AARHUS UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES

ISSN: 1397–4076

SPECTRAL THEORY OF THE LAPLACIAN ON THE MODULAR JACOBI GROUP MANIFOLD

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Preprint Series No. 3

August 2011

Publication date: 2011/08/26

Published by

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Spectral theory of the Laplacian on the modular Jacobi group manifold

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Abstract

The reduced modular Jacobi group is a semidirect product of $\mathrm{SL}_2(\mathbb{Z})$ with the lattice $\mathbb{Z}^2$. We develop the spectral theory of the invariant Laplacian L on the associated group manifold. The operator L is decomposed by Fourier analysis as a direct sum of operators L_{kl} corresponding to frequencies k related to the lattice and l to translations. L_{00} is the Selberg Laplacian for $\mathrm{SL}_2(\mathbb{Z})$. For $k, l \geq 1$, L_{kl} has a purely discrete spectrum, while L_{k0} has a purely continuous spectrum for $k \geq 1$. The set of all eigenvalues of L satisfies a Weyl law. The results are extended to subgroups of the modular Jacobi group of finite index.

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Introduction

The present paper deals with the Jacobi group Γ_J which is the semidirect product of $\mathrm{SL}_2(\mathbb{R})$ with the Heisenberg group, the group of upper triangular, idempotent 3×3 matrices ([1], [3]). Dividing out the center of the Heisenberg group and restricting to integers, we obtain the reduced modular Jacobi group Γ_{1J}. The group Γ_{1J} is isomorphic to the semidirect product $\Gamma_1 \ltimes \mathbb{Z}^2$ of the modular group Γ_1 with the additive group $\mathbb{Z}^2$. We study the spectral theory of the invariant Laplacian on the group manifold Γ_{1J} and its subgroups of finite index.

In section 1 we develop the spectral theory of the Γ_{1J}-invariant Laplacian L. We obtain a decomposition of L as a direct sum of operators L_k, $k = 0, 1, 2, \ldots$, where L_0 is the usual automorphic Laplacian $A = -y^2\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)$, while L_k for $k \geq 1$ is unitarily equivalent to the operator $L_k^0 = A + 4\pi^2\frac{k^2}{y}$ in $L^2(F_{\Gamma_\infty}; d\mu)$. For $k \geq 1$, separation of variables leads to a further decomposition of L_k^0 as a sum of ordinary differential operators $\tilde{L}_{kl} = -y^2\frac{d^2}{dy^2} + 4\pi^2\big(\frac{k^2}{y} + l^2y^2\big)$, $l = 0, 1, 2, \ldots$ The operator $\tilde{L}_{k0}$ has a purely continuous, simple spectrum without resonances. For $l \geq 1$ the spectrum of $\tilde{L}_{kl}$ consists of a sequence of simple eigenvalues $\lambda_{kl}^n \to \infty$ as $n \to \infty$. This provides a complete spectral decomposition of L, formulated in Theorem 1.

In section 2 we study the counting function $N(\lambda)$ for the eigenvalues of the operator L. Based on a result of Titchmarsh [7] on the asymptotics of the counting function for ordinary differential operators we obtain the asymptotics of $N(\lambda)$ by summing the counting functions $N_{kl}(\lambda)$ of $\tilde{L}_{kl}$ over k and l. We obtain the Weyl law for the operator L, expressed in Theorem 2.

In section 3 we extend these results to normal subgroups Γ of Γ_1 of finite index I. We obtain a decomposition of L_{Γ_J} as a direct sum of operators $L_{\Gamma k}$, $k = 0, 1, 2, \ldots$, where $L_{\Gamma 0}$ is the Γ-automorphic Laplacian and for each $k \geq 1$ the operator $L_{\Gamma k}$ splits into a sum of I/β operators $L_{\Gamma k}^i$, where β is the width of Γ and $L_{\Gamma k}^i$ is unitarily equivalent to the operator $L_{\Gamma k}^0$ in $L^2(\Gamma_\infty; d\mu)$. Thus, the spectrum of $L_{\Gamma k}$ depends both on I and β. For $k \geq 1$ the eigenvalues of $L_{\Gamma k}$ are the eigenvalues of the ordinary differential operators $\tilde{L}_{\beta kl} = -y^2\frac{d^2}{dy^2} + 4\pi^2\big(\frac{k^2}{y} + \beta^{-2}l^2y^2\big)$, $l = 1, 2, \ldots$ From this we obtain the asymptotic counting function $IN(\lambda)$, which proves the Weyl law for L_{Γ_J}. The continuous spectrum of $L_{\Gamma k}$ has multiplicity I/β for each $k = 0, 1, 2, \ldots$ The results are formulated in Theorem 3.

In section 4 we study the perturbation of $\Gamma(2)_J$ by characters $\chi(\alpha)$ defined by a holomorphic modular form of weight 2 (Eisenstein series). For each $k = 0, 1, 2, \ldots$, two cusps are closed, and the multiplicity of the continuous spectrum is reduced from 3 to 1 for $\alpha \neq 0$. Eigenvalues λ_{kl}^n of L_{Γ_J} continue smoothly as eigenvalues $\lambda_{kl}^n(\alpha)$ of $L_{\Gamma(2),J}(\alpha)$ for $\alpha \neq 0$. Moreover, in the two closed susps new sequences of eigenvalues $\lambda_{k0}^n(\alpha)$ appear, converging to $\frac{1}{4}$ as $\alpha \to 0$ and replacing the continuous spectrum. The Weyl law remains valid for all α. The results are given in Theorem 4.

In section 5 we consider a few important examples of non-normal subgroups Γ of Γ_1. We establish in Theorem 5.1 the spectral decomposition of L_{Γ_J} and in particular the Weyl law for the three conjugate groups $\Gamma_0(2)$, $\Gamma^0(2)$, $\Gamma_\vartheta(2)$ of index 3 in the modular group. In Theorem 5.2 we obtain the corresponding results for the conjugate groups $\Gamma_0(4)$, $\Gamma^0(4)$, $\Gamma_\vartheta(4)$ of index 6 in Γ_1. It is interesting here that $\Gamma_0(4)_J$ is not isospectral to $\Gamma(2)_J$. The groups $\Gamma(2)$ and $\Gamma_0(4)$ are conjugate through

$\Gamma(2) = 2\Gamma_0(4)\frac{1}{2}$, but this is not a conjugaton of $\Gamma(2)_J$ and $\Gamma_0(4)_J$. In Theorem 5.3 we consider two conjugate groups of index 6 generated by 3 elliptic elements of order 3 [5]. These groups also have width 6 and their Jacobi groups are therefore for $k \geq 1$ isospectral to the normal subgroup Γ'_J.

1 Spectral theory of the $\Gamma_{1,J}$-invariant Laplacian

We denote by $\Gamma_{1,J}$ the reduced Jacobi group $\Gamma_1 \ltimes \mathbb{Z}^2$ with the elements (g, c), $g \in \Gamma_1$, $c = \binom{a}{b} \in \mathbb{Z}^2$ and

$$(g_1, c_1) \ltimes (g_2, c_2) = (g_1 g_2, g_1^{-1t} c_2 + c_1)$$

for $g_1, g_2 \in \Gamma_1$ and $c_1, c_2 \in \mathbb{Z}^2$.

The 4-dimensional reduced Jacobi manifold M_J has coordinates (z, w) with $z \in h = \{x + iy \mid x \in \mathbb{R}, y > 0\}$, $w = \binom{u}{v} \in \mathbb{R}^2$.

The action of $\Gamma_{1,J}$ on M_J is given by

$$(g, c)(z, w) = (z', w'), \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1, \quad c = \binom{a}{b} \in \mathbb{Z}^2$$

where

$$z' = \left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \quad w' = g^{-1t} w + c$$

In the group $\Gamma_{1,J}$ we identify $\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \binom{a}{b}\right)$ with $\left(\begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}, \binom{-a}{-b}\right)$. So all Γ_1-invariant functions on M_J satisfy $f(z, u, v) = f(z, -u, -v)$.

The fundamental domain of Γ_J can be chosen as

$$F_{\Gamma_J} = F_{\Gamma_1} \times \{(u, v) \mid -\tfrac{1}{2} < u \leq \tfrac{1}{2}, \ -u < v \leq \tfrac{1}{2}\}.$$

We define $T_{(g,c)} f$ for functions f on $h \times \mathbb{R}^2$ by

$$\big(T_{(g,c)} f\big)(z, w) = f\big((g, c)(z, w)\big).$$

For $g \in \Gamma_1$ and f a function on h we set

$$\big(T_g f\big)(z) = f(gz).$$

For f a function on $h \times \mathbb{R}^2$ we set

$$\big(\widetilde{T}_g f\big)(z, w) = \big(T_{g,0} f\big)(z, w) = f\big((g, 0)(z, w)\big).$$

For $k \in \mathbb{N} \setminus \{0\}$, $m \in \mathbb{Z}$ we define $e_{km}(u, v)$ by

$$e_{km}(u, v) = e^{2\pi iku} e^{2\pi imv} + e^{-2\pi iku} e^{-2\pi imv}.$$

The $\Gamma_{1,J}$-invariant Laplacian L on M_J is given [2] by

$$L = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{y} \left\{\frac{\partial^2}{\partial u^2} - 2x \frac{\partial^2}{\partial u \, \partial v} + (x^2 + y^2) \frac{\partial^2}{\partial v^2}\right\} \tag{1.1}$$

L is a self-adjoint operator in $\mathcal{H}_{\Gamma_{1,J}} = L^2(F_{\Gamma_J})$.

Let

$$A = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad \text{in } \mathcal{H}_{\Gamma_1} = L^2(F_{\Gamma_1}; d\mu)$$

Lemma 1.1. *Let f be a $\Gamma_{1,J}$-invariant, continuous function of z, w, C^1 in u, v for fixed z. Then*

$$f(z,u,v) = \sum_{k,m} f_{km}(z) e_{km}(u,v) \tag{1.2}$$

where the functions $f_{km}(z)$ are related by

$$f_{g^{-1}\binom{k}{m}}(z) = \big(T_g f_{km}\big)(z).$$

Proof. Since f is Γ_J-invariant, it is for fixed z $\mathbb{Z}^2$-invariant, so

$$f(z,u,v) = \sum_{k,m} f_{km}(z) e_{km}(u,v) \tag{1.3}$$

and the series is absolutely and uniformly convergent for $u, v \in \mathbb{R}$.

For $g \in \Gamma_1$ we have

$$\begin{aligned} \widetilde{T}_g\big(e^{2\pi iku} e^{2\pi imv}\big) &= e^{2\pi iku'} e^{2\pi ikv'} = e^{2\pi ik(\delta u - \gamma v)} e^{2\pi im(-\beta u + \alpha v)} \\ &= e^{2\pi ik'u} e^{2\pi im'v}, \quad k' = k\delta - m\beta, \;\; m' = -k\gamma + m\alpha \end{aligned}$$

or

$$\binom{k'}{m'} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\binom{k}{m} = g^{-1}\binom{k}{m} \tag{1.4}$$

From (1.3) and (1.4) we obtain for $g \in \Gamma_1$

$$\big(\widetilde{T}_g f\big)(z,u,v) = \sum_{k,m} \big(T_g f_{km}\big)(z) e_{k'm'}(u,v). \tag{1.5}$$

The invariance of f under $\widetilde{T}_g$ means by (1.3) and (1.5)

$$\sum_{k,m} \big(T_g f_{km}\big)(z) e_{k'm'}(u,v) = \sum_{k,m} f_{km}(z) e_{km}(u,v). \tag{1.6}$$

Since $\binom{k}{m} \to \binom{k'}{m'} = g^{-1}\binom{k}{m}$ is a bijection of $\mathbb{Z}^2$ the r.h.s. of (1.6) equals

$$\sum_{k',m'} f_{k'm'}(z) e_{k'm'}(u,v) \tag{1.7}$$

and it follows from (1.6) and (1.7) that for all $g \in \Gamma_1$, $\binom{k}{m} \in \mathbb{Z}^2$

$$f_{k'm'}(z) = \big(T_g f_{km}(z)\big), \quad \binom{k'}{m'} = g^{-1}\binom{k}{m}. \tag{1.8}$$

The Lemma is proved. $\square$

To further analyze the series (1.1) representing the invariant function $f(z,u,v)$ we determine the equivalence classes in $\mathbb{Z}^2$ under the action of Γ_1 as follows.

Let $\binom{k'}{m'} \sim \binom{k''}{m''}$ if $\binom{k'}{m'} = g\binom{k''}{m''}$ for some $g \in \Gamma_1$.

Lemma 1.2. *For each $k \in \mathbb{N}$, the equivalence class of $\binom{k}{0}$ is*

$$\mathbb{Z}_k^2 = \left\{ \binom{k'}{m'} \,\middle|\, (k', m') = k \right\}.$$

The stabilizer of $\binom{k}{0}$,

$$\left\{ g \in \Gamma_1 \,\middle|\, g\binom{k}{0} = \binom{k}{0} \right\}$$

is the translation group $\Gamma_\infty = \left\{ \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) \,\middle|\, \beta \in \mathbb{Z} \right\}$.

Proof. Since $\binom{k}{m} \sim \binom{-k}{-m}$ via $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we restrict ourselves to $k \geq 0$, $m \in \mathbb{Z}$. We determine the equivalence class of $\binom{k}{m}$ for $k \geq 0$.

1) If $(k,m) = 1$, there exist β, δ such that $k\delta - m\beta = 1$. Setting $\alpha = k$, $\gamma = m$, we get $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_1$, such that

$$\binom{k}{m} = g\binom{1}{0}, \quad \text{so } \binom{k}{m} \sim \binom{1}{0}.$$

On the other hand, if $\binom{k}{m} \sim \binom{1}{0}$, there exists $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_1$, such that

$$\binom{k}{m} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\binom{1}{0}, \quad \text{so } \alpha = k,\ \gamma = m$$

and

$$k\delta - m\beta = 1, \quad \text{so } (k,m) = 1.$$

Thus, $\binom{k}{m} \sim \binom{1}{0}$ if and only if $(k,m) = 1$.

2) Let $(k', m') = k$, $k > 1$, $k' = pk$, $m' = qk$, $p \geq 1$, $(p,q) = 1$. Then by 1), $\binom{p}{q} \sim \binom{1}{0}$, and there exists $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_1$, such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\binom{1}{0} = \binom{p}{q}.$$

Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\binom{k}{0} = \binom{pk}{qk} = \binom{k'}{m'}$$

so

$$\binom{k'}{m'} \sim \binom{k}{0}.$$

Conversely, if $\binom{k'}{m'} \sim \binom{k}{0}$, for some $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_1$, $\alpha k = k'$, $\gamma k = m'$, $(\alpha, \gamma) = 1$ so $(k', m') = k$.

The Lemma is proved. □

From Lemma 1.1 and Lemma 1.2 we obtain

Lemma 1.3. *Let $f(z,u,v)$ be a $\Gamma_{1,J}$-invariant continuous function of z,u,v, C^1 in u and v. Then*

$$f(z,u,v) = f_o(z) + \sum_{k=1}^{\infty} f_k(z,u,v) \tag{1.9}$$

where

$$f_0(z) \text{ is } \Gamma_1\text{-invariant}$$

and for $k \in \mathbb{N}$

$$f_k(z,u,v) = \sum_{g\in\Gamma_1\backslash\Gamma_\infty} \big(T_g f_{k0}\big)(z) e_{k(\delta,-\gamma)}(u,v) \tag{1.10}$$

where

$$\tbinom{k\delta}{-k\gamma} = g^{-1}\tbinom{k}{0} \quad \text{and } f_{k0} \text{ is } \Gamma_\infty\text{-invariant.}$$

Proof. The set of all terms of the series (1.3) is the union of all equivalence classes under the equivalence relation

$$f_{k'm'}(z)e_{k'm'}(u,v) \sim f_{k''m''}(z)e_{k''m''}(u,v)$$

iff there exists $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_1$ such that

$$f_{k''m''}(z) = \big(T_g f_{k'm'}\big)(z), \quad \tbinom{k''}{m''} = g^{-1}\tbinom{k'}{m''}. \tag{1.11}$$

By Lemma 1.2 this holds iff $(k'',m'') = (k',m')$.

Therefore the equivalence classes M_k are given for $k \in \mathbb{N}$ by

$$M_k = \{f_{k'm'}(z)e_{k'm'}(u,v) \,|\, (k',m') = k\}$$

and

$$M_0 = \{f \,|\, \big(T_g f\big)(z) = f(z) \text{ for } g \in \Gamma_1\}$$

Since the series (1.3) is absolutely convergent, we can rearrange it as follows,

$$f(z,u,v) = \sum_{k=0}^{\infty} f_k(z,u,v) \tag{1.12}$$

where

$$f_k(z,u,v) = \sum_{(k',m')=k} f_{k'm'}(z)e_{k'm'}(u,v) \tag{1.13}$$

$$f_0(z) = \big(T_g f_0\big)(z) \quad \text{for } g \in \Gamma_1.$$

For each k the function $f_k(z,u,v)$ is Γ_J-invariant.

For any term of (1.13)

$$f_{k'm'}(z)e_{k'm'}(u,v)$$

let $g \in \Gamma_1$ be such that $\tbinom{k'}{m'} = g^{-1}\tbinom{k}{0}$ and let

$$f_{k'm'}(z) = \big(T_g f_{k0}(z), \quad f_{k0}(z) = \big(T_{g^{-1}} f_{k'm'}\big)(z).$$

Since $f_k(z,u,v)$ is invariant under Γ_∞ and for $g_0 \in \Gamma_\infty$

$$\widetilde{T}_{g_0}\big(f_{k0}(z)e_{k0}(u,v)\big) = \big(T_{g_0}f_{k0}\big)(z)e_{k0}(u,v),$$

the function $f_{k0}(z)$ is Γ_∞-invariant. Therefore

$$f_k(z,u,v) \sum_{g\in\Gamma_1\setminus\Gamma_\infty} \big(T_g f_{k0}\big)(z)e_{k(\delta,-\gamma)}(u,v)$$

and Lemma 1.3 is proved. □

Lemma 1.4. *For all $g \in \Gamma_1$ and $k,m \in \mathbb{Z}$*

$$\widetilde{T}_g\big[\tfrac{1}{y}\{k^2 - 2kmx + m^2(x^2+y^2)\}\big] = \tfrac{1}{y}\{k'^2 - 2k'm'x + m'^2(x^2+y^2)\}$$

where

$$\binom{k'}{m'} = g^{-1}\binom{k}{m}.$$

Proof. We have

$$\begin{aligned}&L\big(f_{km}(z)e_{km}(u,v)\big)\\ &\quad= \Big[Af_{km}(z) + \frac{4\pi^2}{y}\{k^2 - 2kmx + m^2(x^2+y^2)\}f_{km}(z)\Big]e_{km}(u,v)\end{aligned}$$

For the $\Gamma_{1,J}$-invariant function $f(z,u,v)$ on M_J given by (1.2) this yields

$$\begin{aligned}&(Lf)(z,u,v)\\ &\quad= \sum_{km} \big(Af_{km}\big)(z) + \frac{4\pi^2}{y}\{k^2 - 2kmx + m^2(x^2+y^2)\}f_{km}(z)e_{km}(u,v).\end{aligned}$$

For $g \in \Gamma_1$

$$\begin{aligned}\big(\widetilde{T}_g Lf\big)(z,u,v) &= \sum_{km} \big(T_g Af_{km}\big)(z)\\ &+ T_g\Big[\frac{4\pi^2}{y}\{k^2 - 2mx + m^2(x^2+y^2)\}\big(T_g f_{km}\big)(z)\Big]e_{k'm'}(u,v)\end{aligned} \tag{1.14}$$

where $\binom{k'}{m'} = g^{-1}\binom{k}{m}$.

On the other hand,

$$\big(T_g f\big)(z,u,v) = \sum_{km} \big(T_g f_{km}\big)(z)e_{k'm'}(u,v)$$

and

$$\begin{aligned}\big(LT_g f\big)(z,u,v) &= \sum_{km} \big(AT_g f_{km}\big)(z)\\ &+ \frac{4\pi^2}{y}\{k'^2 - 2k'm'x + m'^2(x^2+y^2)\}e_{k'm'}(u,v).\end{aligned} \tag{1.15}$$

By the J-invariance of L, the series in the r.h.s. of (1.14) and (1.15) are identical and A is Γ_1-invariant.

The Lemma follows. □

Lemma 1.5. *For each $k = 0, 1, 2, \ldots$, the set of functions f_k defined for $k = 1, 2, \ldots$ by (1.10) and for $k = 0$ by f_0 being Γ_1-invariant form a subspace $\mathcal{H}'_k$ of the Hilbert space $\mathcal{H}_{1J}$ of square-integrable, $\Gamma_{1,J}$-invariant functions on $F_{\Gamma_{1,J}}$ with measure $\frac{dx\,dy}{y^2} du\,dv$.*

Let $\mathcal{H}_k$ be the closure of $\mathcal{H}'_k$. Then

$$\mathcal{H}_{1J} = \sum_{k=0}^{\infty} \bigoplus \mathcal{H}_k, \tag{1.16}$$

the subspaces $\mathcal{H}_k$ of $\mathcal{H}_{1J}$ are invariant under the Laplacian L, and

$$L = \sum_{k=0}^{\infty} \bigoplus L_k, \quad L_k = L\big|_{\mathcal{H}_k \cap \mathcal{D}(L)}. \tag{1.17}$$

Proof. Let $1 \le k_1 < k_2$, $k'_1 = k_1\delta_1$, $m'_1 = -k_1\gamma_1$, $(\gamma_1, \delta_1) = 1$, $k'_2 = k_2\delta_2$, $m'_2 = -k_2\gamma_2$, $(\gamma_2, \delta_2) = 1$. Then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i k_1 \delta_1 u} e^{-2\pi i k_1 \gamma_1 v} e^{-2\pi i k_2 \delta_2 u} e^{2\pi i k_2 \gamma_2 v}\, du\, dv \ne 0$$

if and only if

$$k_1\gamma_1 = k_2\delta_2, \quad k_1\gamma_1 = k_2\gamma_2.$$

But $(k_1\gamma_1, k_1\delta_1) = k_1$, $(k_2\gamma_2, k_2\delta_2) = k_2$, a contradiction, so

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i k_1 \delta_1 u} e^{-2\pi i k_1 \gamma_1 v} e^{-2\pi i k_2 \delta_2 u} e^{2\pi i k_2 \gamma_2 v}\, du\, dv = 0$$

for $1 \le k_1 < k_2$ and all $\gamma \in \Gamma_1$.

Similarly it is shown that the other three terms of $(e_{k'_1 m'_1}, e_{k'_2 m'_2})$ are 0. It follows that $\big(f_{k_1}(z, u, v), f_{k_2}(z, u, v)\big)_{\mathcal{H}_{1J}} = 0$ for $1 \le k_1 < k_2$. Clearly,

$$\big(f_0(z), f_k(z, u, v)\big) = 0 \quad \text{for } k \ge 1.$$

Now it follows from Lemma 1.3 that

$$\mathcal{H}_{1J} = \sum_{k=0}^{\infty} \bigoplus \mathcal{H}_k.$$

Let $\mathcal{H}'_k$ be the space of continuous functions $f(z, u, v)$ in $\mathcal{H}_k$ such that f is continuous and C^1 in u and v.

We shall prove that $\mathcal{H}_k$ is invariant under L by proving that $L\big(\mathcal{H}'_k \cap \mathcal{D}(L)\big) \subset \mathcal{H}_k$. Let $f_{k0} \in \mathcal{H}'_k \cap \mathcal{D}(L)$. Then, with $\binom{k'}{m'} = g^{-1}\binom{k}{0}$ we obtain, using Lemma 1.3 and

Lemma 1.4,

$$\begin{aligned}
\big(Lf_k\big)(z,u,v) &= L\Big\{\sum_{g\in\Gamma_1\backslash\Gamma_\infty}\big(T_g f_{k0}\big)(z)e_{k'm'}(u,v)\Big\}\\
&= \sum_{g\in\Gamma_1\backslash\Gamma_\infty}\Big\{\big(AT_g f_{k0}\big)(z)\\
&\qquad + \big(T_g f_{k0}\big)(z)\frac{4\pi^2}{y}\big[k'^2-2k'm'x+m'^2(x^2+y^2)\big]\Big\}e_{k'm'}(u,v)\\
&= \sum_{g\in\Gamma_1\backslash\Gamma_\infty}\Big\{\big(AT_g f_{k0}\big)(z)+\big(T_g f_{k0}\big)(z)\widetilde{T}_g\Big[\frac{4\pi^2}{y}k^2\Big]\Big\}e_{k'm'}(u,v)\\
&= \sum_{g\in\Gamma_1\backslash\Gamma_\infty}\Big\{T_g\Big[\Big(A+\frac{4\pi^2k^2}{y}\Big)f_{k0}\Big]\Big\}(z)e_{k'm'}(u,v).
\end{aligned}$$

So $Lf_k \in \mathcal{H}'_k$ provided Af_{k0} is continuous, and $(Lf_k)(z,u,v)$ has the series expansion (1.10) with $f_{k0}(z)$ replaced by $\big(A+\frac{4\pi^2k^2}{y}f_{k0}\big)(z)$.

The Lemma is proved. □

By Lemma 1.5, the $\Gamma_{1,J}$-invariant Laplacian L is decomposed into a direct sum of operators L_k in invariant subspaces $\mathcal{H}_k$.

For $k=0$, L_0 is the Γ_1-invariant Laplacian A in $\mathcal{H}_{\Gamma_1}$.

For $k\ge 1$, let $\mathcal{H}^0_k=\mathcal{H}_{\Gamma_\infty}=L^2(F_{\Gamma_\infty};y^{-2}\,dx\,dy)$, where

$$F_{\Gamma_\infty}=\{z=x+iy\,|\,-\tfrac12<x\le\tfrac12,\ y>0\},$$

and let L^0_k be the Γ_∞-invariant, self-adjoint operator in $\mathcal{H}_{\Gamma_\infty}$

$$L^0_k=A+4\pi^2\frac{k^2}{y}.$$

Let Σ_k be the map from $\mathcal{H}^0_k$ into $\mathcal{H}_k$ defined for $f\in\mathcal{H}^0_k$ by

$$\Sigma_k f=\sum_{g\in\Gamma_1\backslash\Gamma_\infty}\big(T_g f\big)(z)e_{k(\delta,-\gamma)}(u,v),\quad g=\big(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\big). \tag{1.18}$$

Lemma 1.6. *For each $k\ge1$, Σ_k is a unitary operator from $\mathcal{H}^0_k$ onto $\mathcal{H}_k$, and L_k is unitarily equivalent to L^0_k,*

$$L_k\Sigma_k=\Sigma_k L^0_k.$$

Proof. Let $f_i\in\mathcal{H}^0_k$ and let $\Sigma_k f_i$ be defined by (1.18), $g_i=\big(\begin{smallmatrix}\alpha_i&\beta_i\\ \gamma_i&\delta_i\end{smallmatrix}\big)$, $i=1,2$. Since $g_1g_2^{-1}\notin\Gamma_\infty$ implies $\gamma_1\ne\gamma_2$ or $\delta_1\ne\delta_2$, for $g_1\ne g_2 \mod \Gamma_\infty$

$$\Big(T_{g_1}f_1(z)e_{k(\delta_1,-\gamma_1)}(u,v),\big(T_{g_2}f_2\big)(z)e_{k(\delta_2,-\gamma_2)}(u,v)\Big)_{\mathcal{H}_{\Gamma_{1,J}}}=0,$$

so

$$\Big(\Sigma_k f_1,\Sigma_k f_2\Big)_{\mathcal{H}_k}=\sum_{g\in\Gamma_1\backslash\Gamma_\infty}\Big(\big(T_g f_1\big)(z),\big(T_g f_2\big)(z)\Big)_{\mathcal{H}_{\Gamma_1}}=(f_1,f_2)_{\mathcal{H}_{\Gamma_\infty}},$$

unfolding the integral, and Σ_k is unitary from $\mathcal{H}_k^0 = \mathcal{H}_{\Gamma_\infty}$ onto $\mathcal{H}_k$. By the last part of the proof of Lemma 1.5, for $f \in \mathcal{D}(L_k^0)$

$$L_k \Sigma_k f = \Sigma_k L_k^0 f$$

and L_k is unitarily equivalent to L_k^0. □

We proceed to analyze for $k \geq 1$ the operators

$$L_k^0 = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 4\pi^2 \frac{k^2}{y} \quad \text{in } \mathcal{H}_k^0 = L^2\Big(F_{\Gamma_\infty}; \frac{dx\,dy}{y^2} \Big)$$

with the condition

$$f(\tfrac{1}{2} + iy) = f(-\tfrac{1}{2} + iy) \quad \text{for } 0 < y < \infty.$$

Lemma 1.7. *Let $\mathcal{H} = L^2(0, \infty; y^{-2}\, dy)$. Then $\mathcal{H}_k^0$ can be decomposed as a direct sum of subspaces invariant under L_k^0,*

$$\mathcal{H}_k^0 = \sum_{l \in \mathbb{Z}} \bigoplus \mathcal{H}_{kl}^0, \quad \mathcal{H}_{kl}^0 = \{ e^{2\pi i l x} \varphi_{kl}(y) \mid \varphi_{kl} \in \mathcal{H} \}$$

with

$$L_k^0 \big(e^{2\pi i l x} \varphi_{kl}(y) \big) = e^{2\pi i l x} \cdot \Big\{ -y^2 \frac{\partial^2}{\partial y^2} + 4\pi^2 \Big(\frac{k^2}{y} + l^2 y^2 \Big) \Big\} \varphi_{kl}(y).$$

Let

$$\widetilde{L}_{kl} = -y^2 \frac{\partial^2}{\partial y^2} + 4\pi^2 \Big(\frac{k^2}{y} + l^2 y^2 \Big) \quad \text{in } \mathcal{H}$$

and

$$L_{kl}^0 = L_k^0 \big|_{\mathcal{H}_{kl}^0}, \quad L_k^0 = \sum_{l \in \mathbb{Z}} \bigoplus L_{kl}^0.$$

Then L_{kl}^0 is unitarily equivalent to $\widetilde{L}_{kl}$ via the map $\varphi_{kl}(y) \to e^{2\pi i l x} \varphi_{kl}(y)$.

For $k \geq 1$, $l \neq 0$, the operator $\widetilde{L}_{kl}$ has a purely discrete, simple spectrum consisting of a sequence of eigenvalues

$$\tfrac{1}{4} < \lambda_{kl}^1 < \lambda_{kl}^2 < \cdots < \lambda_{kl}^n < \cdots, \quad \lambda_{kl}^n \xrightarrow[n \to \infty]{} \infty \tag{1.19}$$

with orthonormal eigenfunctions

$$\varphi_{kl}^1(y), \varphi_{kl}^2(y), \ldots, \varphi_{kl}^n(y), \ldots \tag{1.20}$$

The operator L_{kl}^0 has the same eigenvalues λ_{kl}^n with eigenfunctions

$$e^{2\pi i l x} \varphi_{kl}^n(y)$$

or

$$\cos(2\pi l x) \varphi_{kl}^n(y), \; \sin(2\pi l x) \varphi_{kl}^n(y).$$

For $k \geq 1$, $l = 0$ we obtain the operator

$$L^0_{k0} = L_{k0} = -y^2 \frac{\partial^2}{\partial y^2} + 4\pi^2 \frac{k^2}{y}.$$

This operator has a simple, purely continuous spectrum, to be discussed in detail in the following Lemma.

Proof. The proof is straightforward by separation of variables. □

Lemma 1.8. *The operator L_{k0} is self-adjoint in $\mathcal{H}$ with a simple, purely continuous spectrum. The generalized eigenfunctions $h_k(y,s)$ are given for $s \in \mathbb{C}$, $y > 0$, $k \geq 1$ by the Bessel functions,*

$$h_k(y,s) = \sqrt{y} K_{2s-1}(4\pi k y^{-1/2})$$

which are the solutions of the Bessel equation

$$-y^2 \frac{d^2 h_k(y,s)}{dy^2} + 4\pi^2 \frac{k^2}{y} h_k(y,s) = s(1-s) h_k(y,s) \tag{1.21}$$

with the asymptotics

$$\begin{aligned} K_\nu(y) &\sim \left(\frac{\pi}{2y}\right)^{\frac{1}{2}} e^{-y} && \text{for } y \to \infty \\ K_\nu(y) &\sim \frac{\Gamma(\nu)}{2} \left(\frac{y}{2}\right)^{-\nu} && \text{for } y \to 0,\ \nu \neq 0 \\ K_0(y) &\sim -\log y && \text{for } y \to 0. \end{aligned}$$

Also

$$K_\nu(y) = K_{-\nu}(y)\,.$$

The other solution $I_{2s-1}(\pi k y^{-1/2})$ grows exponentially as $y \to 0$ and so does not contribute to the continuous spectrum.

The functions $h_k(y;s)$ are entire functions of s, and

$$h_k(y, 1-s) = h_k(y,s).$$

Moreover, for $k \in \mathbb{N}$

$$L_{k0} = U(k) L_{10} U^{-1}(k), \quad h_k(y;s) = \big(U(k)h_1\big)(y;s), \quad \big(U(k)f\big)(y) = k f(k^{-2}y).$$

Proof. This follows from well known properties of the Bessel functions. □

From Lemmas 1.5–1.8 we obtain

Theorem 1. *For $k \in \mathbb{N}$, $l \in \mathbb{Z}$, let*

$$\mathcal{H}_{kl} = \Sigma_k \mathcal{H}^0_{kl}$$

where Σ_k is given by (1.18).

Then

$$\mathcal{H}_{\Gamma J} = \sum_{k=1}^{\infty} \bigoplus \Big\{ \sum_{l=-\infty}^{\infty} \bigoplus \mathcal{H}_{kl} \Big\},$$

$\mathcal{H}_{kl}$ is invariant under L and

$$L = \sum_{k=1}^{\infty} \bigoplus \Big\{ \sum_{l=-\infty}^{\infty} \bigoplus L_{kl} \Big\},$$

where

$$L_{kl}\Sigma_k\big(e^{2\pi i l x}\varphi_{kl}(y)\big) = \Sigma_k L^0_{kl}\big(e^{2\pi i l x}\varphi_{kl}(y)\big) = \Sigma_k\big(e^{2\pi i l x}\widetilde{L}_{kl}\varphi_{kl}(y)\big).$$

For $l \neq 0$ the spectrum of L_{kl} is the sequence of simple eigenvalues λ^n_{kl} of $\widetilde{L}_{kl}$ given by (1.19) *with orthonormal eigenfunctions*

$$\begin{aligned}\Psi_{kl}(z,u,v) &= \Sigma_k\big(e^{2\pi i l x}\varphi^n_{kl}(y)\big) \\ &= \sum_{g \in \Gamma_1 \backslash \Gamma_\infty} \Big\{ T_g\big(e^{2\pi i l x}\varphi^n_{kl}(y)\big)\Big\}(z) e_{k(\delta,-\gamma)}(u,v), \quad g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right).\end{aligned}$$

Each λ^n_{kl} is a 2-dimensional eigenvalue of L_k with eigenfunctions $\sum_k e^{2\pi i l x}\varphi^n_{kl}(y)$ and $\sum_k e^{-2\pi i l x}\varphi^n_{kl}(y)$.

The operators $\tilde{L}_{k_1 l_1}$ and $\tilde{L}_{k_2 l_2}$ with $k_1^2 l_1 = k_2^2 l_2$ are unitarily equivalent via the dilation $\varphi(y) \to \big(\frac{l_1}{l_2}\big)^{1/2}\varphi\big(\frac{l_1}{l_2}y\big)$.

For $l = 0$ the spectrum of L_{k0} is purely continuous with generalized eigenfunctions

$$\begin{aligned}E_k(z,u,v;s) &= \Sigma_k h_k(y;s) \\ &= \sum_{g \in \Gamma_1 \backslash \Gamma_\infty} \Big\{ T_g\big(\sqrt{y}K_{2s-1}(4\pi k y^{-1/2})\big)\Big\}(z) e_{k(\delta,-\gamma)}(u,v), \quad g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)\end{aligned} \tag{1.22}$$

The series (1.22) *is absolutely convergent for all $s \in \mathbb{C}$ and defines an entire function of s for any $k \in \mathbb{N}$, $(z,u,v) \in F_{\Gamma_J}$.*

The function $E_k(z,u,v;s)$ satisfies a functional equation

$$E_k(z,u,v;s) = E_k(z,u,v;1-s).$$

There is no scattering and no resonances associated with L_{k0}.

Moreover,

$$E_k(z,u,v;s) = \big(\Sigma_k U(k)\Sigma_1^{-1}E_1\big)(z,u,v;s).$$

2 The counting function for eigenvalues of L and the Weyl law

We now discuss the asymptotic counting function for the eigenvalues of L. We make use of the following result of Titchmarsh ([7] Ch. VII Theorem 7.5) where the uniform bound on the remainder is obtained by keeping track of the constants in the proof.

Lemma 2.1. *Let $q \in C^1(-\infty,\infty)$ be downward convex with $q'(x)$ increasing on $(-\infty,\infty)$, $q(x) \to \infty$ as $x \to \pm\infty$.*

Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

be the (simple) eigenvalues of the operator

$$M = -\frac{d^2}{dx^2} + q(x) \quad \text{in } L^2(-\infty,\infty)$$

with eigenfunctions $y_n(x)$,

$$y_n''(x) + \{\lambda - q(x)\}y_n(x) = 0.$$

Then

$$\left| n - \frac{1}{\pi}\int_{x_1}^{x_2} \{\lambda_n - q(x)\}^{\frac{1}{2}}\, dx \right| < 4 + \frac{14}{3\pi} = K_1$$

where

$$q(x_1) = q(x_2) = \lambda$$

and

$$N_q(\lambda) = \#\{\lambda_n \le \lambda\} = \frac{1}{\pi}\int_{x_1}^{x_2} \{\lambda - q(x)\}^{\frac{1}{2}}\, dx + O(1), \quad |O(1)| \le 4 + \frac{14}{3\pi}$$

for all q in the above class.

Let T be the unitary operator from $L^2(0,\infty,;y^{-2}\,dy)$ to $L^2(-\infty,\infty;dt)$ defined for $g \in L^2(0,\infty,;y^{-2}\,dy)$ by

$$f(t) = \big(Tg\big)(t) = g(e^t)e^{-\frac{t}{2}}.$$

Then

$$TL_{kl}T^{-1} = M_{kl} = -\frac{d^2}{dt^2} + \tfrac{1}{4} + 4\pi^2(k^2e^{-t} + l^2e^{2t})$$

with the simple eigenvalues

$$\tfrac{1}{4} < \lambda_{kl}^1 < \lambda_{kl}^2 < \cdots < \lambda_{kl}^n < \cdots$$

Let

$$N_{kl}(\lambda) = \#\{\lambda_{kl}^n < \lambda\}$$

be the counting function for M_{kl}. By Lemma 2.1,

$$N_{kl}(\lambda) = \frac{1}{\pi}\int_{J_{kl}(\lambda)} \left\{\lambda - \tfrac{1}{4} - 4\pi^2(k^2e^{-t} + l^2e^{2t})\right\}^{\frac{1}{2}} dt + O_{kl}(\lambda)$$

where

$$J_{kl}(\lambda) = \{t \,|\, \tfrac{1}{4} + 4\pi^2(k^2e^{-t} + l^2e^{2t}) < \lambda\} \quad \text{for } \lambda > m_{kl}$$

with

$$m_{kl} = \min\{\tfrac{1}{4} + 4\pi^2(k^2e^{-t} + l^2e^{2t}) \,|\, t \in \mathbb{R}\} = \tfrac{1}{4} + 4\pi^2k^{4/3}l^{2/3}3 \cdot 2^{-2/3}$$

and

$$|O_{kl}(\lambda)| \leq K_1 \quad \text{for all } k, l, \lambda.$$

We have

$$N_{kl}(\lambda) = 2\int_{\widetilde{J}_{kl}(\lambda)} \left\{\frac{\lambda - \frac{1}{4}}{4\pi^2} - \left(k^2e^{-t} + l^2e^{2t}\right)\right\}^{\frac{1}{2}} dt + O_{kl}(\lambda)$$

and

$$\widetilde{J}_{kl}(\lambda) = \left\{t \,\middle|\, k^2e^{-t} + l^2e^{2t} < \frac{\lambda - \frac{1}{4}}{4\pi^2}\right\}.$$

To simplify the calculations we replace λ by $\lambda' = (\lambda - \frac{1}{4})/(4\pi^2)$, prove the asymptotic estimates with λ' and at the end substitute $\lambda' = (\lambda - \frac{1}{4})/(4\pi^2)$ and introduce the factor 2. For simpler notation we replace λ' by λ until then. So we study the scaled problem with

$$N_{kl}(\lambda) = \int_{J'_{kl}(\lambda)} \left\{\lambda - (k^2e^{-t} + l^2e^{2t})\right\}^{\frac{1}{2}} dt + O_{kl}(\lambda)$$

where

$$J'_{kl}(\lambda) = \{t \,|\, k^2e^{-t} + l^2e^{2t} < \lambda\}$$

for $\lambda > k^{4/3}l^{2/3}3 \cdot 2^{-2/3}$ and

$$|O_{kl}(\lambda)| < K_1 \quad \text{for all } k, l, \lambda.$$

Setting $u = e^t$, we get

$$N_{kl}(\lambda) = I_{kl}(\lambda) + O_{kl}(\lambda) \tag{2.1}$$

where

$$I_{kl}(\lambda) = \int_{J_{kl}(\lambda)} \left\{\lambda - (k^2u^{-1} + l^2u^2)\right\}^{\frac{1}{2}} u^{-1}\, du \tag{2.2}$$

and $J_{kl}(\lambda) = \{u = e^t \,|\, t \in J'_{kl}(\lambda)\}$. By Theorem 1, this implies

$$N(\lambda) = \sum_{\substack{k,l \geq 1 \\ m_{kl} < \lambda}} I_{kl}(\lambda) + \sum_{\substack{k,l \geq 1 \\ m_{kl} < \lambda}} O_{kl}(\lambda) \tag{2.3}$$

where we have not counted the double multiplicity of the λ^n_{kl} as eigenvalues of L. It will be restored at the end, when we prove the Weyl law.

We estimate the last term of (2.3) as follows.

Lemma 2.2.
$$\Big|\sum_{\substack{k,l\geq 1\\ m_{kl}<\lambda}} O_{kl}(\lambda)\Big| = O(\lambda^{3/2}). \tag{2.4}$$

Proof. With $\mu = \lambda 3^{-1}2^{2/3}$ we have

$$\Big|\sum_{\substack{k,l\geq 1\\ m_{kl}<1}} O_{kl}(\lambda)\Big| \leq K_1 \sum_{\substack{k,l\geq 1\\ k^{4/3}l^{2/3}<\mu}} 1 = K_1\Big\{\sum_{\substack{k,l\geq 2\\ k^{4/3}l^{2/3}<\mu}} 1 + \sum_{\substack{k\geq 1\\ k^{4/3}<\mu}} 1 + \sum_{\substack{l\geq 1\\ l^{2/3}<\mu}} 1\Big\}.$$

We have

$$\sum_{\substack{k,l\geq 2\\ k^{4/3}l^{2/3}<\mu}} 1 \leq \iint_{\substack{k,l\geq 1\\ k^{4/3}l^{2/3}<\mu}} 1\, dk\, dl = \int_1^{\mu^{3/4}} dk \int_1^{\mu^{3/2}k^{-2}} 1 dl$$
$$= \int_1^{\mu^{3/4}} \big\{\mu^{3/2}k^{-2}(1-\mu^{-3/4}) - (\mu^{3/4}-1)\big\}\, dk < \mu^{3/2}$$

and

$$\sum_{\substack{k\geq 1\\ k^{4/3}<\mu}} 1 \leq \mu^{3/4}, \qquad \sum_{\substack{l\geq 1\\ l^{2/3}<\mu}} 1 \leq \mu^{3/2}.$$

The Lemma is proved. □

We introduce

$$\widetilde{N}_k(\lambda) = \sum_{1\leq l<\lambda^{3/2}k^{-2}3^{-3/2}2} I_{kl}(\lambda) \qquad \text{for } 1\leq k<\lambda^{3/4}3^{-3/4}2^{1/2}$$
$$\widetilde{N}(\lambda) = \sum_{1\leq k<\lambda^{3/4}3^{-3/4}2} \widetilde{N}_k(\lambda)$$
$$N_k(\lambda) = \sum_{1\leq l<\lambda^{3/2}k^{-2}3^{-3/2}2} N_{kl}(\lambda)$$
$$N(\lambda) = \sum_{1\leq k<\lambda^{3/4}3^{-3/4}2^{1/2}} N_k(\lambda).$$

From (2.1), Lemma 2.2 and Theorem 1 follows

Lemma 2.3.

$$\sum_{1\leq k\leq\lambda^{3/4}3^{-3/4}2^{1/2}} N_{k1}(\lambda) = \sum_{1\leq k\leq\lambda^{3/4}3^{-3/4}2^{1/2}} I_{k1}(\lambda) + O(\lambda^{3/2})$$
$$N_k(\lambda) = \widetilde{N}_k(\lambda) + O(\lambda^{3/2}), \qquad \big|O(\lambda^{3/2})\big| \leq K\lambda^{3/2} \quad \textit{for all } k$$
$$N(\lambda) = \widetilde{N}(\lambda) + O(\lambda^{3/2}).$$

We now study the asymptotics of $\widetilde{N}_k(\lambda)$ and $\widetilde{N}(\lambda)$ through approximation of sums over k and l by integrals.

Lemma 2.4.

$$\sum_{2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} I_{k1}(\lambda)$$
$$\le \int_1^{\lambda^{3/4}3^{-3/4}2^{1/2}} dk' \int_{J_{k'1}} \big\{\lambda - (k'^2u^{-1}+u^2)^{\frac{1}{2}}u^{-1}\big\}^{\frac{1}{2}}u^{-1}\,du \le K_1\lambda^{5/4},$$
$$\sum_{2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} N_{k1}(\lambda) = O(\lambda^{3/2}).$$

Proof. For $2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}$

$$I_{k1}(\lambda) = \int_{k^2u^{-1}+u^2<\lambda} \big\{\lambda-(k^2u^{-1}+u^2)\big\}^{\frac{1}{2}}u^{-1}\,du$$
$$\le \int_{k-1}^{k} dk' \int_{k'^2u^{-1}+u^2<\lambda} \big\{\lambda-(k'^2u^{-1}+u^2)\big\}^{\frac{1}{2}}u^{-1}\,du$$

Adding over k, we get the left inequality of the Lemma. We estimate the integral

$$\widetilde{I} = \int_1^{\lambda^{3/4}3^{-3/4}2^{1/2}} dk' \int_{J_{k'1}} \big\{\lambda-(k'^2u^{-1}+u^2)\big\}^{1/2}u^{-1}\,du.$$

Set $u=v\lambda^{1/2}$, $k'=x\lambda^{3/4}$. Then

$$\widetilde{I} = \lambda^{5/4} \int\limits_{\substack{\lambda^{-3/4}\le x\le 3^{-3/4}2^{1/2}\\ x^2v^{-1}+v^2<1}} dx\,dv\big\{1-v^2-x^2v^{-1}\big\}^{\frac{1}{2}}v^{-1}.$$

The positive solution $x(v)$ of the equation

$$1-v^2-x^2v^{-1}=0$$

is

$$x=(v-v^{-3})^{1/2},\quad 0\le v\le 1.$$

The function

$$g(v)=v-v^3 \quad \text{with } g'(v)=1-3v^2 \text{ has}$$
$$\max_{0\le v\le 1} g(v) = g(3^{-1/2}) = 2\cdot 3^{-3/2}$$

so

$$x(v)=g^{\frac{1}{2}}(v) \quad \text{has} \max_{0\le v\le 1} x(v) = 3^{-3/4}2^{1/2}$$

Therefore

$$\widetilde{I} = \lambda^{5/4}\int_{\lambda^{-3/4}}^{(v-v^3)^{1/2}} dx \int_{x^2v^{-1}+v^2<1} dv\big\{1-v^2-x^2v^{-1}\big\}^{\frac{1}{2}}v^{-1}.$$

Setting $x=(v-v^3)^{\frac{1}{2}}x_1$, we get

$$\widetilde{I}=\lambda^{5/4}\int_{v_1}^{v_2}dv(1-v^2)v^{-\frac{1}{2}}\int_{\lambda^{-3/4}(v-v^3)^{-1/2}}^{1}\left(1-x_1^2\right)^{\frac{1}{2}}dx_1,\quad 0<v_1<v_2<1.$$

With $x_1=\sin\varphi$ we get

$$\begin{aligned}\int_{\lambda^{-3/4}(v-v^3)^{-1/2}}^{1}\left(1-x_1^2\right)^{1/2}dx_1&=\int_{\arcsin[\lambda^{-3/4}(v-v^3)^{-1/2}]}^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi\\ &=\left[\tfrac{1}{2}+\tfrac{1}{4}\sin 2\varphi\right]_{\arcsin[\lambda^{-3/4}(v-v^3)^{-1/2}]}^{\frac{\pi}{2}}\\ &=\tfrac{\pi}{4}-\tfrac{1}{2}\arcsin\left[\lambda^{-3/4}(v-v^3)^{-1/2}\right]\\ &\quad-\tfrac{1}{2}\lambda^{-3/4}(v-v^3)^{-1/2}\left[1-\lambda^{-3/2}(v-v^3)^{-1}\right]^{1/2}\\ &<\frac{\pi}{4}.\end{aligned}$$

So

$$\widetilde{I}\le\lambda^{5/4}\frac{\pi}{4}\int_0^1 dv(1-v^2)v^{-1/2}=\lambda^{5/4}K_1. \tag{2.5}$$

It now follows from (2.5) and Lemma 2.3 that

$$\sum_{2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}}N_{k1}(\lambda)=O(\lambda^{3/2}).$$

This completes the proof of the Lemma. □

Next we approximate for each k, $\widetilde{N}_k(\lambda)$ by the integral

$$I_k(\lambda)=\int_1^{\lambda^{3/2}k^{-2}3^{-3/2}2}dl'\int_{J_{kl'}(\lambda)}\left\{\lambda-(k^2u^{-1}+l'^2u^2)\right\}^{-\frac{1}{2}}u^{-1}\,du. \tag{2.6}$$

Lemma 2.5. $\widetilde{N}_k(\lambda)=I_k(\lambda)+O(\lambda^{5/4})$ *uniformly in* k.

$$\widetilde{N}(\lambda)=\sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}}I_k(\lambda)+O(\lambda^{5/4}).$$

Proof. For fixed $k\in\mathbb{N}$, $l=2,3,\dots$ we have

$$\begin{aligned}&\int_l^{l+1}dl'\int_{J_{kl'}(\lambda)}\left\{\lambda-(k^2u^{-1}+l'^2u^2)\right\}^{\frac{1}{2}}u^{-1}\,du\\ &\quad\le\int_{J_{kl}(\lambda)}\left\{\lambda-(k^2u^{-1}+l^2u^2)\right\}^{\frac{1}{2}}u^{-1}\,du\\ &\quad\le\int_{l-1}^{l}dl'\int_{J_{kl'}(\lambda)}\left\{\lambda-(k^2u^{-1}+l'^2u^2)\right\}^{\frac{1}{2}}u^{-1}\,du.\end{aligned} \tag{2.7}$$

Summing over $l = 2, 3, \dots$ we get

$$
\begin{aligned}
&\int\limits_{2}^{\lambda^{3/2}k^{-2}3^{-3/2}2} dl' \int_{J_{kl'}(\lambda)} \{\lambda - (k^2u^{-1} + l'^2u^2)\}^{\frac{1}{2}} u^{-1}\, du \\
&\qquad \le \sum_{l\in\mathbb{N}, l\gg 2} \int_{J_{kl}(\lambda)} \{\lambda - (k^2u^{-1} + l^2u^2)\}^{\frac{1}{2}} u^{-1}\, du \\
&\qquad \le \int\limits_{1}^{\lambda^{3/2}k^{-2}3^{-3/2}2} dl' \int_{J_{kl'}(\lambda)} \{\lambda - (k^2u^{-1} + l'^2u^2)\}^{\frac{1}{2}} u^{-1}\, du.
\end{aligned} \tag{2.8}
$$

The difference of the r.h.s. and the l.h.s. of (2.8) is

$$
\begin{aligned}
&\int_1^2 dl' \int_{J_{kl'}(\lambda)} \{\lambda - (k^2u^{-1} + l'^2u^2)\}^{\frac{1}{2}} u^{-1}\, du \\
&\qquad \le \int_{J_{k1}(\lambda)} \{\lambda - (k^2u^{-1} + u^2)\}^{\frac{1}{2}} u^{-1}\, du.
\end{aligned} \tag{2.9}
$$

Summing (2.8) over $k \in \mathbb{N}$, we get

$$
\begin{aligned}
&\sum_{1<k<\lambda^{3/4}3^{-3/4}2^{1/2}} \int\limits_{2}^{\lambda^{3/2}k^{-2}3^{-3/2}2} dl' \int_{J_{kl'}(\lambda)} \{\lambda - (k^2u^{-1} + l'^2u^2)\}^{\frac{1}{2}} u^{-1}\, du \\
&\quad \le \sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} \sum_{l\in\mathbb{N}, l\ge 2} \int_{J_{kl}(\lambda)} \{\lambda - (k^2u^{-1} + l^2u^2)\}^{\frac{1}{2}} u^{-1}\, du \\
&\quad \le \sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} \int\limits_{1}^{\lambda^{3/2}k^{-2}3^{-3/2}2} dl' \int_{J_{kl'}(\lambda)} \{\lambda - (k^2u^{-1} + l'^2u^2)\}^{\frac{1}{2}} u^{-1}\, du.
\end{aligned} \tag{2.10}
$$

The difference of the right hand side and the l.h.s. if (2.10) is established by summing (2.9) over k and is equal to

$$
\begin{aligned}
&\sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} \int_1^2 dl' \int_{J_{kl'}(\lambda)} \{\lambda - (k^2u^{-1} + l'^2u^2\}^{\frac{1}{2}} u^{-1}\, du \\
&\qquad \le \sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} \int_{J_{k1}} \{\lambda - (k^2u^{-1} + u^2\}^{\frac{1}{2}} u^{-1}\, du \\
&\qquad = I_{11}(\lambda) + \sum_{2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} I_{k1}(\lambda).
\end{aligned} \tag{2.11}
$$

We have

$$
I_{11}(\lambda) = \int_{J_{11}} \{\lambda - (u^{-1} + u^2\}^{\frac{1}{2}} u^{-1}\, du = O(\lambda^{\frac{1}{2}}). \tag{2.12}
$$

By Lemma 2.4,

$$
\sum_{2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}} I_{k1}(\lambda) \le K_1 \lambda^{5/4}. \tag{2.13}
$$

By (2.10)–(2.13) we have

$$\begin{aligned}\sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}}\ \sum_{l\in\mathbb{N},l\gg 2}\int_{J_{kl}(\lambda)}&\left\{\lambda-(k^2u^{-1}+l^2u^2)\right\}^{\frac12}u^{-1}\,du\\ =\sum_{1\le k<\lambda^{3/4}3^{-3/4}2^{1/2}}\ \int_1^{\lambda^{3/2}k^2 3^{-3/2}2}&dl'\int_{J'_{kl}(\lambda)}\left\{\lambda-(k^2u^{-1}+l'^2u^2)\right\}^{\frac12}u^{-1}\,du\\ &+O(\lambda^{5/4}).\end{aligned}\tag{2.14}$$

Also, by Lemma 2.4

$$\sum_{2\le k<\lambda^{3/4}3^{-3/4}2^{1/2}}\int_{J_{k1}(\lambda)}\left\{\lambda-(k^2u^{-1}+u^2)\right\}^{\frac12}u^{-1}\,du=O(\lambda^{5/4}).\tag{2.15}$$

Adding (2.12), (2.14) and (2.15), we conclude the proof of the Lemma. □

We now determine the asymptotics for $\lambda\to\infty$ of the integral $I_k(\lambda)$ for each fixed k.

Lemma 2.6. *$I_k(\lambda)\nearrow\!\!\!\sim \lambda^2k^{-2}\frac{\pi}{8}$ for fixed $k\in\mathbb{N}$.*

Proof. We have

$$I_k(\lambda)=\int\limits_{\substack{1\le l<\lambda^{3/2}k^{-2}3^{-3/2}2\\ k^2u^{-1}+l^2u^2<\lambda}} dl\int du\quad\left\{\lambda-(k^2u^{-1}+l^2u^2)\right\}^{\frac12}u^{-1}.$$

Setting

$$u=v\lambda^{-1}k^2,\quad l=x\lambda^{3/2}k^{-2}$$

we get

$$I_k(\lambda)=\lambda^2k^{-2}\int\limits_{\substack{\lambda^{-3/2}k^2\le x<3^{-3/2}2\\ v^{-1}+x^2v^2<1}} dx\int dv\quad\left\{1-(v^{-1}+x^2v^2)\right\}^{\frac12}v^{-1}.\tag{2.16}$$

We discuss the domain of integration, given by

$$\lambda^{-3/2}k^2\le x<3^{-3/2}2,\quad 0<x<(1-v^{-1})^{\frac12}v^{-1}.$$

Let $f(v)$ be the function defined for $1\le v<\infty$ by

$$\begin{aligned} f(v)&=\frac{1-v^{-1}}{v^2}, &\qquad f(1)&=\lim_{v\to\infty}f(v)=0\\ f'(v)&=\frac{3-2v}{v^4}, & f(\tfrac32)&=\tfrac{4}{27}=\max_{1\le v<\infty}f(v).\end{aligned}$$

Since $3^{-3/2}\cdot 2>\frac{2}{\sqrt{27}}$, x is restricted by

$$\lambda^{-3/2}k^2\le x\le(1-v^{-1})^{\frac12}v^{-1}.\tag{2.17}$$

Since $x \geq \lambda^{-3/2}k^2$, v is restricted by

$$f(v) = \frac{1 - v^{-1}}{v^2} \geq \lambda^{-3}k^4 \tag{2.18}$$

Let $v_1 < v_2$ be the two roots in $(1, \infty)$ of

$$\frac{1 - v^{-1}}{v^2} = \lambda^{-3}k^4. \tag{2.19}$$

When $\lambda \to \infty$ for fixed k, $v_1(\lambda) \searrow 1$, $v_2(\lambda) \nearrow \infty$.

Asymptotically, for fixed $k < \lambda^{3/4}3^{-3/4}2^{1/2}$

$$v_2(\lambda) \sim \lambda^{3/2}k^{-2} \qquad \text{as } \lambda \to \infty \tag{2.20}$$

$$\frac{v_1(\lambda) - 1}{v_1^3(\lambda)} \sim \lambda^{-3}k^4; \quad v_1(\lambda) - 1 \sim \lambda^{-3}k^4 \qquad \text{as } \lambda \to \infty. \tag{2.21}$$

Now we interchange the order of the integration in $I_k(\lambda)$, replacing the limits for x by (2.17) and for v by the roots $v_1 = v_1(\lambda)$ and $v_2 = v_2(\lambda)$ of (2.19). We obtain from (2.16)

$$\begin{aligned} I_k(\lambda) &= \lambda^2 k^{-2} \int_{v_1}^{v_2} dv \int_{\lambda^{-3/2}k^2}^{(1-v^{-1})^{\frac{1}{2}}v^{-1}} dx \left\{ \frac{1 - v^{-1}}{v^2} - x^2 \right\}^{\frac{1}{2}} \\ &= \lambda^2 k^{-2} \int_{v_1}^{v_2} dv \, I_k(v) \end{aligned} \tag{2.22}$$

where

$$I_k(v) = \int_{\lambda^{-3/2}k^2}^{(1-v^{-1})^{\frac{1}{2}}v^{-1}} dx \left\{ \frac{1 - v^{-1}}{v^2} - x^2 \right\}^{\frac{1}{2}}. \tag{2.23}$$

We calculate $I_k(v)$. Setting $x = \frac{(1-v^{-1})^{\frac{1}{2}}}{v} x_1$, we get

$$I_k(v) = \frac{1 - v^{-1}}{v^2} \int_{\lambda^{-3/2}k^2 v(1-v^{-1})^{-1/2}}^{1} (1 - x_1^2)^{\frac{1}{2}} \, dx_1 = \frac{1 - v^{-1})^{\frac{1}{2}}}{v^2} I_{k0}(v). \tag{2.24}$$

where

$$I_{k0}(v) = \int_{\lambda^{-3/2}k^2 v(1-v^{-1})^{-1/2}]}^{1} (1 - x^2)^{\frac{1}{2}} \, dx. \tag{2.25}$$

Setting $x = \sin\varphi$, we get

$$\begin{aligned} I_{k0}(v) &= \int_{\arcsin[\lambda^{-3/2}k^2 v(1-v^{-1})^{-1/2}]}^{\pi/2} \cos^2\varphi \, d\varphi \\ &= \tfrac{1}{2} \int_{\arcsin[\lambda^{-3/2}k^2 v(1-v^{-1})^{-1/2}]}^{\pi/2} (1 + \cos 2\varphi) \, d\varphi \\ &= \left[\tfrac{\varphi}{2} + \tfrac{1}{4} \sin 2\varphi \right]_{\arcsin[\lambda^{-3/2}k^2 v(1-v^{-1})^{-1/2}]}^{\pi/2} \\ &= \tfrac{\pi}{4} - \tfrac{1}{2} \arcsin\left[\lambda^{-3/2}k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}} \right] \\ &\quad - \tfrac{1}{2} \Big[\lambda^{-3/2}k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}} \Big] \Big\{ 1 - \lambda^{-3}k^4 \frac{v^2}{1 - v^{-1}} \Big\}^{\frac{1}{2}} \end{aligned} \tag{2.26}$$

From (2.24)–(2.26) we obtain

$$\int_{v_1}^{v_2} dv\, I_k(v) = F_{k1}(v_1, v_2) - F_{k2}(v_1, v_2) - F_{k3}(v_1, v_2) \tag{2.27}$$

where

$$F_{k1}(v_1, v_2) = \int_{v_1}^{v_2} \left(\frac{1}{v^2} - \frac{1}{v^3}\right) \frac{\pi}{4}\, dv = \frac{\pi}{4}\left(\frac{1}{v_1} - \frac{1}{v_2}\right) - \frac{\pi}{8}\left(\frac{1}{v_1^2} - \frac{1}{v_2^2}\right), \tag{2.28}$$

$$F_{k2}(v) = \int_{v_1}^{v_2} \left(\frac{1}{v^2} - \frac{1}{v^3}\right) \tfrac{1}{2} \arcsin\left[\lambda^{-3/2} k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}}\right] dv \tag{2.29}$$

$$F_{k3}(v) = \int_{v_1}^{v_2} \left(\frac{1}{v^2} - \frac{1}{v^3}\right) \tfrac{1}{2}\left[\lambda^{-3/2} k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}}\right]\left\{1 - \lambda^{-3} k^4 \frac{v^2}{1 - v^{-1}}\right\}^{\frac{1}{2}} \tag{2.30}$$

Using the asymptotic limits (2.20), (2.21) of $v_1(\lambda)$ and $v_2(\lambda)$ we get from (2.28) for fixed $k \in \mathbb{N}$

$$F_{k1}(v_1, v_2) \sim \frac{\pi}{4}\left(\frac{1}{1 + \lambda^{-3}k^4} - \lambda^{-3/2}k^2\right) - \frac{\pi}{8}\left(\frac{1}{[1 + \lambda^{-3}k^4]^2} - \lambda^{-3}k^4\right) \quad \text{as } \lambda \to \infty$$

and a calculation using that $v_1(\lambda) \searrow 1$, $v_2(\lambda) \nearrow \infty$ shows that for fixed $k \geq 1$

$$F_{k1}(v_1, v_2) \nearrow \frac{\pi}{8} \quad \text{for } \lambda \to \infty. \tag{2.31}$$

We consider next $F_{k2}(v_1, v_2)$ and $F_{k3}(v_1, v_2)$. For fixed k and v in both cases the integrand converges to 0 for $\lambda \to \infty$. We have by (2.18)

$$\lambda^{-3/2} k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}} \leq 1.$$

Also,

$$0 \leq \frac{1}{v^2} - \frac{1}{v^3} = f(v) \leq \tfrac{4}{27}$$

so for all k and $v \in (1, \infty)$

$$\left(\frac{1}{v^2} - \frac{1}{v^3}\right) \arcsin\left[\lambda^{-3/2} k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}}\right] < \left(\frac{1}{v^2} - \frac{1}{v^3}\right) \frac{\pi}{2}$$

and

$$\left(\frac{1}{v^2} - \frac{1}{v^3}\right)\left[\lambda^{-3/2} k^2 \frac{v}{(1 - v^{-1})^{\frac{1}{2}}}\right]\left\{1 - \lambda^{-3} k^4 \frac{v^2}{1 - v^{-1}}\right\}^{\frac{1}{2}} < \frac{1}{v^2} - \frac{1}{v^3}.$$

By Lebesgue's dominated convergence theorem and the asymptotic limits (2.20), (2.21), for fixed $k \in \mathbb{N}$

$$F_{k2}\big(v_1(\lambda), v_2(\lambda)\big) \xrightarrow[\lambda \to \infty]{} 0 \tag{2.32}$$

$$F_{k3}\big(v_1(\lambda), v_2(\lambda)\big) \xrightarrow[\lambda \to \infty]{} 0. \tag{2.33}$$

Moreover, $F_{k2}(v_1, v_2) > 0$, $F_{k3}(v_1, v_2) > 0$, so by (2.27) and (2.31)

$$\int_{v_1(\lambda)}^{v_2(\lambda)} dv I_k(v) \leq \frac{\pi}{8} \quad \text{as } \lambda \to \infty$$

and by (2.22) for each $k \geq 1$

$$I_k(\lambda) \leq \lambda^2 k^{-2} \tfrac{\pi}{8}.$$

Inserting (2.31), (2.32), (2.33) in (2.27), we get for fixed $k \in \mathbb{N}$

$$\int_{v_1(\lambda)}^{v_2(\lambda)} I_k(v)\,(v)\,dv \nearrow \frac{\pi}{8} \quad \text{as } \lambda \to \infty \tag{2.34}$$

and

$$I_k(\lambda) \gtrsim \lambda^2 k^{-2} \tfrac{\pi}{8} \qquad \text{for } \lambda \to \infty. \tag{2.35}$$

The Lemma is proved. □

Theorem 2. *The counting function $N(\lambda)$ for the eigenvalues of the operator L satisfies the Weyl law,*

$$N(\lambda) \sim \frac{1}{192\pi} \lambda^2 \quad \textit{for } \lambda \to \infty.$$

Proof. 1)

$$N(\lambda) = \sum_{\substack{k \geq 1 \\ m_{k1} < \lambda}} N_{k1}(\lambda) + \sum_{\substack{k \geq 1 \\ m_{kl} < \lambda}} \sum_{l \geq 2} N_{kl}(\lambda). \tag{2.36}$$

By Lemmas 2.4, 2.5,

$$\sum_{\substack{k \geq 1 \\ m_{k1} < \lambda}} N_{k1}(\lambda) = O(\lambda^{3/2}) \tag{2.37}$$

and

$$\sum_{\substack{k \geq 1 \\ m_{kl} < \lambda}} \sum_{l \geq 2} N_{kl}(\lambda) = \sum_{k=1}^{\infty} I_k(\lambda) + O(\lambda^{3/2}). \tag{2.38}$$

From Lemma 2.6 and (2.36)–(2.38) follows

$$N(\lambda) \lesssim \lambda^2 \tfrac{\pi}{8} \sum_{k=1}^{\infty} k^{-2} = \lambda^2 \tfrac{\pi}{8} \cdot \tfrac{\pi^2}{6} = \tfrac{\pi^3}{48} \lambda^2.$$

2) Let $C = \frac{\pi^3}{48} - \varepsilon$. We prove that for $\lambda > \lambda_0$, $N(\lambda) > C\lambda^2$. Choose K such that

$$\sum_{k=K+1}^{\infty} \frac{1}{k^2} < \varepsilon.$$

As in 1) we see that

$$N'_k(\lambda) = \#\big\{\lambda^n_{kl} \leq \lambda \,\big|\, l \in \mathbb{N}, k \geq K+1\big\} \lesssim \tfrac{\pi}{8} \varepsilon \lambda^2. \tag{2.39}$$

Also, by Lemmas 2.5 and 2.6 there exists λ_0 such that for $\lambda > \lambda_0$

$$\sum_{1\le k\le K} N_k(\lambda) = \#\{\lambda_{kl}^n \le \lambda \,|\, l \in \mathbb{N}, 1 \le k \le K\} > \frac{\pi}{8}\Big(\sum_{k=1}^{\infty} \frac{1}{k^2} - \varepsilon\Big)\lambda^2. \qquad (2.40)$$

From (2.39) and (2.40) we get for $\lambda > \lambda_0$

$$N(\lambda) \gtrsim \frac{\pi}{4}\Big(\sum_{k=1}^{\infty} \frac{1}{k^2} - 2\varepsilon\Big)\lambda^2 = \left(\frac{\pi^3}{48} - \frac{\pi}{4}\varepsilon\right)\lambda^2. \qquad (2.41)$$

The Weyl law. By 1) and 2),

$$N(\lambda) \sim \tfrac{\pi^3}{48}\lambda^2 \qquad \text{for } \lambda \to \infty.$$

Replacing λ by $\lambda' = (\lambda - \frac{1}{4})/(4\pi^2)$, introducing the factor 2 in the formula for $N_{kl}(\lambda)$, and taking into account the double multiplicity of λ_{kl}^n as eigenvalue of L, we obtain for the counting function of the operator L

$$N(\lambda) \sim \tfrac{1}{192\pi}\lambda^2 \qquad \text{for } \lambda \to \infty.$$

The constant C_W of the Weyl law equals $\frac{\pi}{6}/(32\pi^2) = 1/(192\pi)$ where $\frac{\pi}{6}$ is the volume of the Jacobi manifold M_J and $32\pi^2$ is the area of the unit sphere in $\mathbb{R}^4$.

This concludes the proof of the Weyl law for L_{Γ_J}. □

3 Normal subgroups of finite index

Let Γ be a normal subgroup of Γ_1 of index I and let $\Gamma_J = \Gamma \ltimes \mathbb{Z}$ be the reduced Jacobi group associated with Γ, a normal subgroup of the modular Jacobi group $\Gamma_{1,J}$ of index I.

Let β be the width of Γ and let

$$\Gamma_\infty = \big\{\left(\begin{smallmatrix}1 & \beta l\\ 0 & 1\end{smallmatrix}\right) \,\big|\, l \in \mathbb{Z}\big\}$$

be the lranslation group of Γ..

The Γ_J-invariant Laplacian

$$L_{\Gamma_J} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{y}\left(\frac{\partial^2}{\partial u^2} - 2x\frac{\partial}{\partial u}\frac{\partial}{\partial v} + \big(x^2+y^2\big)\frac{\partial^2}{\partial v^2}\right)$$

is a self-adjoint operator in the Hilbert space

$$H_{\Gamma_J} = L^2(F_{\Gamma_J}; y^{-2}\,dx\,dy), \quad F_{\Gamma_J} = F_\Gamma \times \{(u,v) \,|\, -\tfrac{1}{2} < u \le \tfrac{1}{2}, -u < v \le \tfrac{1}{2}\}.$$

Lemma 3.1. *Let f be a Γ_J-invariant, continuous function of z, u, v, C^1 in u and v for fixed z. Then*

$$f(z,u,v) = \sum_{k,m} f_{km}(z) e_{km}(u,v)$$

where
$$f_{k'm'}(z) = \big(T_g f_{km}\big)(x), \qquad \binom{k'}{m'} = g^{-1}\binom{k}{m} \quad \textit{for } g \in \Gamma.$$

Proof. This is proved as Lemma 1.1. ☐

Let $\binom{k}{m} \underset{\Gamma}{\sim} \binom{k'}{m'}$ if $\binom{k}{m} = g\binom{k'}{m'}$ for some $g \in \Gamma$. Let $\hat\Gamma = \Gamma_1/\Gamma$, $\hat g = g\Gamma \in \hat\Gamma$,
$$U = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \quad \hat\Gamma_{\infty\beta} = \{\widehat{U}^i \mid i = 0,1,\dots,\beta-1\}.$$

Choose $g_1, g_2, \dots, g_{I/\beta} \in \Gamma_1$ with $g_1 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, such that
$$\{\hat g_i \hat\Gamma_{\infty\beta} \mid i = 1,2,\dots,I/\beta\} = \hat\Gamma/\hat\Gamma_{\infty\beta}$$

or
$$\Gamma_1/\Gamma = \{g_i U^j \Gamma \mid i - 1,2,\dots,I/\beta,\ j = 0,1,\dots,\beta-1\} \tag{3.1}$$

By Lemma 1.2, the equivalence classes of $\mathbb{Z}^2$ under $\underset{\Gamma_1}{\sim}$ are given by
$$\mathbb{Z}^2_k = \left\{\binom{k'}{m'} \,\middle|\, (k',m') = k\right\}, \quad k = 1,2,\dots$$

Lemma 3.2. *For fixed $k = 1,2,\dots$ the equivalence classes of $\mathbb{Z}^2_k$ under $\underset{\Gamma}{\sim}$ are given by*
$$\Gamma g_i\binom{k}{0} = \Gamma g_i U^j\binom{k}{0}, \quad i = 1,\dots,I/\beta,\ j = 0,\dots,\beta-1.$$

Let
$$\binom{k_i}{m_i} = g_i\binom{k}{0}.$$

The stabilizer of $\binom{k_i}{m_i}$ is $\Gamma^i_\infty = g_i\Gamma_\infty g_i^{-1}$.

The parabolic subgroup Γ^i_∞ of Γ is the stabilizer of the cusp $g_i(\infty)$.

Proof. By definition of the g_i, for $i \neq j$ $g_i\binom{k}{0} \underset{\Gamma}{\nsim} g_j\binom{k}{0}$, so $\{\Gamma g_i\binom{k}{0}\}$ are distinct classes under $\underset{\Gamma}{\sim}$ for $i = 1,\dots,I/\beta$.

On the other hand, $U^j\binom{k}{0} = \binom{k}{0}$ for $j = 0,1,\dots,\beta-1$, so
$$\Gamma g_i U^j\binom{k}{0} = \Gamma g_i\binom{k}{0} \quad \text{for } j = 0,1,\dots,\beta-1.$$

By (3.1) the distinct equivalence classes $\Gamma g_i\binom{k}{0}$, $i = 1,\dots,I/\beta$ are all the equivalence classes under $\underset{\Gamma}{\sim}$. ☐

We proceed to characterize the Γ_J-invariant functions as it was done in Lemma 1.3 for $\Gamma = \Gamma_1$.

Lemma 3.3. *Let* $f_{k0} \in \mathcal{H}_{\Gamma_\infty} = L^2(F_{\Gamma_\infty}; y^{-2}\,dx\,dy)$ *and for* $i = 1, 2, \ldots, I/\beta$

$$\begin{aligned}
f^i_{k0}(z,u,v) &= \widetilde{T}_{g_i}\big(f_{k0}(z)e^{2\pi iku}\big) \\
&= \big(T_{g_i}f_{k0}\big)(z)e_{k_im_i}(u,v), \quad \tbinom{k_i}{m_i} = g_i^{-1}\tbinom{k}{0} \\
f^i_k(z,u,v) &= \sum_{\tilde g\in\Gamma/\Gamma^i_\infty}\big(\widetilde{T}_{\tilde g}f^i_{k0}\big)(z,u,v) \qquad (3.2)\\
&= \sum_{\tilde g\in\Gamma/\Gamma^i_\infty}(T_{\tilde g}T_{g_i}f_{k0})(z)e_{k'm'}(u,v), \quad \tbinom{k'}{m'} = \tilde g^{-1}\tbinom{k_i}{m_i}
\end{aligned}$$

Then the functions $f^i_k(z,u,v)$ *are* Γ_J*-invariant and*

$$f^i_k(z,u,v) = \big(\widetilde{T}_{g_i}f^1_k\big)(z,u,v), \quad i = 1,\ldots,I/\beta$$

where

$$\big(\widetilde{T}_{g_i}f^1_k\big)(z,u,v) = \sum_{g\in\Gamma/\Gamma_\infty}\big(T_{g_i}T_gf_{k0}\big)(z)e_{k''m''}(u,v), \quad \tbinom{k''}{m''} = g_i^{-1}g^{-1}\tbinom{k}{0}$$

and by the choice $g_1 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$

$$f^1_k(z,u,v) = \sum_{g\in\Gamma/\Gamma_\infty}\big(T_gf_{k0}\big)(z)e_{k'm'}(u,v), \quad \tbinom{k'}{m'} = g^{-1}\tbinom{k}{0}. \qquad (3.3)$$

Proof.

$$\begin{aligned}
f^i_k(z,u,v) &= \sum_{\tilde g\in\Gamma/\Gamma^i_\infty}\big(T_{\tilde g}T_{g_i}f_{k0}\big)(z)e_{k'm'}(u,v) && \Big(\tbinom{k'}{m'} = \tilde g^{-1}g_i^{-1}\tbinom{k}{0}\Big)\\
&= \sum_{\tilde g\in\Gamma/\Gamma^i_\infty}\big[T_{g_i}\big(T_{g_i^{-1}}T_{\tilde g}T_{g_i}\big)f_{k0}\big](z)e_{k'm'} && \Big(\tbinom{k'}{m'} = g_i^{-1}\big(g_i\tilde g^{-1}g_i^{-1}\big)\tbinom{k}{0}\Big)
\end{aligned}$$

Since Γ is normal, setting $g = g_i^{-1}\tilde g g_i$, this equals

$$\begin{aligned}
&\sum_{g\in\Gamma/\Gamma_\infty}\big(T_{g_i}T_gf_{k0}\big)(z)e_{k''m''}(u,v) && \Big(\tbinom{k''}{m''} = g_i^{-1}g^{-1}\tbinom{k}{0}\Big)\\
&\quad = \widetilde{T}_{g_i}\sum_{g\in\Gamma/\Gamma_\infty}\big(T_gf_{k0}\big)(z)e_{k''m''}(u,v) && \Big(\tbinom{k''}{m''} = g^{-1}\tbinom{k}{0}\Big)\\
&\quad = \big(\widetilde{T}_{g_i}f^1_k\big)(z,u,v), \quad i = 1,\ldots,I/\beta
\end{aligned}$$

The Lemma is proved. □

Lemma 3.4. *For each* $k \geq 1$ *and* $i = 1,\ldots,I/\beta$ *the map*

$$f^1_k \longrightarrow f^i_k = \widetilde{T}_{g_i}f_{k1}$$

given by (3.2) and (3.3) is unitary from the Hilbert space $\mathcal{H}^1_{\Gamma k}$ *of* Γ_J*-invariant functions of the form given by (3.3) onto* $\mathcal{H}^i_{\Gamma k} = \widetilde{T}_{g_i}\mathcal{H}^1_{\Gamma k}$.

For $k_1 \neq k_2$ and for $k_1 = k_2$, $i_1 \neq i_2$, the Hilbert spaces $\mathcal{H}^{i_1}_{\Gamma k_1}$ and $\mathcal{H}^{i_2}_{\Gamma k_2}$ are orthogonal. The subspaces $\mathcal{H}^{i}_{\Gamma k}$ are invariant under L, and for each $k \geq 1$ the operators

$$L^i_{\Gamma k} = L|_{\mathcal{H}^i_{\Gamma k}}, \quad i = 1, \ldots, I/\beta \tag{3.4}$$

are unitarily equivalent.

With

$$L_{\Gamma k} = \sum_{i=1}^{I/\beta} \bigoplus L^i_{\Gamma k} \quad \text{for } k \geq 1 \tag{3.5}$$

and

$$L_{\Gamma 0} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad \text{in } L^2(F_\Gamma) \tag{3.6}$$

we have

$$L_{\Gamma_J} = \sum_{k=0}^{\infty} \bigoplus L_{\Gamma k}. \tag{3.7}$$

Proof. Fix $k \geq 1$, $i = 1, \ldots, I/\beta$.

Then for $f^i_{k_j} \in L^2(F_{\Gamma_J})$, $j = 1, 2$, by Lemma 3.3, with $D = \{(u,v) \mid -\frac{1}{2} < u \leq \frac{1}{2},\ -u < v \leq \frac{1}{2}\}$

$$\begin{aligned}
\left(f^i_{k1}, f^i_{k2}\right)_{L^2(F_{\Gamma_J})} &= \int_{F_\Gamma \times D} \big(\widetilde{T}_{g_i} f^1_{k1}\big)(z,u,v)\big(\overline{\widetilde{T}_{g_i} f^1_{k2}}\big)(z,u,v)\, d\mu(z)\, du\, dv \\
&= \int_{F_\Gamma \times D} \Big\{ \sum_{g \in \Gamma/\Gamma_\infty} \big(T_{g_i} T_g f_{k0,1}\big)(z) e_{k'm'}(u,v) \Big\} \\
&\qquad \cdot \Big\{ \sum_{\tilde{g} \in \Gamma/\Gamma_\infty} \big(\overline{T_{g_i} T_{\tilde{g}} f_{k0,2}}\big)(z) e_{-k'',-m''}(u,v) \Big\} d\mu(z)\, du\, dv \\
&\qquad \tbinom{k'}{m'} = g_i^{-1} g^{-1} \tbinom{k}{0}, \quad \tbinom{k''}{m''} = g_i^{-1} \tilde{g}^{-1} \tbinom{k}{0}
\end{aligned} \tag{3.8}$$

Since $\binom{k'}{m'} \neq \binom{k''}{m''}$ iff $g \neq \tilde{g} \mod \Gamma_\infty$, this equals

$$\begin{aligned}
&\int_{F_\Gamma} \sum_{g \in \Gamma/\Gamma_\infty} \big(T_{g_i} T_g f_{k0,1}\big)(z)\big(\overline{T_{g_i} T_g f_{k0,2}}\big)(z)\, d\mu(z) \\
&\quad = \int_{T_{g_i} F_\Gamma} \sum_{g \in \Gamma/\Gamma_\infty} \big(T_g f_{k0,1}\big)(z)\big(\overline{T_g f_{k0,2}}\big)(z)\, d\mu(z) \\
&\quad = \int_{F_\Gamma} \sum_{g \in \Gamma/\Gamma_\infty} \big(T_g f_{k0,1}\big)(z)\big(\overline{T_g f_{k0,2}}\big)(z)\, d\mu(z) \\
&\quad = \left(f^1_{k1}, f^2_{k2}\right)_{L^2(F_{\Gamma_J})}
\end{aligned}$$

since $T_{g_i} F_\Gamma = F_\Gamma$, which can be seen as follows. We have a fundamental domain F_Γ of the form

$$F_\Gamma = \bigcup_{i=1}^{I} T_{g_i} F_{\Gamma_1}$$

and

$$T_{g_j} F_\Gamma = T_{g_j}\Big(\bigcup_{i=1}^{I} T_{g_i} F_{\Gamma_1}\Big) = \bigcup_{i=1}^{I}\big(T_{g_i} T_{g_j} F_{\Gamma_1}\big) = \bigcup_{i=1}^{I} T_{g_k} F_{\Gamma_1} = F_\Gamma .$$

This proves that the map $f_k^1 \to f_k^i = \widetilde{T}_{g_i} f_{k1}$ is unitary from $\mathcal{H}^1_{\Gamma k}$ onto $\mathcal{H}^i_{\Gamma k}$ for $i = 1, \dots, I/\beta$.

If $k_1 \neq k_2$ and $i.j = 1, \dots, I/\beta$, then $(k', m') = k_1 \neq k_2 = (k'', m'')$ for all pairs in the series

$$\begin{aligned}
&\big(f^i_{k_1}, f^j_{k_2}\big)_{L^2(F_{\Gamma_J})}\\
&\quad = \int_{F_{\Gamma_J}} \Big\{ \sum_{g\in\Gamma/\Gamma_\infty} \big(T_{g_i} T_g f_{k_1 0}\big)(z) e_{k'm'}(u,v)\Big\}\\
&\qquad \cdot \Big\{ \sum_{\tilde g\in\Gamma/\Gamma_\infty} \big(\overline{T_{g_j} T_{\tilde g} f_{k_2 0}}\big)(z) e_{-k_2''-m_2''}(u,v)\Big\} d\mu(z)\, du\, dv\\
&\qquad \binom{k'}{m'} = g_i^{-1} g^{-1}\binom{k}{0}, \quad \binom{k''}{m''} = g_j^{-1}\tilde g^{-1}\binom{k}{0}.
\end{aligned} \tag{3.9}$$

Therefore all terms in the series (3.9) are 0, so $\big(f^i_{k_1}, f^j_{k_2}\big) = 0$, and $\mathcal{H}^i_{\Gamma k_1}$ and $\mathcal{H}^j_{\Gamma k_2}$ are orthogonal.

To prove that $\big(f^i_{k1}, f^j_{k2}\big) = 0$ for $k \geq 1$, $i \neq j$, we write

$$\begin{aligned}
\big(f^i_{k1}, f^j_{k2}\big) &= \int_{F_\Gamma\times T} \Big\{ \sum_{g\in\Gamma/\Gamma_\infty} \big(T_{g_i} T_g f_{k0,1}\big)(z) e_{k'm'}(u,v)\Big\}\\
&\qquad \cdot \Big\{ \sum_{\tilde g\in\Gamma/\Gamma_\infty} \big(\overline{T_{g_j} T_{\tilde g} f_{k0,2}}\big)(z) e_{-k_2''-m_2''}(u,v)\Big\} d\mu(z)\, du\, dv\\
&\qquad \binom{k'}{m'} = g_i^{-1} g^{-1}\binom{k}{0}, \quad \binom{k''}{m''} = g_j^{-1}\tilde g^{-1}\binom{k}{0}.
\end{aligned}$$

We have

$$U^i\binom{k}{0} = \binom{k}{0}, \quad i = 0, \dots, \beta - 1$$

and

$$g_j g_i^{-1} g^{-1}\binom{k}{0} \neq \tilde g^{-1}\binom{k}{0}, \quad \text{since } g^{-1}\binom{k}{0} \underset{\Gamma}{\sim} \tilde g^{-1}\binom{k}{0}$$

so

$$\binom{k''}{m''} \neq \binom{k'}{m'} \quad \text{for all terms, and } (f^i_{k1}, {}^j_{k2}) = 0$$

so the Hilbert spaces $\mathcal{H}^i_{\Gamma k}$ and $\mathcal{H}^j_{\Gamma k}$ are orthogonal for $i \neq j$.

The unitary equivalence of the operators $L^i_{\Gamma k}$, $i = 1, \dots, I/\beta$ then follows from the fact that L_Γ commutes with $\widetilde{T}_g$ for all $g \in \Gamma$,

$$L^i_{\Gamma k} \widetilde{T}_{g_i} f^1_k = L^i_{\Gamma k} f^i_k = L_\Gamma \widetilde{T}_{g_i} f_{k1} = \widetilde{T}_{g_i} L_\Gamma f_{k1} = \widetilde{T}_{g_i} L^1_{\Gamma k} f_{k1}, \quad i = 1, \dots, I/\beta.$$

The Lemma is proved except (3.7) which will be proved after the proof of Lemma 3.6. □

Lemma 3.5. *For $k \geq 1$ the operator $L^1_{\Gamma k}$ is unitarily equivalent to the operator*

$$L^0_{\beta k} = A + 4\pi^2 \frac{k^2}{y} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 4\pi^2 \frac{k^2}{y}$$

in $\mathcal{H}^0_{\beta k} = \mathcal{H}_{\Gamma_\infty} = L^2\big(F_{\Gamma_\infty}; y^{-2}\, dx\, dy\big)$, $F_{\Gamma_\infty} = \{x + iy \,|\, o \leq x < \beta, y > 0\}$ *via the map*

$$\mathcal{H}^o_{\beta k} \ni f(z) \longrightarrow \big(\Sigma_{\Gamma k} f\big)(z, u, v) = f^1_k(z, u, v)$$

where

$$\big(\Sigma_{\Gamma k} f\big)(z, u, v) = \sum_{g \in \Gamma / \Gamma_\infty} \big(T_g f\big)(z) e_{k(\delta, -\gamma)}(u, v), \quad g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right),$$

and

$$L^1_{\Gamma k} \Sigma_{\Gamma k} f = \Sigma_{\Gamma k} L^0_{\beta k} f \quad \text{for } f \in \mathcal{D}\big(L^0_{\beta k}\big).$$

Proof. This is proved as Lemma 1.8, replacing Γ_1 by Γ. □

We notice that $\mathcal{H}^0_{\beta k}$ and $L^0_{\beta k}$ are common to all Γ with the same width β, while $\Sigma_{\Gamma k}$ and $L^i_{\Gamma k}$ depend on Γ.

Lemma 3.6. *For $k \geq 1$ the space $\mathcal{H}^0_{\beta k}$ can be decomposed as a direct sum of subspaces*

$$\mathcal{H}^0_{\beta k} = \sum_{l \in \mathbb{Z}} \bigoplus \mathcal{H}^0_{\beta k l}, \quad \mathcal{H}^0_{\beta k l} = \big\{ e^{2\pi i \beta^{-1} l x} \varphi_{\beta k l}(y) \,\big|\, \varphi_{\beta k l}(y) \in \mathcal{D}\big(\tilde{L}_{\beta k l}\big) \big\}$$

where the spaces $\mathcal{H}^0_{\beta k l}$ are invariant under $L^0_{\beta k}$, and

$$L^0_{\beta k} = \sum_{l \in \mathbb{Z}} \bigoplus L^0_{\beta k l}$$

where

$$L^0_{\beta k l} = L^0_{\beta k}|_{\mathcal{H}^0_{\beta k l}},$$
$$L^0_{\beta k l}\big(e^{2\pi i \beta^{-1} l x} \varphi_{\beta k l}(y)\big) = e^{2\pi i \beta^{-1} l x} \big(\tilde{L}_{\beta k l} \varphi_{\beta k l}\big)(y)$$

and

$$\big(\tilde{L}_{\beta k l} \varphi_{\beta k l}\big)(y) = \left\{ -y^{-2} \frac{d^2}{dy^2} + 4\pi^2 \left(\frac{k^2}{y} \beta^{-2} l^2 y^2 \right) \right\} \varphi_{\beta k l}(y).$$

The operator $\tilde{L}_{\beta k l}$ is unitarily equivalent via the map $g(y) \to f(t) = g(e^t) e^{-t/2}$ to the operator

$$M_{\beta k l} = -\frac{d^2}{dt^2} + \frac{1}{4} + 4\pi^2 (k^2 e^{-t} + \beta^{-2} l^2 e^{2t}) \quad \text{in } L^2(-\infty, \infty).$$

For $k \geq !$, $l \geq 1$, the operator $\tilde{L}_{\beta k l}$ has a simple, discrete spectrum

$$\lambda^1_{\beta k l} < \lambda^2_{\beta k l} < \cdots < \lambda^n_{\beta k l} < \cdots \tag{3.10}$$

$$\tfrac{1}{4} + 4\pi^2 k^{4/3} l^{2/3} \beta^{-2/3} \cdot 3 \cdot 2^{-2/3} < \lambda^1_{\beta k l}, \quad \lambda^n_{\beta k l} \xrightarrow[n \to \infty]{} \infty, \tag{3.11}$$

with real, orthonormal eigenfunctions $\varphi^n_{\beta kl}(y)$ giving rise to even and odd eigenfunctions of $L^0_{\beta kl}$,

$$\varphi^n_{\beta kl}(y)\cos 2\pi\beta^{-1}lx, \quad \varphi^n_{\beta kl}(y)\sin 2\pi\beta^{-1}lx. \tag{3.12}$$

For $k \geq 1$, $l = 0$, the operator $\tilde{L}_{\beta k0}$ has a simple, purely continuous spectrum, identical with that of L_{k0} analyzed in Lemma 1.8, in fact

$$\tilde{L}_{\Gamma k0} = L_{k0}.$$

Proof. Let $\mathcal{D}(L^{0\prime}_{\Gamma k})$ be the space of continuous functions in $\mathcal{H}^0_{\Gamma k}$, C^1 in (u,v) and let f_k be a function in $\mathcal{D}(L^{0\prime}_{\Gamma k})$. Then f_k has an expansion

$$f_k(x,y) = \sum_{l\in\mathbb{Z}} \varphi_{\beta kl}(y) e^{2\pi i\beta^{-1}lx}$$

and

$$\begin{aligned}\left(L^{01}_{\Gamma k} f_k\right)(x,y) &= \sum_{l\in\mathbb{Z}} \left\{-y^2\frac{d^2}{dy^2} + 4\pi^2\left(\frac{k^2}{y} + \beta^{-2}l^2\right)\right\}\varphi_{\beta kl}(y)e^{2\pi i\beta^{-1}lx} \\ &= \sum_{l\in\mathbb{Z}} \left(\tilde{L}_{\beta kl}\varphi_{\beta kl}\right)(y)e^{2\pi i\beta^{-1}lx}.\end{aligned}$$

Since $L^0_{\Gamma k}$ is the closure of $L^{01}_{\Gamma k}$, the first part of the Lemma follows.

A calculation shows that $\tilde{L}_{\beta kl}$ is unitarily equivalent to $M_{\beta kl}$.

For $l \neq 0$ the operators $\tilde{L}_{\beta kl}$ have purely discrete simple spectra. The function $k^2e^{-t} + \beta^{-2}l^2e^{2t}$ has minimum $k^{4/3}l^{2/3}\beta^{-2/3}\cdot 3\cdot 2^{-2/3}$. It follows that the spectrum of $\tilde{L}_{\beta kl}$ is a sequence of simple eigenvalues $\lambda^n_{\beta kl}$ satisfying (3.10) and (3.11).

For $l = 0$, $\tilde{L}_{\beta k0} = L_{k0}$ stemming from $L_{\Gamma_1,J}$ which is treated in Lemma 1.8. □

Proof of (3.7) *of Lemma 3.4.* Fix $k \geq 1$, $l \neq 0$. Consider for $j = 1\ldots,\beta-1$ the functions obtained from (3.2) by replacing g_i by g_iT^j,

$$f^{ij}_k(z,u,v) = \sum_{\tilde{g}\in\Gamma/\Gamma^i_\infty} \left(T_{\tilde{g}}T_{g_i}T_{U^j}f_{k0}\right) \tag{3.13}$$

with

$$f_{k0}(z) = \varphi^n_{\beta kl}(y)e^{2\pi i\beta^{-1}lx}.$$

Then

$$\left(T_U f_{k0}\right)(z) = \varphi^n_{\beta kl}(y)e^{2\pi i\beta^{-1}l(x+j)} = f_{k0}(z)e^{2\pi i\beta^{-1}lj},$$

so

$$f^{ij}_k(z,u,v) = e^{2\pi i\beta^{-1}lj}f^i_k(z,u,v) \quad \text{for all } f_{k0}(z)$$

with fixed $k \geq 1$, $l \neq 0$ and $j = 1,\ldots,\beta-1$.

For $l = 0$,

$$f^{ij}_k(z,u,v) = f^i_k(z,u,v), \quad j = 1,\ldots,\beta-1.$$

It follows that the series (3.13) contribute the same functions as the functions $f^i_k(z,u,v)$ given for $j = 0$ by (3.2) and (3.7) follows. □

We conclude this section by proving the Weyl law for normal subgroups of Γ_1.

Lemma 3.7. *The counting function $N_\Gamma(\lambda)$ for the eigenvalues of the operator L_Γ with index I of Γ in Γ_1 satisfies the Weyl law*

$$N_\Gamma(\lambda) \sim \frac{I}{192\pi}\lambda^2 \qquad \textit{for } \lambda \to \infty.$$

Proof. We have

$$l < l + \frac{i}{\beta} \leq l+1 \qquad\qquad \text{for } l = 1, 2, \ldots,\ i = 1, \ldots, \beta \tag{3.14}$$

$$\frac{1}{\beta} \leq \frac{i}{\beta} \leq 1 \qquad\qquad \text{for } i = 1, \ldots, \beta. \tag{3.15}$$

The set of eigenvalues of L_Γ is the union over k, l and β of the sequences of eigenvalues of $L^0_{\beta kl}$,

$$\{\lambda^n_{\beta kl}\}_{n=1}^\infty = \{\lambda^n_{k,l+\frac{i}{\beta}}\}_{n=1}^\infty, \qquad l = 0, 1, 2, \ldots,\ \ i = 1, \ldots, \beta. \tag{3.16}$$

By Lemmas 2.1 and 2.2

$$\#\{\lambda^n_{k,l+\frac{i}{\beta}} \leq \lambda\} = I_{k,l+\frac{i}{\beta}}(\lambda) + O_{kli}(1) \tag{3.17}$$

where

$$I_{k,l+\frac{i}{\beta}}(\lambda) = \int_{J_{k,l+i/\beta}(\lambda)} \left\{\lambda - 4\pi^2\left(\frac{k^2}{y} + \left(l+\frac{i}{\beta}\right)^2 y^2\right)\right\}^{1/2} y^{-1}\, dy$$

and

$$\sum_{kli} O_{kli}(1) = O(\lambda^{3/2}). \tag{3.18}$$

It therefore suffices to estimate $\sum_{kli} I_{k,l+\frac{i}{\beta}}(\lambda)$.

We consider first the sum over $l \geq 1$ and then $l = 0$.

For fixed $k, l \in \mathbb{N}$ we have

$$I_{k,l+1}(\lambda) < I_{k,l+\frac{\beta-1}{\beta}}(\lambda) < \cdots < I_{k,l+\frac{2}{\beta}}(\lambda) < I_{k,l+\frac{1}{\beta}}(\lambda) < I_{k,l}(\lambda).$$

It follows that for fixed $k, l \in \mathbb{N}$

$$\beta I_{k,l+1}(\lambda) < \sum_{i=1}^{\beta} I_{k,l+\frac{i}{\beta}}(\lambda) < \beta I_{k,l}(\lambda)$$

and hence

$$\beta \sum_{l\geq 2} I_{k,l}(\lambda) < \sum_{l\geq 1}\sum_{i=1}^{\beta} I_{k,l+\frac{i}{\beta}}(\lambda) < \beta \sum_{l \geq 1} I_{k,l}(\lambda). \tag{3.19}$$

By Lemma 2.4

$$\sum_{k\geq 2} I_{k,1}(\lambda) = O(\lambda^{5/4}). \tag{3.20}$$

Adding (3.19) over k and using (3.20) and $I_{11}(\lambda) = O(\lambda^{1/2})$, we get

$$\sum_{k\geq 1}\sum_{l\geq 1}\sum_{i]1}^{\beta} I_{k,l+\frac{i}{\beta}}(\lambda) = \beta \sum_{k\geq 1}\sum_{l\geq 1} I_{kl}(\lambda) + O(\lambda^{5/4}). \tag{3.21}$$

From (3.17), (3.18), (3.21) and Lemma 2.3 we get

$$\#\{\lambda^n_{k,l+\frac{i}{\beta}} \leq \lambda \,|\, k \geq 1, l \geq 1, i = 1, \ldots, \beta\} = \beta N(\lambda) + O(\lambda^{3/2}). \tag{3.22}$$

For $l = 0$ we get as in the proof of Lemma 2.4

$$\sum_{i=1}^{\beta}\sum_{k\geq 1} I_{k,\frac{i}{\beta}} = O(\lambda^{5/4}). \tag{3.23}$$

Adding (3.22) and (3.23), we obtain the counting function

$$\widetilde{N}_\Gamma(\lambda) = \beta N(\lambda) + O(\lambda^{3/2}). \tag{3.24}$$

In the asymptotic formula (3.24) we have taken into account the fact that each eigenvalue $\lambda^n_{k,l+\frac{i}{\beta}}$ of $L^0_{\beta kl}$ is double with eigenfunctions $\varphi_{\beta kl}(y)e^{\pm 2\pi i\beta^{-1}lx}$ as in the case of Γ_1. By Lemma 3.4, $\lambda^n_{k,l+\frac{i}{\beta}}$ as an eigenvalue of L_k is further degenerate by the factor I/β.

Therefore we obtain for the total counting function $N_\Gamma(\lambda)$ the asymptotics

$$N_\Gamma(\lambda) = I/\beta \cdot \beta N(\lambda) + O(\lambda^{3/2}) = IN(\lambda) + O(\lambda^{3/2}). \tag{3.25}$$

From Theorem 2 and (3.25) follows Lemma 3.7. □

From Lemmas 3.4– 3.7 we obtain

Theorem 3. *The Hilbert space $\mathcal{H}_{\Gamma_J}$ can be decomposed into a direct sum of invariant subspaces*

$$\mathcal{H}_{\Gamma_J} = \sum_{k=1}^{\infty}\bigoplus\Big\{\sum_{l=-\infty}^{\infty}\bigoplus\Big(\sum_{i=1}^{I/\beta}\bigoplus \mathcal{H}^i_{\Gamma kl}\Big)\bigoplus\sum_{i=1}^{I/\beta}\mathcal{H}^i_{\Gamma k0}\Big\}$$

where

$$\mathcal{H}^i_{\Gamma kl} = \widetilde{T}_{g_i}\Sigma_{\Gamma k}\mathcal{H}^0_{\Gamma kl}$$

and

$$L_\Gamma = \sum_{k=1}^{\infty}\bigoplus\Big\{\sum_{l=-\infty}^{\infty}\bigoplus\Big(\sum_{i=1}^{I/\beta}\bigoplus L^i_{\Gamma kl}\Big) + \sum_{i=1}^{I/\beta}\bigoplus L^i_{\Gamma k0}\Big\}$$

where $L^i_{\Gamma kl}$ is unitarily equivalent to $\tilde{L}_{\beta kl}$, $k \geq 1$, $l \geq 1$, $i = 1, \ldots, I$,

$$L^i_{\Gamma kl}\widetilde{T}_{g_i}\Sigma_{\Gamma k}\big(e^{2\pi i\beta^{-1}lx}\varphi_{\Gamma kl}(y)\big) = \widetilde{T}_{g_i}L^1_{\Gamma kl}\Sigma_{\Gamma k}\big\{e^{2\pi i\beta^{-1}lx}(\tilde{L}_{\beta kl}\varphi_{\Gamma kl})(y)\big\}.$$

For $l \geq 1$ the spectrum of $L^i_{\Gamma kl}$ is discrete with eigenvalues λ^n_{kl} given by (3.1) *and eigenfunctions*

$$\Psi^{ni}_{kl}(z,u,v) = \widetilde{T}_{g_i} \sum_{g\in\Gamma/\Gamma_\infty} T_g\big(\varphi^n_{kl}(y)e^{2\pi i\beta^{-1}lx}\big)e_{k(\delta u,-\gamma v)}(u,v).$$

For each $k \geq 1$ and $l = 0$ the spectrum of $L_{\Gamma k0}$ is continuous of multiplicity I/β.

The counting function $N_\Gamma(\lambda)$ for the eigenvalues of L_Γ satisfies the Weyl law

$$N_\Gamma(\lambda) \approx I\frac{1}{192\pi}\lambda^2.$$

4 Perturbation by modular forms

We consider the subgroup $\Gamma = \Gamma(2)$ of index 6 in Γ_1. The translation subgroup Γ_∞ is

$$\Gamma_\infty = \left\{\left(\begin{smallmatrix}1 & 2l\\ 0 & 1\end{smallmatrix}\right) \,\middle|\, l\in\mathbb{Z}\right\} = \left\{A^l \,\middle|\, l\in\mathbb{Z}\right\}, \quad A = \left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$$

and the width $\beta = 2$.

In Lemma 3.2 we choose for g_1, g_2, g_3 the powers of the elliptic element of third order

$$e = \left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right), \quad e^2 = \left(\begin{smallmatrix}-1 & 1\\ -1 & 0\end{smallmatrix}\right), \quad e^3 = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \sim \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right),$$
$$g_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \quad g_2 = e, \quad g_3 = e^2 = e^{-1}.$$

The group Γ_1/Γ_2 is generated by

$$g_1,\ U,\ g_2,\ g_2U,\ g_3,\ g_3U, \quad U = \left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right).$$

$\Gamma(2)$ is generated by the parabolic elements

$$A = \left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right), \quad B = eAe^{-1} = \left(\begin{smallmatrix}-1 & 0\\ 2 & -1\end{smallmatrix}\right) \simeq \left(\begin{smallmatrix}1 & 0\\ -2 & 1\end{smallmatrix}\right), \quad S = e^2Ae^{-2} = \left(\begin{smallmatrix}1 & -2\\ 2 & -3\end{smallmatrix}\right)$$

with the relation

$$ABS = I.$$

We have

$$A(\infty) = \infty, \quad B(0) = 0, \quad S(1) = 1$$
$$e(\infty) = 0, \quad e^2(\infty) = e(0) = 1.$$

We define the function $F(z)$ by

$$F(z) = P(z) - 3P(2z) + 2P(4z) = E_2(z) - 3E_2(2z) + 2E_2(4z)$$

where

$$P(z) = 1 - 24\sum_{i=1}^{\infty}\sigma(n)e^{2\pi inz} = E_2(z) - \frac{3}{\pi y}, \quad \sigma(n) = \sum_{d|n} d$$

and the Eisenstein series $E_2(z)$ is a modular form of weight 2. The function $F(z)$ is a holomorphic form of weight 2 for the group $\Gamma_0(4)$ such that $F(\infty)=0$.

From the relation

$$\Gamma(2) = 2\Gamma_0(4)\tfrac{1}{2}$$

we obtain the holomorphic form of weight 2 for $\Gamma(2)$ defined by

$$G(z) = F\big(\tfrac{z}{2}\big).$$

Based on $G(z)$ we define a group of characters χ_α on $\Gamma(2)$ as follows. Let

$$I = \int_i^{Bi} G(z)\,d\mu(z) = \int_i^{\frac{i}{-2i+1}} G(z)\,d\mu(z) = I_1 + iI_2.$$

It is easy to check that $I_1 \neq 0$. We normalize $G(z)$ by setting

$$\widetilde{G}(x) = I_1^{-1} G(z).$$

Then

$$\tilde{I} = I_1^{-1}\int_i^{Bi} G(z)\,d\mu(z) = \int_i^{Bi} \widetilde{G}(z)\,d\mu(z) = 1 + iI_1^{-1}I_2$$

and

$$\int_i^{Ai} \widetilde{G}\,d\mu(z) = \int_i^{i+2} \widetilde{G}(z)\,d\mu(z) = 0.$$

We define a group of characters χ_α on $\Gamma(2)$ by

$$\chi_\alpha(g) = \exp\Big\{2\pi i\alpha\,\mathrm{Re}\int_{z_0}^{gz_0} \widetilde{G}(z)\,d\mu(z)\Big\}, \quad \alpha\in\mathbb{R}.$$

The integral is independent of $z_0 \in F_{\Gamma(2)}$, and

$$\chi_\alpha(A) = 1 \quad \text{for all } \alpha$$
$$\chi_\alpha(B) = e^{2\pi i\alpha}, \quad \chi_\alpha(S) = e^{-2\pi i\alpha}.$$

For $\alpha\neq 0$, the character χ_α closes the cusps 0 and 1 and keeps the cusp ∞ open.

A family of Laplacians L_α is defined by

$$L_\alpha = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{y}\left(\frac{\partial^2}{\partial u^2} - 2x\frac{\partial}{\partial u}\frac{\partial}{\partial v} + (x^2+y^2)\frac{\partial^2}{\partial v^2}\right)$$

on Γ_{J,χ_α}-invariant functions,

$$\mathcal{D}(L_\alpha) = \big\{ f\in\mathcal{H}_{\Gamma_J} \,\big|\, L_\alpha f\in\mathcal{H}_{\Gamma_J},\ \big(\widetilde{T}_g f\big)(z,w) = \chi_\alpha(g) f(z,w) \quad \text{for } g\in\Gamma(2),\ (z,w)\in F_{\Gamma_J}\big\}$$

We proceed to analyze the operators L_α and their spectra. An extension of Lemma 3.1 to functions transforming under Γ_J with character χ_α gives

Lemma 4.1. *Let f be a Γ_{J,χ_α}-invariant, continuous function of (z,u,v), C^1 in u and v for fixed z. Then*

$$f(z,u,v)=\sum_{k,m} f_{km}(z)e_{km}(u,v)$$

where

$$f_{k'm'}(z)=\big(T_g f_{km}\big)(z)\chi_\alpha(g),\quad \tbinom{k'}{m'}=g^{-1}\tbinom{k}{m}.$$

We characterize $\Gamma_{J,\alpha}$-invariant functions by extending Lemma 3.3 to the case with character χ_α.

Definition. *Let $f_{k0}\in\mathcal{H}_{\Gamma_\infty}$ and for $k=1,2,3,\dots$, $i=1,2,3$, $\alpha\in\mathbb{R}\setminus\mathbb{Z}$*

$$\begin{aligned} f^i_{k0}(z,u,v)&=\widetilde{T}_{g_i}\big(f_{k0}(z)e^{2\pi iku}\big)=\big(T_{g_i}f_{k0}\big)(z)e_{k_im_i}(u,v),\quad \tbinom{k_i}{m_i}=g_i^{-1}\tbinom{k}{0}\\ f^i_{k\alpha}(z,u,v)&=\big(\Sigma^i_{k,\alpha}f_{k0}\big)(z,u,v)=\sum_{\tilde g\in\Gamma/\Gamma^i_\infty}\big(\widetilde{T}_{\tilde g\alpha}f^i_{k0}\big)(z,u,v)\\ &=\sum_{\tilde g\in\Gamma/\Gamma^i_\infty}\big(T_{\tilde g}T_{g_i}f_{k0}\big)(z)\chi_\alpha(\tilde g)e_{k'_im'_i}(u,v),\qquad \tbinom{k'_i}{m'_i}=\tilde g^{-1}\tbinom{k_i}{m_i}. \end{aligned}\tag{4.1}$$

Set $\tilde g=g_igg_i^{-1}$, $\chi_{i\alpha}(g)=\chi_\alpha(\tilde g)=\chi_\alpha(g_igg_i^{-1})$.

Since $\Gamma^i_\infty=g_i\Gamma_\infty g_i^{-1}$ and Γ is normal, we get from (4.1)

$$\begin{aligned} f^i_{k\alpha}(z,u,v)&=\sum_{g\in\Gamma/\Gamma_\infty}\big(T_{g_i}T_gf_{k0}\big)(z)\chi_\alpha(g_igg_i^{-1})e_{k'_im'_i}(u,v)\quad \tbinom{k'_i}{m'_i}=g_i^{-1}g^{-1}\tbinom{k}{0}\\ &=\widetilde{T}_{g_i}\sum_{g\in\Gamma/\Gamma_\infty}\big(T_gf_{k0}\big)(z)\chi_{i\alpha}(g)e_{k'm'}(u,v)\qquad \tbinom{k'}{m'}=g^{-1}\tbinom{k}{0}. \end{aligned}\tag{4.2}$$

The group Γ_∞ is generated by A, Γ^2_∞ by $B=g_2Ag_2^{-1}$ and Γ^3_∞ by $S=g_3Ag_3^{-1}$ and

$$\chi_{1\alpha}(A)=\chi_\alpha(A)=1,\quad \chi_{2\alpha}(A)=\chi_\alpha(B)=e^{2\pi i\alpha},\quad \chi_{3\alpha}(A)=\chi_\alpha(S)=e^{-2\pi i\alpha}.$$

Lemma 4.2. *The functions $f^i_{k\alpha}(z,u,v)$ are $\Gamma_{J,\alpha}$-invariant. For $k_1\neq k_2$ and for $k_1=k_2$, $i\neq j$, $f_{k_10},f_{k_20}\in\mathcal{H}_{\Gamma_\infty}$*

$$\big(\Sigma^i_{k_1,\alpha}f_{k_i0},\Sigma^j_{k_2,\alpha}h_{k_20}\big)_{\mathcal{H}_{\Gamma_J}}=0$$

while

$$\big(\Sigma^i_{k,\alpha}f_{k0},\Sigma^i_{k,\alpha}h_{k0}\big)_{\mathcal{H}_{\Gamma_J}}=\big(f_{k0},h_{k0}\big)_{\mathcal{H}_{\Gamma_\infty}}.$$

The operators $\Sigma^i_{k,\alpha}$ are unitary from $\mathcal{H}_{\Gamma_\infty}$ to $\overline{\Sigma^i_{k,\alpha}\mathcal{H}_{\Gamma_\infty}}=\mathcal{H}^i_k$ for $\alpha\in\mathbb{R}\setminus\mathbb{Z}$.

Proof. For $k_1\neq k_2$, $\binom{k_1}{0}\not\sim\binom{k_2}{0}$, so $\tilde g^{-1}g_i^{-1}\binom{k_1}{0}\neq\tilde h^{-1}g_j^{-1}\binom{k_2}{0}$ and

$$\big(e_{k'_{1i}m'_{1i}},e_{k'_{2j}m'_{2j}}\big)=0$$

for each pair of terms in $\sum_{k_1\alpha}^i f_{k0}$ and $\sum_{k_2,\alpha}^j h_{k0}$ so $\left(\sum_{k_1}^i f_{k0}, \sum_{k_2}^j h_{k0}\right) = 0$.

For $i \neq j$ and $\tilde{h} \in \Gamma/\Gamma_\infty^j$,

$$\binom{k_i'}{m_i'} = \tilde{g}^{-1} g_i^{-1} \binom{k}{0} \neq \tilde{h}^{-1} g_j^{-1} \binom{k}{0} = \binom{k_j''}{m_j''}$$

since

$$g_i^{-1}\binom{k}{0} \underset{\Gamma}{\nsim} g_j^{-1}\binom{k}{0} \quad \text{and} \quad \tilde{g} \in \Gamma/\Gamma_\infty^i,\ \tilde{h} \in \Gamma/\Gamma_\infty^i.$$

Hence for all $\binom{k_i'}{m_i'}$, $\binom{k_j''}{m_j''}$

$$\left(e_{k_i' m_i'}, e_{k_j'' m_j''}\right) = 0$$

and

$$\left(\Sigma_{k_1,\alpha}^i f_{k0}, \Sigma_{k_2,\alpha}^j h_{k0}\right) = 0.$$

For $i = j$, $i = 1,2,3$ and $f_{k0,1}, f_{k0,2} \in \mathcal{H}_{\Gamma_\infty}$,

$$\begin{aligned}
&\left(\Sigma_{k,\alpha}^i f_{k0,1}, \Sigma_{k,\alpha}^i f_{k0,2}\right)_{\mathcal{H}_{\Gamma_\infty}} \\
&\quad = \int_{F_{\Gamma_\infty}} \Big\{ \sum_{\tilde{g} \in \Gamma_\infty^i} \left(T_{\tilde{g}} T_{g_i} f_{k0,1}\right)(z) \chi_\alpha(\tilde{g}) \left(e^{2\pi i k' u} e^{2\pi i m' v} + e^{-2\pi i k' u} e^{-2\pi i m' v}\right) \Big\} \\
&\qquad\qquad \cdot \Big\{ \sum_{\tilde{h} \in \Gamma_\infty^i} \left(\overline{T_{\tilde{h}} T_{g_i} f_{k0,2}}\right)(z) \overline{\chi}_\alpha(\tilde{h}) \left(e^{2\pi i k'' u} e^{2\pi i m'' v} + e^{-2\pi i k'' u} e^{-2\pi i m'' v}\right\} \\
&\qquad\qquad d\mu(z)\, du\, dv.
\end{aligned}$$

Since $\binom{k'}{m'} \neq \binom{k''}{m''}$ iff $\tilde{g} \neq \tilde{h} \mod \Gamma_\infty$, this equals, setting $\tilde{g} = g_i g g_i^{-1}$, $\tilde{h} = g_i g g_i^{-1}$,

$$\begin{aligned}
&\int_{F_\Gamma} \sum_{g \in \Gamma/\Gamma_\infty} \left(T_{g_i} T_g f_{k0,1}\right)(z) \left(\overline{T_{g_i} T_g f_{k0,2}}\right)(z)\, d\mu(z) \\
&\quad = \int_{T_{g_i} F_\Gamma} \sum_{g \in \Gamma/\Gamma_\infty} \left(T_g f_{k0,1}\right)(z) \left(\overline{T_g f_{k0,2}}\right)(z)\, d\mu(z) \\
&\quad = \int_{F_\Gamma} \sum_{g \in \Gamma/\Gamma_\infty} \left(T_g f_{k0,1}\right)(z) \left(\overline{T_g f_{k0,2}}\right)(z)\, d\mu(z) \\
&\quad = \int_{F_{\Gamma_\infty}} f_{k0,1}(z) \overline{f_{k0,2}(z)}\, d\mu(z),
\end{aligned}$$

using $T_{g_i} F_\Gamma = F_\Gamma$ and unfolding the last integral , and for $i = 1,2,3$, $\alpha \in \mathbb{R}$, $k = 1, 2\ldots$

$$\left(\Sigma_{k,\alpha}^i f_{k0,1}, \Sigma_{k,\alpha}^i f_{k0,2}\right)_{\mathcal{H}_{\Gamma_J}} = \left(f_{k0,1}, f_{k0,2}\right)_{\mathcal{H}_{\Gamma_\infty}}. \qquad \square$$

Definition. *Let $L_{k\alpha}^{0j}$ be the operators in $\mathcal{H}_{\Gamma_\infty}$ defined for $k = 1, 2\ldots$, $j = 1,2,3$, $\alpha \in \mathbb{R}$ by*

$$L_{k\alpha}^{0j} f = \Big\{ -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 4\pi^2 \frac{k^2}{y} \Big\} f \qquad \text{for } f, L_{k\alpha}^{0j} f \in \mathcal{H}_{\Gamma_\infty}$$

and $f(2 + iy) = f(iy)\chi_{j\alpha}(A)$ for $y > 0$.

Lemma 4.3. *The orthogonal Hilbert spaces* $\mathcal{H}_k^j$ *defined for* $k = 1, 2, \ldots$, $j = 1, 2, 3$, *and* $\alpha \in \mathbb{R}$ *by*

$$\mathcal{H}_k^j = \overline{\Sigma_{k,\alpha}^j \mathcal{H}_{\Gamma_\infty}}$$

are invariant under $L_{k\alpha}$, *and*

$$\mathcal{H}_\Gamma = \sum_{k=1}^{\infty} \bigoplus \sum_{j=1,2,3} \bigoplus \mathcal{H}_k^j.$$

The Hilbert spaces $\mathcal{H}_{k\alpha}^j$ *are invariant under* L_α, *and the operators*

$$L_{k\alpha}^j = L_\alpha|_{\mathcal{H}_{k\alpha}^j}$$

are unitarily equivalent to $L_{k\alpha}^{0j}$ *via the maps* $\Sigma_{k\alpha}^j$.

Proof. This follows from Lemma 4.2, (4.2), and the fact that

$$\Sigma_{k,\alpha}^j L_{k\alpha}^{0j} = L_{k\alpha}^j \Sigma_{k,\alpha}^j \qquad \text{on } \mathcal{D}(L_{k\alpha}^{0j}). \qquad \square$$

We further analyze the spectra of the operators $L_{k\alpha}^{0j}$, $j = 1, 2, 3$.

For $j = 1$, $\chi_{1\alpha}(g) = 1$, so $L_{k\alpha}^{01} = L_k^0 = L_{\Gamma k}^0$, and the spectrum is given by Lemma 3.6.

For $j = 2, 3$, $\alpha \in \mathbb{R}$,

$$L_{k\alpha}^{02} f = \left\{ - y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 4\pi^2 \frac{k^2}{y} \right\} f \qquad \text{for } f, L_{k\alpha}^{02} f \in \mathcal{H}_{\Gamma_\infty}$$

and

$$f(2 + iy) = f(iy) e^{2\pi i \alpha} \qquad \text{for } y > 0.$$

We separate variables as for $\alpha = 0$ and obtain

Lemma 4.4.

$$L_{k\alpha}^{02} = \sum_{l \in \mathbb{Z}} \bigoplus L_{kl\alpha}^{02}$$

where

$$\mathcal{D}(L_{kl\alpha}^{02}) = \{ e^{\pi i (l+\alpha) x} \varphi_{kl\alpha}(y) \,|\, \varphi_{kl\alpha} \in \mathcal{D}(L_{kl\alpha}) \}$$

and

$$L_{kl\alpha} = -y^2 \frac{d^2}{dy^2} + 4\pi^2 \left(\frac{k^2}{y} + \left(\frac{l+\alpha}{2} \right)^2 y^2 \right)$$

with domain

$$\mathcal{D}(L_{kl\alpha}) = \left\{ \varphi_{kl\alpha} \in L^2(0, \infty; y^{-2}\, dy) \,\middle|\, L_{kl\alpha} \varphi_{kl\alpha} \in L^2(0, \infty; y^{-2}\, dy) \right\}.$$

Then for $l \in \mathbb{Z}$

$$\begin{aligned} &L_{kl\alpha}^{02} \big(e^{\pi i (l+\alpha) x} \varphi_{kl\alpha}(y) \big) \\ &\qquad = e^{\pi i (l+\alpha) x} \left\{ -y^2 \frac{d^2}{dy^2} + 4\pi^2 \left(\frac{k^2}{y} + \left(\frac{l+\alpha}{2} \right)^2 y^2 \right) \right\} \varphi(y). \end{aligned}$$

We now discuss the spectra of the operators $L_{kl\alpha}$ for $k = 1, 2\dots$, $l \in \mathbb{Z}$, $0 < |\alpha| < 1$ and the limit $\alpha \to 0$.

Lemma 4.5. *For $l \neq 0$, the spectrum of $L_{kl\alpha}$ is discrete and simple, consisting of a sequence of eigenvalues*

$$\lambda_{kl}^1(\alpha) < \lambda_{kl}^2(\alpha) < \cdots < \lambda_{kl}^n(\alpha) < \cdots$$

with eigenfunctions

$$\varphi_{kl}^1(\alpha), \varphi_{kl}^2(\alpha), \dots, \varphi_{kl}^n(\alpha), \dots$$

For $\alpha \to 0$,

$$\begin{aligned} &\lambda_{kl}^n(\alpha) \to \lambda_{kl}^n(0), \\ &\varphi_{kl}^n(\alpha) \to \varphi_{kl}^n(0) = \varphi_{kl}^n \qquad \text{in } L^2(0,\infty; y^{-2}\,dy). \end{aligned}$$

Proof. The fact that for $l \neq 0$, $0 < |\alpha| < 1$, the spectrum of $L_{kl\alpha}$ is a sequence of simple eigenvalues follows as for $\alpha = 0$.

Consider the quadratic form

$$\big(L_{kl\alpha}\varphi, \varphi\big) = \sum_0^\infty \Big\{ -y^2 \frac{d^2}{dy^2}\varphi\bar{\varphi} + 4\pi^2 \Big[\frac{k^2}{y} + \Big(\frac{l+\alpha}{2}\Big)^2 y^2\Big]\varphi\bar{\varphi}\Big\} y^{-2}\,dy.$$

We have

$$c_1\big(L_{kl}\varphi, \varphi\big) < \big(L_{kl\alpha}\varphi, \varphi\big) < c_2\big(L_{kl}\varphi, \varphi\big).$$

This implies that $L_{kl\alpha}^{1/2}$ is self-adjoint on $\mathcal{D}(L_{kl}^{1/2})$ for $|\alpha| < 1$. It follows from general theory (cf. [4]) that eigenvalues $\mu_{kl}^n(\alpha)$ and eigenfunctions $\varphi_{kl}^n(\alpha)$ are analytic in α, and therefore the same holds for the eigenvalues $\lambda_{kl}^n(\alpha) = (\mu_{kl}^n(\alpha))^2$ and eigenfunctions $\varphi_{kl}^n(\alpha)$ of $L_{kl\alpha}$. $\square$

Lemma 4.6. *For $l = 0$, $0 < |\alpha| < 1$, the spectra of $L_{k0\alpha}$ are discrete and simple, consisting of sequences of eigenvalues*

$$\lambda_{k0}^1(\alpha) < \lambda_{k0}^2(\alpha) < \cdots < \lambda_{k0}^n(\alpha) < \cdots$$

with normalized eigenfunctions

$$\varphi_{k0}^1(\alpha), \varphi_{k0}^2(\alpha), \dots, \varphi_{k0}^n(\alpha), \dots$$

For each n, $\lambda_{k0}^n(\alpha)$ is increasing in α, and

$$\begin{aligned} &\lambda_{k0}^n(\alpha) \longrightarrow \tfrac{1}{4} \quad \text{for } \alpha \to 0 \\ &\varphi_{k0}^n(\alpha) \underset{\alpha\to 0}{\longrightarrow} 0 \quad \text{weakly in } L^2(0,\infty; y^{-2}\,dy). \end{aligned}$$

Proof. We transform the operator $L_{k0\alpha}$ by the unitary map

$$U\colon f(g) \to g(t) = f(e^t)e^{-t/2}$$

into the operator

$$M_{k0\alpha} = UL_{k0\alpha}U^{-1} \quad \text{in } L^2(-\infty,\infty;dt)$$

given by

$$M_{k0\alpha} = -\frac{d^2}{dt^2} + \tfrac{1}{4} + 4\pi^2(k^2e^{-t} + \alpha^2 e^{2t})$$

with eigenvalues $\lambda_{k0}^n(\alpha)$ and eigenfunctions $\Psi_{k0}^n(\alpha) = U\varphi_{k0}^n(\alpha)$.

Let $\varepsilon > 0$ and let $f \in C_0^\infty(\mathbb{R})$, $\|f\| = 1$ and

$$\Big(-\frac{d^2}{dt^2}f, f\Big) < \varepsilon.$$

Let t_0 be such that

$$\big(4\pi^2k^2e^{-t}f_{t_0}(t), f_{t_0}(t)\big) < \varepsilon$$

where

$$f_{t_0}(t) = f(t-t_0).$$

Then choose α_0 such that

$$\big(4\pi^2\alpha_0^2e^{2t}f_{t_0}, f_{t_0}\big) < \varepsilon$$

and hence

$$\big(M_{k0\alpha}f_{t_0}, f_{t_0}\big) < \tfrac{1}{4} + 3\varepsilon \quad \text{for } 0 < |\alpha| < \alpha_0.$$

It follows that

$$\lambda_{k0\alpha}^1 < \tfrac{1}{4} + 3\varepsilon \quad \text{for } 0 < |\alpha| < \alpha_0.$$

and we have proved that

$$\lambda_{k0\alpha}^1 \xrightarrow[\alpha\to 0]{} \tfrac{1}{4} \quad \text{for every } k = 1, 2, \ldots.$$

Consider now the subspace

$$\mathcal{H}_{k0\alpha}^1 = L^2(-\infty,\infty;dx) \ominus \{\Psi_{k0\alpha}^1\} \quad \text{of } L^2(-\infty,\in;dx).$$

Then $\mathcal{H}_{k0\alpha}^1$ is invariant under $M_{k0\alpha}$ and

$$\lambda_{k0\alpha}^2 = \min\{\big(M_{k0\alpha}f, f\big) \mid f \in \mathcal{H}_{k0\alpha}^1, M_{k0\alpha}f \in \mathcal{H}_{k0\alpha}^1, \|f\| = 1\}.$$

Let $f \in C_0^\infty(\mathbb{R})$, $\|f\| = 1$, $f_{t_0}(t) = f(t-t_0)$. Then

$$\big(f_{t_0}, \Psi_{k0\alpha}^1\big) \xrightarrow[t_0\to\infty]{} 0.$$

Choose t_0 such that

$$|(f_{t_0}, \Psi_{k0\alpha}^1)| < \varepsilon$$

and

$$\|4\pi^2 k^2 e^{-t} f_{t_0}\| < \varepsilon.$$

Let $P^1_{k0\alpha}$ be the orthogonal projection on $\mathcal{H}^1_{k0\alpha}$. Then

$$P^1_{k0\alpha} f_{t_0} = f_{t_0} - \big(f_{t_0}, \Psi^1_{k0\alpha}\big)\Psi^1_{k0\alpha}$$

and

$$\|\big(f_{t_0}, \Psi^1_{k0\alpha}\big)\Psi^1_{k0\alpha}\| < \varepsilon,$$

$1 - \varepsilon < \|P^1_{k0\alpha} f_{t_0}\| < 1$. Let

$$g_{t_0} = \|P^1_{k0\alpha} f_{t_0}\|^{-1} P^1_{k0\alpha} f_{t_0}.$$

Then

$$g_{t_0} \in \mathcal{H}^1_{k0\alpha}, \quad \|g_{t_0}\| = 1$$

and

$$\begin{aligned}
\big(M_{k0\alpha} g_{t_0}, g_{t_0}\big) &= \|P^1_{k0\alpha} f_{t_0}\|^{-2}\big(M_{k0\alpha} P^1_{k0\alpha} f_{t_0}, P^1_{k0\alpha} f_{t_0}\big) \\
&= \|P^1_{k0\alpha} f_{t_0}\|^{-2}\Big(M_{k0\alpha}\big(f_{t_0} - (f_{t_0}, \Psi^1_{k0\alpha})\Psi^1_{k0\alpha}\big), f_{t_0} - (f_{t_0}, \Psi^1_{k0\alpha})\Psi^1_{k0\alpha}\Big) \\
&= \|P^1_{k0\alpha} f_{t_0}\|^{-2}\Big\{\big(M_{k0\alpha} f_{t_0}, f_{t_0}\big) - \overline{(f_{t_0}, \Psi^1_{k0\alpha})}\big(M_{k0\alpha} f_{t_0}, \Psi^1_{k0\alpha}\big) \\
&\quad - (f_{t_0}, \Psi^1_{k0\alpha})\big(M_{k0\alpha}\Psi^1_{k0\alpha}, f_{t_0}\big) + |(f_{t_0}, \Psi^1_{k0\alpha})|^2\big(M_{k0\alpha}\Psi^1_{k0\alpha}, \Psi^1_{k0\alpha}\big)\Big\}
\end{aligned}$$

We have

$$\big(4\pi^2 k^2 e^{-t} f_{t_0}(t), f_{t_0}(t)\big) \le \|4\pi^2 k^2 e^{-t} f_{t_0}(t)\| < \varepsilon.$$

Choose α_0 such that

$$(4\pi^2 \alpha_0^2 f_{t_0}, f_{t_0}) < \varepsilon.$$

Then

$$\big(M_{k0\alpha} f_{t_0}, f_{t_0}\big) < \tfrac{1}{4} + 3\varepsilon.$$

The remaining terms in the bracket are now estimated, using $M_{k0\alpha}\Psi^1_{k0\alpha} = \lambda^1_{k0\alpha}\Psi^1_{k0\alpha}$, by

$$\begin{aligned}
\big|\overline{(f_{t_0}, \Psi^1_{k0\alpha})}\big(M_{k0\alpha} f_{t_0}, \Psi^1_{k0\alpha}\big)\big| &< \varepsilon\lambda^1_{k0\alpha} \\
\big|(f_{t_0}, \Psi^1_{k0\alpha})\big(M_{k0\alpha}\Psi^1_{k0\alpha}, f_{t_0}\big)\big| &< \varepsilon\lambda^1_{k0\alpha} \\
\big|(f_{t_0}, \Psi^1_{k0\alpha})\big|^2\big(M_{k0\alpha}\Psi^1_{k0\alpha}, \Psi^1_{k0\alpha}\big) &< \varepsilon^2\lambda^1_{k0\alpha}.
\end{aligned}$$

Adding these inequalities, we get, given $\varepsilon_1 > 0$

$$\big(M_{k0\alpha}, g_{t_0}\big) < (1-\varepsilon)^{-2}\{\tfrac{1}{4} + 3\varepsilon + \lambda^1_{k0\alpha}\varepsilon(2+\varepsilon)\} < \tfrac{1}{4} + \varepsilon_1 \quad \text{for } \varepsilon < \varepsilon_0$$

and $\lambda^2_{k0\alpha} \to \frac{1}{4}$ for $\alpha \to 0$.

Repeating this procedure we prove by induction on n that

$$\lambda^n_{k0\alpha} \to \tfrac14 \quad \text{for } \alpha \to 0 \text{ for every } k \geq 1 \text{ and all } n.$$

It remains to prove that the eigenfunctions $\Psi^n_{k0}(\alpha)$ of $M_{k0\alpha}$ converge weakly to 0 as $\alpha \to 0$ for every $k \geq 1$, $n \geq 1$.

We introduce the operators

$$\widetilde{M}_{k0\alpha} = M_{k0\alpha} - \tfrac14 = -\frac{d^2}{dt^2} + 4\pi^2(k^2e^{-t} + \alpha^2 e^{2t})$$
$$\widetilde{M}_{k0} = \widetilde{M}_{k00} = -\frac{d^2}{dt^2} + 4\pi^2k^2e^{-t}.$$

$\widetilde{M}_{k0\alpha}$ has the eigenvalues $\lambda^n_{k0\alpha}(\alpha) - \frac14$ with eigenfunctions $\Psi^n_{k0}(\alpha)$, $\|\Psi^n_{k0}(\alpha)\| = 1$.

$\widetilde{M}_{k0}$ has the purely continuous spectrum $[0,\infty)$.

Since 0 is not an eigenvalue of $\widetilde{M}_{k0}$, $\widetilde{M}_{k0}C_0^\infty(\mathbb{R})$ is dense in $L^2(-\infty,\infty;dx)$.

Let $\theta \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}\big(\Psi^n_{k0}(\alpha), \widetilde{M}_{k0}\theta\big) &= \Big(\Psi^n_{k0}(\alpha), \big(\widetilde{M}_{k0\alpha} - 4\pi^2\alpha^2e^{2t}\big)\theta\Big)\\ &= \big(\widetilde{M}_{k0\alpha}\Psi^n_{k0}(\alpha), \theta\big) - 4\pi\alpha^2\big(\Psi^n_{k0}(\alpha), e^{2t}\theta\big)\\ &= \big(\lambda^n_{k0}(\alpha) - \tfrac14\big)\big(\Psi^n_{k0}(\alpha),\theta\big) - 4\pi^2\alpha^2\big(\Psi^n_{k0}(\alpha), e^{2t}\theta\big)\\ &\xrightarrow[\alpha\to 0]{} 0.\end{aligned}$$

Since $\widetilde{M}_{k0}C_o^\infty$ is dense, we have

$$\big(\Psi^n_{k0}(\alpha), f\big) \xrightarrow[\alpha\to0]{} 0 \quad \text{for all } f \in L^2(-\infty,\infty;dx)$$

and the last statement is proved. □

As in the proof of (3.7) of Lemma 3.4 we show that the functions given by (3.13) for $i = 1,2,3$ and $j = 1$ contribute the same functions as for $j = 0$.

Moreover, for fixed $\alpha \neq 0$ Lemma 3.7 is proved for the operator L_α in the same way as for $\alpha = 0$.

We summarize the results of Lemmas 4.3– 4.6 in

Theorem 4. *$\mathcal{H}_\Gamma$ can be decomposed into a direct sum of subspaces*

$$\mathcal{H}_\Gamma = \sum_{k=1}^\infty \bigoplus \sum_{j=1,2,3} \bigoplus \mathcal{H}^j_k$$

where each space $\mathcal{H}^j_k$ is invariant under L_α for $\alpha \in \mathbb{R}$ and the operators

$$L^j_{k\alpha} = L_\alpha|_{\mathcal{H}^j_k}$$

are unitarily equivalent to $L^{0j}_{k\alpha}$ via the maps $\Sigma^j_{k\alpha}$, where

$$L^{0j}_{k\alpha} = \left\{-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 4\pi^2\frac{k^2}{y}\right\} \quad \textit{in } \mathcal{H}_{\Gamma_\infty}$$

with the characters

$$f(2+iy) = \begin{cases} f(iy) & \text{for } j = 1 \\ e^{2\pi i\alpha} f(iy) & \text{for } j = 2 \\ e^{-2\pi i\alpha} f(iy) & \text{for } j = 3. \end{cases}$$

For $j = 1$, $L^{01}_{k\alpha} = L^0_{\Gamma k}$, and the spectrum is given by Lemma 3.6. In particular, the continuous spectrum for each k is simple, equal to $[\frac{1}{4}, \infty)$.

For $j = 2, 3$ the continuous spectrum disappears for $0 < |\alpha| < 1$ (the cusps 0 and 1 are closed by $\chi(\alpha)$), and the eigenvalues λ^n_{kl} of L_{k0} are perturbed into eigenvalues $\lambda^n_{kl\alpha}$ for $k, l \in \mathbb{N}$.

In addition to this a new sequence of eigenvalues $\lambda^n_{k0\alpha}$ appear for $\alpha \neq 0$, replacing the continuous spectrum. For each $n, k \in \mathbb{N}$ and $\alpha \to 0$

$$\lambda^n_{k0\alpha} \to \tfrac{1}{4}, \quad \varphi^n_{k0\alpha} \to 0 \qquad \text{weakly.}$$

For each α the Weyl law holds for L_α:

$$N_\Gamma(\lambda) \sim I \cdot \tfrac{1}{192\pi}\lambda^2 = \tfrac{1}{32\pi}\lambda^2 \quad \text{for } \lambda \to \infty.$$

5 Non-normal subgroups of Γ_1 of small index

We develop the spectral theory of L_Γ for some important non-normal subgroup of Γ_1.

I. We consider the three conjugate subgroups of Γ_1 of index 3, $\Gamma_U(2)$, $\Gamma_\vartheta(2)$, $\Gamma_W(2)$ ([6]) defined by

$$\begin{aligned} \Gamma_U &= \Gamma_U(2) = \Gamma_0(2) = \{g \in \Gamma_1 \,|\, g \overset{2}{\equiv} U \text{ or } g \overset{2}{\equiv} I\}, && U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \beta = 1 \\ \Gamma_V &= \Gamma_V(2) = \Gamma_\vartheta(2) = \{g \in \Gamma_1 \,|\, g \overset{2}{\equiv} V \text{ or } g \overset{2}{\equiv} I\}, && V = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \beta = 2 \\ \Gamma_W &= \Gamma_W(2) = \Gamma^0(2) = \{g \in \Gamma_1 \,|\, g \overset{2}{\equiv} W \text{ or } g \overset{2}{\equiv} I\}, && W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right), \beta = 2 \end{aligned}$$

$$\Gamma_V = P^{-1}\Gamma_U P, \quad \Gamma_W = P^{-1}\Gamma_V P, \quad \Gamma_U = P^{-1}\Gamma_W P, \qquad P = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right),\ P^3 = I.$$

Γ_{UJ}, Γ_{VJ}, Γ_{WJ} are the associated Jacobi groups.

We derive expressions for the Jacobi-invariant functions for the groups Γ_{UJ}, Γ_{VJ}, Γ_{WJ}.

(1) For Γ_V we take I, U, P^2 as right coset representatives.

(I) Let $f_{k0}(z) \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$F^V_{k1}(z,u,v) = \big(\Sigma^V_{k1} f_{k0}\big)(z,u,v) = \sum_{g \in \Gamma_\vartheta/\Gamma_{2\infty}} (T_g f_{k0})(z) e_{k'm'}(u,v), \quad \left(\begin{smallmatrix} k' \\ m' \end{smallmatrix}\right) = g^{-1}\left(\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right).$$

$\boldsymbol{(U)}$ Let $f_{k0} \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$F^V_{k2}(z,u,v) = \big(\Sigma^V_{k2} f_{k0}\big)(z,u,v) \sum_{g\in\Gamma_\vartheta/U\Gamma_{2\infty}U^{-1}} \big(T_g T_V f_{k0}\big)(z) e_{k''m''}(u,v), \quad \binom{k''}{m''} = g^{-1}U^{-1}\binom{k}{0}$$

$$= \sum_{g\in\Gamma_V/\Gamma_{2\infty}} \big(T_g h_{k0}\big)(z) e_{k'm'}(u,v), \quad \binom{k'}{m'} = g^{-1}\binom{k}{0}, \quad h_{k0}(z) = \big(T_U f_{k0}\big)(z) \in \mathcal{H}_{\Gamma_{2\infty}}.$$

$$= \big(\Sigma^V_{k1}\big(T_U f_{k0}\big)\big)(z,u,v).$$

$\boldsymbol{(P^2)}$ Let $f_{k0}(z) \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$F^V_{k3}(z,u,v) = \big(\Sigma^V_{k3} f_{k0}\big)(z,u,v) = \sum_{g\in\Gamma_V/P^2\Gamma_{1\infty}P^{-2}} \big(T_g T_{P^2} f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \binom{k'''}{m'''} = g^{-1}p^{-2}\binom{k}{0}$$

$$= \sum_{g\in\Gamma_V/P^2\Gamma_{1\infty}P^{-2}} \big(T_{P^2} T_{P^{-2}gP^2} f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \binom{k'''}{m'''} = P^{-2}P^2 g^{-1}P^{-2}\binom{k}{0}$$

$(g_1 = P^{-2}gP^2 \in U)$

$$= \sum_{g_1\in\Gamma_U/\Gamma_{1\infty}} \big(T_{P^2} T_{g_1} f_{k0}\big)(z) e_{k'''m'''}, \quad \binom{k'''}{m'''} = P^{-2}g_1^{-2}\binom{k}{0}$$

$$= \widetilde{T}_{P^2} \sum_{g_1\in\Gamma_U/\Gamma_{1\infty}} \big(T_{g_1} f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \binom{k'''}{m'''} = g_1^{-1}\binom{k}{0}$$

$$= \widetilde{T}_{P^2}\big(\Sigma^U_{k1}\big)(z,u,v) = \big(\widetilde{T}_{P^2} F^U_{k1}\big)(z,u,v).$$

(2) For Γ_W we take I, U, P as right coset representatives.

$\boldsymbol{(I)}$ Let $f_{k0} \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$F^W_{k1}(z,u,v) = \big(\Sigma^W_{k1} f_{k0}\big)(z,u,v) = \sum_{g\in\Gamma_W/\Gamma_{2\infty}} \big(T_g f_{k0}\big)(z) e_{k'm'}(u,v), \ \binom{k'}{m'} = g^{-1}\binom{k}{0}.$$

$\boldsymbol{(U)}$ Let $f_{k0} \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$F^W_{k2}(z,u,v) = \big(\Sigma^W_{k2} f_{k0}\big)(z,u,v)$$

$$= \sum_{g\in\Gamma_W/U\Gamma_{2\infty}U^{-1}} \big(T_g T_U f_{k0}\big)(z) e_{k''m''}(u,v), \quad \binom{k''}{m''} = g^{-1}U^{-1}\binom{k}{0}$$

$$= \sum_{g\in\Gamma_W/\Gamma_{2\infty}} \big(T_g h_{k0}\big) e_{k'm'}(u,v),$$

$$\binom{k'}{m'} = g^{-1}\binom{k}{0}, \ h_{k0}(z) = \big(T_U f_{k0}\big)(z) \in \Gamma_{2\infty}.$$

$\boldsymbol{(P)}$ Let $f_{k0}(z) \in \mathcal{H}_{\Gamma_{1\infty}}$ and

$$\begin{aligned} F^W_{k3}(z,u,v) &= \big(\Sigma^W_{k3} f_{k0}\big)(z,u,v) \\ &= \sum_{g\in\Gamma_W/P\Gamma_{1\infty}P^{-1}} \big(T_g T_P f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \tbinom{k'''}{m'''} = g^{-1}P^{-1}\tbinom{k}{0} \end{aligned}$$

$(g_1 = P^{-1}gP \in \Gamma_U)$

$$\begin{aligned} &= \sum_{g\in\Gamma_U/\Gamma_{1\infty}} \big(T_P T_{g_1} f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \tbinom{k'''}{m'''} P^{-1}g_1^{-1}\tbinom{k}{0} \\ &= \widetilde{T}_P\big(\Sigma^U_{k1} f_{k0}\big)(z,u,v) = \big(\widetilde{T}_P F^U_{k1}\big)(z,u,v). \end{aligned}$$

(3) For Γ_U we take I, P, P^2 as right coset representatives.

$\boldsymbol{(I)}$ Let $f_{k0} \in \mathcal{H}_{\Gamma_{1\infty}}$ and

$$\begin{aligned} F^U_{k1}(z,u,v) &= \big(\Sigma^U_{k1} f_{k0}\big)(z,u,v) \\ &= \sum_{g\in\Gamma_U/\Gamma_{1\infty}} \big(T_g f_{k0}\big)(z) e_{k'm'}(u,v), \quad \tbinom{k'}{m'} = g^{-1}\tbinom{k}{0}. \end{aligned}$$

$\boldsymbol{(P)}$ Let $f_{k0} \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$\begin{aligned} F^U_{k2}(z,u,v) &= \big(\Sigma^U_{k2} f_{k0}\big)(z,u,v) \\ &= \sum_{g\in\Gamma_U/P\Gamma_{2\infty}P^{-1}} \big(T_g T_p f_{k0}\big)(z) e_{k''m''}(u,v), \quad \tbinom{k''}{m''} = g^{-1}P^{-1}\tbinom{k}{0} \end{aligned}$$

$(g_1 = P^{-1}gP \in \Gamma_\vartheta)$

$$\begin{aligned} &= \sum_{g\in\Gamma_V/\Gamma_{2\infty}} \big(T_P T_{g_1} f_{k0}\big)(z) e_{k''m''}(u,v), \quad \tbinom{k''}{m''} = P^{-1}g_1^{-1}\tbinom{k}{0} \\ &= \big(\widetilde{T}_P F^V_{k1}\big)(z,u,v). \end{aligned}$$

$\boldsymbol{(P^2)}$ Let $f_{k0} \in \mathcal{H}_{\Gamma_{2\infty}}$ and

$$\begin{aligned} F^U_{k3}(z,u,v) &= \big(\Sigma^U_{k3} f_{k0}\big)(z,u,v) \\ &= \sum_{g\in\Gamma_U/P^2\Gamma_{2\infty}P^{-2}} \big(T_g T_{P^2} f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \tbinom{k'''}{m'''} = g^{-1}P^{-2}\tbinom{k}{0} \end{aligned}$$

$(g_2 = P^{-2}gP^2 \in \Gamma_W)$

$$\begin{aligned} &= \sum_{g_2\in\Gamma_W/\Gamma_{2\infty}} \big(T_{P^2} T_{g_2} f_{k0}\big)(z) e_{k'''m'''}(u,v), \quad \tbinom{k'''}{m'''} = P^{-2}g_2^{-2}\tbinom{k}{0} \\ &= \big(\widetilde{T}_{P^2} F^W_{k1}\big)(z,u,v). \end{aligned} \tag{5.1}$$

Replacing U by P in the calculation of $F^V_{k2}(z,u,v)$, we get

(P)

$$\widetilde{F}^V_{k2}(z,u,v) = \big(\widetilde{\Sigma}_{k2} f_{k0}\big)(z,u,v)$$
$$= \sum_{g\in\Gamma_\vartheta/P\Gamma_{2\infty}P^{-1}} \big(T_g T_P f_{k0}\big)(z) e_{k''m''}(u,v), \quad \tbinom{k''}{m''} = g^{-1}P^{-1}\tbinom{k}{0}$$

$(P^{-1}gP = g_1 \in \Gamma_W)$

$$= \sum_{g_1\in\Gamma_W/\Gamma_{2\infty}} \big(T_p T_{g_1} f_{k0}\big)(z) e_{k''m''}(u,v), \quad \tbinom{k''}{m''} = P^{-1}g_1^{-1}\tbinom{k}{0}$$
$$= \big(\widetilde{T}_P F^W_{k1}\big)(z,u,v). \tag{5.2}$$

By (5.1) and (5.2),

$$F^U_{k3}(z,u,v) = \big(\widetilde{T}_P F^V_{k2}\big)(z,u,v).$$

By (1),

$$\big(\Sigma^V_{k2} f_{k0}\big)(z,u,v) = \big(\Sigma^V_{k1}(T_U f_{k0})\big)(z,u,v)$$

and we obtain by (3)

$$F^U_{k3}(z,u,v) = \widetilde{T}_p\big(\Sigma^V_{k1}(T_U f_{k0})\big)(z,u,v) = \big(\Sigma^U_{k2}(T_U f_{k0})\big)(z,u,v) = F^U_{k2}(z,u,v).$$

Definition. *The subspaces $\mathcal{H}^i_{\Gamma_V k}$ of $\mathcal{H}_{\Gamma_V}$, $\mathcal{H}^i_{\Gamma_W k}$ of $\mathcal{H}_{\Gamma_W}$, and $\mathcal{H}^i_{\Gamma_U k}$ of $\mathcal{H}_{\Gamma_U}$, are given by*

$$\begin{aligned}
\mathcal{H}^1_{\Gamma_V k} &= \Sigma^V_{k1}\mathcal{H}_{\Gamma_{2\infty}}, & \mathcal{H}^2_{\Gamma_V k} &= \Sigma^V_{k2}\mathcal{H}_{\Gamma_{2\infty}}, & \mathcal{H}^3_{\Gamma_V k} &= \Sigma^V_{k3}\mathcal{H}_{\Gamma_{1\infty}},\\
\mathcal{H}^1_{\Gamma_W k} &= \Sigma^W_{k1}\mathcal{H}_{\Gamma_{2\infty}}, & \mathcal{H}^2_{\Gamma_W k} &= \Sigma^W_{k2}\mathcal{H}_{\Gamma_{2\infty}}, & \mathcal{H}^3_{\Gamma_W k} &= \Sigma^W_{k3}\mathcal{H}_{\Gamma_{1\infty}},\\
\mathcal{H}^1_{\Gamma_U k} &= \Sigma^U_{k1}\mathcal{H}_{\Gamma_{1\infty}}, & \mathcal{H}^2_{\Gamma_U k} &= \Sigma^U_{k2}\mathcal{H}_{\Gamma_{2\infty}}, & \mathcal{H}^3_{\Gamma_U k} &= \Sigma^U_{k3}\mathcal{H}_{\Gamma_{2\infty}}.
\end{aligned}$$

Theorem 5.1. *Σ^V_{k1} is unitary from $\mathcal{H}_{\Gamma_{2\infty}}$ to $\mathcal{H}^1_{\Gamma_V k}$, Σ^V_{k2} is unitary from $\mathcal{H}_{\Gamma_{2\infty}}$ to $\mathcal{H}^2_{\Gamma_V k}$, Σ^V_{k3} is unitary from $\mathcal{H}_{\Gamma_{1\infty}}$ to $\mathcal{H}^3_{\Gamma_V k}$.*

$\mathcal{H}^1_{\Gamma_V k}$, $\mathcal{H}^2_{\Gamma_V k}$, $\mathcal{H}^3_{\Gamma_V k}$ are pairwise orthogonal and invariant under the operator

$$L_{\Gamma_V J} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{y}\left\{\frac{\partial^2}{\partial x^2} - 2x\frac{\partial^2}{\partial x\,\partial y} + (x^2+y^2)\frac{\partial^2}{\partial y^2}\right\}.$$

Let

$$L^i_{\Gamma_V k} = L_{\Gamma_V J}|_{\mathcal{H}^i_{\Gamma_V k}}, \quad i = 1,2,3.$$

Then

$$L_{\Gamma_V k} = L^1_{\Gamma_V k} \oplus L^3_{\Gamma_V k}.$$

$L^1_{\Gamma_V k}$ is unitarily equivalent to the operator $L_{\Gamma_{2\infty}}$ in $\mathcal{H}_{\Gamma_{2\infty}}$ via Σ^V_{k1},

$$L^1_{\Gamma_V k}\Sigma^V_{k1} = \Sigma^V_{k1} L_{\Gamma_{2\infty}}.$$

$L^3_{\Gamma_V k}$ *is unitarily equivalent to the operator* $L_{\Gamma_{1\infty}}$ *in* $\mathcal{H}_{\Gamma_{1\infty}}$ *via* Σ^V_{k3},

$$L^3_{\Gamma_V k}\Sigma^V_{k3} = \Sigma^V_{k3} L_{\Gamma_{1\infty}}.$$

The continuous spectrum of L_{Γ_V} *is of multiplicity* 2*, and there are no resonances.*

The point spectrum of $L^1_{\Gamma_V k}$ *is the union of* $\{\lambda^n_{kl}\}^\infty_{n=1}$, $l = 1, 2, \dots$ *and* $\{\lambda_{k,l+1/2}\}^\infty_{n=1}$, $l = 0, 1, 2, \dots$

The point spectrum of $L^3_{\Gamma_V k}$ *is the union of* $\{\lambda^n_{kl}\}^\infty_{n=1}$, $l = 1, 2, \dots$ *Each* λ^n_{kl}, $l = 1, 2, \dots$ *is a* 4*-dimensional eigenvalue of* $L_{\Gamma_V k}$*. Each* $\lambda^n_{k,l+1/2}$, $l = 0, 1, 2, \dots$ *is a* 2*-dimensional eigenvalue of* $L_{\Gamma_V k}$*.*

Similarly

$$L_{\Gamma_W k} = L^1_{\Gamma_W k} \oplus L^3_{\Gamma_W k},$$

where

$$L^1_{\Gamma_W k}\Sigma^W_{k1} = \Sigma^W_{k1} L_{\Gamma_{2\infty}}$$

and

$$L^3_{\Gamma_W k}\Sigma^W_{k3} = \Sigma^W_{k3} L^3_{\Gamma_{1\infty}}$$

with the same spectral properties as Γ_{Vk}*.*

Also

$$L_{\Gamma_U k} = L^1_{\Gamma_U k} \oplus L^2_{\Gamma_U k},$$

where

$$L^1_{\Gamma_U k}\Sigma^U_{k1} = \Sigma^U_{k1} L_{\Gamma_{1\infty}}$$

and

$$L^2_{\Gamma_U k}\Sigma^U_{k2} = \Sigma^U_{k2} L_{\Gamma_{2\infty}}.$$

with the same spectral properties as Γ_{Vk}*, replacing* Γ^1_{Vk} *by* Γ^2_{Uk} *and* Γ^3_{Vk} *by* Γ^1_{Uk}*.*

The operators $L_{\Gamma_U J}$, $L_{\Gamma_V J}$ *and* $L_{\Gamma_W J}$ *have the same eigenvalues with he same multiplicities in agreement with the fact that they are conjugate as Jacobi groups. Their counting function is guven asymptotically by*

$$N_{\Gamma_V J}(\lambda) = N_{\Gamma_U J}(\lambda) = N_{\Gamma_W J} \sim 3\frac{1}{192\pi}\lambda^2$$

which is the Weyl law for these groups.

Proof. This is proved, using our expressions for the invariant functions for Γ_{UJ}, Γ_{VJ}, Γ_{WJ}, in the same way as the analogous results on normal subgroups are proved in section 3. □

II. We consider next the three conjugate groups $\Gamma_0(4)$, $\Gamma^0(4)$ and $\Gamma_\vartheta(4)$ ([6]) where

$$\Gamma_\vartheta(4) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a + b - c - d \overset{4}{\equiv} 0\}$$
$$\Gamma_\vartheta(4) = P^{-1}\Gamma_0(4)P, \quad \Gamma^0(4) = P^{-1}\Gamma_\vartheta(4)P, \quad \Gamma_0(4) = P^{-1}\Gamma^0(4)P.$$

$\Gamma_0(4)$ is normal in $\Gamma_0(2)$, $\Gamma_\vartheta(4)$ normal in $\Gamma_\vartheta(2)$, $\Gamma^0(4)$ normal in $\Gamma^0(2)$, all of index 2, so their index in Γ_1 is 6.

The width of $\Gamma_0(4)$ is 1 and that of $\Gamma^0(4)$ and $\Gamma_\vartheta(4)$ is 4.

We determine the Jacobi-invariant functions for $\Gamma_0(4)$, $\Gamma^0(4)$, $\Gamma_\vartheta(4)$.

As in the previous case we can calculate Jacobi-invariant functions for these three groups. We consider $\Gamma^0(4)$, the others are calculated along the same lines as for the previous groups.

(1) Let $\Gamma^0(4)$ have coset representatives I, U, U^2, U^3, P, V. For $f_{k0}(z) \in \mathcal{H}_{\Gamma_{4\infty}}$ we set

(I)

$$F_{k1}^{\Gamma^0(4)}(z,u,v) = \big(\Sigma_{k1}^{\Gamma^0(4)} f_{k0}\big)(z,u,v) = \sum_{g\in\Gamma^0(4)/\Gamma_{4\infty}} \big(T_g f_{k0}\big)(z) e_{k'm'}(u,v),$$
$$\tbinom{k'}{m'} = g^{-1}\tbinom{k}{0}.$$

For $t = 1,2,3$ and
$$f_{k0}(z) = \varphi^n_{k,l+t/4}(y) e^{2\pi i(l+t/4)x}$$
we have
$$\big(T_u f_{k0}\big)(z) = e^{2\pi i t/4} f_{k0}(z)$$
and we obtain

(U)

$$\begin{aligned} F_{k2}^{\Gamma^0(4)}(z,u,v) &= \big(\Sigma_{k2}^{\Gamma^0(4)} f_{k0}\big)(z,u,v) \\ &= \sum_{g\in\Gamma^0(4)/\Gamma_{4\infty}} \big(T_g T_U f_{k0}\big)(z) e_{k'm'}(u,v), \quad \tbinom{k'}{m'} = g^{-1}\tbinom{k}{0} \\ &= iF_{k1}^{\Gamma^0(4)}(z,u,v). \end{aligned}$$

(U²)

$$F_{k3}^{\Gamma^0(4)} = (z,u,v) = -F_{k1}^{\Gamma^0(4)}(z,u,v).$$

(U³)

$$F_{k4}^{\Gamma^0(4)} = (z,u,v) = -iF_{k1}^{\Gamma^0(4)}(z,u,v).$$

(*P*) For $f_{k0} \in \mathcal{H}_{\Gamma_{1\infty}}$

$$\begin{aligned}
F^{\Gamma^0(4)}_{k5}(z,u,v) &= \big(\Sigma^{\Gamma^0(4)}_{k5} f_{k0}\big)(z,u,v)\\
&= \sum_{g\in\Gamma^0(4)/P\Gamma_{1\infty}P^{-1}} \big(T_g T_P f_{k0}\big)(z) e_{k''m''}(u,v),\\
&\qquad\qquad \tbinom{k''}{m''} = g^{-1}P^{-1}\tbinom{k}{0}
\end{aligned}$$

$(g_1 \in P^{-1}gP \in \Gamma^0(4))$

$$\begin{aligned}
&= \sum_{g_1\in\Gamma^0(4)/P\Gamma_{1\infty}P^{-1}} \big(T_P T_{g_1} f_{k0}\big)(z) e_{k''m''}(u,v),\\
&\qquad\qquad \tbinom{k''}{m''} = P^{-1}g_1^{-1}\tbinom{k}{0}\\
&= \big(\widetilde{T}_P F^{\Gamma^0(4)}_{k1}\big)(z,u,v).
\end{aligned}$$

(*V*)

$$\begin{aligned}
F^{\Gamma^0(4)}_{k6}(z,u,v) &= \big(\Sigma^{\Gamma^0(4)}_{k6} f_{k0}\big)(z,u,v)\\
&= \sum_{g\in\Gamma^0(4)/V\Gamma_{1\infty}V^{-1}} \big(T_g T_V f_{k0}\big)(z) e_{k'''m'''}(u,v),\\
&\qquad\qquad \tbinom{k'''}{m'''} = g^{-1}V^{-1}\tbinom{k}{0}
\end{aligned}$$

$(V^{-1}gV = g_1 \in \Gamma^0(4))$

$$\begin{aligned}
&= \sum_{g_1\in\Gamma^0(4)/V\Gamma_{1\infty}V^{-1}} \big(T_V T_{g_1} f_{k0}\big)(z) e_{k'''m'''}(u,v),\\
&\qquad\qquad \tbinom{k'''}{m'''} = V^{-1}g_1^{-1}\tbinom{k}{0}\\
&= \big(\widetilde{T}_V F^{\Gamma^0(4)}_{k1}\big)(z,u,v).
\end{aligned}$$

(2) Similarly we obtain for the group $\Gamma_\vartheta(4)$ with coset representatives I, U, U^2, U^3, P^2, P^2U.

($\boldsymbol{I}$, $\boldsymbol{U}$, $\boldsymbol{U^2}$, $\boldsymbol{U^3}$)

$$\begin{aligned}
F^{\Gamma_\vartheta}_{k1}(z,u,v) &= \big(\Sigma^{\Gamma_\vartheta}_{k1} f_{k0}\big)(z,u,v)\\
&= \sum_{g\in\Gamma_\vartheta/\Gamma_{4\infty}} \big(T_g f_{k0}\big)(z) e_{k'm'}(u,v), \quad \tbinom{k'}{m'} = g^{-1}\tbinom{k}{0}.
\end{aligned}$$

($\boldsymbol{P^2}$) For $f_{k0} \in \Gamma_{1\infty}$,

$$\begin{aligned}
F^{\Gamma^0(4)}_{k5}(z,u,v) &= \big(\Sigma^{\Gamma^0(4)}_{k5} f_{k0}\big)(z,u,v)\\
&= \sum_{g\in\Gamma_\vartheta/P^2\Gamma_{1\infty}P^{-2}} \big(T_g T_{P^2} f_{k0}\big)(z) e_{k''m''}(u,v),\\
&\qquad\qquad \tbinom{k''}{m''} = g^{-1}P^{-2}\tbinom{k}{0}
\end{aligned}$$

$(g_1 = P^{-2}gP^2 \in \Gamma^0(4))$

$$\begin{aligned}
&= \sum_{g_1\in\Gamma^0(4)/\Gamma_{1\infty}} \big(T_{P^2} T_g f_{k0}\big)(z) e_{k'''m'''}(u,v),\\
&\qquad\qquad \tbinom{k'''}{m'''} = P^{-2}g_1^{-1}\tbinom{k}{0}\\
&= \big(\widetilde{T}_{P^2} F^{\Gamma^0(4)}_{k1}\big)(z,u,v).
\end{aligned}$$

$(\boldsymbol{P^2U})$ For $f_{k0} \in \Gamma_{1\infty}$,

$$\Sigma_{k6}^{\Gamma_\vartheta}(z,u,v) = \sum_{g\in\Gamma_\vartheta/P^2U\Gamma_{1\infty}U^{-1}P^{-2}} \big(T_g T_{P^2U} f_{k0}\big)(z) e_{k'''m'''}(u,v),$$
$$\binom{k'''}{m'''} = g^{-1}P^{-2}U^{-1}\binom{k}{0} = g^{-1}P^{-2}\binom{k}{0}$$

$(U^{-1}P^{-2}gP^2U = g_2 \in \Gamma^0(4))$

$$= \sum_{g_2\in\Gamma^0(4)/\Gamma_{1\infty}} \big(T_{P^2U} T_{g_2} f_{k0}\big)(z) e_{k'''m'''}(u,v)$$
$$\binom{k'''}{m'''} = P^{-2}g_2^{-1}\binom{k}{0}$$
$$= \big(\widetilde{T}_{P^2U} F_{k1}^{\Gamma^0(4)}\big)(z,u,v).$$

(2) For the Group $\Gamma_0(4)$ we obtain in a way similar to for $\Gamma_0(2)$ expressions similar to those of $\Gamma^0(4)$ and $\Gamma_\vartheta(4)$,

$$F_{k1}^{\Gamma_0(4)}(z,u,v) = \big(\Sigma_{k1}^{\Gamma_\vartheta(4)} f_{k0}\big)(z,u,v), \quad f_{k0} \in \mathcal{H}_{\Gamma_{1\infty}}$$
$$F_{k2}^{\Gamma_0(4)}(z,u,v) = \big(\Sigma_{k2}^{\Gamma_\vartheta(4)} f_{k0}\big)(z,u,v), \quad f_{k0} \in \mathcal{H}_{\Gamma_{1\infty}}$$
$$F_{k3}^{\Gamma_0(4)}(z,u,v) = \big(\Sigma_{k3}^{\Gamma_\vartheta(4)} f_{k0}\big)(z,u,v), \quad f_{k0} \in \mathcal{H}_{\Gamma_{4\infty}}$$

with $F_{ki}^{\Gamma_o(4)}(z,u,v) = c_i F_{k3}^{\Gamma_0(4)}(z,u,v)$, $i = 4,5,6$.

From this expression for the $\Gamma_{0J}(4)$-, $\Gamma_J^0(4)$-, and $\Gamma_{\vartheta J}(4)$-invariant functions we obtain with $\mathcal{H}^i_{\Gamma^0(4)}$, $\mathcal{H}^i_{\Gamma_0(4)}$, $\mathcal{H}^i_{\Gamma_\vartheta(4)}$ defined as above, $i = 1,2,3$.

Theorem 5.2. *$\Sigma_{k1}^{\Gamma^0(4)}$ is unitary from $\mathcal{H}_{\Gamma_{4\infty}}$ to $\mathcal{H}^1_{\Gamma^0(4)}$. $\Sigma_{ki}^{\Gamma^0(4)}$ are unitary from $\mathcal{H}_{\Gamma_{1\infty}}$ to $\mathcal{H}^i_{\Gamma^0(4)}$, $i = 5,6$.*

$\mathcal{H}^1_{\Gamma^0(4),k}$, $\mathcal{H}^5_{\Gamma^0(4),k}$, $\mathcal{H}^6_{\Gamma^0(4),k}$ are pairwise orthogonal and invariant under the operator $L_{\Gamma^0(4)J}$.

Let

$$L^i_{\Gamma^0(4)k} = L_{\Gamma^0(4)k}\big|_{\mathcal{H}^i_{\Gamma^0 k}}, \quad i = 1,5,6.$$

Then

$$L_{\Gamma^0(4)k} = L^1_{\Gamma^0(4)k} \oplus L^5_{\Gamma^0(4)k} \oplus L^6_{\Gamma^0(4)k}.$$

$L^1_{\Gamma^0(4)k}$ is unitarily equivalent to the operator $L_{\Gamma_{4\infty}}$ in $\mathcal{H}_{\Gamma_{4\infty}}$ via $\Sigma_{k1}^{\Gamma^0(4)}$,

$$L^i_{\Gamma^0(4)k}\Sigma_{k1}^{\Gamma^0(4)} = \Sigma_{k1}^{\Gamma^0(4)} L_{\Gamma_{4\infty}}.$$

$L^i_{\Gamma^0(4)k}$ is unitarily equivalent to the operator $L_{\Gamma_{1\infty}}$ in $\mathcal{H}_{\Gamma_{1\infty}}$ via $\Sigma_{ki}^{\Gamma^0(4)}$,

$$L^i_{\Gamma^0(4)k}\Sigma_{ki}^{\Gamma^0(4)} = \Sigma_{ki}^{\Gamma^0(4)} L_{\Gamma_{1\infty}}, \quad i = 5,6.$$

The continuous spectrum of $L_{\Gamma^0(4)k}$ is of multiplicity 3, and there are no resonances.

The point spectrum of $L^1_{\Gamma^0(4)k}$ is the union of the sets $\{\lambda^n_{kl}\}_{n=1}^\infty$, $l = 1,2,\ldots$; $\{\lambda^n_{k,l+1/4}\}_{n=1}^\infty$, $\{\lambda^n_{k,l+1/1}\}_{n=1}^\infty$, $\{\lambda^n_{k,l+3/4}\}_{n=1}^\infty$, $l = 0,1,2,\ldots$

The point spectrum of $L^i_{\Gamma^0(4)k}$ *is the union of the sets* $\{\lambda^n_{kl}\}^\infty_{n=1}$, $l = 1, 2, \ldots$ *Similarly*

$$L_{\Gamma_\vartheta(4)k} = L^1_{\Gamma_\vartheta(4)k} \oplus L^5_{\Gamma_\vartheta(4)k} \oplus L^6_{\Gamma_\vartheta(4)k}$$

where

$$L^1_{\Gamma_\vartheta k}\Sigma^{\Gamma_\vartheta(4)}_{k1} = \Sigma^{\Gamma_\vartheta(4)}_{k1} L_{\Gamma_{4\infty}}$$

and

$$L^i_{\Gamma_\vartheta k}\Sigma^{\Gamma_\vartheta(4)}_{k1} = \Sigma^{\Gamma_\vartheta(4)}_{k1} L_{\Gamma_{1\infty}}, \quad i = 5, 6$$

with the same spectral properties as $L_{\Gamma^0(4)k}$.

A similar result holds for $L_{\Gamma_0(4)k}$, *replacing* $L^1_{\Gamma^0(4)k}$ *by* $L^3_{\Gamma_0(4)k}$ *and* $L^5_{\Gamma^0(4)k}$, $L^6_{\Gamma^0(4)k}$ *by* $L^1_{\Gamma_0(4)k}$, $L^2_{\Gamma_0(4)k}$.

The operators $\Gamma_0(4)$, $L_{\Gamma_\vartheta(4)}$ *and* $L_{\Gamma^0(4)}$ *have the same eigenvalues with the same multiplicity in agreement with the fact that they are conjugate as Jacobi groups. Their counting function is given asymptotically by*

$$N_{\Gamma_0(4)_J}(\lambda) = N_{\Gamma_\vartheta(4)_J}(\lambda) = N_{\Gamma^0(4)_J}(\lambda) \sim 6 \cdot \frac{1}{192\pi}\lambda^2$$

which is the Weyl law for these groups.

Proof. This is proved, using our expressions for the invariant functions for $\Gamma_0(4)_J$, $\Gamma_\vartheta(4)_J$ and $\Gamma^0(4)_J$ as it was proved for normal subgroups in section 3. □

Remark. *The operators* $\Gamma(2)$ *and* $\Gamma_0(4)$ *are conjugate in* $\mathrm{SL}_2(\mathbb{R})$,

$$\Gamma(2) = 2\Gamma_0(4)\tfrac{1}{2}.$$

Therefore the operators $L_{\Gamma(2)}$ *in* $\mathcal{H}_{\Gamma(2)}$ *and* $L_{\Gamma_0(4)}$ *in* $\mathcal{H}_{\Gamma_0(4)}$, *are isospectral, they have the same eigenvalues with the same multiplicities.*

The operators $L_{\Gamma_{2J}}$ *and* $L_{\Gamma_0(4)_J}$ *are not isospectral. This follows from Theorem 3 and Theorem 5.2. However, the above conjugation is not a conjugation in the Jacobi group, so there is no contradiction.*

III. We consider two conjugate subgroups Γ_{E_1} and Γ_{E_2} of index 6, generated by 3 elliptic elements of order 3 ([5]). These groups are normal in Γ^2 and are conjugate by U (and by V). The group Γ_{E_1} and Γ_{E_2} are generated by the following elliptic elements with their fix points indicated, where $\Gamma_{E_2} = U\Gamma_{E_1}U^{-1}$,

$$E_1 = \left(\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right),\ \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}; \qquad F_1 = \left(\begin{smallmatrix}-2 & -7\\ 1 & -3\end{smallmatrix}\right),\ \tfrac{5}{2} + i\tfrac{\sqrt{3}}{2}; \qquad G_1 = \left(\begin{smallmatrix}2 & -21\\ 1 & -5\end{smallmatrix}\right),\ \tfrac{9}{2} + i\tfrac{\sqrt{3}}{2}$$
$$E_2 = \left(\begin{smallmatrix}1 & -3\\ 1 & -2\end{smallmatrix}\right),\ \tfrac{3}{2} + i\tfrac{\sqrt{3}}{2}; \qquad F_2 = \left(\begin{smallmatrix}3 & -13\\ 1 & -4\end{smallmatrix}\right),\ \tfrac{7}{2} + i\tfrac{\sqrt{3}}{2}; \qquad G_2 = \left(\begin{smallmatrix}5 & -31\\ 1 & -6\end{smallmatrix}\right),\ \tfrac{11}{2} + i\tfrac{\sqrt{3}}{2}.$$

We have

$$E_1F_1G_1 = E_2F_2G_2 = I, \quad A = U^2$$
$$F_i = AE_iA^{-1}, \quad G_i = AF_iA^{-1}, \quad i = 1, 2.$$

Γ^2 is generated by E_i and A, $\Gamma^2/\Gamma_{E_i} = \{I, A, E_i\}$. Thus, $A^3 \in \Gamma_{E_i}$, but $A \notin \Gamma_{E_i}$ hence $A^4 \notin \Gamma_{E_i}$, $i = 1, 2$. Also $U \notin \Gamma_{E_i}$ so $U^3 \notin \Gamma_{E_i}$, $i = 1, 2$. It follows that the width of Γ_{E_i} is 6, $i = 1, 2$. Then I, U, U^2, U^3, U^4, U^5 are right coset representatives of Γ_{E_i} in Γ_1, $i = 1, 2$.

Theorem 5.3. *Let $f_0(z) \in \mathcal{H}_{\Gamma_{6\infty}}$ and for $i = 1, 2$*

$$\begin{aligned} F_k^{\Gamma_{E_i}}(z,u,v) &= \big(\Sigma_k^{\Gamma_{E_i}} f_{k0}\big)(z,u,v) \\ &= \sum_{g\in\Gamma_{E_i}} \big(T_g f_{k0}\big)(z) e_{k'm'}(u,v), \quad \tbinom{k'}{m'} = g^{-1}\tbinom{k}{0}. \end{aligned}$$

For $i = 1, 2$, the operators $\Sigma_k^{\Gamma_{E_i}}$ are unitary from $\mathcal{H}_{\Gamma_{6\infty}}$ to $\mathcal{H}_{\Gamma_{E_i},k} = \Sigma_k^{\Gamma_{E_i}} \mathcal{H}_{\Gamma_{6\infty}}$, these Hilbert spaces are invariant under the Jacobi Laplacian $L_{\Gamma_{E_i},J}$ and its restrictions

$$L_{\Gamma_{E_i},k} = L_{\Gamma_{E_i}}|_{\mathcal{H}_{\Gamma_{E_i}}}$$

satisfy

$$L_{\Gamma_{E_i},k}\Sigma_k^{\Gamma_{E_i}} = \Sigma_k^{\Gamma_{E_i}} L_{\Gamma_{6\infty}}.$$

The set of eigenvalues of $L_{\Gamma_{E_i},k}$ is the union of the sets

$$\big\{\lambda_{k,l+i/6}\big\}_{n=1}^{\infty}, \quad l = 0, 1, 2, \ldots, \quad i = 1, 2, 3, 4, 5, 6.$$

The Weyl law holds for the counting function

$$N_{\Gamma_{E_i}}(\lambda) \sim 6 \cdot \frac{1}{192\pi}\lambda^2, \quad i = 1, 2.$$

The continuous spectrum has multiplicity 1.

Remark. *The operators $L_{\Gamma_{E_1},k}$ and $L_{\Gamma_{E_2},k}$ are isospectral with the operator $L_{\Gamma',k}$ associated with the normal commutator group, since this also has index* 6 *and width* 6.

Acknowledgment

I would like to thank Alexei Venkov for many valuable discussions over the years. The results on the continuous spectrum (Lemma 1.8 and the last part of Theorem 1) are due to him. I thank Erik Skibsted for the idea of the proof of the last part of Lemma 4.6 (eigenfunctions converging weakly to 0). I also thank Lars Madsen for typing the manuscript and for pointing out some mistakes and omissions for my correction.

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