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A Taylor-like Expansion of a Commutator with a Function of Self-adjoint, Pairwise Commuting Operators

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Abstract

Let A be a ν-vector of self-adjoint, pairwise commuting operators and B a bounded operator of class $C^{n_0}(A)$. We prove a Taylor-like expansion of the commutator $[B, f(A)]$ for a large class of functions $f\colon \mathbb{R}^\nu \to \mathbb{R}$, generalising the one-dimensional result where A is just a self-adjoint operator. This is done using almost analytic extensions and the higher-dimensional Helffer-Sjöstrand formula.

Keywords: Commutator expansions, functional calculus, almost analytic extensions, Helffer-Sjöstrand formula.

Mathematical Subject Classification (2010): 47B47

1 Introduction

It is well-known that if A is a self-adjoint operator, B is a bounded operator of class $C^{n_0}(A)$ in the sense of [1] and f satisfies $|f^{(n)}(x)| \leq C_n \langle x\rangle^{s-n}$ for all n, then for $0 \leq t_1 \leq n_0$, $0 \leq t_2 \leq 1$ with $s + t_1 + t_2 < n_0$,

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} f^{(k)}(A)\, \mathrm{ad}_A^k(B) + R_{n_0}(A, B)$$

where $\mathrm{ad}_A^k(B)$ is the k'th iterated commutator, $R_{n_0}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}; \mathcal{H}_A^{t_1})$ and $\mathcal{H}_A^t$ is defined as $\mathcal{D}(\langle A\rangle^t)$ equipped with the graph-norm $\|v\|_t = \|\langle A\rangle^t v\|$ for $t \geq 0$ and $\mathcal{H}_A^{-t}$ is the dual space of $\mathcal{H}_A^t$. This follows relatively easily from using the (one-dimensional) Helffer-Sjöstrand formula

$$f(A) = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(A - z)^{-1} dz, \tag{1}$$

where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\tilde{f}$ is an almost analytic extension of f, and the identity

$$\begin{aligned}[B, f(A)] = & \sum_{k=1}^{n_0-1} \frac{1}{k!}\frac{k!}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(-1)^k (A - z)^{-k-1} dz \\ & + \frac{(-1)^{n_0}}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(A - z)^{-n_0}\, \mathrm{ad}_A^{n_0}(B)(A - z)^{-1} dz\end{aligned}$$

when $\frac{k!}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(-1)^k(A - z)^{-k-1}dz$ is recognised as $f^{(k)}(A)$ using (1). See e.g. [4] for details. Due to the higher complexity of the general Helffer-Sjöstrand formula, these calculations do not lead directly to the generalised result where A is a vector of self-adjoint, pairwise commuting operators. However, we will follow the same idea.

The theorem may be viewed as an abstract analogue of pseudo-differential calculus. The one-dimensional version is an often used result, see e.g. [2] and [4]. Apart from the obvious interest in generalising the result to higher dimensions, our improvement has proven useful in the treatment of models in quantum field theory, see [6]. In particular, a lemma in [6] whose proof depends on our result, extends the results of [5] to a larger class of models.

2 The setting and result

In the following, $A = (A_1, \ldots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [1]. For notational convenience, we adobt the following convention: If $0 \leq j \leq \nu$, then δ_j denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the j'th entry.

Definition 1. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^\alpha(B) = [\mathrm{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where $\alpha \geq \delta_j$ is a multi-index and $1 \leq j \leq \nu$, can be represented by a bounded operator also denoted by $\mathrm{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0 + 1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

Remark 2. The definition of $\mathrm{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\mathrm{ad}_A^\alpha(B)$.

In the following, $\mathcal{H}_A^s := D(|H|^s)$ for $s \geq 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathcal{H}_A^s := (\mathcal{H}_A^{-s})^*$.

Theorem 3. *Assume that $B \in C^{n_0}(A)$ for some $n_0 \geq n+1 \geq 1$, $0 \leq t_1 \leq n+1$, $0 \leq t_2 \leq 1$ and that $\{f_\lambda\}_{\lambda \in I}$ satisfies*

$$\forall \alpha \, \exists C_\alpha \colon |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$$

uniformly in λ for some $s \in \mathbb{R}$ such that $t_1 + t_2 + s < n+1$. Then

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \, \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A,B)$$

as an identity on $\mathcal{D}(\langle A \rangle^s)$, where $R_{\lambda,n}(A,B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$ and there exist a constant C independent of A, B and λ such that

$$\|R_{\lambda,n}(A,B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \leq C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

Remark 4. A similar statement holds with the $\mathrm{ad}_A^\alpha(B)$ and $\partial^\alpha f_\lambda(A)$ interchanged at the cost of a sign correction given by $(-1)^{|\alpha|-1}$, and the corresponding remainder term $R'_{\lambda,n}(A,B) \in \mathcal{B}(\mathcal{H}_A^{-t_1}, \mathcal{H}_A^{t_2})$. This can be seen either by proving it analogously or by taking the adjoint equation and replacing B by $-B$.

Remark 5. If $k \leq t_1$ and $n_0 \geq n+1+k$, then $R_{\lambda,n}(A,B)$ can be replaced by $R^k_{\lambda,n}(A,B) \in \mathcal{B}(\mathcal{H}_A^{-t_2+k}, \mathcal{H}_A^{t_1-k})$. This can be seen by commuting $|A-z|^{-2}$ and $\mathrm{ad}_A^\alpha(B)$ in the terms of the remainder, see page 8.

3 The Proof

Let $z \in \mathbb{C}^\nu$, $\mathrm{Im}\, z \neq 0$, $1 \leq \ell \leq \nu$ and $g, g_\ell \colon \mathbb{R}^\nu \to \mathbb{C}$ be given as $g(t) = |t-z|^{-2}$ and $g_\ell(t) = t_\ell - \bar{z}_\ell$. Write for $2\beta \leq \alpha$

$$T_\alpha^\beta(t,z) := \tfrac{(-2)^{|\alpha-\beta|}|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t - \mathrm{Re}\, z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|}.$$

Lemma 6. *Let g be as above and α be any multi-index. Then*

$$\partial^\alpha g(t) = \sum_{2\beta \leq \alpha} \alpha! T_\alpha^\beta(t,z)|t-z|^{-2}.$$

Proof. For brevity, we will write α^i or β^i for $\alpha + \delta_i$ or $\beta + \delta_i$, respectively. The formula is obviously true for $|\alpha| \leq 1$. Now assume that we have proven the formula for $|\alpha| \leq k$. Let $|\alpha| = k$ and $0 \leq i \leq \nu$ be arbitrary. It suffices to prove the formula for α^i. One easily verifies using the chain rule that

$$(\partial^{\delta_i} g^n)(t) = -2n(t_i - \mathrm{Re}\, z_i)|t-z|^{-2n-2}. \tag{2}$$

Now by the induction hypothesis, we see that

$$\begin{aligned}\partial^{\alpha+\delta_i} g(t) &= \partial_t^{\delta_i} \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|-2} \\ &= \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\partial_t^{\delta_i}(t-\operatorname{Re} z)^{\alpha-2\beta})|t-z|^{-2|\alpha-\beta|-2} \qquad (3)\\ &\quad+\sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(\partial_t^{\delta_i}|t-z|^{-2|\alpha-\beta|-2}). \qquad (4)\end{aligned}$$

For the sake of clarity, we will now consider each sum independently.

$$\begin{aligned}(3) &= \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\alpha_i-2\beta_i)(t-\operatorname{Re} z)^{\alpha-2\beta-\delta_i}|t-z|^{-2|\alpha-\beta|-2} \\ &= \sum_{\substack{2\beta\le\alpha\\ 2\beta_i<\alpha_i}} 2(\beta_i+1)\tfrac{(-2)^{|\alpha^i-\beta^i|}\alpha!|\alpha^i-\beta^i|!}{2^{|\beta^i|}\beta^i!(\alpha^i-2\beta^i)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta^i}|t-z|^{-2|\alpha^i-\beta^i|-2} \\ &= \sum_{2\beta\le\alpha+\delta_i} 2\beta_i \tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \qquad (5)\end{aligned}$$

Using (2), we see that (4) equals

$$\begin{aligned}&\sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(-2)(|\alpha-\beta|+1)(t_i-\operatorname{Re} z_i)|t-z|^{-2|\alpha-\beta|-4} \\ &= \sum_{2\beta\le\alpha}(\alpha_i+1-2\beta_i)\tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2} \\ &= \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2} \qquad (6)\\ &\quad-\sum_{2\beta\le\alpha} 2\beta_i\tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \qquad (7)\end{aligned}$$

Now (7) cancels (5) except for possible terms with $2\beta=\alpha+\delta_i$:

$$(5)+(7) = \sum_{2\beta=\alpha+\delta_i} \tfrac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \qquad (8)$$

Adding (6) and (8) finishes the induction. □

Lemma 7. *Let $B\in C^{n_0}(A)$ for some $n_0\ge 1$ and let $n\in\mathbb{N}_0$ and α_0 be a multi-index satisfying $|\alpha_0|+n+1\le n_0$. Then*

$$[\operatorname{ad}_A^{\alpha_0}(B), g(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha g(A)\operatorname{ad}_A^{\alpha_0+\alpha}(B) + R_n^g(A, \operatorname{ad}_A^{\alpha_0}(B)), \qquad (9)$$

where

$$
\begin{aligned}
&R_n^g(A, \operatorname{ad}_A^{\alpha_0}(B)) \\
&\quad = \sum_{\substack{|\alpha|=n-1 \\ 2\beta\le\alpha}} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} && (10)\\
&\qquad + \sum_{\substack{|\alpha|=n \\ 2\beta\le\alpha}} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)(A_i-\bar z_i)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2} && (11)\\
&\qquad + \sum_{\substack{|\alpha|=n \\ 2\beta\le\alpha}} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)(A_i-z_i)|A-z|^{-2}. && (12)
\end{aligned}
$$

Proof. The proof goes by induction. One may check by inspection of the following identity that the statement is true for $n = 0$.

$$
\begin{aligned}
[\operatorname{ad}_A^{\alpha_0}(B), |A-z|^{-2}] = &-\sum_{i=1}^{\nu}|A-z|^{-2}(A_i-\bar z_i)\operatorname{ad}_A^{\alpha_0+\delta_i}(B)|A-z|^{-2} \\
&-\sum_{i=1}^{\nu}|A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\delta_i}(B)(A_i-z_i)|A-z|^{-2}.
\end{aligned} \tag{13}
$$

Now assume that we have proven the formula for $k \le n$, $|\alpha_0| + n + 2 \le n_0$. We will now show that this implies that the formula holds for $k = n+1$. We begin by noting two useful identities.

$$T_\alpha^\beta(t,z)|t-z|^{-2} = -\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(t,z). \tag{14}$$

$$(\beta_i+1)T_{\alpha+2\delta_i}^{\beta+\delta_i}(t,z)2(t_i - \operatorname{Re} z_i) = (\alpha_i+1-2\beta_i)T_{\alpha+\delta_i}^{\beta}(t,z). \tag{15}$$

Now using (13) and (14) we see that

$$
\begin{aligned}
(10) = &\sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)|A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B) && (16)\\
&+\sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A,z) \\
&\qquad \times (A_j-\bar z_j)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)|A-z|^{-2} && (17)\\
&+\sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A,z) \\
&\qquad \times \operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)(A_j-z_j)|A-z|^{-2}, && (18)
\end{aligned}
$$

and by reordering and reindexing the sum in (16), (17) and (18), we get

$$(16) = \sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1 \\ \alpha_i\ge 2}}\sum_{\substack{2\beta\le\alpha \\ \beta_i\ge 1}} \tfrac{\beta_i}{|\alpha-\beta|} T_\alpha^\beta(A,z)|A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\alpha}(B), \tag{19}$$

and (17) equals

$$\sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\geq 2}}\sum_{\substack{2\beta\leq\alpha\\ \beta_i\geq 1}}\sum_{j=1}^{\nu}\tfrac{\beta_i}{|\alpha-\beta|}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)(A_j-\bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2} \tag{20}$$

and similarly for (18) with the factor $(A_j-\bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)$ replaced by the factor $\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)$. Note that we may relax the extra conditions on α and β in the above statements, as a term with $\beta_i=0$ contributes nothing.

Instead of continuing in the same fashion with (11) and (12), we note using (15) that

$$(11)+(12)=\sum_{|\alpha|=n}\sum_{2\beta\leq\alpha}\sum_{i=1}^{\nu}\tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} \tag{21}$$

$$+\sum_{|\alpha|=n}\sum_{2\beta\leq\alpha}\sum_{i=1}^{\nu}\tfrac{\alpha_i+1-2\beta_i}{|\alpha+\delta_i-\beta|}T^{\beta}_{\alpha+\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2}, \tag{22}$$

so we may focus our attention on (22):

$$(22)=\sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\geq 1}}\sum_{\substack{2\beta\leq\alpha\\ 2\beta_i<\alpha_i}}\tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|}T^{\beta}_{\alpha}(A,z)|A-z|^{-2}\operatorname{ad}_A^{\alpha_0+\alpha}(B) \tag{23}$$

$$\begin{aligned}&+\sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\geq 1}}\sum_{\substack{2\beta\leq\alpha\\ 2\beta_i<\alpha_i}}\sum_{j=1}^{\nu}\tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)\\ &\quad\times(A_j-\bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}.\end{aligned} \tag{24}$$

$$\begin{aligned}&+\sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\geq 1}}\sum_{\substack{2\beta\leq\alpha\\ 2\beta_i<\alpha_i}}\sum_{j=1}^{\nu}\tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)\\ &\quad\times\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)|A-z|^{-2}\end{aligned} \tag{25}$$

We note again that the additional conditions on α and β are superfluous.

We may now recollect the terms. First we see using Lemma 6:

$$\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha g(A)\operatorname{ad}_A^{\alpha_0+\alpha}(B)+(19)+(23)=\sum_{|\alpha|=1}^{n+1}\frac{1}{\alpha!}\partial^\alpha g(A)\operatorname{ad}_A^{\alpha_0+\alpha}(B), \tag{26}$$

then

$$(20)+(24)=\sum_{\substack{|\alpha|=n+1\\ 2\beta\leq\alpha}}\sum_{j=1}^{\nu}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)(A_j-\bar{z}_j)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}, \tag{27}$$

and

$$(18)+(25)=\sum_{\substack{|\alpha|=n+1\\ 2\beta\leq\alpha}}\sum_{j=1}^{\nu}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)|A-z|^{-2}, \tag{28}$$

so adding up, we have proved that (9) equals the sum of (26), (21), (27) and (28) as stated. □

The following lemma plays the same role for g_ℓ as Lemma 7 plays for g, but contrary to Lemma 7, the proof is trivial.

Lemma 8. *Let $B \in C^{n_0}(A)$ for some $n_0 \geq 1$ and let $n \in \mathbb{N}_0$ and α_0 be a multi-index satisfying $|\alpha_0| + n + 1 \leq n_0$. Then*

$$[\mathrm{ad}_A^{\alpha_0}(B), g_\ell(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha g_\ell(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B) + R_n^{g_\ell}(A, \mathrm{ad}_A^{\alpha_0}(B)),$$

where $R_n^{g_\ell}(A, \mathrm{ad}_A^{\alpha_0}(B)) = 0$ for $n \geq 1$, $R_0^{g_\ell}(A, \mathrm{ad}_A^{\alpha_0}(B)) = \mathrm{ad}_A^{\alpha_0+\delta_\ell}(B)$.

The following lemma also follows by induction.

Lemma 9. *Let $B \in C^{n_0}(A)$ for some $n_0 \geq 1$. Assume that $h_i \in C^\infty(\mathbb{R}^\nu)$, $1 \leq i \leq k$, satisfies*

$$[\mathrm{ad}_A^{\alpha_0}(B), h_i(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha h_i(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B) + R_n^{h_i}(A, \mathrm{ad}_A^{\alpha_0}(B)),$$

where $R_n^{h_i}(A, \mathrm{ad}_A^{\alpha_0}(B))$ is bounded for all $n \in \mathbb{N}_0$ and multi-indices α_0 satisfying $|\alpha_0| + n + 1 \leq n_0$ and $\partial^\alpha h_i(A)$ is bounded for all $1 \leq |\alpha| \leq n_0 - 1$. Then

$$\begin{aligned}\Big[B, \prod_{i=1}^{k} h_i(A)\Big] &= \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{k} h_i\Big)(A)\,\mathrm{ad}_A^{\alpha}(B) \\ &\quad + \sum_{j=1}^{k}\sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{j-1} h_i\Big)(A) R_{n-|\alpha|}^{h_j}(A, \mathrm{ad}_A^{\alpha}(B)) \prod_{i=j+1}^{k} h_i(A).\end{aligned}$$

Let $n+1 \leq n_0$. If we put $k = \nu+1$, $h_i = g$ for $i \neq \nu$, $h_\nu = g_\ell$ and apply Lemma 7, 8 and 9 we see that

$$\begin{aligned}&[B, |A-z|^{-2\nu}(A_\ell - \bar{z}_\ell)] \\ &\qquad = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha\big(|\cdot - z|^{-2\nu}(\cdot_\ell - \bar{z}_\ell)\big)(A)\,\mathrm{ad}_A^{\alpha}(B) + R_{\ell,n}(A,B),\end{aligned} \tag{29}$$

where

$$\begin{aligned}&R_{\ell,n}(A,B) \\ &\quad = \sum_{j=1}^{\nu-1}\sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\partial^\alpha(g^{j-1})(A) R_{n-|\alpha|}^{g}(A, \mathrm{ad}_A^{\alpha}(B))|A-z|^{-2(\nu-j)}(A_\ell - \bar{z}_\ell) &(30)\\ &\qquad + \sum_{|\alpha|=n} \frac{1}{\alpha!}\partial^\alpha(g^{\nu-1})(A)\,\mathrm{ad}_A^{\alpha+\delta_\ell}(B)|A-z|^{-2} &(31)\\ &\qquad + \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\partial^\alpha(g^{\nu-1}g_\ell)(A) R_{n-|\alpha|}^{g}(A, \mathrm{ad}_A^{\alpha}(B)) &(32)\end{aligned}$$

In the following, we will refer to the terms of $R_{\ell,n}(A,B)$ as the remainder terms. Let $0 \le t_1 \le n+1$ and $0 \le t_2 \le 1$. By Hadamard's three-line lemma and using (10–12), (30–32), Lemma 6 and the identity

$$\partial^\alpha\Big(\prod_{i=1}^{j} f_i\Big) = \sum_{\sum \alpha_i = \alpha} \frac{\alpha!}{\prod_{i=1}^{j} \alpha_i!} \prod_{i=1}^{j} \partial^{\alpha_i} f_i,$$

we may inspect that each remainder term (with $R_{\ell,n}(A,B)$ replaced by the remainder term) and hence $R_{\ell,n}(A,B)$ satisfies the inequality

$$\|\langle A\rangle^{t_1} R_{\ell,n}(A,B)\langle A\rangle^{t_2}\| \le C\langle z\rangle^{t_1+t_2}|\mathrm{Im}\, z|^{-n-2\nu}. \tag{33}$$

We will now use the functional calculus of almost analytic extensions. See e.g. [3] for details. In the following, we write $\bar\partial = (\bar\partial_1, \dots, \bar\partial_\nu)$ where $\bar\partial_j = \frac{1}{2}(\partial_{u_j} + i\partial_{v_j})$ and $u_j + v_j = z_j \in \mathbb{C}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^\nu$. The following proposition is inspired by [4] and [7, Chap. X.2].

Proposition 10. *Let $s \in \mathbb{R}$ and $\{f_\lambda\}_{\lambda\in I} \subset C^\infty(\mathbb{R}^\nu)$ satisfy*

$$\forall\alpha\ \exists C_\alpha\colon |\partial^\alpha f_\lambda(x)| \le C_\alpha\langle x\rangle^{s-|\alpha|}.$$

There exists a family of almost analytic extensions $\{\tilde f_\lambda\}_{\lambda\in I} \subset C^\infty(\mathbb{C}^\nu)$ satisfying

(i) $\mathrm{supp}(\tilde f_\lambda) \subset \{u + iv \mid u \in \mathrm{supp}(f_\lambda), |v| \le C\langle u\rangle\}$.

(ii) $\forall \ell \ge 0\ \exists C_\ell\colon |\bar\partial \tilde f_\lambda(z)| \le C_\ell\langle z\rangle^{s-\ell-1}|\mathrm{Im}\, z|^\ell$.

Proof. We define a mapping $C^\infty(\mathbb{R}^\nu) \ni f \mapsto \tilde f \in C^\infty(\mathbb{C}^\nu)$ in the following way. Choose a function $\kappa \in C_0^\infty(\mathbb{R})$ which equals 1 in a neighbourhood of 0 and put $\lambda_0 = C_0$, $\lambda_k = \max\{\max_{|\alpha|=k} C_\alpha, \lambda_{k-1}+1\}$ for $k \ge 1$. Writing $z = u + iv \in \mathbb{R}^\nu \oplus i\mathbb{R}^\nu$, we now define

$$\tilde f(z) = \sum_\alpha \frac{\partial^\alpha f(u)}{\alpha!}(iu)^\alpha \prod_{j=1}^{\nu} \kappa\Big(\frac{\lambda_{|\alpha|} v_j}{\langle u\rangle}\Big).$$

One can now check that the properties hold. □

Remark 11. Note that if we for a $\chi \in C_0^\infty(\mathbb{R}^\nu; [0,1])$ with $\chi(0) = 1$ define a sequence of functions by $f_{k,\lambda}(x) = \chi(\frac{x}{k})f_\lambda(x)$, then

$$[B, f_\lambda(A)] = \lim_{k\to\infty}[B, f_{k,\lambda}(A)]$$

as a form identity on $\mathcal{D}(\langle A\rangle^s)$ and we have the dominated pointwise convergence

$$\bar\partial \tilde f_{k,\lambda}(x) \to \bar\partial \tilde f_\lambda(x) \text{ for } k \to \infty.$$

Let $\{f_\lambda\}_{\lambda\in I}$ satisfy the assumption of Proposition 10 with $s < 0$. Then the almost analytic extensions provide a functional calculus via the formula

$$f_\lambda(A) = C_\nu \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)(A_\ell - \bar z_\ell)|A - z|^{-2\nu} dz, \tag{34}$$

where C_ν is a positive constant (again we refer to [3] for details). Note that the integrals are absolutely convergent by Proposition 10(ii).

Multiplying $\langle A\rangle^{t_1}R_{\ell,n}(A,B)\langle A\rangle^{t_2}$ with $\bar\partial\tilde f_\lambda(z)$, we get from (33) and Proposition 10 (ii) that

$$\|\langle A\rangle^{t_1}\bar\partial\tilde f_\lambda(z)R_{\ell,n}(A,B)\langle A\rangle^{t_2}\| \le C\langle z\rangle^{t_1+t_2+s-n-1-2\nu}. \tag{35}$$

Hence, if $t_1+t_2+s<n+1$, $\langle A\rangle^{t_1}\bar\partial\tilde f_\lambda(z)R_{\ell,n}(A,B)\langle A\rangle^{t_2}$ is integrable over $\mathbb{C}^\nu$. Using (29), (34) and (35), we see that

$$\begin{aligned}
[B,f_\lambda(A)] &= C_\nu\sum_{\ell=1}^{\nu}\int_{\mathbb{C}^\nu}\bar\partial_\ell\tilde f_\lambda(z)[B,(A_\ell-\bar z_\ell)|A-z|^{-2\nu}]\,dz\\
&= C_\nu\sum_{\ell=1}^{\nu}\int_{\mathbb{C}^\nu}\bar\partial_\ell\tilde f_\lambda(z)\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha\big(|\cdot-z|^{-2\nu}(\cdot_\ell-\bar z_\ell)\big)(A)\,dz\ \mathrm{ad}_A^\alpha(B)\\
&\quad + C_\nu\sum_{\ell=1}^{\nu}\int_{\mathbb{C}^\nu}\bar\partial_\ell\tilde f_\lambda(z)R_{\ell,n}(A,B)\,dz.
\end{aligned}\tag{36}$$

We denote (36) by $R_{\lambda,n}(A,B)$. Note that

$$\begin{aligned}
&\sum_{\ell=1}^{\nu}\int_{\mathbb{C}^\nu}\bar\partial_\ell\tilde f_\lambda(z)\frac{1}{\alpha!}\partial_t^\alpha\big(|t-z|^{-2\nu}(t_\ell-\bar z_\ell)\big)\,dz\\
&\qquad=\frac{1}{\alpha!}\partial_t^\alpha\sum_{\ell=1}^{\nu}\int_{\mathbb{C}^\nu}\bar\partial_\ell\tilde f_\lambda(z)|t-z|^{-2\nu}(t_\ell-\bar z_\ell)\,dz=\frac{1}{\alpha!}\partial^\alpha f_\lambda(t),
\end{aligned}$$

which implies

$$[B,f_\lambda(A)]=\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha f_\lambda(A)\ \mathrm{ad}_A^\alpha(B)+R_{\lambda,n}(A,B).$$

We have now proved Theorem 3 in the case $s<0$. For the general case, we use Remark 11 to see that $[B,f_\lambda(A)]=\lim_{k\to\infty}[B,f_{k,\lambda}(A)]$ and clearly, $f_{k,\lambda}$ satisfies the assumption of Proposition 10 with the same s, so the estimate corresponding to (35) is now uniform in k and λ. The pointwise convergence and Lebesgue's theorem on dominated convergence now finishes the argument.

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