Abel–Jacobi map was also investigated and Abel's theorem proved by Hitchin [15] and Chatterjee [2], Theorem 6.4.2, in the context of 2–gerbes. Moreover, Hitchin in [15], Theorem 3.2 proves a moduli theorem for families of special Lagrangian 3–tori in a Calabi–Yau 3–fold via the Abel–Jacobi map. Again, these were motivating examples for the present work.

Theorem 6.14 (Moduli Theorem)**.** *The following statements are equivalent and hold for any compact connected oriented Riemannian manifold X of dimension $n \geq 2$, $d = 0,\dots,n-1$:*

(1) *The Picard map $\overline{\alpha}\colon \mathcal{M}_d^\circ(X) \to \mathrm{Pic}^{n-d-1}(X)$ is an isomorphism.*

(2) *The Abel–Jacobi map $\overline{J}\colon \mathcal{M}_d^\circ(X) \to \mathrm{Jac}^{d+1}(X)$ is an isomorphism.*

(3) *The mapping $\overline{\Lambda}\colon \mathcal{M}_d(X) \to \mathcal{H}^{n-d}_{\mathcal{D}}(X,\mathbb{Z})$ is an isomorphism.*

(4) *Every equivalence class $[\Lambda]$ of $(n-d-1)$–gerbes in the harmonic Deligne cohomology $\mathcal{H}^{n-d}_{\mathcal{D}}(X,\mathbb{Z})$ can be realized by a unique (up to linear equivalence) Abel gerbe Λ_Z.*

Proof. (1) and (2) are equivalent by diagram (6.18). To prove that (3) is equivalent to (1), we observe thst (6.12), (6.15) and (6.16) determine a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_d^\circ(X) & \longrightarrow & \mathcal{M}_d(X) & \longrightarrow & H_d(X) & \longrightarrow & 0 \\ & & \downarrow \overline{\alpha} & & \downarrow \overline{\Lambda} & & \cong \downarrow PD & & \\ 0 & \longrightarrow & \mathrm{Pic}^{n-d-1}(X) & \longrightarrow & \mathcal{H}_{\mathcal{D}}^{n-d}(X,\mathbb{Z}) & \xrightarrow{d_*} & H^{n-d}(X,\mathbb{Z}) & \longrightarrow & 0. \end{array} \tag{6.20}$$

The result follows now from the 5–lemma. (4) is a restatement of (3). Thus it suffices to prove (2).

We choose an (orthonormal) basis $\{\theta_1,\dots,\theta_k\}$, $k = \dim H^{d+1}(X,\mathbb{R})$ in the integral lattice $\mathcal{H}^{d+1}(X,\mathbb{Z}) \subset \mathcal{H}^{d+1}(X)$ of harmonic forms and observe that the Jacobi vector

$$J_{\partial\Gamma} = \left(\textstyle\int_\Gamma \theta_1,\dots,\int_\Gamma \theta_k\right) \tag{6.21}$$

determines the element $\overline{J}_{\partial\Gamma} \in \mathrm{Jac}^{d+1}(X)$ via the expansion $\theta = \sum_{i=1}^k a_i\theta_i$, $a_i \in \mathbb{Z}$ for any $\theta \in \mathcal{H}^{d+1}(X,\mathbb{Z})$. Next, we choose a dual basis $\{\overline{Y}_i\}$ in the integral lattice $j_*H_{d+1}(X,\mathbb{Z}) \subset H_{d+1}(X,\mathbb{R})$, represented by cycles $Y_i \in Z_{d+1}(X), i = 1,\dots,k$; that is, we have

$$\int_{Y_i} \theta_j = \delta_{ij}. \tag{6.22}$$

Now, we write $Y_i = \sum_\ell \Gamma_{i,\ell}$, where the $\Gamma_{i,\ell}$ are regular $(d+1)$–simplices for a triangulation K of X. Expanding

$$\det\left(\textstyle\int_{Y_i}\theta_j\right)_{(i,j=1,\dots,k)} = 1, \tag{6.23}$$

we see that for every $i = 1,\dots,k$, there is an ℓ_i such that for $\Gamma_i = \Gamma_{i,\ell_i}$ we have

$$\det\left(\textstyle\int_{\Gamma_i}\theta_j\right)_{(i,j=1,\dots,k)} > 0. \tag{6.24}$$

We now deform the regular simplices Γ_i according to the deformation in the proof of Lemma 6.5; so we consider $\Gamma_i(r)\colon \Delta^{d+1} \xrightarrow{\phi_r} \Delta^{d+1} \xrightarrow{\Gamma_i} X$, $r \in [0,1]$; that is $\Gamma_i\ (r) = \Gamma_i \circ \phi_r$. First we have from Lemma 6.5 (5)

$$\lim_{r\downarrow 0} \int_{\Gamma_i(r)} \theta = 0,\ i = 1,\dots,k,$$

for every $\theta \in \mathcal{H}^{d+1}(X,\mathbb{Z})$, since $\Gamma_i\ (r)$ degenerates to a d–simplex as $r \downarrow 0$. This is equivalent to

$$\lim_{r\downarrow 0} J_{\partial\Gamma_i(r)} = 0,\ i = 1,\dots,k. \tag{6.25}$$

From (6.24) it follows that we can choose $\varepsilon > 0$ sufficiently small, such that the smooth mapping

$$r = (r_1,\dots,r_k) \mapsto \left(\textstyle\int_{\Gamma_i(r_i)}\theta_j\right)_{(i,j=1,\dots,k)} = \begin{pmatrix} \int_{\Gamma_1(r_1)}\theta_1 & \cdots & \int_{\Gamma_1(r_1)}\theta_k \\ \vdots & \vdots & \vdots \\ \int_{\Gamma_k(r_k)}\theta_1 & \cdots & \int_{\Gamma_k(r_k)}\theta_k \end{pmatrix} \tag{6.26}$$

has positive determinant for $r_i \in (1-\varepsilon, 1]$. Moreover, by passing to a subtriangulation of K if necessary, we can achieve that our construction takes place in the interior of the fundamental domain in the universal cover $\mathrm{Hom}(\mathcal{H}^{d+1}(X,\mathbb{Z}),\mathbb{R})$ of $\mathrm{Jac}^{d+1}(X)$. Therefore the Jacobi vectors (the row vectors in (6.26))

$$J_{\partial\Gamma_i(r_i)} = \left(\textstyle\int_{\Gamma_i(r_i)}\theta_1,\dots,\int_{\Gamma_i(r_i)}\theta_k\right) \tag{6.27}$$

are linearly independent in $\operatorname{Hom}(\mathcal{H}^{d+1}(X,\mathbb{Z}),\mathbb{R})$ for $r_i \in (1-\varepsilon, 1]$ and $i = 1,\dots,k$. Setting $\Gamma(r_1,\dots,r_k) = \sum_{i=1}^k \Gamma_i(r_i)$ and taking the linear combination of the Jacobi vectors

$$J_{\partial\Gamma(r_1,\dots,r_k)} = \sum_{i=1}^{k} J_{\partial\Gamma_i(r_i)} = \sum_{i=1}^{k} \left(\int_{\Gamma_i(r_i)} \theta_1, \dots, \int_{\Gamma_i(r_i)} \theta_k\right) \tag{6.28}$$

gives a mapping $\Phi\colon D \to \operatorname{Hom}(\mathcal{H}^{d+1}(X,\mathbb{Z}),\mathbb{R}) \cong \mathbb{R}^k$ defined by

$$\Phi\colon D \ni r = (r_1,\dots,r_k) \mapsto J_{\partial\Gamma(r_1,\dots,r_k)} = \sum_{i=1}^{k} J_{\partial\Gamma_i(r_i)}. \tag{6.29}$$

Here $D \subset \mathbb{R}^k$ is the hypercube (prism) given by $r_i \in [0,1], i = 1,\dots,k$. Φ is continuous on D and smooth on the interior $B \subset D$, given by $r_i \in (0,1)$, $i = 1,\dots,k$.

The following lemma asserts that the Jacobian $D\Phi$ has positive determinant in the neighborhood of an inner point $r_0 = (r_{0,1},\dots,r_{0,k}) \in B$. By the inverse function theorem, Φ is a local diffeomorphism near r_0 and therefore our theorem follows from Proposition 6.6 (3). □

Lemma 6.15. *The Jacobian matrix $D\Phi$ is given by*

$$D\Phi = \left(\frac{\partial \Phi_j}{\partial r_i}\right)_{(i,j=1,\dots,k)} = \left(\frac{\partial \int_{\Gamma_i(r_i)} \theta_j}{\partial r_i}\right)_{(i,j=1,\dots,k)}. \tag{6.30}$$

Further, there exists $r_0 = (r_{0,1},\dots,r_{0,k}) \in B$, such that $\det D\Phi(r_0) > 0$.

Proof. The form of the Jacobian matrix $D\Phi$ follows immediately from the fact that each Jacobi vector $J_{\partial\Gamma_i(r_i)}$ in (6.26) and in the definition (6.28), (6.29) of Φ depends only on one variable r_i. In what follows, we will use this fact repeatedly. We now inductively use partial differentiation to pass from (6.26) to (6.30), using just the intermediate value theorem of calculus. Writing the matrix in (6.26) as a column of Jacobi vectors, we know that the determinant is positive for r_i in the indicated region, while it goes to 0 for $r_1 \downarrow 0$ by (6.25). Therefore there is a $r_{0,1} \in (0,1)$, such that

$$\frac{\partial}{\partial r_1}\Big|_{r_{0,1}} \det \begin{pmatrix} J_{\partial\Gamma_1(r_1)} \\ J_{\partial\Gamma_2(r_2)} \\ \vdots \\ J_{\partial\Gamma_k(r_k)} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial J_{\partial\Gamma_1(r_{0,1})}}{\partial r_1} \\ J_{\partial\Gamma_2(r_2)} \\ \vdots \\ J_{\partial\Gamma_k(r_k)} \end{pmatrix} > 0. \tag{6.31}$$

Proceeding inductively, we assume that we have $r_{0,1},\dots,r_{0,j-1} \in (0,1), 1 < j \le k$, such that the determinant

$$\det \begin{pmatrix} \frac{\partial J_{\partial\Gamma_1(r_{0,1})}}{\partial r_1} \\ \vdots \\ \frac{\partial J_{\partial\Gamma_{j-1}(r_{0,j-1})}}{\partial r_{j-1}} \\ J_{\partial\Gamma_j(r_j)} \\ \vdots \\ J_{\partial\Gamma_k(r_k)} \end{pmatrix} > 0 \tag{6.32}$$

is positive for $r_j, \dots, r_k \in (1-\varepsilon, 1]$. Since (6.32) goes to zero as $r_j \downarrow 0$ by (6.25), there is a $r_{0,j} \in (0,1)$, such that

$$\frac{\partial}{\partial r_j}|_{r_{0,j}} \det \begin{pmatrix} \frac{\partial J_{\partial\Gamma_1(r_{0,1})}}{\partial r_1} \\ \vdots \\ \frac{\partial J_{\partial\Gamma_{j-1}(r_{0,j-1})}}{\partial r_{j-1}} \\ J_{\partial\Gamma_j(r_j)} \\ J_{\partial\Gamma_{j+1}(r_{j+1})} \\ \vdots \\ J_{\partial\Gamma_k(r_k)} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial J_{\partial\Gamma_1(r_{0,1})}}{\partial r_1} \\ \vdots \\ \frac{\partial J_{\partial\Gamma_{j-1}(r_{0,j-1})}}{\partial r_{j-1}} \\ \frac{\partial J_{\partial\Gamma_j(r_{0,j})}}{\partial r_j} \\ J_{\partial\Gamma_{j+1}(r_{j+1})} \\ \vdots \\ J_{\partial\Gamma_k(r_k)} \end{pmatrix} > 0, \tag{6.33}$$

with $r_{j+1}, \dots, r_k$ as above. This completes the induction. So for $j = k$ we have $r_0 = (r_{0,1}, \dots, r_{0,k}) \in B$ such that (6.33) is positive at r_0. But for $j = k$, (6.33) is the determinant of the Jacobian (6.30) and the proof is complete. □

7. Euler and Thom gerbes

In this section we construct the *Euler gerbe* and the *Thom gerbe* of an orthogonal bundle, based on the 'gerbe approach' in [10]. In Section 8 we will investigate the relationship between the Euler–, Thom– and the Abel gerbe. We briefly recall the main properties of the construction of *characteristic gerbes* from section 5 of [10] , which generalizes the classical constructions of secondary characteristic classes and 'characters' for connections on principal G-bundles in terms of simplicial forms. For the classical constructions we refer to Kamber–Tondeur [17], Chern–Simons [4], Cheeger–Simons [3] or Dupont–Kamber [9].

In the following $p\colon P \to X$ is a smooth principal G-bundle, G a Lie-group with only finitely many components and $K \subseteq G$ is the maximal compact subgroup. As in Section 5 of [10], we fix an invariant homogeneous polynomial $Q \in I^{n+1}(G)$, $n \geq 0$, such that one of the following 2 cases occur:

Case I: $Q \in \ker(I^{n+1}(G) \to I^{n+1}(K))$.

Case II: $Q \in I^{n+1}_{\mathbb{Z}}(G)$, that is, there exists an integral class $u \in H^{2n+2}(BK, \mathbb{Z})$ representing the Chern-Weil image of Q in $H^*(BG, \mathbb{R}) \cong H^*(BK, \mathbb{R})$.

With this notation the *secondary characteristic class* associated to Q (case I) or (Q, u) (case II) for a connection A on $P \to X$ is a class

$$\begin{aligned} [\Lambda(Q, A)] &\in H^{2n+2}_{\mathcal{D}}(X) && \text{in case I,} \\ [\Lambda(Q, u, A)] &\in H^{2n+2}_{\mathcal{D}}(X, \mathbb{Z}) && \text{in case II.} \end{aligned} \tag{7.1}$$

Note that that the characteristic classes in $H^*_{\mathcal{D}}(X)$ are defined by global forms, whereas the classes in $H^*_{\mathcal{D}}(X, \mathbb{Z})$ are defined by simplicial forms.

(1) The classes in (7.1) are natural with respect to bundle maps and compatible coverings.
(2) *Curvature formula* :

$$\begin{aligned} d\Lambda(Q, A) &= Q(F_A^{n+1}) && \text{in case I} \\ d\Lambda(Q, u, A) &= \varepsilon^* Q(F_A^{n+1}) - \gamma && \text{in case II} \end{aligned} \tag{7.2}$$

where $\gamma \in \Omega_{\mathbb{Z}}(|N\mathcal{U}|)$ represents the characteristic class $u(P)$ associated with u and F_A is the curvature of A.

(3) If $Q(F_A^{n+1}) = 0$, then

$$\begin{aligned} &[\ \Lambda(Q, A)\ ] \in H^{2n+1}(X, \mathbb{R}) && \text{in case I} \\ &[\ \Lambda(Q, u, A)\ ] \in H^{2n+1}(X, \mathbb{R}/\mathbb{Z}) && \text{in case II,} \end{aligned} \tag{7.3}$$

and

$$\beta_*[\Lambda(Q, u, A)] = -u(P) \tag{7.4}$$

where $\beta_*\colon H^{2n+1}(X, \mathbb{R}/\mathbb{Z}) \to H^{2n+2}(X, \mathbb{Z})$ is the Bockstein homomorphism.

We shall now use these classes for the case $G = SO(2m)$, $\mathrm{Pf} \in I^m_{\mathbb{Z}}(SO(2m))$ the Pfaffian polynomial and $u = e \in H^{2m}(BSO(2m), \mathbb{Z})$ the Euler class. That is, for Y any smooth manifold and $\pi\colon E \to Y$ a $2m$-dimensional oriented vector bundle with Riemannian metric and metric connection A we obtain a characteristic class $[\Lambda(\mathrm{Pf}, e, A)] \in H^{2m}_{\mathcal{D}}(Y, \mathbb{Z})$ represented by a simplicial form

$$\Lambda(\mathrm{Pf}, e, A) \in \Omega^{2m-1}(\|N\mathcal{U}\|)$$

for a suitable covering $\mathcal{U}$ of Y, satisfying

$$d\Lambda(\mathrm{Pf}, e, A) = \varepsilon^* \,\mathrm{Pf}(F_A) - e(E). \tag{7.5}$$

We shall refer to this form $\Lambda(\mathrm{Pf}, e, A)$ as the *Euler gerbe* associated to E.

As is the case for the primary Euler class $e(E)$, there is an alternative definition of $\Lambda(\mathrm{Pf}, e, A)$, using the Thom space of E:

Let $(\mathbb{B}(E), \mathbb{S}(E))$ denote the ball and sphere bundle of radius 1. Then up to the choice of a 'bump function' in the radial direction, the volume form and connection determine a 'canonical' representative form $U_E \in \Omega^{2m}_c(\mathbb{B}(E))$, with support inside $\mathbb{S}(E)$, for the Thom class in $H^{2m}(\mathbb{B}(E), \mathbb{S}(E))$. The restriction of U_E to a neighborhood of the image of the zero section $s\colon Y \to E$ is independent of the choice of the bump function. For F_A the curvature form of A, we have

$$s^* U_E = \mathrm{Pf}(F_A) \in \Omega^{2m}(Y). \tag{7.6}$$

For a suitable open covering $\mathcal{V}$ of $\mathbb{B}(E)$, let $\beta_E \in \Omega^{2m}_{\mathbb{Z}}(\|N\mathcal{V}\|)$ represent the Thom class of E in $H^{2m}(\mathbb{B}(E), \mathbb{S}(E), \mathbb{Z})$; that is, β_E vanishes when restricted to $\|N\mathcal{V} \cap \mathbb{S}(E)\|$. Then there exists a simplicial form $\mu_E \in \Omega^{2m-1}(\|\mathcal{V}\|)$, also with μ_E vanishing in $\|N\mathcal{V} \cap \mathbb{S}(E)\|$, such that

$$d\mu_E = \varepsilon^* U_E - \beta_E. \tag{7.7}$$

Since $H^{2m-1}(\mathbb{B}(E), \mathbb{S}(E)) = 0$, the form μ_E is unique modulo $\Omega^{2m-1}_{\mathbb{Z}}(\|N\mathcal{V}\|) + d\Omega^{2m-2}(\mathbb{B}(E))$ and hence the Deligne class $[\mu_E] \in H^{2m}_{\mathcal{D}}(\mathbb{B}(E), \mathbb{Z})$ is well–defined. We shall call μ_E the*Thom gerbe* of E.

Proposition 7.1. *The Thom gerbe determines the characteristic Euler gerbe by the formula:*

$$[s^* \mu_E] = [\Lambda(\mathrm{Pf}, e, A)] \in H^{2m}_{\mathcal{D}}(Y, \mathbb{Z}). \tag{7.8}$$

Proof. In the 'universal' case (cf. [10], Proposition 5.3), the differential of both sides of the equation is $\varepsilon^* \,\mathrm{Pf}(F_A) - e(E)$ by (7.5), (7.6) and (7.7). Hence the result follows from the fact that $H^{2m-1}(BSO(2m), \mathbb{R}) = 0$. □

We shall now study the Euler and Thom gerbe in particular for $E = \nu_M = \nu$, where $\nu \to M$ is the normal bundle of a submanifold $M^d \subset X^n$, which we now assume to be of even codimension $n - d = 2m$. Here X as usual is a compact oriented Riemannian manifold. In this case, we identify ν with a tubular neighborhood V of $M \subset X$ and let $V_0 = \mathbb{B}(\nu) \subset V$.

Now both U_ν and β_ν define (ordinary, respectively simplicial) forms on V with respect to $\overline{V}_0$ and hence $\mu_\nu \in \Omega^{2m-1}(\|N\mathcal{V} \cap \mathbb{B}(\nu)\|)$ extends (non–canonically) to a simplicial form $\widetilde{\mu}_\nu \in \Omega^{2m-1}(\|N\mathcal{U}\|)$ for a suitable covering $\mathcal{U}$ of X, extending $\mathcal{V}$ on V. Hence we have

$$[\widetilde{\mu}_\nu] \in H^{2m-1}(X, \mathbb{Z}). \tag{7.9}$$

which we shall call the *extended Thom gerbe*. Once $U_\nu \in \Omega^{2m}(V)$ is chosen, $[\widetilde{\mu}_\nu]$ is well-defined, independent of the choice of $\mathcal{U}$ and the choice of μ_ν. But it does depend on the 'scaling' of $U_\nu \in \Omega^{2m}(V)$. This of course is not the case for $[\mu_\nu] \in H^{2m}_{\mathcal{D}}(M)$, since $U_\nu|_{V_0}$ has a canonical form.

8. Comparison of the Abel gerbe and the Euler gerbe

Continuing with the situation in Section 7 of a submanifold $M^d \subset X$, of even codimension, we want to compare the Euler gerbe with the Abel gerbe associated to M. Thus let $M \subset V_0 \subset \overline{V}_0 \subseteq V$ be a tubular neighborhood of M and let $U_\nu \in \Omega^{n-d}_c(V_0)$ be the 'canonical' Thom class representative. Extend it to X by 0 outside V_0 (also denoted by U_ν). With this we can define a *topologically trivial* gerbe, called the *difference gerbe*

$$[\tau_M] \in H^{n-d}_{\mathcal{D}}(X)$$

as follows:
From the beginning of Section 4, recall that $F = F_Z|_{X-|Z|} = * \, d * H_Z|_{X-|Z|}$ is smooth and satisfies (4.3); that is $dF = \eta_Z|_{X-|Z|}$. Triangulate $M \subset X$ and choose the covering $\mathcal{U}$ as in Section 4; choose a partition of unity $\{\varphi_i\}_{i=1,\dots,N}$ subordinate to $\mathcal{U}$ and define *smooth* forms in $\Omega^{n-d}(X)$:

$$\begin{aligned}
\zeta_0 &= \sum_{i\le m} \varphi_i \, \eta_M \quad , \quad \zeta_1 = \sum_{i>m} \varphi_i \, \eta_M,\\
\zeta_2 &= \sum_{i>m} d\varphi_i \wedge F = -\sum_{i\le m} d\varphi_i \wedge F,\\
G_1 &= \sum_{i>m} \varphi_i \, F.
\end{aligned}$$

Again, we have

$$\begin{aligned}
dG_1 &= \zeta_1 + \zeta_2 \quad , \quad \eta_M = \zeta_0 + \zeta_1,\\
\eta_M &= (\zeta_0 - \zeta_2) + dG_1,
\end{aligned}$$

with $\operatorname{supp}(\zeta_0 - \zeta_2) \subseteq V$. Then $d(\zeta_0 - \zeta_2) = 0$ and hence $\zeta_0 - \zeta_2 = U_\nu + d\lambda$, for $\lambda \in \Omega^{n-d-1}(X)$, $\operatorname{supp}(\lambda) \subseteq V$. Then we put $\tau_M = \lambda + G_1 \in \Omega^{n-d-1}(X)$, so that $d\tau_M = d\lambda + dG_1 = (\zeta_0 - \zeta_2 - U_\nu) + dG_1 = \eta_M - U_\nu$; that is, τ_M satisfies

$$\begin{aligned}
\tau_M &= \lambda + G_1 \in \Omega^{n-d-1}(X),\\
d\tau_M &= \eta_M - U_\nu.
\end{aligned} \tag{8.1}$$

Hence we get $[\tau_M] \in H^{n-d}_{\mathcal{D}}(X)$, and again this is well-defined (even independent of the choice of $\mathcal{U}$ and $\{\varphi_i\}_{i=1,\dots,N}$) once U_ν is chosen. Again, since $U_\nu|_{V_0}$ has a canonical form, we have of course that $\tau_M|_{V_0} \in H^{n-d}_{\mathcal{D}}(V_0)$ and $\tau_M|_M \in H^{n-d}_{\mathcal{D}}(M)$ are well-defined. Also note that in a neighborhood (say V_0), we have $G_1|_{V_0} = 0$, so that $\tau|_{V_0} = \lambda$.

Theorem 8.1. (1) *In $H^{n-d}_{\mathcal{D}}(X,\mathbb{Z})$, we have*

$$[\Lambda_M] = [\widetilde{\mu}_\nu] + \iota_*\,[\tau_M] \tag{8.2}$$

and all three are well-defined except that they depend on a 'scaling' of U_ν. Further, the characteristic class of Λ_M and $\widetilde{\mu}_\nu$ is $[\beta_M] = [\beta_\nu] \in H^{n-d}(X,\mathbb{Z})$ and τ_M is a topologically trivial gerbe with curvature $\eta_M - U_\nu$, where $\eta_M \in \mathcal{H}^{n-d}(X,\mathbb{Z})$.
(2) *In particular, in $H^{n-d}_{\mathcal{D}}(M,\mathbb{Z})$, we have*

$$[\Lambda_M|_M] = [\Lambda(\mathrm{Pf}, e, A)] + \iota_*\,[\tau_M|_M]. \tag{8.3}$$

Proof. First we observe that the integral simplicial form $\beta_M = \beta_{Z_M}$, representing the Poincaré dual of $[M] \in H_d(X)$, which was constructed at the beginning of the proof of Theorem 4.1 and in Remark 4.3, can now be chosen to be the integral representative β_ν of the Thom class for the normal bundle ν. Since

$$\eta_M - \beta_M = \eta_M - \beta_\nu = (\eta_M - U_\nu) + (U_\nu - \beta_\nu),$$

we have

$$d\Lambda_M = d\tau_M + d\widetilde{\mu}_\nu;$$

that is

$$d(\Lambda_M - \tau_M - \widetilde{\mu}_\nu) = d(\gamma + F_1 - (\lambda + G_1) - \widetilde{\mu}_\nu) = 0,$$

where γ, F_1 are as in the proof of Theorem 4.1.

Now, choosing $\mathcal{U}$ suitable, we can assume $(F_1 - G_1)|_{X-\overline{W}} = 0$, and since again

$$H^{n-d-1}(\overline{W}, \partial\overline{W}) \cong H^{n-d-1}(\mathbb{B}(\nu), \mathbb{S}(\nu)) = 0,$$

we get that

$$(\gamma - \lambda) + (F_1 - G_1) - \widetilde{\mu}_\nu \in d\Omega^{n-d-2}(\|\mathcal{U}\|).$$

The theorem is proved. □

Corollary 8.2. *Suppose that $[\Lambda_M] = 0 \in H^{n-d}_{\mathcal{D}}(X,\mathbb{Z})$; that is, $M \subset X$ is linearly equivalent to zero. Then the Euler gerbe $[\Lambda(\mathrm{Pf}, e, A)] \in H^{n-d}_{\mathcal{D}}(M,\mathbb{Z})$ is topologically trivial, given by the global gerbe $[\Lambda(\mathrm{Pf}, e, A)] = -\iota_*[\tau_M|_M]$.*

Proof. This follows directly from Theorem 8.1 (2). □

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