AARHUS UNIVERSITY
DEPARTMENT OF MATHEMATICS

THE TRANSLATION INVARIANT MASSIVE NELSON MODEL: II. THE CONTINUOUS SPECTRUM BELOW THE TWO-BOSON THRESHOLD

by Jacob Schach Møller and Morten Grud Rasmussen

The translation Invariant Massive Nelson Model: II. The Continuous Spectrum Below the Two-boson Threshold

Jacob Schach Møller[1] and Morten Grud Rasmussen[*2]

[1]Department of Mathematics, Aarhus University
email: jacob@imf.au.dk
[2]Department of Engineering, Aarhus University
email: mgr@ase.au.dk

Abstract

In this paper we continue the study of the energy-momentum spectrum of a class of translation invariant, linearly coupled, and massive Hamiltonians from non-relativistic quantum field theory. The class contains the Hamiltonians of E. Nelson [42] and H. Fröhlich [14]. In [38, 40] one of us previously investigated the structure of the ground state mass shell and the bottom of the continuous energy-momentum spectrum. Here we study the continuous energy-momentum spectrum itself up to the two-boson threshold, the threshold for energetic support of two-boson scattering states. We prove that non-threshold embedded mass shells have finite multiplicity and can accumulate only at thresholds. We furthermore establish the non-existence of singular continuous energy-momentum spectrum. Our results hold true for all values of the particle-field coupling strength but only below the two-boson threshold. The proof revolves around the construction of a certain relative velocity vector field used to construct a conjugate operator in the sense of Mourre.

Contents

1 Introduction and Results

The present paper is a sequel to [38], where the ground state mass shell and the bottom of the continuous energy-momentum spectrum of the translation invariant massive Nelson model was studied. The massive Nelson model was introduced in [42] as a toy model for nucleon-meson interactions. It is similar in structure to the large polaron model of H. Fröhlich [14], describing electrons in polar crystals, interacting with longitudinal optical phonons. In [40], the results of [38] were in fact extended to cover a larger class of models encompassing both the massive Nelson model and the large polaron model.

With the pioneering works of Hübner-Spohn [32, 33] and Bach-Fröhlich-Sigal [5, 6] in the mid 90's, it became apparent that many of the techniques developed to deal with spectral and scattering problems for many-body quantum mechanics were in fact also applicable to models of quantized matter interacting with second quantized fields, sometimes called non-relativistic QFT. These models range from finite level systems interacting with scalar fields, e.g. the spin-boson model, to models of atoms and molecules minimally coupled to a second quantized Maxwell field. They include models from solid state physics describing electrons interacting with vibrational modes of crystals, i.e. phonons. We recall that acoustic phonons are modeled by massless fields, and optical phonons by massive fields.

The purpose of this sequel to [38, 40] is to study the structure of the continuous energy-momentum regime. More precisely the region supporting at most one asymptotic boson, i.e. the region below the threshold for energetic support of states with

two (or more) asymptotic bosons. This is what is meant with 'two-boson threshold' in the title. In particular we prove that fiber Hamiltonians in this energy regime have isolated thresholds, non-threshold eigenvalues have finite multiplicity and can only accumulate at thresholds, and there is no singular continuous spectrum. Our results do not depend on the strength of the particle-field coupling. The main tool is the construction of an energy-momentum dependent relative velocity field, describing at fixed total momentum the difference of velocities of a single asymptotically free boson and an interacting effective particle (e.g. polaron). This velocity field goes into the construction of a modified generator of dilation, which induces a second quantized conjugate operator in the sense of Mourre, admitting a positive commutator with the fiber Hamiltonian. Our work can be seen as a fusion of the spectral theory part of [11] by Derezinski-Gérard and the paper [25] by Gérard-Nier on analytically fibered operators. A simpler version of the construction and results of this paper formed a part of the Ph.D. thesis of the second author [44].

We distinguish between the bare particle entering into the model via its dispersion relation $\Omega \colon \mathbb{R}^\nu \to [0, \infty)$, and the effective particles described by the interacting model. In the polaron model, this is even hammered home by the word 'polaron' used to refer to the effective particle associated with the ground state, and the word 'electron' reserved for the (bare) band electron entering into the Fröhlich Hamiltonian. While only the bare quantities enter into the construction of the Hamiltonian, for an observer the bare particle is a mythical entity which never appears in a scattering experiment. Only effective particles are manifest as identifiable quantities.

In relativistically invariant field theories, the particle content of a theory is determined by eigenvalues of the mass operator. That only the mass characterizes the effective particles is due to Poincaré invariance, which ensures that the dispersion relations of the effective particles are forced to be of the form $\sqrt{k^2 + M^2}$, where M is an eigenvalue of the mass operator. In our setup, the model is non-relativistic and only invariance under translations and spatial rotations remains. This means that the dispersion relations of the effective particles are not determined by a single number, but a priori by a function on (a subinterval of) the half-line. Very little is known about the general structure of the effective dispersion relations, or mass shells, even for the ground state. This is a source of complications since we have to take into account the following features: **(A)** There may be multiple species of effective particles, i.e. mass shells. **(B)** The effective dispersions may not be convex, nor are they a priori forced to be radially increasing. **(C)** Excited mass shells may cross making it ambiguous how to assign velocity to a state constructed by energy-momentum localizations.

Let us give a heuristic explanation for the role of the effective dispersions in the analysis of the continuous energy-momentum spectrum. The continuous spectrum pertains to scattering states of the Hamiltonian, and scattering states should at large times look like (superpositions of) an interacting bound effective particle plus a number of free asymptotic bosons, with the sum of momenta and energies of the constituents summing up to the total momentum and energy of the initial state. The dynamics for such compound asymptotic systems at total momentum ξ is governed by a kinetic energy of the form

$$S^{(n)}(\xi; k_1, \ldots, k_n) := S(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n),$$

where S is the dispersion relation for an effective particle, ω is that of the bosons,

and $k_1, \cdots, k_n$ labels the momenta of n asymptotic bosons. We observe from this expression that we get thresholds at energies where $(k_1, \ldots, k_n) \to S^{(n)}(\xi; k_1, \ldots, k_n)$ has critical points. Computing the gradient we see that this happens when $\nabla S(\xi - k_1 - \cdots - k_n) = \nabla \omega(k_j)$ for all j, i.e. when all the asymptotic bosons has velocity equal to the velocity of the effective particle. As a first step we ensure that the threshold set is small. Secondly starting from a given total momentum and energy (ξ, E) below the two-boson threshold, we ensure that we can unambiguously assign a non-vanishing relative velocity field to the scattering states sitting in a small energy-momentum region near (ξ, E). To translate non-vanishing of the relative velocity field into a positive commutator estimate, we develop a method to extract from the Hamiltonian, expressed in terms of the bare particle dispersion only, the effective dispersions used to construct the relative velocity field.

We remark that in the weak coupling regime there are results in the literature about the structure of the continuous energy-momentum spectrum both for massive and massless bosons. Angelescu-Minlos-Zagrebnov and Minlos [4, 37] studies polaron type models and proves absence of embedded mass shells below the two-boson threshold at small coupling using a Feshbach reduction to a (generalized) Friedrichs model also studied in [1]. For what appears to be technical reasons only, the papers [1, 4, 37] cover neither the Nelson nor the polaron model, due to field energy and form factor restrictions respectively. The Friedrichs model itself, corresponding to cutting the Fock space down to the vacuum and one-particle sectors, was studied for all couplings in [19, 27]. For massless bosons (photons) De Roeck-Fröhlich-Pizzo [9] shows that in the weak coupling regime, (necessarily) interacting 'large and regular' embedded mass shells must lie close to the bottom of the continuous energy-momentum spectrum, and outside a natural cylinder around zero total momentum. In a narrower cylinder Chen-Faupin-Fröhlich-Sigal proves in [8] the absence of singular continuous spectrum. For sufficiently small energies and momenta, these results were previously established in [17] under the additional assumption that soft bosons are non-interacting.

We stress that our results are valid also outside a weak coupling regime, which necessitates – to put it in somewhat poetic terms – a final goodbye to the electron and a full embrace of the polaron.

Together with Wojciech Dybalski, the authors are currently working on applying the constructions of this paper to prove asymptotic completeness below the two-boson threshold for models of the type considered in this paper.

In the remaining part of Chapter 1 we introduce the Hamiltonian and its energy-momentum spectrum, formulate our main results, and at the end we give a geometric picture describing our central construction in a nutshell. In Chapter 2 we introduce the $C^k(A)$ classes of self-adjoint operators, cf [3], and prove that the fiber Hamiltonians of our model is of class $C^2(A)$, whenever A is a second quantized (modified) generator of dilation. In Chapter 3 we prove our main theorems. We begin with an analysis of the threshold set, followed by a geometric analysis of level crossings needed to patch together the relative velocity fields of potentially several effective particle species. Finally we prove a Mourre estimate for fiber Hamiltonians, first for a comparison Hamiltonian describing an interacting system plus a free boson, and subsequently for the Hamiltonian itself.

Acknowledgments

1.1 The Hamiltonian and its Energy-Momentum Spectrum

We consider a bare quantum particle, moving in $\mathbb{R}^\nu$ and linearly coupled to a scalar field of massive bosons. The particle Hilbert space is

$$\mathcal{K} := L^2(\mathbb{R}^\nu_y)$$

where y is the particle position. The bare particle Hamiltonian is $\Omega(D_y)$, where $D_y := -\mathrm{i}\nabla_y$.

The one-boson Hilbert space is

$$\mathfrak{h}_{\mathrm{ph}} := L^2(\mathbb{R}^\nu_k)$$

where k is the photon momentum, and the one-photon dispersion relation is $\omega(k)$. See Condition 1.2 below for the conditions we impose on the dispersion relations Ω and ω.

The Hilbert space for the field is the bosonic Fock space

$$\mathcal{F} = \Gamma(\mathfrak{h}_{\mathrm{ph}}) := \bigoplus_{n=0}^\infty \mathcal{F}^{(n)}, \tag{1.1}$$

$$\text{where} \qquad \mathcal{F}^{(n)} = \Gamma^{(n)}(\mathfrak{h}_{\mathrm{ph}}) := \mathfrak{h}_{\mathrm{ph}}^{\otimes_s n}. \tag{1.2}$$

Here $\mathfrak{h}_{\mathrm{ph}}^{\otimes_s n}$ is the symmetric tensor product of n copies of $\mathfrak{h}_{\mathrm{ph}}$. We write $|0\rangle = (1, 0, 0, \dots)$ for the vacuum state. The creation and annihilation operators $a^*(k)$ and $a(k)$ satisfy the following distributional form identities, known as the canonical commutation relations.

$$\begin{aligned}
[a^*(k), a^*(k')] &= [a(k), a(k')] = 0, \\
[a(k), a^*(k')] &= \delta(k - k') \quad \text{and} \\
a(k)|0\rangle &= 0.
\end{aligned} \tag{1.3}$$

The free field energy is the second quantization of the one-boson dispersion relation,

$$\mathrm{d}\Gamma(\omega) = \int_{\mathbb{R}^\nu} \omega(k) a^*(k) a(k) \, \mathrm{d}k. \tag{1.4}$$

The Hilbert space of the combined system is

$$\mathcal{H} := \mathcal{K} \otimes \mathcal{F}. \tag{1.5}$$

The free and coupled Hamiltonians for the combined system are

$$H_0 := \Omega(D_y) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) \quad \text{and} \tag{1.6}$$

$$H := H_0 + V \tag{1.7}$$

where the interaction V is given by

$$V := \int_{\mathbb{R}^\nu} \left(e^{-ik\cdot y}\, g(k)\, \mathbb{1}_\mathcal{K} \otimes a^*(k) + e^{ik\cdot y}\, \overline{g(k)}\, \mathbb{1}_\mathcal{K} \otimes a(k) \right) dk. \tag{1.8}$$

Here $g \in \mathfrak{h}_{\mathrm{ph}} = L^2(\mathbb{R}^\nu)$ is a coupling function.

The total momentum of the combined system is given by

$$P = D_y \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{K} \otimes d\Gamma(k). \tag{1.9}$$

The operators H_0 and H commute with P, i.e. H_0 and H are translation invariant. This implies that H_0 and H are fibered operators. Using the unitary transform I_{LLP} first introduced by Lee-Low-Pines in [35] and given by

$$I_{\mathrm{LLP}} := (F \otimes \mathbb{1}_\mathcal{F}) \circ \Gamma(e^{-ik\cdot y}) \tag{1.10}$$

we can identify the fibers of H_0 respectively H. Here F is the Fourier transform and Γ the second quantization functor. We get

$$I_{\mathrm{LLP}}\, H_0\, I_{\mathrm{LLP}}^* = \int_{\mathbb{R}^\nu}^\oplus H_0(\xi)\, d\xi \quad \text{and}$$

$$I_{\mathrm{LLP}}\, H\, I_{\mathrm{LLP}}^* = \int_{\mathbb{R}^\nu}^\oplus H(\xi)\, d\xi,$$

where $H_0(\xi)$ and $H(\xi)$ are operators on $\mathcal{F}$ and given by

$$H_0(\xi) = d\Gamma(\omega) + \Omega(\xi - d\Gamma(k)) \quad \text{and}$$
$$H(\xi) = H_0(\xi) + \phi(g).$$

Here $\phi(g)$ is the field operator evaluated at $y = 0$

$$\phi(g) = \int_{\mathbb{R}^\nu} \left(g(k)\, a^*(k) + \overline{g(k)}\, a(k) \right) dk.$$

See also [46, 47] and [11] for general constructions related to bosonic Fock space.

Remark 1.1. Above we introduced the unitary operator $\Gamma(e^{-ik\cdot y})$ on $\mathcal{H}$. This is in fact a slight abuse of notation since the functor Γ a priori only maps contractions on $\mathfrak{h}$ to contractions on $\mathcal{F}$. Here $q = e^{-ik\cdot y}$ is a contraction on $\mathcal{K} \otimes \mathfrak{h}$.

Suppose now that q is a contraction on $\mathcal{K} \otimes \mathfrak{h}$. Introduce for each $n \geq 2$ and $j = 1, \ldots, n$ a unitary operator $\mathcal{E}_j^{(n)}$ on $\mathcal{G}^{(n)} := \mathcal{K} \otimes \mathfrak{h}^{\otimes n}$ (full n-fold tensor product) by the following prescription on simple tensors

$$\mathcal{E}_j^{(n)}\left(f \otimes u_1 \otimes \cdots \otimes u_j \otimes \cdots \otimes u_n \right) = f \otimes u_j \otimes u_1 \otimes \cdots \otimes u_{j-1} \otimes u_{j+1} \cdots \otimes u_n.$$

Note that $\mathcal{E}_1^{(n)} = \mathbb{1}_{\mathcal{G}^{(n)}}$. We extend q to contractions on $\mathcal{G}^{(n)}$ setting $q_j = \mathcal{E}_j^{(n)*} q \otimes \mathbb{1}_{\mathfrak{h}^{\otimes n-1}} \mathcal{E}_j^{(n)}$. Using this construction we can define a contraction $G^{(n)}(q) = q_1 \cdots q_n$ on $\mathcal{G}^{(n)}$. Let $G^{(0)}(q) = \mathbb{1}_\mathcal{K}$, $G^{(1)}(q) = q$, and construct the direct sum $G(q) = \bigoplus_{n=0}^\infty G^{(n)}(q)$ to get a contraction on $\mathcal{G} = \mathcal{K} \otimes \bigoplus_{n=0}^\infty \mathfrak{h}^{\otimes n}$. If q is unitary, the contraction $G(q)$ is in fact unitary on $\mathcal{G}$.

Letting P_s denote the projection onto the symmetric Fock space $\mathcal{F}$ inside $\bigoplus_{n=0}^\infty \mathfrak{h}^{\otimes n}$ we can now define $\Gamma(q) = (\mathbb{1}_\mathcal{K} \otimes P_s) G(q) (\mathbb{1}_\mathcal{K} \otimes P_s)$ as a contraction on $\mathcal{H}$. We warn the reader that for unitary q, if q_i and q_j do not commute, the contraction $\Gamma(q)$ may not be unitary! $\diamond$

The following minimal conditions will be imposed on the dispersion relations and coupling function throughout the paper, and often without explicit reference. We will in particular formulate and use results from the literature under these minimal conditions although they may in fact hold true under weaker assumptions. The reader is asked to consult the literature for optimal formulations of known results. The notation $\langle k \rangle$ is an abbreviation of the function $\sqrt{1 + |k|^2}$. We will use the same notation for numbers, vectors, and self-adjoint operators.

Condition 1.2 (Minimal Conditions). There exist $s_\Omega \in [0, 2]$ and $C > 0$ such that the dispersion relations and coupling satisfy:

(MC1) $\omega \in C(\mathbb{R}^\nu)$, $\Omega \in C^2(\mathbb{R}^\nu)$ and $g \in L^2(\mathbb{R}^\nu)$.

(MC2) $m := \inf_{k \in \mathbb{R}^\nu} \omega(k) > 0$.

(MC3) $\forall k \in \mathbb{R}^\nu$ we have $\omega(k) \leq C\langle k \rangle$, $\Omega(k) \geq C^{-1}\langle k \rangle^{s_\Omega} - C$.

(MC4) $|\partial_\eta^\alpha \Omega(\eta)| \leq C\langle k \rangle^{s_\Omega - |\alpha|}$, for all multi-indicies α with $0 \leq |\alpha| \leq 2$.

(MC5) $\forall k_1, k_2 \in \mathbb{R}^\nu$ we have $\omega(k_1 + k_2) < \omega(k_1) + \omega(k_2)$.

(MC6) Either $\lim_{|k| \to \infty} \omega(k) = \infty$ or: $\sup_{k \in \mathbb{R}^\nu} \omega(k) < \infty$ and $\lim_{|k| \to \infty} \Omega(k) = \infty$.

We remark that the translation invariant massive Nelson model as well as Fröhlich's polaron model satisfies the above conditions, both with non-relativistic and relativistic electron dispersion relation. For both models we are required to impose a UV cutoff on the physical interaction. However there does not seem to be a fundamental obstacle to consider also the UV-renormalized models (if $\Omega(\eta) = \eta^2$) as in [2]. We recall that the physical interactions g, up to a constant multiple, are $1/\sqrt{\omega(k)}$, with $\omega(k) = \sqrt{k^2 + m^2}$, for the Nelson model, and $1/|k|$ for the polaron model in three dimensions. The phonon dispersion relation in the polaron model is taken to be a positive (material dependent) constant function.

In the remaining part of this section we list a number of known properties of H and its energy-momentum spectrum

$$\Sigma = \left\{ (\xi, E) \in \mathbb{R}^\nu \times \mathbb{R} \,\middle|\, E \in \sigma(H(\xi)) \right\}. \tag{1.11}$$

These properties have a long history, see e.g. [7, 15, 26, 36, 38, 40, 42, 49], with the most complete results in [40], where the reader can also find a comprehensive discussion of the literature on the subject.

Let

$$\mathcal{C} := \Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^\nu)) \subset \mathcal{F}, \tag{1.12}$$

where $\Gamma_{\mathrm{fin}}(\mathcal{V})$ denotes the algebraic direct sum of the algebraic tensor products $\mathcal{V}^{\otimes_s n}$, where $\mathcal{V} \subset \mathfrak{h}$. In fact, when tensor products appear between spaces not all of which are complete (as Hilbert spaces) an algebraic tensor product is implicitly understood. The operator H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu) \otimes \mathcal{C}$, and V is an infinitesimally small perturbation in the sense of Kato-Rellich. Hence H is also essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu) \otimes \mathcal{C}$, and the domain of the closures, which we as usual denote by the same symbols, coincide.

Similarly $H_0(\xi)$ is essentially self-adjoint on $\mathcal{C}$ and $\phi(g)$ is an infinitesimally small perturbation, hence $H(\xi)$ is also essentially self-adjoint on $\mathcal{C}$. Not only do their

domains coincide, they are independent of total momentum ξ, and we denote the common domain of self-adjointness by

$$\mathcal{D} := \mathcal{D}(H_0(\xi)) = \mathcal{D}(H(\xi)). \tag{1.13}$$

One can easily verify that $\xi \to (H(\xi) - i)^{-1}$ is norm continuous, and hence we observe by a norm resolvent convergence argument, cf [46, Theorem VIII.23], that Σ is a closed set.

We pause to introduce some notation. We denote the bottom of the spectrum of the fiber Hamiltonians by

$$\Sigma_0(\xi) := \inf \sigma(H(\xi)), \tag{1.14}$$

and the bottom of the spectrum of the full operator by

$$\Sigma_0 := \inf_{\xi \in \mathbb{R}^\nu} \Sigma_0(\xi) > -\infty. \tag{1.15}$$

Let $n \in \mathbb{N}$ be some positive integer and $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{R}^{n\nu}$. We introduce the least energy of a composite system consisting of a copy of an interacting system at momentum $\xi - \sum_{j=1}^n k_j$ and n non-interacting photons with momenta k_j

$$\Sigma_0^{(n)}(\xi; \underline{k}) := \Sigma_0(\xi - \sum_{j=1}^n k_j) + \sum_{j=1}^n \omega(k_j). \tag{1.16}$$

The following functions are the so called n-boson thresholds, i.e. the least energy needed to support an interacting state and n free bosons at a given total momentum

$$\Sigma_0^{(n)}(\xi) := \inf_{\underline{k} \in \mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi; \underline{k}). \tag{1.17}$$

Abusing notation, we write $\Sigma_0^{(n)}$ both for the function and for its graph. We should warn the reader that the terminology 'threshold' carries a dual meaning. The use of '*the n-boson threshold*' to describe $\Sigma_0^{(n)}(\xi)$ refers to its literal meaning as the lowest energy supporting an interacting system and n free bosons. It is in fact also '*an n-boson threshold*' in the physical sense of the word threshold as an energy at which the system can form an interacting bound state plus n free bosons, with zero break up velocity. We stress that these are in general not the only (physical) thresholds of the system.

With the above notation the HVZ Theorem takes the form

$$\sigma_{\text{ess}}(H(\xi)) = [\Sigma_0^{(1)}(\xi), \infty), \tag{1.18}$$

and below $\Sigma_0^{(1)}(\xi)$ the spectrum of $H(\xi)$ consists of locally finitely many eigenvalues all of finite multiplicity, that may only accumulate at $\Sigma_0^{(1)}(\xi)$. We will often write $\Sigma_{\text{ess}}(\xi) = \Sigma_0^{(1)}(\xi)$ to emphasize the role of the one-boson threshold as the bottom of the essential energy-momentum spectrum. We remark that the assumption (MC6) ensures that the essential energy-momentum spectrum does not have holes.

Due to the subadditivity assumption (MC5) on ω, the n-boson thresholds are increasing in n, i.e.

$$\forall n > m : \quad \Sigma_0^{(n)}(\xi) \geq \Sigma_0^{(m)}(\xi). \tag{1.19}$$

If $\lim_{|k| \to \infty} \omega(k) = \infty$ the inequality is strict. If $M = \sup_{k \in \mathbb{R}^\nu} \omega(k) < \infty$, then the inequality remains strict under the extra assumption $2 \liminf_{|k| \to \infty} \omega(k) > M$,

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satisfied obviously by the constant polaron dispersion, cf. [40]. This can be considered a remark on non-triviality of our results, since we work in the energy-momentum region between the graphs of $\Sigma_0^{(1)}$ and $\Sigma_0^{(2)}$.

Finally we remark that isolated ground states of $H(\xi)$ are non-degenerate, in particular the ground state mass shell does not cross any possibly existing isolated excited mass shells. Very little is known about the structure of the discrete spectrum when we are away from a weak coupling regime. In fact, we rely only on some symmetry observations and Kato's general analytic perturbation theory, which applies to the family $\{H(\xi)\}_{\xi \in \mathbb{R}^\nu}$. In the weak coupling regime one can compare with the uncoupled model and derive stronger results [4]. We identify distinct mass shells, as functions of total momentum $\xi \to S(\xi)$, with effective particles, with dispersion relation given by S. In the case of the polaron model, it is the ground state which in the literature is referred to as the Fröhlich polaron.

1.2 Extended Objects

In this section we introduce a new Hamiltonian which plays the role of the generator of the dynamics for a system of one interacting particle and a number of free bosons. The interacting particle and the free bosons are not coupled. Operators of this type were also used in [2, 11, 12, 16, 17, 38, 40]. This is a natural object in the context of scattering theory, where one expects scattering states to decay into interacting bound states under emission of asymptotically free bosons.

We abbreviate

$$\mathcal{F}^{\mathrm{x}} = \mathcal{F} \otimes \mathcal{F} \quad \text{and} \quad \mathcal{H}^{\mathrm{x}} = \mathcal{H} \otimes \mathcal{F} = \mathcal{K} \otimes \mathcal{F}^{\mathrm{x}}.$$

For a self-adjoint operator a on $\mathfrak{h}$, we extend the second quantization operation to $\mathcal{F}^{\mathrm{x}}$ by the construction

$$\mathrm{d}\Gamma^{\mathrm{x}}(a) = \mathrm{d}\Gamma(a) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma(a).$$

Note that $\mathrm{d}\Gamma(a)$ is essentially self-adjoint on $\Gamma_{\mathrm{fin}}^{\mathrm{x}}(D) = \Gamma_{\mathrm{fin}}(D) \otimes \Gamma_{\mathrm{fin}}(D)$, if $D \subset \mathfrak{h}$ is a domain of essential self-adjointness for a, cf. [46]. We can now define the Hamiltonian describing an interacting system together with free (asymptotic) bosons. It is given by

$$H^{\mathrm{x}} = H \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{H}} \otimes \mathrm{d}\Gamma(\omega) = \Omega(D_y) \otimes \mathbb{1}_{\mathcal{F}^{\mathrm{x}}} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma^{\mathrm{x}}(\omega) + V \otimes \mathbb{1}_{\mathcal{F}} \quad (1.20)$$

as an operator on the Hilbert space $\mathcal{H}^{\mathrm{x}}$. The free operator, with $g = 0$, is essentially self-adjoint on

$$\mathcal{C}^{\mathrm{x}} = \mathcal{C} \otimes \Gamma_{\mathrm{fin}}\big(C_0^\infty(\mathbb{R}^\nu)\big), \quad (1.21)$$

and so is H^{x} by a Kato-Rellich argument.

We adopt the terminology from [11] and call H^{x} the *extended Hamiltonian* and $\mathcal{H}^{\mathrm{x}}$ the *extended Hilbert space*. We remark that H^{x} commutes with the extended total momentum operator

$$P^{\mathrm{x}} = P \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{H}} \otimes \mathrm{d}\Gamma(k) = D_y \otimes \mathbb{1}_{\mathcal{F}^{\mathrm{x}}} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma^{\mathrm{x}}(k). \quad (1.22)$$

We extend the functor Γ from Remark 1.1 as follows. Denote by $\mathcal{E}$ the exchange involution on $\mathcal{H}^{\mathrm{x}}$ defined on simple tensors as $\mathcal{E}(f \otimes \psi \otimes \varphi) = f \otimes \varphi \otimes \psi$, where $f \in \mathcal{K}$ and $\psi, \varphi \in \mathcal{F}$. For a contraction q on $\mathcal{K} \otimes \mathfrak{h}$ we define

$$\Gamma^{\mathrm{x}}(q) = \left(\Gamma(q) \otimes \mathbb{1}_{\mathcal{F}}\right)\mathcal{E}\left(\Gamma(q) \otimes \mathbb{1}_{\mathcal{F}}\right)\mathcal{E}.$$

This is only a good definition if the q_i's commute, cf. Remark 1.1. Denote by $P^{(n)}$ the projection of $\mathcal{H}^{\mathrm{x}}$ onto $\mathcal{H} \otimes \mathcal{F}^{(n)}$ and observe that $P^{(n)}\Gamma^{\mathrm{x}}(q) = \Gamma^{\mathrm{x}}(q)P^{(n)}$. Abbreviate $\Gamma^{(n)}(q) = P^{(n)}\Gamma^{\mathrm{x}}(q)P^{(n)}$ as a contraction on $\mathcal{H} \otimes \mathcal{F}^{(n)}$.

We now build the extended Hamiltonian H^{x}, cf. (1.20), from the inside out as an explicitly fibered operator. Recall that H^{x} commutes with the extended total momentum P^{x}, cf. (1.22). First we introduce fiber Hamiltonians for an interacting system at total momentum ξ, and n asymptotic free bosons with momenta $\underline{k} = (k_1, \ldots, k_n)$. These are self-adjoint operators on $\mathcal{F}$ given by

$$H^{(n)}(\xi; \underline{k}) = H(\xi - \textstyle\sum_{j=1}^{n} k_j) + \left(\textstyle\sum_{j=1}^{n} \omega(k_j)\right) \mathbb{1}_{\mathcal{F}}.$$

From these operators we construct self-adjoint fiber operators on $L^2_{\mathrm{sym}}(\mathbb{R}^{n\nu}; \mathcal{F}) \simeq \mathcal{F} \otimes \mathcal{F}^{(n)}$ by the direct integral construction

$$H^{(n)}(\xi) = \int_{\mathbb{R}^{n\nu}}^{\oplus} H^{(n)}(\xi; \underline{k}) \, \mathrm{d}k.$$

Here the subscript 'sym' indicates that the functions are symmetric under permutation of the n variables, reflecting Bose statistics. Finally, by another direct integral construction and an application of an extended version of I_{LLP}, cf. (1.10), we can define

$$H^{(n)} = I_{\mathrm{LLP}}^{(n)*} \left(\int_{\mathbb{R}^{\nu}}^{\oplus} H^{(n)}(\xi) \, \mathrm{d}\xi\right) I_{\mathrm{LLP}}^{(n)},$$

as an operator on the Hilbert space $\mathcal{H}^{(n)} = \mathcal{H} \otimes \mathcal{F}^{(n)}$. Here, as operators from $\mathcal{H} \otimes \mathcal{F}^{(n)}$ to $L^2(\mathbb{R}; \mathcal{F} \otimes \mathcal{F}^{(n)})$, $I_{\mathrm{LLP}}^{(n)} = (F \otimes \mathbb{1}_{\mathcal{F} \otimes \mathcal{F}^{(n)}})\Gamma^{(n)}(\mathrm{e}^{\mathrm{i}k \cdot y})$. The full extended Hamiltonian can now be expressed as a direct sum

$$H^{\mathrm{x}} = H \oplus \left(\bigoplus_{n=1}^{\infty} H^{(n)}\right)$$

as an operator on the extended Hilbert space $\mathcal{H}^{\mathrm{x}} = \mathcal{H} \otimes \mathcal{F} = \mathcal{H} \oplus (\oplus_{n=1}^{\infty} \mathcal{H}^{(n)})$. Similarly we can introduce fiber operators

$$H^{\mathrm{x}}(\xi) = H(\xi) \oplus \left(\bigoplus_{n=1}^{\infty} H^{(n)}(\xi)\right)$$

as an operator on $\mathcal{F}^{\mathrm{x}}$. From this construction we can directly identify $H^{\mathrm{x}}(\xi)$ as the fiber operators of H^{x} and we have the fibration

$$H^{\mathrm{x}} = I_{\mathrm{LLP}}^{\mathrm{x}*} \left(\int_{\mathbb{R}^{\nu}}^{\oplus} H^{\mathrm{x}}(\xi) \, \mathrm{d}\xi\right) I_{\mathrm{LLP}}^{\mathrm{x}},$$

where $I_{\mathrm{LLP}}^{\mathrm{x}} = (F \otimes \mathbb{1}_{\mathcal{F}^{\mathrm{x}}})\Gamma^{\mathrm{x}}(\mathrm{e}^{\mathrm{i}k \cdot y}) = I_{\mathrm{LLP}} \oplus (\oplus_{n=1}^{\infty} I_{\mathrm{LLP}}^{(n)})$.

Note that these constructions tie in well with the notion of thresholds for supporting states with free bosons, i.e. the functions $\Sigma_0^{(n)}(\xi; \underline{k})$ and $\Sigma_0^{(n)}(\xi)$ introduced in (1.16) and (1.17). More precisely we have

$$\inf \sigma\left(H^{(n)}(\xi; \underline{k})\right) = \Sigma_0^{(n)}(\xi; \underline{k}) \quad \text{and} \quad \inf \sigma\left(H^{(n)}(\xi)\right) = \Sigma_0^{(n)}(\xi). \tag{1.23}$$

1.3 The Results

To formulate our main results on the structure of the energy-momentum spectrum below the two-boson threshold we need an extra set of assumptions. The condition below depends on a natural number n_0 encoding the amount of control required. The condition will be used with $n_0 = 0$ for our result on the structure of the threshold set, with $n_0 = 1$ for our result on the structure of embedded point spectrum, and with $n_0 = 2$ for our result on absence of singular continuous spectrum.

Condition 1.3 (Spectral Theory). Let $n_0 \in \mathbb{N}$. We impose

(ST1) ω and Ω are real analytic functions.

(ST2) g admits n_0 distributional derivatives with $\partial_k^\alpha g \in L^2_{\mathrm{loc}}(\mathbb{R}^\nu \backslash \{0\})$, for all $1 \leq |\alpha| \leq n_0$.

(ST3) For all orthogonal matrices $O \in \mathrm{O}(\nu)$ and all $k \in \mathbb{R}^\nu$ we have $\omega(Ok) = \omega(k)$, $\Omega(Ok) = \Omega(k)$ and $g(Ok) = g(k)$ almost everywhere.

(ST4) $\sup_{k \in \mathbb{R}^\nu} |\partial^\alpha \omega(k)| < \infty$ for $|\alpha| \geq 1$ and $\sup_{\eta \in \mathbb{R}^\nu} |\partial^\beta \Omega(\eta)| < \infty$ for $|\beta| \geq 2$.

Remark 1.4. The assumptions of real analyticity (ST1) and rotation invariance (ST3) serve a combined purpose. The rotation invariance ensures that the energy-momentum spectrum Σ (and all its components, i.e. pure point, absolutely and singular continuous spectrum), are rotation invariant, i.e. $(\xi, E) \in \Sigma$ and $O \in \mathrm{O}(\nu)$ implies $(O\xi, E) \in \Sigma$. In particular, the n-boson thresholds $\Sigma_0^{(n)}$ are rotation invariant, cf. (1.17). The functions $\Sigma_0^{(n)}(\xi; \underline{k})$, cf. (1.16), however only retain invariance under simultaneous rotation of all k_j's around the ξ axis.

From the point of view of the models discussed so far, these are reasonable assumptions. However, one should keep in mind that dispersion relations in solid state physics are material dependent functions and more realistic ones are not likely to carry any more symmetry than discrete symmetries of an underlying lattice. We do not consider (ST3) to be an essential assumption, cf. the discussion in Section 1.4. $\diamond$

The above remark, together with Kato's analytic perturbation theory [34], enables a precise description of the isolated part of the energy momentum spectrum

$$\Sigma_{\mathrm{iso}} = \left\{ (\xi, E) \in \Sigma \,\middle|\, E < \Sigma_{\mathrm{ess}}(\xi) \right\}, \tag{1.24}$$

as a collection of real analytic mass shells and level crossings. The set Σ_{iso} forms a subset of the full pure-point energy-momentum spectrum

$$\Sigma_{\mathrm{pp}} = \left\{ (\xi, E) \in \Sigma \,\middle|\, E \in \sigma_{\mathrm{pp}}(H(\xi)) \right\}. \tag{1.25}$$

While the general analytic structure of Σ_{iso} is understood, the only thing we can a priori say about Σ_{pp} is that it is a Borel subset of $\mathbb{R}^{\nu+1}$, cf. Appendix A.

We introduce the set of level crossings for isolated mass shells:

$$\mathcal{X} := \{ (\xi, E) \in \Sigma_{\mathrm{iso}} \,|\, \forall n \in \mathbb{N} : \ \Sigma_{\mathrm{iso}} \cap B((\xi, E); 1/n) \text{ is not a graph} \}. \tag{1.26}$$

The connected components of $\mathcal{X}$ are $S^{\nu-1}$-spheres of the form $\partial B(0; R) \times \{E\}$, or as a possibly degenerate case, of the form $\{0\} \times \{E\}$. The spheres forming the connected

components of $\mathcal{X}$ will also be called *level crossings*. They are isolated $S^{\nu-1}$-spheres, possibly accumulating either at infinity or at the bottom of the essential energy-momentum spectrum Σ_{ess}. In particular, elements $(\xi, E) \in \mathcal{X}$ represent eigenvalues E of $H(\xi)$ with a given finite multiplicity. The connected components of $\mathcal{X}$ are connected in Σ_{iso} by real analytic manifolds, each carrying a finite multiplicity, in such a way that the sum of the multiplicities of shells emanating from the same crossing, should equal the multiplicity of the crossing. We denote the collection of such real analytic manifolds by $\mathcal{S}$. To be more precise, by a shell we understand a pair $(\mathcal{A}, S) \in \mathcal{S}$, where $\mathcal{A}$ is an open annulus $\{\xi \in \mathbb{R}^{\nu} \,|\, r < |\xi| < R\}$, with $0 \le r < R$, or an open ball centered at 0. The function $S \colon \mathcal{A} \to \mathbb{R}$ is real analytic and rotation invariant, with $\Sigma_0(\xi) \le S(\xi) < \Sigma_{\mathrm{ess}}(\xi)$ and such that the graphs of the shells together with the level crossings cover the entire isolated spectrum in energy-momentum space. For $(\mathcal{A}, S) \in \mathcal{S}$, denote by

$$\mathcal{G}_S = \Big\{ (\xi, S(\xi)) \,\Big|\, \xi \in \mathcal{A} \Big\} \tag{1.27}$$

its graph in energy-momentum space. We have $\mathcal{G}_S \cap \mathcal{G}_{S'} = \emptyset$, for all distinct shells $(\mathcal{A}, S) \ne (\mathcal{A}', S')$, and $\mathcal{G}_S \cap \mathcal{X} = \emptyset$ for all $(\mathcal{A}, S) \in \mathcal{S}$. In addition, to ensure we have all shells covered, we demand that

$$\Sigma_{\mathrm{iso}} = \mathcal{X} \cup \Big(\bigcup_{(\mathcal{A}, S) \in \mathcal{S}} \mathcal{G}_S \Big).$$

We remark that due to rotation invariance, the mass shells continue analytically through level crossings. The reader can consult [34] for the analytic structure of isolated eigenvalues of holomorphic families of self-adjoint operators. We remark that for a fixed unit vector $u \in \mathbb{R}^{\nu}$, the map $\kappa \to H(\kappa u)$ defines a 'Type A' family of operators. See [15].

For a given element $(\mathcal{A}, S) \in \mathcal{S}$, the graph $\mathcal{G}_S$ may have 0, 1 or 2 finite boundaries that are $S^{\nu-1}$-spheres (perhaps of radius 0). The case of no boundary, indicates a mass shell that without crossings extends to infinity in total momentum. An example of such a shell would be the ground state mass shell in dimensions one and two, cf. [38, 40, 49]. A boundary $S^{\nu-1}$-sphere can be one of two things. Either it is a connected component of $\mathcal{X}$, i.e. a level crossing, or it is a subset of $\Sigma_{\mathrm{ess}} = \Sigma_0^{(1)}$, the boundary of the continuous energy-momentum spectrum.

Unless a mass shell $(\mathcal{A}, S)$ is constant, its gradient ∇S can at most vanish on isolated $S^{\nu-1}$-spheres that can only accumulate at infinity. We remark that we do not know the manner in which mass shells, ground state or excited, dip into the continuous spectrum. One could speculate that it does so at worst as a branch of a Puiseaux series, something which may have useful consequences. See [39].

Having discussed the structure of the isolated spectrum, we now turn to the subset of the continuous energy-momentum spectrum below the two-boson threshold

$$\mathcal{E}^{(1)} = \Big\{ (\xi, \lambda) \in \mathbb{R}^{\nu} \times \mathbb{R} \,\Big|\, \Sigma_0^{(1)}(\xi) \le \lambda < \Sigma_0^{(2)}(\xi) \Big\}. \tag{1.28}$$

Write $\mathcal{E}^{(1)}(\xi) = [\Sigma_0^{(1)}(\xi), \Sigma_0^{(2)}(\xi))$ such that $\mathcal{E}^{(1)} = \{ (\xi, \lambda) \in \mathbb{R}^{\nu} \times \mathbb{R} \,|\, \lambda \in \mathcal{E}^{(1)}(\xi) \}$.

Our first result is concerned with the structure of possibly embedded point spectrum inside $\mathcal{E}^{(1)}$. To formulate the theorem, we need to carefully formalize the

notion of thresholds. We should identify energy-momenta inside $\mathcal{E}^{(1)}$ where emitted bosons fail to break free from the remaining interacting system with a non-zero relative velocity, thus preventing them from becoming asymptotic field particles. The threshold set pertaining to one-boson emission processes $\mathcal{T}^{(1)}$ has three components which we now discuss.

The first, and perhaps most obvious, is the set of one-boson thresholds where the remaining interacting system after boson emission ends up inside an isolated mass shell $(\mathcal{A}, S) \in \mathcal{S}$. We define $S^{(1)}(\xi; \cdot) \colon \mathcal{A} + \xi \to \mathbb{R}$ by

$$S^{(1)}(\xi; k) = S(\xi - k) + \omega(k). \tag{1.29}$$

This extends the construction (1.16) to (possibly existing) excited mass shells, and is the post-emission effective dispersion relation governing the composite interacting system plus emitted boson. The mass shell contribution to one-boson thresholds is

$$\mathcal{T}_{\mathcal{S}}^{(1)} = \left\{ (\xi, E) \in \mathbb{R}^{\nu+1} \,\middle|\, E \in \mathcal{T}_{\mathcal{S}}^{(1)}(\xi) \right\}, \tag{1.30}$$

$$\mathcal{T}_{\mathcal{S}}^{(1)}(\xi) = \left\{ E \in \mathbb{R} \,\middle|\, \exists (\mathcal{A}, S) \in \mathcal{S}, k \in \mathcal{A} + \xi : \ E = S^{(1)}(\xi; k), \nabla_k S^{(1)}(\xi; k) = 0 \right\}.$$

We emphasize that $\nabla_k S^{(1)}(\xi; k) = 0$ is the same as $\nabla S(\xi - k) = \nabla \omega(k)$, i.e. the asymptotic boson and the remaining interacting system have identical velocities. This defines one contribution to the one-boson threshold set. One can similarly define n-body thresholds, which however will sit above the (lowest) two-boson threshold $\Sigma_0^{(2)}$ and therefore we disregard them here, cf. (1.19).

To understand the next two contributions to the threshold set we need to explain the dynamics at level crossings. Suppose we are at an energy E and total momentum ξ, with one free boson at momentum k such that $(\xi - k, E - \omega(k)) \in \chi$. The only direction in momentum space we can control is where $(\xi - k, E - \omega(k))$ moves inside level crossings, which form $S^{\nu-1}$-spheres. Inside such spheres the energy of the bound system stays constant, due to being constrained to a crossing, so the effective dispersion only varies through the contribution from $k \to \omega(k)$. The effective dispersion therefore have critical momenta where the tangential derivative of ω, with respect to the $S^{\nu-1}$-sphere, vanishes. Since ω is rotation invariant this can happen in two ways. Either k is parallel to ξ in which case $\nabla \omega(k)$ is normal to the sphere, or it can happen if $\nabla \omega(k) = 0$.

The next contribution comes from the need to avoid landing on a level crossing with k parallel to ξ after emission of one boson with momentum k. Given $\xi \in \mathbb{R}^\nu$, let $u \in \mathbb{R}^\nu$ be a unit vector such that $\xi = su$ for some $s \in \mathbb{R}$. We introduce the set

$$\mathcal{T}_{\parallel}^{(1)}(\xi) := \left\{ E \in \mathbb{R} \,\middle|\, \exists r \in \mathbb{R} : \ \big(\xi - ru, E - \omega(ru)\big) \in \mathcal{X} \right\}. \tag{1.31}$$

If $\xi = 0$, the unit vector u can be chosen arbitrarily and we observe, since ω and the set $\mathcal{X}$ are rotation invariant, that

$$\mathcal{T}_{\parallel}^{(1)}(0) = \left\{ E \in \mathbb{R} \,\middle|\, \exists k \in \mathbb{R}^\nu : \ \big(k, E - \omega(k)\big) \in \mathcal{X} \right\}. \tag{1.32}$$

The final contribution to the threshold set consists of energies at which it is possible to emit a boson of momentum k with $\nabla \omega(k) = 0$ and the remaining interacting system at a level crossing.

$$\mathcal{T}_{\nparallel}^{(1)}(\xi) := \left\{ E \in \mathbb{R} \,\middle|\, \exists k \in \mathbb{R}^\nu : \ \big(\xi - k, E - \omega(k)\big) \in \mathcal{X} \text{ and } \nabla \omega(k) = 0 \right\}. \tag{1.33}$$

The reader can safely on a first reading disregard this contribution since in typical models $\mathcal{T}_{\nparallel}^{(1)}(\xi)$ will be a subset of $\mathcal{T}_{\parallel}^{(1)}(\xi)$. This happens of course in dimension 1, if $\nabla\omega(k) \neq 0$ for $k \neq 0$, and finally in the case of the polaron model. Note that we always have $\mathcal{T}_{\nparallel}^{(1)}(0) \subset \mathcal{T}_{\parallel}^{(1)}(0)$, cf. (1.32).

The total threshold set can now be defined to be

$$\mathcal{T}^{(1)} = \mathcal{T}_{\mathcal{S}}^{(1)} \cup \mathcal{T}_{\parallel}^{(1)} \cup \mathcal{T}_{\nparallel}^{(1)},$$

where $\mathcal{T}_{\parallel}^{(1)} = \{(\xi, E) \mid E \in \mathcal{T}_{\parallel}^{(1)}(\xi)\}$ and $\mathcal{T}_{\nparallel}^{(1)} = \{(\xi, E) \mid E \in \mathcal{T}_{\nparallel}^{(1)}(\xi)\}$.

The first theorem we present establishes the structure of the threshold set below the two-boson threshold.

Theorem 1.5. *Assume Conditions 1.2 and 1.3, with $n_0 = 0$. Let $\xi \in \mathbb{R}^\nu$. The following holds*

(i) *$\mathcal{T}^{(1)} \cap \mathcal{E}^{(1)}$ is a relatively closed subset of $\mathcal{E}^{(1)}$.*

(ii) *The set $\mathcal{T}^{(1)}(\xi) \cap \mathcal{E}^{(1)}(\xi)$ is a discrete subset of $\mathcal{E}^{(1)}(\xi)$, i.e. it is at most countable and can accumulate only at $\Sigma_0^{(2)}(\xi)$.*

In fact Theorem 1.5 holds for each of the three types of thresholds sets individually. This is obvious for (ii), and follows for (i) from its proof.

The final energy-momenta we need to avoid come from our desire to handle the infrared singular interaction in the polaron model. It consists simply of the set $(0, \omega(0)) + \Sigma_{\text{iso}}$. When localizing away from $(0, \omega(0)) + \Sigma_{\text{iso}}$, we cannot emit a boson with zero momentum, hence we will never meet the infrared singularity. This contribution can be omitted if the coupling function g behaves no worse than $|k|^\beta$ at zero, with $\beta > 2 - \nu/2$. In order not to introduce a superfluous exceptional set we define

$$\text{Exc} = \begin{cases} (0, \omega(0)) + \Sigma_{\text{iso}}, & \partial_{k_j} g \notin L^2_{\text{loc}}(\mathbb{R}^\nu), \text{ for some } j \in \{1, \dots, \nu\} \\ \emptyset, & \partial_{k_j} g \in L^2_{\text{loc}}(\mathbb{R}^\nu), \text{ for all } j \in \{1, \dots, \nu\}. \end{cases}$$

We write $\text{Exc}(\xi)$ as usual for the fixed total momentum fibers of the set Exc. Observe that $\text{Exc}(\xi)$, $\xi \in \mathbb{R}^\nu$, are discrete sets and that $\text{Exc} \cap \mathcal{E}^{(1)}$ is a relatively closed subset of $\mathcal{E}^{(1)}$. The latter is a consequence of the HVZ theorem.

Our second theorem is concerned with the structure of the embedded pure point spectrum below the two-boson threshold. That is, the set $\Sigma_{\text{pp}} \cap \mathcal{E}^{(1)}$, cf. (1.25) and (1.28).

Theorem 1.6. *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. Let $\xi \in \mathbb{R}^\nu$. The following holds*

(i) *All eigenvalues in $\sigma_{\text{pp}}(H(\xi)) \cap \mathcal{E}^{(1)}(\xi) \setminus (\mathcal{T}^{(1)}(\xi) \cup \text{Exc}(\xi))$ have finite multiplicity.*

(ii) *The set $\sigma_{\text{pp}}(H(\xi)) \cap \mathcal{E}^{(1)}(\xi)$ is at most countable, with accumulation points at most in $\mathcal{T}^{(1)}(\xi) \cup \text{Exc}(\xi) \cup \{\Sigma_0^{(2)}(\xi)\}$.*

(iii) *The set $(\Sigma_{\text{pp}} \cup \mathcal{T}^{(1)} \cup \text{Exc}) \cap \mathcal{E}^{(1)}$ is a relatively closed subset of $\mathcal{E}^{(1)}$.*

The above theorem follows from standard arguments once we have established a so-called Mourre estimate, cf. Theorem 3.16, away from $\mathcal{T}^{(1)}$ and Exc. An additional consequence of a Mourre estimate is a limiting absorption principle and hence in particular:

Theorem 1.7. *Assume Conditions 1.2 and 1.3, with $n_0 = 2$. Then the fiber Hamiltonians $H(\xi)$ have no singular continuous spectrum below the two-boson threshold, i.e.*

$$\forall \xi \in \mathbb{R}^\nu : \quad \sigma_{\mathrm{sc}}\big(H(\xi)\big) \cap \big(-\infty, \Sigma_0^{(2)}(\xi) \big) = \emptyset.$$

1.4 A Stratification Point of View

The paper is build around the construction of a vector field $v_\xi \in C_0^\infty(\mathbb{R}^\nu)$, from which we construct a self-adjoint one-body operator $a_\xi = \mathrm{i}(v_\xi \cdot \nabla_k + \nabla_k \cdot v_\xi)/2$ and a second quantized observable $A_\xi = \mathrm{d}\Gamma(a_\xi)$ on $\mathcal{F}$. The physical interpretation of v_ξ is that of a relative velocity field, assigning to a momentum k the difference of the velocity of a bound state at total momentum $\xi - k$ and the velocity of an asymptotic boson at momentum k. In Chapter 2 we argue that under our assumptions the fiber Hamiltonians $H(\xi)$ are of class $C^k(A_\xi)$, for $k = 1, 2$, provided (ST2) holds with $n_0 = k$.

In Chapter 3 we construct the vector field v_ξ locally in energy in $\mathcal{E}^{(1)}(\xi)$ and away from thresholds $\mathcal{T}^{(1)}(\xi)$, in such a way that we can deduce at the end of the chapter a Mourre estimate for the pair $H(\xi)$ and A_ξ. From our Mourre estimate, Theorems 1.6 and 1.7 will follow. Theorem 1.5 will be proved in Section 3.1, and ensures that the construction of v_ξ can be done in a sufficiently large energy region inside $\mathcal{E}^{(1)}(\xi)$.

The rest of this section is devoted to an explanation of the construction of the threshold set and the vector field v_ξ, from the point of view of stratifications of proper maps. We will not make any attempt to properly introduce the notions we refer to here, which are entirely standard. For literature on the subject we refer the reader to [30, Section 3] as well as [25, 29]. All stratifications discussed here will satisfy Whitney's regularity condition and the so-called frontier condition: Two strata either have disjoint closures, or one is contained in the closure of the other.

Consider a (real analytic) ambient space $X \subset \mathbb{R}^{2\nu+1}$ given as the following open set

$$X = \Big\{ (k, \xi, E) \in \mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathbb{R} \, \Big| \, E < \Sigma_0^{(2)}(\xi) \Big\}.$$

Along with this we consider fibered ambient spaces

$$X_\xi = \Big\{ (k, E) \in \mathbb{R}^\nu \times \mathbb{R} \, \Big| \, (k, \xi, E) \in X \Big\}.$$

We define (real analytic) projections $\Pi \colon X \to \mathbb{R}^{\nu+1}$ and $\Pi_\xi \colon X_\xi \to \mathbb{R}$ by

$$\Pi(k, \xi, E) = (\xi, E) \quad \text{and} \quad \Pi_\xi(k, E) = E.$$

The projections in fact take values inside the (real analytic) target spaces $Y = \{(\xi, E) \mid E < \Sigma_0^{(2)}(\xi)\}$ and $Y_\xi = (-\infty, \Sigma_0^{(2)}(\xi))$ respectively.

We now introduce what turns out to be semi-analytic subsets of the ambient spaces just defined. Let

$$\mathcal{A} = \Big\{ (k, \xi, E) \in X \, \Big| \, (\xi - k, E - \omega(k)) \in \Sigma_{\mathrm{iso}} \Big\}$$

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and
$$\mathcal{A}_\xi = \Big\{ (k, E) \in X_\xi \,\Big|\, (k, \xi, E) \in \mathcal{A} \Big\}.$$

To see that these sets are semi-analytic we first remark that Σ_{iso} is semi-analytic as a subset of the ambient space $\{(\xi, E) \,|\, E < \Sigma_{\mathrm{ess}}(\xi)\}$. This follows from the analysis of Gérard and Nier [25]. It now follows that $\mathcal{A}$ and $\mathcal{A}_\xi$ are semi-analytic as subsets of X and X_ξ respectively. Here one makes use of $E - \omega(k) < \Sigma_{\mathrm{ess}}(\xi - k)$ provided $(k, \xi, E) \in X$.

That the projections Π and Π_ξ, when restricted to $\mathcal{A}$ and $\mathcal{A}_\xi$ respectively, are proper (preimages of compact sets are compact) is a consequence of [40, Theorem 2.4]. See also (3.1) below.

The splitting of Σ_{iso} into graphs of mass shells $\mathcal{G}_S$ and level crossings, as $S^{\nu-1}$-spheres, is a stratification of Σ_{iso} with strata being graphs of real analytic functions of total momentum. This induces a stratification of X and X_ξ into strata which are again graphs of real analytic functions of (k, ξ) and k respectively.

The thresholds sets $\mathcal{T}^{(1)}(\xi) \cap \mathcal{E}^{(1)}(\xi)$ and $\mathcal{T}^{(1)} \cap \mathcal{E}^{(1)}$ can be interpreted as coming from a Hironaka-stratification of the maps $\Pi_{|\mathcal{A}}$ and $\Pi_{\xi|\mathcal{A}_\xi}$ as follows. The threshold set $\mathcal{T}^{(1)}(\xi) \cap \mathcal{E}^{(1)}(\xi)$ are zero-strata in a stratification of the target space Y_ξ, and $\mathcal{T}^{(1)} \cap \mathcal{E}^{(1)}$ is the union of zero-strata and those d-strata, with $1 \leq d \leq \nu$, transverse to each $\{\xi\} \times Y_\xi$ inside Y.

The strata of the compatible stratification of $\mathcal{A}$ and $\mathcal{A}_\xi$ will again be graphs of real analytic functions and the strata not projecting into the threshold sets are exactly those for which the function, e.g. $S^{(1)}$ from (1.29), defining the strata has nowhere vanishing gradient with respect to k.

The vector field v_ξ, used at total momentum ξ, will be constructed by gluing together k-gradients of the functions generating non-threshold strata in $\mathcal{A}_\xi$, which plays the physical role of a vector field of relative break up velocity of a compound system consisting of an asymptotic boson at momentum k, and an interacting system at momentum $\xi - k$.

In fact we expect/conjecture that a Hironaka-stratification of the projections Π and Π_ξ can be used also without the assumption (ST3) on rotation invariance to construct the threshold sets, and a subsequent analysis of the resulting Whitney stratification of $\mathcal{A}$ and $\mathcal{A}_\xi$ should make it possible, along the same lines as employed in Chapter 3 of this paper, to build a vector field v_ξ that works in a Mourre estimate. However, at this stage where there are still many questions remaining about scattering theory as well as the structure of high energy sectors of the energy-momentum spectrum, we prefer the home-cooked and completely explicit stratification from Chapter 3, where we have full control over all the nuts and bolts. We remark that our insistence on constructing v_ξ as a vector field necessitates some geometrical/technical considerations not met in [24] and [25], where v_ξ was allowed to be a more complicated object. Again, with a view towards the future, we prefer to keep v_ξ as concrete as possible.

We remark that it is a consequence of the analysis in Section 3.1 that $\mathcal{T}^{(1)} \cap \mathcal{E}^{(1)}$ is a semi-analytic subset of the ambient space $\mathcal{E}^{(1)}$. However, we cannot conclude that $\Sigma_{\mathrm{iso}} \cup (\mathcal{T}^{(1)} \cap \mathcal{E}^{(1)})$ is a semi-analytic subset of the ambient space $\{(\xi, E) \,|\, E < \Sigma_0^{(2)}(\xi)\}$. The reason being that we have no control over the manner in which isolated mass shells may hit the continuous energy-momentum spectrum. Such a statement together

with control of possibly embedded non-threshold mass shells, would be a natural input for investigating higher energy sectors.

Another, perhaps more serious, obstacle to analyzing the spectrum above the two-boson threshold, is the possible existence of embedded mass shells. Embedded mass shells below the two-boson threshold would give rise to one-boson scattering states between the two- and three-boson thresholds. Controlling the induced thresholds, in a manner similar to what is done here, necessitates that embedded mass shells are real analytic. Proving this is well beyond current technology [13, 31, 41]. One solution would be to pass to a weak coupling regime where the work of [4, 37] can be used to rule out embedded mass shells below the two-boson threshold altogether.

2   Regularity with Respect to a Conjugate Operator

In this chapter we recall the property of a Hamiltonian being of class $C^k(A)$, with respect to a self-adjoint conjugate operator A. In addition, we verify that our fiber Hamiltonians are of class $C^2(A)$, for conjugate operators on the general form constructed here. We remark that the particular model studied in this paper is in fact quite singular in terms of the $C^k(A)$ classes, in that the free operator $H_0(\xi) = \mathrm{d}\Gamma(\sqrt{k^2 + m^2}) + (\xi - \mathrm{d}\Gamma(k))^2$ is of class $C^2(A)$ but fails to be of class $C^3(A)$, if one chooses A to be e.g. a second quantized generator of dilation. While this does not become a serious issue in the present paper, it will be a more serious obstacle when possible embedded mass shells are analyzed, since the most advanced results to date only hold under a $C^2(A)$ assumption [13, 41]. There are in particular no results allowing one to follow degenerate embedded eigenvalues under perturbations without stronger regularity assumptions.

Additionally, while $H_0(\xi)$ is of class $C^1(A)$, it does not satisfy a Mourre type regularity condition on the first commutator, which manifests itself in the fact that the group W_t generated by the generator of dilation does not preserve the domain of any positive power of $H_0(\xi)$.

The class of conjugate operators we consider in this paper are build from one-body operators of the form

$$a = \tfrac{1}{2}\big\{v \cdot \mathrm{i}\nabla_k + \mathrm{i}\nabla_k \cdot v\big\}, \quad \text{where } v \in C_0^\infty(\mathbb{R}^\nu\backslash\{0\}). \tag{2.1}$$

It is well-known that such a are essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$. Furthermore, the second quantization

$$A = \mathrm{d}\Gamma(a) \tag{2.2}$$

is essentially self-adjoint on $\mathcal{C}$, cf. (1.12). Being self-adjoint, the operator a generates a unitary group $w_t = \mathrm{e}^{\mathrm{i}ta}$ which can be expressed in terms of the flow ψ_t of the autonomous ODE $\dot{\psi}_t = v(\psi_t)$, with $\psi_0(k) = k$. We have the formula

$$(w_t f)(k) = \sqrt{J_t(k)}f(\psi_t(k)), \tag{2.3}$$

where J_t is the determinant of the Jacobi matrix $D_k\psi_t$. By Liouville's formula we have the equation

$$J_t(k) = \mathrm{e}^{\int_0^t \mathrm{Tr}[Dv(\psi_s(k))]\mathrm{d}s}, \tag{2.4}$$

which is uniformly bounded in k. By the functorial properties of second quantization we find that the group $\mathrm{e}^{\mathrm{i}tA}$ generated by A is $\Gamma(w_t)$.

Note that $\psi_t(k) = k$ for $k \notin \mathrm{supp}(v)$ and by boundedness of v we have

$$\sup_{k\in\mathbb{R}^\nu} \|\psi_t(k) - k\| \leq \sup_{k\in\mathbb{R}^\nu} \int_0^t \|v(\psi_s(k))\| \, \mathrm{d}s \leq t\|v\|_\infty. \tag{2.5}$$

Unfortunately we use k here both as a momentum variable and as an integer power for the class $C^k(A)$. Both are standard notation that we prefer to adhere to and trust the reader to distinguish from the context when k denotes momentum and when it denotes an integer power.

2.1 The $C^k(A)$ Classes of Operators

Let A be a self-adjoint operator on a complex Hilbert space $\mathcal{H}$. We recall the notion of $C^k(A)$, $k = 1, 2, \ldots$, regularity from [3].

Definition 2.1 (The $C^k(A)$ class of operators). Let A be a self-adjoint operator on $\mathcal{H}$, with domain $\mathcal{D}(A)$.

(i) Let $B \in \mathcal{B}(\mathcal{H})$ be a bounded operator and $k \in \mathbb{N}$. We say that $B \in C^k(A)$ if, for all $\phi \in \mathcal{H}$, the map $\mathbb{R} \ni s \mapsto \mathrm{e}^{-\mathrm{i}sA} B \mathrm{e}^{\mathrm{i}sA} \phi \in \mathcal{H}$ is k times continuously differentiable.

(ii) Let H be a self-adjoint operator on $\mathcal{H}$. We say that H is *of class $C^k(A)$* if there exists $z \in \mathbb{C}\backslash\sigma(H)$ such that $(H - z)^{-1} \in C^k(A)$.

Note that $C^k(A)$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, cf. [3, 21]. Lets make some remarks. The requirement that A and H be self-adjoint can be relaxed considerably [21], something we will however not need. The requirement in (ii) that $(H - z)^{-1} \in C^k(A)$ for *some* z in the resolvent set of H, is equivalent to $(H - z)^{-1} \in C^k(A)$ for *all* such z. Finally, we note that if the bounded operator B is itself self-adjoint then $B \in C^k(A)$ if and only if B is of class $C^k(A)$.

The results in this section are recalled from the literature without proofs, for which we refer the reader to [3, 18, 20, 21].

We remind the reader that there are several equivalent formulations for a bounded operator B to be of class $C^1(A)$. We collect some as a lemma.

Lemma 2.2. *Let $B \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

(i) $B \in C^1(A)$.

(ii) *It holds that $\liminf\limits_{s\to 0^+} \frac{1}{s}\|\mathrm{e}^{-\mathrm{i}sA} B \mathrm{e}^{\mathrm{i}sA} - B\| < \infty$.*

(iii) *There is a constant C such that for all $\psi, \phi \in \mathcal{D}(A)$,*

$$|\langle A\psi, B\phi\rangle - \langle B\psi, A\phi\rangle| \leq C\|\phi\|\|\psi\|. \tag{2.6}$$

(iv) *B maps $\mathcal{D}(A)$ into itself and $AB - BA\colon \mathcal{D}(A) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$.*

If $B \in C^1(A)$, the commutator $[A, B]$, which is a priori only defined as a form on $\mathcal{D}(A)$, can by Lemma 2.2 be extended to $\mathcal{H}$. We write $[A, B]^\circ$ for the unique bounded operator on $\mathcal{H}$ extending the quadratic form $[A, B]$.

If $B = (H - z)^{-1}$, with H being self-adjoint and of class $C^1(A)$, then we can compute the form $[B, A]$ on $\mathcal{D}(A)$ and find that $[B, A] = -(H - z)^{-1}[H, A](H - z)^{-1}$, which is meaningful due to Lemma 2.2 (iv). Here $[H, A]$ is read as a form on $\mathcal{D}(H) \cap \mathcal{D}(A)$. Since the left-hand side extends by continuity to the bounded operator $[A, B]^\circ$, we observe that $[H, A]$ extends from $\mathcal{D}(A) \cap \mathcal{D}(H)$ to a bounded form on $\mathcal{D}(H)$, which we can and will identify with an operator $[H, A]^\circ \in \mathcal{B}(\mathcal{H}_1; \mathcal{H}_{-1})$. Here we used the standard scale of space associated with H. That is $\mathcal{H}_s$, $|s| \leq 1$, is the completion of $\mathcal{D}(H)$ with respect to the norm $\|\psi\|_s = \|(|H| + 1)^s \psi\|$. We remark that if H is of class $C^1(A)$, then

$$\mathcal{D}(H) \cap \mathcal{D}(A) \text{ is dense in } \mathcal{D}(H) \tag{2.7}$$

and hence, the extension $[H, A]^\circ$ of the form $[H, A]$ is unique.

We will need the following well-known lemma

Lemma 2.3. *If H is a self-adjoint operator of class $C^1(A)$ and $W_t = \mathrm{e}^{\mathrm{i}tA}$ is the unitary group associated to the self-adjoint operator A, then we have*

$$\forall \psi, \phi \in \mathcal{D}(H): \quad \langle \psi, \mathrm{i}[H, A]^\circ \phi \rangle = \lim_{s \to 0} \tfrac{1}{s} \big(\langle H\psi, W_s \phi \rangle - \langle \psi, W_s H \phi \rangle \big).$$

2.2 $H(\xi)$ is of Class $C^2(A)$

In this section we state and prove a $C^2(A)$ regularity result for the fiber Hamiltonians with respect to conjugate operators of the type (2.2). Since this is of independent interest, we formulate precise conditions under which our results hold, conditions that are implied by a combination of Conditions 1.2 and 1.3.

Condition 2.4. We say that (ω, Ω, g) satisfies a C^k-condition, $k = 1, 2$, if there exists $s_\Omega \in [0, 2]$ such that

(Ck1) $\omega, \Omega \in C^k(\mathbb{R}^\nu)$.

(Ck2) $\Omega \geq 0$ and $\inf_{\eta \in \mathbb{R}^\nu} \omega(\eta) > 0$.

(Ck3) $\exists C > 0$ such that $\Omega(\eta) \geq C^{-1}\langle \eta \rangle^{s_\Omega} - C$ and $|\partial^\alpha \Omega(\eta)| \leq C\langle \eta \rangle^{s_\Omega - |\alpha|}$, $|\alpha| \leq k$.

(Ck4) $g \in L^2(\mathbb{R}^\nu)$ admits k distributional derivatives with $\partial^\alpha g \in L^2_{\mathrm{loc}}(\mathbb{R}^\nu)$, $|\alpha| \leq k$.

Note that due to the ξ- and g-independence of the domain of $H(\xi)$, and the equivalence of the associated $\|\cdot\|_s$ norms, the scale of spaces $\mathcal{H}_s$ are ξ- and g-independent. To avoid ambiguity we use $H_0(0)$ to define the s-norms.

Proposition 2.5. *Suppose (ω, Ω, g) satisfies a C^k-condition with*

$k = 1$: *Then for all $\xi \in \mathbb{R}^\nu$ the fiber Hamiltonian $H(\xi)$ is of class $C^1(A)$ and we have the explicit form of the commutator*

$$\mathrm{i}[H(\xi), A]^\circ = \mathrm{d}\Gamma(v \cdot \nabla\omega) - \mathrm{d}\Gamma(v) \cdot \nabla\Omega(\xi - \mathrm{d}\Gamma(k)) - \phi(\mathrm{i}ag).$$

Furthermore

$$\forall t \in \left[0, \tfrac{1}{2}\right]: \quad [H(\xi), A]^\circ \in \mathcal{B}\big(\mathcal{H}_{1-t}, \mathcal{H}_{-\frac{1}{2}-t}\big). \tag{2.8}$$

$k = 2$: *Then for all $\xi \in H(\xi)$ the fiber Hamiltonian $H(\xi)$ is of class $C^2(A)$.*

Proof. Fix a $\xi \in \mathbb{R}^\nu$. For the purpose of this proof we abbreviate $H_0 = H_0(\xi)$ and $H = H(\xi)$. Recall the notation $\mathcal{D}$ (1.13) for the common domain of $H_0(\xi)$ and $H(\xi)$, $\xi \in \mathbb{R}^\nu$, and $\mathcal{C}$ (1.12) for the common core of $H(\xi)$ and A.

We begin with the case '$k = 1$' and observe that

$$\forall z \in \mathbb{C} \backslash \sigma(H_0) : \quad (H_0 - z)^{-1} \mathcal{C} = \mathcal{C}, \tag{2.9}$$

which ensures that the following computation, for $\psi, \varphi \in \mathcal{C}$,

$$\left\langle \psi, \left[(H_0 + 1)^{-1}, A \right] \varphi \right\rangle = -\left\langle (H_0 + 1)^{-1} \psi, [H_0, A](H_0 + 1)^{-1} \varphi \right\rangle$$

is meaningful. As a form on $\mathcal{C}$ one can easily compute that

$$F_\mathcal{C} := \mathrm{i}[H_0, A]_{|\mathcal{C}} = \mathrm{d}\Gamma(v \cdot \nabla\omega) - \mathrm{d}\Gamma(v) \cdot \nabla\Omega(\xi - \mathrm{d}\Gamma(k)).$$

Since $v \cdot \nabla\omega$ is uniformly bounded, cf. (Ck1), we can bound the first term by a number operator, and hence due to (Ck2) by H_0. Likewise, we can bound $\mathrm{d}\Gamma(v)$ by H_0, and the operator $\nabla\Omega(\xi - \mathrm{d}\Gamma(k))$ can due to (Ck3) be controlled by $H_0^{1/2}$. Recall that $s_\Omega \leq 2$. This yields the following bound for all $\tilde\psi, \tilde\varphi \in \mathcal{C}$

$$\left| \left\langle \tilde\psi, \mathrm{i}[H_0, A]\tilde\varphi \right\rangle \right| \leq C \left(\left\| (H_0 + 1)\tilde\psi \right\|^2 + \left\| (H_0 + 1)^{\frac{1}{2}}\tilde\varphi \right\|^2 \right).$$

Hence we find that

$$\forall \psi, \varphi \in \mathcal{C} : \quad \left| \left\langle \psi, \mathrm{i}\left[(H_0 + 1)^{-1}, A \right] \varphi \right\rangle \right| \leq C \left(\left\| \psi \right\|^2 + \left\| (H_0 + 1)^{-\frac{1}{2}}\varphi \right\|^2 \right).$$

Since $\mathcal{C}$ is a core for A, this proves that H_0 is of class $C^1(A)$ and hence $\mathrm{i}[H_0, A]$ has a unique extension by continuity from $\mathcal{D} \cap \mathcal{D}(A)$ to a bounded form $\mathrm{i}[H_0, A]^\circ$ on $\mathcal{D}$. We now observe, appealing to the bound, that the form $F_\mathcal{C}$ extends continuously to a bounded form $F_\mathcal{D}$ on $\mathcal{D}$, defined by the same expression. Since $F_\mathcal{D}$ and $\mathrm{i}[H_0, A]^\circ$ coincide on $\mathcal{C}$, they must also be identical as forms on $\mathcal{D}$. Finally we observe by symmetry and interpolation that

$$\forall t \in \left[0, \tfrac{1}{2} \right] : \quad \mathrm{i}[H_0, A]^\circ \in \mathcal{B}\left(\mathcal{H}_{1-t}, \mathcal{H}_{-\frac{1}{2}-t} \right). \tag{2.10}$$

We now turn to the full fiber Hamiltonian H. Since $\phi(g)$ is $H_0^{1/2}$ bounded, we can choose $\lambda > 0$ large enough such that $\| \phi(g)R_0(\lambda) \| < 1$, where $R_0(\lambda) = (H_0 + \lambda)^{-1}$. We can now write

$$R(\lambda) := (H - \lambda)^{-1} = R_0(\lambda)(\mathbb{1} + \phi(g)R_0(\lambda))^{-1}. \tag{2.11}$$

Recall that $C^1(A)$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, and $S \in C^1(A)$ invertible implies $S^{-1} \in C^1(A)$ (see [21, Corollary 2.10]). Hence it suffices to show that $\phi(v)R_0(\lambda) \in C^1(A)$ in order to prove that H is of class $C^1(A)$.

Using Lemma 2.2 (iv) we conclude that for $\varphi \in \mathcal{C} \subset \mathcal{D}(A)$ we have $R_0(\lambda)A\varphi = AR_0(\lambda)\varphi - [R_0(\lambda), A]^\circ\varphi$. Calculate for $\psi, \varphi \in \mathcal{C}$

$$\begin{aligned}
&\big\langle \phi(g)\psi, R_0(\lambda)\mathrm{i}A\varphi \big\rangle - \big\langle A\psi, \mathrm{i}\phi(g)R_0(\lambda)\varphi \big\rangle \\
&= \big\langle \phi(g)\psi, \mathrm{i}AR_0(\lambda)\varphi \big\rangle - \big\langle A\psi, \mathrm{i}\phi(g)R_0(\lambda)\varphi \big\rangle \\
&\quad - \big\langle \phi(g)\psi, R_0(\lambda)\mathrm{i}[H_0, A]R_0(\lambda)\varphi \big\rangle \\
&= -\big\langle \psi, \phi(\mathrm{i}ag)R_0(\lambda)\varphi \big\rangle - \big\langle \psi, \phi(g)R_0(\lambda)\mathrm{i}[H_0, A]^\circ R_0(\lambda)\varphi \big\rangle.
\end{aligned} \tag{2.12}$$

By using (2.10) and $H_0^{1/2}$-boundedness of $\phi(g)$, it follows that for all $\psi, \varphi \in \mathcal{C}$

$$\left| \big\langle \psi, \phi(g)R_0(\lambda)\mathrm{i}A\varphi \big\rangle - \big\langle A\psi, \mathrm{i}\phi(g)R_0(\lambda)\varphi \big\rangle \right| \le C\big(\|\psi\|^2 + \|\varphi\|^2 \big),$$

for some $C > 0$. Since $\mathcal{C}$ is a core for A this bound extends to $\mathcal{D}(A)$ and hence by Lemma 2.2 we conclude that $\phi(g)R_0(\lambda) \in C^1(A)$. To verify the formula for $\mathrm{i}[H, A]^\circ$ it now suffices to verify the formula as a form on $\mathcal{C}$ as we did for $[H_0, A]^\circ$. The perturbation contributes an H-bounded term, so it is (2.10) that is the most singular contribution and hence (2.8) holds true. This completes the proof for the case '$k = 1$'.

We turn to the case '$k = 2$'. Having established that $R(\lambda) \in C^1(A)$, one can repeat the argument around (2.12) above to conclude that $\phi(g)R(\lambda) \in C^1(A)$. Since $(\phi(g)R(\lambda))^* = \overline{R(\lambda)\phi(g)}$, the closure of $R(z)\phi(g)$ defined a priori on $\mathcal{D}$, we get

$$\phi(g)R(\lambda) \text{ and } \overline{R(\lambda)\phi(g)} \text{ are of class } C^1(A). \tag{2.13}$$

Compute as an identity between bounded operators

$$\begin{aligned}
\mathrm{i}[R(\lambda), A]^\circ &= -R(\lambda)\big\{ \mathrm{i}[H_0, A]^\circ - \phi(\mathrm{i}ag) \big\}R(\lambda) \\
&= -R(\lambda)(H_0 + \lambda)R_0(\lambda)\mathrm{i}[H_0, A]^\circ R(\lambda) + R(\lambda)\phi(\mathrm{i}ag)R(\lambda) \\
&= -\overline{R(\lambda)\phi(g)}\,\mathrm{i}[R_0(\lambda), A]^\circ(\mathbb{1} + \phi(g)R_0(\lambda))^{-1} + R(\lambda)\phi(\mathrm{i}ag)R(\lambda) \\
&\quad + \mathrm{i}[R_0(\lambda), A]^\circ(\mathbb{1} + \phi(g)R_0(\lambda))^{-1},
\end{aligned}$$

where the last equality made use of (2.11). We conclude that to show that H is of class $C^2(A)$, it suffices to show that $[R_0(\lambda), A]^\circ$ and $R(\lambda)\phi(\mathrm{i}ag)R(\lambda)$ are both of class $C^1(A)$.

We begin with $[R_0(\lambda), A]^\circ$. Compute for $\psi, \varphi \in \mathcal{C}$

$$\begin{aligned}
&\big\langle \psi, [R_0(\lambda), A]^\circ A\varphi \big\rangle - \big\langle A\psi, [R_0(\lambda), A]^\circ\varphi \big\rangle \\
&= \big\langle \psi, R_0(\lambda)[[H_0, A]^\circ, A]R_0(\lambda)\varphi \big\rangle \\
&\quad - 2\big\langle \psi, R_0(\lambda)[H_0, A]^\circ R_0(\lambda)[H_0, A]^\circ R_0(\lambda)\varphi \big\rangle,
\end{aligned} \tag{2.14}$$

where we used again (2.9) and $A\mathcal{C} \subset \mathcal{C}$ to perform the computations. The form $[[H_0, A]^\circ, A]$ should be understood as a form on $\mathcal{C}$ where it can be computed to be

$$\begin{aligned}
F'_{\mathcal{C}} &:= [[H_0, A]^\circ, A] \\
&= -\mathrm{d}\Gamma(\langle v, (\nabla^2\omega)v \rangle) - \mathrm{d}\Gamma(\langle (\nabla v)v, \nabla\omega \rangle) \\
&\quad - \langle \mathrm{d}\Gamma(v), \nabla^2\Omega(\xi - \mathrm{d}\Gamma(k))\mathrm{d}\Gamma(v) \rangle + \mathrm{d}\Gamma((\nabla v)v) \cdot \nabla\Omega(\xi - \mathrm{d}\Gamma(k)).
\end{aligned}$$

The two first terms in $F'_\mathcal{C}$ are controlled by the number operator, cf. (Ck1), and hence by H_0. The third term is the most singular and require a square of the number operator to bound, cf. (Ck3), and hence is just bounded as a form on $\mathcal{D}$. The fourth and final term can be controlled by $H_0^{3/2}$. In conclusion we find the existence of a $C > 0$ such that

$$\forall \tilde\psi, \tilde\varphi \in \mathcal{C}: \quad \left|\left\langle \tilde\psi, F'_\mathcal{C} \tilde\varphi \right\rangle\right| \leq C\left(\left\|(H_0 + \lambda)\tilde\psi\right\|^2 + \left\|(H_0 + \lambda)\tilde\varphi\right\|^2\right)$$

We can now estimate the left-hand side in (2.14), cf. also (2.10), and find that

$$\forall \psi, \varphi \in \mathcal{C}: \quad \left|\left\langle \psi, [[R_0, A]^\circ, A]\varphi \right\rangle\right| \leq C\left(\|\psi\|^2 + \|\varphi\|^2\right),$$

for some $C > 0$. Since $\mathcal{C}$ is a core for A, we have thus established that $[R_0(\lambda), A]^\circ \in C^1(A)$. Note that to control the last term in (2.14) using (2.10), one has to make full use of all the free resolvents.

It remains to consider $R(\lambda)\phi(\mathrm{i}ag)R(\lambda)$. Writing

$$R(\lambda)\phi(\mathrm{i}ag)R(\lambda) = R_0(\lambda)\phi(\mathrm{i}ag)R_0(\lambda) + 2\mathrm{Re}\left\{R(\lambda)\phi(g)R_0(\lambda)\phi(\mathrm{i}ag)R_0(\lambda)\right\}$$
$$+ R(\lambda)\phi(g)R_0(\lambda)\phi(\mathrm{i}ag)R_0(\lambda)\phi(g)R(\lambda),$$

we appeal to (2.13) and conclude that it suffices to show that $R_0(\lambda)\phi(\mathrm{i}ag)R_0(\lambda)$ is of class $C^1(A)$. Here we can compute as a form on $\mathcal{C}$ for one last time

$$[R_0(\lambda)\phi(\mathrm{i}ag)R_0(\lambda), A] = -\mathrm{i}R_0(\lambda)\phi(a^2g)R_0(\lambda)$$
$$+ [R_0(\lambda), A]^\circ \phi(\mathrm{i}ag)R_0(\lambda) + R_0(\lambda)\phi(\mathrm{i}ag)[R_0(\lambda), A]^\circ.$$

By (Ck4) $ag, a^2g \in L^2(\mathbb{R}^\nu)$ and the right-hand side clearly extends to a bounded operator and we are done. $\qquad\square$

2.3 Extended Operators

Below we will make use of the following two simple observations, the proofs of which are left to the reader.

Lemma 2.6. *Let H, A_1, A_2, A_3 be self-adjoint operators such that $H \in C^1(A_j)$, $j = 1, 2$. Suppose furthermore that there exists a dense set $\mathcal{D}_0$, with the following properties:*

(i) *$\mathcal{D}_0 \subset \mathcal{D}(A_j)$, $j = 1, 2, 3$.*

(ii) *$\mathcal{D}_0$ is a core for A_3.*

(iii) *For all $\psi \in \mathcal{D}_0$ we have $A_3\psi = A_1\psi + A_2\psi$.*

Then H is of class $C^1(A_3)$ and $[H, A_3]^\circ = [H, A_1]^\circ + [H, A_2]^\circ$ as an identity between elements of $\mathcal{B}(\mathcal{H}_1; \mathcal{H}_{-1})$.

Lemma 2.7. *Let $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$ be a family of Hilbert spaces, and suppose that for each $n \in \mathbb{N}$ we are given two self-adjoint operators H_n and A_n on $\mathcal{H}_n$, with H_n of class $C^1(A_n)$. Then $H = \oplus_{n=1}^\infty H_n$ is of class $C^1(A)$, with $A = \oplus_{n=1}^\infty A_n$, as self-adjoint operators on $\mathcal{H} = \oplus_{n=1}^\infty \mathcal{H}_n$. Furthermore $[H, A]^\circ = \oplus_{n=1}^\infty [H_n, A_n]^0$ under the identification $\mathcal{B}(\mathcal{H}_1; \mathcal{H}_{-1}) = \oplus_{n=1}^\infty \mathcal{B}(\mathcal{H}_{n;1}; \mathcal{H}_{n;-1})$.*

In the proposition below $v, \tilde{v} \in C_0^\infty(\mathbb{R}^\nu \backslash \{0\})$, $\tilde{a} = (\tilde{v} \cdot i\nabla_k + i\nabla_k \cdot \tilde{v})/2$ and $\widetilde{A}^{(\ell)} = A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} + \mathbb{1}_{\mathcal{F}} \otimes d\Gamma^{(\ell)}(\tilde{a})$. The tilde-free versions are as usual constructed using v.

Proposition 2.8. *Suppose (ω, Ω, g) satisfies a C^1-condition. Then for all $\ell \in \mathbb{N}$ and $\xi \in \mathbb{R}^\nu$ the following holds: $H^{(\ell)}(\xi)$ is of class $C^1(A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}})$, $C^1(\mathbb{1}_{\mathcal{F}} \otimes d\Gamma^{(\ell)}(\tilde{a}))$ and $C^1(\widetilde{A}^{(\ell)})$, with*

$$i\big[H^{(\ell)}(\xi), A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}}\big]^\circ = \int_{\mathbb{R}^{\ell\nu}}^\oplus i\big[H^{(\ell)}(\xi; \underline{k}), A\big]^\circ dk,$$

$$i\big[H^{(\ell)}(\xi), \mathbb{1}_{\mathcal{F}} \otimes d\Gamma^{(\ell)}(\tilde{a})\big]^\circ = \int_{\mathbb{R}^{\ell\nu}}^\oplus \sum_{j=1}^\ell \tilde{v}(k_j)$$

$$\cdot \Big(\nabla\omega(k_j) - \nabla\Omega\big(\xi - \textstyle\sum_{j=1}^\ell k_j - d\Gamma(k)\big)\Big)dk,$$

$$i\big[H^{(\ell)}(\xi), \widetilde{A}^{(\ell)}\big]^\circ = i\big[H^{(\ell)}(\xi), A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}}\big]^\circ + i\big[H^{(\ell)}(\xi), \mathbb{1}_{\mathcal{F}} \otimes d\Gamma^{(\ell)}(\tilde{a})\big]^\circ.$$

Furthermore, $H^{\mathrm{x}}(\xi)$ is of class $C^1(A^{\mathrm{x}})$ and

$$i\big[H^{\mathrm{x}}(\xi), A^{\mathrm{x}}\big]^\circ = [H(\xi), A]^\circ \oplus \Big\{\bigoplus_{\ell=1}^\infty i\big[H^{(\ell)}(\xi), A^{(\ell)}\big]^\circ\Big\}$$

$$= d\Gamma^{\mathrm{x}}(v \cdot \nabla\omega) - d\Gamma^{\mathrm{x}}(v) \cdot \nabla\Omega\big(\xi - d\Gamma^{\mathrm{x}}(k)\big) - \phi(iag) \otimes \mathbb{1}_{\mathcal{F}}.$$

Remark 2.9. For the purpose of the proof below we abbreviate $k^{(\ell)} = k_1 + \cdots + k_\ell$, for vectors $\underline{k} = (k_1, \ldots, k_\ell) \in \mathbb{R}^{\ell\nu}$. Note that $[H^{(\ell)}(\xi; \underline{k}), A]^\circ = [H(\xi - k^{(\ell)}), A]^\circ$ can be computed using Proposition 2.5 '$k = 1$'. $\diamond$

Proof. We only prove that $H^{(\ell)}(\xi)$ is of class $C^1(A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}})$ and $C^1(\mathbb{1}_{\mathcal{F}} \otimes d\Gamma^{(\ell)}(\tilde{a}))$. The $C^1(\widetilde{A}^{(\ell)})$ property then follows from Lemma 2.6 and that $H^{\mathrm{x}}(\xi)$ is of class $C^1(A^{\mathrm{x}})$ follows from Lemma 2.7 after choosing $\tilde{v} = v$. The expressions can subsequently be easily confirmed by computations on a suitable core for $H^{(\ell)}(\xi)$.

Let $\ell \in \{1, 2, 3, \ldots\}$ and $\xi \in \mathbb{R}^\nu$. We begin by showing that $H^{(\ell)}(\xi)$ is of class $C^1(A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}})$, where we identify $\mathcal{F} \otimes \mathcal{F}^{(\ell)}$ with $L^2_{\mathrm{sym}}(\mathbb{R}^{\ell\nu}; \mathcal{F})$.

Let
$$\mathcal{C}^{(\ell)} = \Big\{f \in C_{0,\mathrm{sym}}(\mathbb{R}^{\ell\nu}; \mathcal{F}) \,\big|\, \forall \underline{k} \in \mathbb{R}^{\ell\nu} : \ f(\underline{k}) \in \mathcal{C}\Big\}.$$

Here $C_{0,\mathrm{sym}}(\mathbb{R}^{\ell\nu}; \mathcal{F})$ denotes the continuous and compactly supported $\mathcal{F}$-valued functions, symmetric under permutation of the ℓ variables. Clearly $\mathcal{C}^{(\ell)}$ is a core for $A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}}$. Pick a $\lambda < \Sigma_0$, cf. (1.15). Since

$$(H^{(\ell)}(\xi) - \lambda)^{-1} = \int_{\mathbb{R}^{\ell\nu}}^\oplus (H^{(\ell)}(\xi; \underline{k}) - \lambda)^{-1} dk,$$

we observe that for $f \in \mathcal{C}^{(\ell)}$ we have

$$\Big((H^{(\ell)}(\xi) - \lambda)^{-1}f\Big)(\underline{k}) = \Big(H(\xi - k^{(\ell)}) + \sum_{j=1}^\ell \omega(k_j) - \lambda\Big)^{-1} f(\underline{k}).$$

Hence by Lemma 2.2 (iv) we conclude that

$$(H^{(\ell)}(\xi) - \lambda)^{-1} \mathcal{C}^{(\ell)} \subset \left\{ f \in C_{0,\mathrm{sym}}(\mathbb{R}^{\ell\nu}; \mathcal{F}) \,\middle|\, \forall \underline{k} \in \mathbb{R}^{\ell\nu} : \ f(\underline{k}) \in \mathcal{D}(A) \cap \mathcal{D} \right\}$$
$$=: \widetilde{\mathcal{C}}^{(\ell)}. \tag{2.15}$$

For $f, g \in \widetilde{\mathcal{C}}^{(\ell)}$ we compute

$$\left\langle f, \left[H^{(\ell)}(\xi), A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} \right] g \right\rangle = \int_{\mathbb{R}^{\ell\nu}} \left\langle f(\underline{k}), \left[H(\xi - k^{(\ell)}) + \sum_{j=1}^{\ell} \omega(k_j), A \right] g(\underline{k}) \right\rangle \mathrm{d}k$$
$$= \int_{\mathbb{R}^{\ell\nu}} \left\langle f(\underline{k}), \left[H(\xi - k^{(\ell)}), A \right]^{\circ} g(\underline{k}) \right\rangle \mathrm{d}k.$$

Since

$$M_1 := \sup_{\eta \in \mathbb{R}^{\nu}} \left\| (H(\eta) - \lambda)^{-1} [H(\eta), A]^{\circ} (H(\eta) - \lambda)^{-1} \right\| < \infty,$$
$$M_2 := \sup_{\xi \in \mathbb{R}^{\nu}, \underline{k} \in \mathbb{R}^{\ell\nu}} \left\| (H(\xi - k^{(\ell)}) - \lambda)(H^{(\ell)}(\xi; \underline{k}) - \lambda)^{-1} \right\| < \infty, \tag{2.16}$$

we can finally estimate for $f, g \in \mathcal{C}^{(\ell)}$

$$\left| \left\langle f, \left[(H^{(\ell)}(\xi) - \lambda)^{-1}, A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} \right] g \right\rangle \right|$$
$$\leq \int_{\mathbb{R}^{\ell\nu}} \left| \left\langle (H^{(\ell)}(\xi; \underline{k}) - \lambda)^{-1} f(\underline{k}), [H(\xi - k^{(\ell)}), A]^{\circ} (H^{(\ell)}(\xi; \underline{k}) - \lambda)^{-1} g(\underline{k}) \right\rangle \right| \mathrm{d}k$$
$$\leq M_1 M_2^2 \|f\| \|g\|.$$

That $\mathcal{C}^{(\ell)}$ is a core for $A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}}$ now implies that $H^{(\ell)}(\xi)$ is of class $C^1(A \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}})$.

By Lemmata 2.6 and 2.7 it now suffices to show that the fiber Hamiltonian $H^{(\ell)}(\xi)$ is of class $C^1(\mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma^{(\ell)}(\tilde{a}))$. Denote by $\widetilde{w}_t$ the group $\mathrm{e}^{\mathrm{i}t\tilde{a}}$ generated by $\tilde{a}$. Then $\widetilde{w}_t^{(\ell)} = \mathbb{1}_{\mathcal{F}} \otimes \Gamma^{(\ell)}(\widetilde{w}_t)$ is the group generated by $\mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma^{(\ell)}(\tilde{a})$. If we denote by $\tilde{\psi}_t$ the globally defined flow generated by the ODE $\dot{\psi}_t = \tilde{v}(\psi_t)$ we can write $(\widetilde{w}_t f)(y) = \sqrt{\widetilde{J}_t} f(\tilde{\psi}_t(y))$, where $\widetilde{J}_t$ is the Jacobi determinant. See (2.3) and (2.4).

We introduce a bit of notation. Given $\underline{k} \in \mathbb{R}^{\ell\nu}$ we write $\tilde{\psi}_t^{(\ell)}(\underline{k}) = \tilde{\psi}_t(k_1) + \cdots + \tilde{\psi}_t(k_\ell)$. We compute as a form on $\widetilde{\mathcal{C}}^{(\ell)}$, cf. (2.15),

$$\widetilde{w}_{-t}^{(\ell)} \left[H^{(\ell)}(\xi), \widetilde{w}_t^{(\ell)} \right]$$
$$= \widetilde{w}_{-t}^{(\ell)} \left(\Omega\left(\xi - k^{(\ell)} - \mathrm{d}\Gamma(k) \right) + \sum_{j=1}^{\ell} \omega(k_j) \right) \widetilde{w}_t^{(\ell)} - \Omega\left(\xi - k^{(\ell)} - \mathrm{d}\Gamma(k) \right) - \sum_{j=1}^{\ell} \omega(k_j)$$
$$= \Omega\left(\xi - \tilde{\psi}_t^{(\ell)}(\underline{k}) - \mathrm{d}\Gamma(k) \right) - \Omega\left(\xi - k^{(\ell)} - \mathrm{d}\Gamma(k) \right) + \sum_{j=1}^{\ell} \left(\omega\left(\tilde{\psi}_t(k_j) \right) - \omega(k_j) \right)$$
$$= - \int_0^t \nabla\Omega\left(\xi - \psi_t^{(\ell)}(\underline{k}) - \mathrm{d}\Gamma(k) \right) \cdot \sum_{j=1}^{\ell} \tilde{v}\left(\tilde{\psi}_s(k) \right) \mathrm{d}s$$
$$+ \sum_{j=1}^{\ell} \int_0^t \nabla\omega\left(\tilde{\psi}_s(k_j) \right) \cdot \tilde{v}\left(\tilde{\psi}_s(k_j) \right) \mathrm{d}s. \tag{2.17}$$

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Estimate, as a fiber operator pointwise in $\underline{k}$,

$$\left\| \nabla\Omega\big(\xi - \psi_s^{(\ell)}(\underline{k}) - \mathrm{d}\Gamma(k)\big)\big(\Omega\big(\xi - k^{(\ell)} - \mathrm{d}\Gamma(k)\big) + 1\big)^{-\frac{1}{2}} \right\|$$

$$\le C \sup_{\eta\in\mathbb{R}^\nu} \frac{\big\langle \xi - \psi_s^{(\ell)}(\underline{k}) - \eta \big\rangle^{s_\Omega - 1}}{\big\langle \xi - k^{(\ell)} - \eta \big\rangle^{s_\Omega/2}} \le (1 + |s|)\widetilde{C}$$

uniformly in s and k. Here we used (2.5) in the last step. Appealing to (2.5) again and the C^1-condition, cf. Condition 2.4, we observe that the right-hand side of (2.17) is $(H(\xi - k^{(\ell)}) - \lambda)^{1/2}$-bounded uniformly in $\underline{k}$. From this observation it is now clear that as a form on $\mathcal{C}^{(\ell)}$

$$\tfrac{1}{t}\big[(H^{(\ell)}(\xi) - \lambda)^{-1}, \widetilde{w}_t^{(\ell)}\big] = -(H^{(\ell)}(\xi) - \lambda)^{-1}\big\{\tfrac{1}{t}\big[H^{(\ell)}(\xi), \widetilde{w}_t^{(\ell)}\big](H^{(\ell)}(\xi) - \lambda)^{-1}\big\},$$

and the term in brackets extends to a bounded operator uniformly bounded in $0 < |t| \le 1$. Cf. (2.16). It thus follows from Lemma 2.2 (ii) that $H^{(\ell)}(\xi)$ is of class $C^1(\mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma^{(\ell)}(\tilde{a}))$. $\qquad\square$

We end this section by formulating and proving a virial theorem which will be used to extract the effective free dynamics induced by mass shells. Similar virial theorems were used in [9, 25]. In the following $\mathcal{U} \subset \mathbb{R}^m$ is open and $\mathcal{H}$ a complex separable Hilbert space, with dense subspace $\mathcal{D}_0$. Suppose $\{H(x)\}_{x\in\mathcal{U}}$ is a family of N-measurable operators, essentially self-adjoint on $\mathcal{D}_0$, cf. Appendix A. Then $H := \int_{\mathcal{U}}^{\oplus} H(x)\mathrm{d}x$, a priori defined on $\{f \in L^2(\mathcal{U}; \mathcal{H}) \mid x \in \mathcal{U} : f(x) \in \mathcal{D}_0 \text{ a.e.}\}$, is essentially self-adjoint. Let A be a self-adjoint operator on $\mathcal{H}$, and $a = \mathbb{1}\otimes\frac{1}{2}\{v \cdot \mathrm{i}\nabla_x + \mathrm{i}\nabla_x \cdot v\}$, with $v \in C_0^\infty(\mathcal{U}; \mathbb{R}^m)$. Then $A^{(1)} = A \otimes \mathbb{1}_{L^2(\mathcal{U})} + \mathbb{1}_{\mathcal{H}} \otimes a$ is self-adjoint as an operator on $\mathcal{H} \otimes L^2(\mathcal{U})$, which we identify as usual with $L^2(\mathcal{U}; \mathcal{H})$.

Theorem 2.10. *Let $E \in C^1(\mathcal{U})$, with $E(x) \in \sigma_{\mathrm{pp}}(H(x))$ for all $x \in \mathcal{U}$. Suppose H is of class $C^1(A^{(1)})$ and the commutator fibers, i.e. $\mathrm{i}[H, A^{(1)}]^\circ = \int_{\mathcal{U}}^{\oplus} \mathrm{i}[H, A^{(1)}]^\circ(x)\mathrm{d}x$. Then for almost every $x \in \mathcal{U}$*

$$\mathbb{1}_{\{E(x)\}}(H(x))\mathrm{i}[H, A^{(1)}]^\circ(x)\,\mathbb{1}_{\{E(x)\}}(H(x)) = v(x) \cdot \nabla E(x)\,\mathbb{1}_{\{E(x)\}}(H(x)). \quad (2.18)$$

Remark 2.11. By the assumption $\mathrm{i}[H, A^{(1)}]^\circ$ being fibered is meant the existence of a family of operators $x \to \mathrm{i}[H, A^{(1)}]^\circ(x) \in \mathcal{B}(\mathcal{H}_{x,1}; \mathcal{H}_{x,-1})$, with

$$\mathcal{U} \ni x \to B(x) = (H(x) - \mathrm{i})^{-1}\mathrm{i}[H, A^{(1)}]^\circ(x)(H(x) - \mathrm{i})^{-1}$$

weakly measurable, and $(H - \mathrm{i})^{-1}\mathrm{i}[H, A^{(1)}]^\circ(H - \mathrm{i})^{-1} = \int_{\mathcal{U}}^{\oplus} B(x)\mathrm{d}x$. Note that it follows from the discussion in Appendix A that both sides of (2.18) are weakly measurable. $\qquad\diamond$

Proof. Let $\psi, \tilde{\psi} \in C_0^\infty(\mathcal{U})$ with $\psi\tilde{\psi} = \psi$ and observe, cf. Lemma 2.3, that in the sense of forms on $\mathcal{D}(H)$ we have

$$\mathrm{i}[H, A^{(1)}]^\circ = \lim_{t\to 0} \tfrac{1}{t}[H, W_t],$$

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with $W_t = \mathrm{e}^{\mathrm{i}tA^{(1)}} = \mathrm{e}^{\mathrm{i}tA} \otimes \mathrm{e}^{\mathrm{i}ta}$. Abbreviate

$$P_\psi := \int_{\mathcal{U}}^{\oplus} \psi(x)\, \mathbb{1}_{\{E(x)\}}(H(x))\, \mathrm{d}x.$$

Note that $x \to \mathbb{1}_{\{E(x)\}}(H(x))$ is weakly measurable, and hence strongly measurable, cf. Appendix A. Since P_ψ preserves, in fact has range inside, $\mathcal{D}(H)$ we can compute as a form on $\mathcal{D}(H)$

$$P_\psi[H, W_t]P_\psi = P_\psi[E, W_t]P_\psi = P_\psi[\tilde{\psi}E, W_t]P_\psi,$$

where E and $\tilde{\psi}E$ should be read as multiplication operators in the base space, or equivalently as $\mathbb{1}_{\mathcal{H}} \otimes E$ and $\mathbb{1}_{\mathcal{H}} \otimes \tilde{\psi}E$. Since $\tilde{\psi}E \in C_0^1(\mathcal{U})$ we clearly have $\tilde{\psi}E \in C^1(A^{(1)})$ with $\mathrm{i}[\tilde{\psi}E, A^{(1)}]^\circ = \mathrm{i}[\tilde{\psi}E, a]^\circ = v \cdot \nabla(\tilde{\psi}E)$. Hence

$$P_\psi \mathrm{i}[H, A^{(1)}]^\circ P_\psi = \lim_{t \to 0} \tfrac{1}{t} P_\psi[H, W_t]P_\psi = \lim_{t \to 0} \tfrac{1}{t} P_\psi[\tilde{\psi}E, W_t]P_\psi$$
$$= v \cdot \nabla(\tilde{\psi}E)P_\psi^2 = v \cdot \nabla E P_\psi^2.$$

We conclude the theorem since ψ was arbitrary and when fibered the above identity reads

$$\int_{\mathcal{U}}^{\oplus} \psi(x)^2\, \mathbb{1}_{\{E(x)\}}(H(x))\mathrm{i}[H, A^{(1)}]^\circ(x)\, \mathbb{1}_{\{E(x)\}}(H(x))\, \mathrm{d}x$$
$$= \int_{\mathcal{U}}^{\oplus} \psi(x)^2 v(x) \cdot \nabla E(x)\, \mathbb{1}_{\{E(x)\}}(H(x))\, \mathrm{d}x. \qquad \square$$

3 The Commutator Estimate

In this section we analyze the geometry of the threshold set $\mathcal{T}^{(1)}$, construct vector fields $v \in C_0^\infty(\mathbb{R}^\nu)$ going into the one-body conjugate operator a, cf. (2.1), and finally prove a Mourre estimate for the fiber Hamiltonians below the two-boson threshold $\Sigma_0^{(2)}$ and away from threshold energies (and the set Exc).

We remark that in the literature, this type of analysis [2, 11, 12, 16, 38] has made essential use of the property $\omega(k) \to \infty$, $|k| \to \infty$, something we do not want to assume here in view of the polaron model. In [40] this assumption was avoided, by instead using that for bounded ω the gap between the ground state energy $\Sigma_0(\xi)$ and the bottom of the essential spectrum $\Sigma_{\mathrm{ess}}(\xi)$ closes at large total momentum. More precisely, under Condition 1.2, and the additional assumption $\sup_{k \in \mathbb{R}^\nu} \omega(k) < \infty$, the second case in (MC6), we have

$$\lim_{|\xi| \to \infty} \Sigma_{\mathrm{ess}}(\xi) - \Sigma_0(\xi) = 0. \tag{3.1}$$

We refer the reader to [40] for a proof. This result is crucial for treating the polaron model, and its importance is encoded in Lemma 3.1 below.

3.1 Structure of the Threshold Set

Recall from (1.28) the notation $\mathcal{E}^{(1)}$ for the energy-momentum region between the 1- and 2-boson thresholds.

Lemma 3.1. *Assume Condition 1.2. Let $C \subset \mathcal{E}^{(1)}$ be a compact set and $\mathcal{K} \subset \mathbb{R}^\nu$. The following holds*

(i) $\mathcal{K}_C := \{k \in \mathbb{R}^\nu \mid \exists (p, e) \in \Sigma_{\mathrm{iso}}, \ \text{s.t.} \ (p + k, e + \omega(k)) \in C\}$ *is compact.*

(ii) *If $\mathcal{K} \cap \mathcal{K}_C$ is closed, then the set $\Sigma_C(\mathcal{K}) = \left\{(p, e) \in \Sigma_{\mathrm{iso}} \mid \exists k \in \mathcal{K} \ \text{s.t.} \ (p + k, e + \omega(k)) \in C\right\}$ is compact.*

Remark 3.2. Observe that $\Sigma_C(\mathcal{K}) = \Sigma_C(\mathcal{K}_C \cap \mathcal{K})$. In particular we abbreviate $\Sigma_C := \Sigma_C(\mathbb{R}^\nu) = \Sigma_C(\mathcal{K}_C)$.

The set $\mathcal{K}_C$ consists of asymptotic momenta available to states localized in C for one-boson emission, due to energy and momentum conservation. The set Σ_C ($\Sigma_C(\mathcal{K})$) contains the interacting bound states reachable from states localized in C after emission of one boson (with asymptotic momentum in $\mathcal{K}$). $\diamond$

Proof. We divide the proof into three steps.

Step I: Reducing the problem to compactness of a single set. Let $X = \mathbb{R}^\nu \times \Sigma_{\mathrm{iso}}$. Define a map $\Psi \colon X \to \mathbb{R}^{\nu+1}$ by

$$\Psi(k, p, e) = (p + k, e + \omega(k)).$$

Denote by $\Pi_1 \colon X \to \mathbb{R}^\nu$ the projection onto the k coordinate and by $\Pi_2 \colon X \to \mathbb{R}^{\nu+1}$ the projection onto the (p, e) coordinate. With this notation we can write $\mathcal{K}_C = \Pi_1(\Psi^{-1}(C))$ and $\Sigma_C(\mathcal{K}) = \Pi_2(\Psi^{-1}(C) \cap (\mathcal{K} \times \mathbb{R}^{\nu+1}))$. Hence it suffices to prove that $C' = \Psi^{-1}(C)$ is a compact subset of $\mathbb{R}^{2\nu+1}$.

Step II: There exists $\varepsilon > 0$ such that $C' \subset \mathbb{R}^\nu \times \{(p, e) \in \Sigma_{\mathrm{iso}} \mid e \leq \Sigma_{\mathrm{ess}}(p) - \varepsilon\}$. Indeed, let $\varepsilon = d(C, \Sigma_0^{(2)}) > 0$, the distance from C to the two-boson threshold. Suppose $(k, p, e) \in C'$ satisfies that $e \in (\Sigma_{\mathrm{ess}}(p) - \varepsilon, \Sigma_{\mathrm{ess}}(p))$. Then

$$e + \omega(k) > \Sigma_{\mathrm{ess}}(p) + \omega(k) - \varepsilon = \Sigma_0^{(1)}((p + k) - k) + \omega(k) - \varepsilon \geq \Sigma_0^{(2)}(p + k) - \varepsilon.$$

This contradicts the choice of ε, since $(p + k, e + \omega(k)) \in C$.

Step III: C' is compact. Since Ψ is continuous, the preimage C' is closed as a subset of X. By Step II, it is in fact closed as a subset of $\mathbb{R}^{2\nu+1}$ as well. It remains to argue that C' is bounded.

Assume C' is unbounded. Then there must exist a sequence $(k_n, p_n, e_n) \in C'$ with $|k_n| + |p_n| \to \infty$. Since $(p_n + k_n, e_n + \omega(k_n))$ is in the compact set C, $p_n + k_n$ is a bounded sequence. Hence $|k_n|$ and $|p_n|$ both diverge to ∞. Let $\varepsilon = d(C, \Sigma^{(2)})$ as in Step II. By (3.1) there exists N such that for $n \geq N$ we have $\Sigma_0(p_n) > \Sigma_0^{(1)}(p_n) - \varepsilon$. For $n \geq N$ we can now estimate

$$e_n + \omega(k_n) \geq \Sigma_0(p_n) + \omega(k_n) > \Sigma^{(1)}(p_n) + \omega(k_n) - \varepsilon$$
$$= \Sigma_0^{(1)}((p_n + k_n) - k_n) + \omega(k_n) - \varepsilon \geq \Sigma_0^{(2)}(p_n + k_n) - \varepsilon.$$

This contradicts the choice of ε and we are done. $\square$

The remainder of this section is devoted to the geometry of the threshold sets $\mathcal{T}_{\mathcal{S}}^{(1)}$, $\mathcal{T}_{\parallel}^{(1)}$ and $\mathcal{T}_{\nparallel}^{(1)}$, cf. (1.30), (1.31) and (1.33).

Lemma 3.3. *Assume Conditions 1.2 and 1.3, with $n_0 = 0$. We have the following two properties*

(i) *The sets $\mathcal{T}_{\parallel}^{(1)}(\xi) \cap \mathcal{E}^{(1)}(\xi)$ and $\mathcal{T}_{\nparallel}^{(1)}(\xi) \cap \mathcal{E}^{(1)}(\xi)$ are locally finite, with possible accumulation points only at the upper boundary $\Sigma_0^{(2)}(\xi)$, the 2-boson threshold.*

(ii) *The sets $\mathcal{T}_{\parallel}^{(1)} \cap \mathcal{E}^{(1)}$ and $\mathcal{T}_{\nparallel}^{(1)} \cap \mathcal{E}^{(1)}$ are (relatively) closed subsets of $\mathcal{E}^{(1)}$.*

Remark 3.4. The set $\mathcal{T}_{\parallel}^{(1)}$ is precisely the union of graphs of ω centered above each crossing point, i.e. a graph of $k \to e + \omega(p+k)$ for each $(p,e) \in \mathcal{X}$. $\diamond$

Proof. We begin with (i) and take first the set $\mathcal{T}_{\parallel}^{(1)}(\xi)$. Fix $\xi \in \mathbb{R}^\nu$ and a matching collinear unit vector u. Let $r \in \mathbb{R}$ be such that $\xi = ru$.

Suppose $\{E_n\} \subset \mathcal{T}_{\parallel}^{(1)}(\xi)$, $E_n < \Sigma_0^{(2)}(\xi)$, with $E_n \to E < \Sigma_0^{(2)}(\xi)$. We need to argue that the sequence $\{E_n\}$ is eventually constant. Let $E' = (E + \Sigma_0^{(2)}(\xi))/2$, such that

$$C = \{\xi\} \times \left[\Sigma_0^{(1)}(\xi), E'\right] \tag{3.2}$$

is a compact subset of $\mathcal{E}^{(1)}$. For n large enough we have $(\xi, E_n) \in C$.

There exists $r_n \in \mathbb{R}$, for each $n \in \mathbb{N}$, such that $(\xi - r_n u, E_n - \omega(r_n u)) \in \mathcal{X}$ for all n. Observe that $(\xi - r_n u, E_n - \omega(r_n u)) \in \mathcal{X} \cap \Sigma_C$ for large n. Since Σ_C is compact, cf. Lemma 3.1, and the set $\mathcal{X}$ consists of isolated $S^{\nu-1}$-spheres centered at $\xi = 0$, we conclude that $|\xi - r_n u| = |r - r_n||u|$, and hence also r_n, only take finitely many values. But then $E_n - \omega(r_n u)$ must also take only finitely many values and hence E_n is eventually constantly equal to E.

As for the set $\mathcal{T}_{\nparallel}^{(1)}(\xi)$ we assume again that $\{E_n\} \subset \mathcal{T}_{\nparallel}^{(1)}(\xi)$, $E_n < \Sigma_0^{(2)}(\xi)$, and $E_n \to E < \Sigma_0^{(2)}(\xi)$. There exists k_n, for each n, such that $(\xi - k_n, E - \omega(k_n)) \in \mathcal{X}$ and $\nabla\omega(k_n) = 0$.

Let $\varepsilon > 0$ and the compact subset C of $\mathcal{E}^{(1)}$ be as before, cf. (3.2). The sequence $\{(\xi - k_n, E_n - \omega(k_n))\}$ must, for n large, again run inside the compact set Σ_C and thus since $\mathcal{X}$ consists of isolated spheres, we must have a subsequence $\{k_{n_j}\}$ such that $|\xi - k_{n_j}| = R_c$ and $E_{n_j} - \omega(k_{n_j}) = E_c$ are constant, signifying that we are on the same level crossing $\partial B(0; R_c) \times \{E_c\} \subset \mathcal{X}$. If $\xi = 0$ or ω is constant, we are done since in either case $\omega(k_{n_j})$ is a constant sequence, and hence $E_{n_j} = E$.

If we are in dimension $\nu = 1$ we are also done, since this will force k_{n_j} to only attain the two values $\xi + R_c$ and $\xi - R_c$. Hence $E_{n_j} = E$, for j sufficiently large.

We can thus assume that $\nu \geq 2$ and ω is not a constant function. Fix another unit vector v, with $v \cdot u = 0$. By symmetry in the hyperspace orthogonal to u, we can assume that $k_{n_j} \in \text{span}\{u, v\}$. Using that $|\xi - k_{n_j}| = R_c$, we can write the momenta as

$$k_{n_j} = k(\vartheta_j) = \xi - R_c\left(\cos(\vartheta_j)u + \sin(\vartheta_j)v\right),$$

with $\vartheta_j \in \mathbb{R}$ a bounded sequence. But since $\vartheta \to |\nabla\omega(k(\vartheta))|^2$ is a non-zero real analytic function the sequence ϑ_j can only attain finitely many values. Again we conclude that E_{n_j} can only attain finitely many values, and hence must be constantly equal to E for j large.

As for (ii), let $\{(\xi_n, E_n)\}_{n \in \mathbb{N}} \subset \mathcal{T}_{\parallel}^{(1)} \cap \mathcal{E}^{(1)}$ be a convergent sequence with $(\xi_n, E_n) \to (\xi, E) \in \mathcal{E}^{(1)}$. Let $0 < R < d((\xi, E), \Sigma_0^{(2)})$, i.e. R is chosen smaller than the distance from (ξ, E) to the upper boundary of $\mathcal{E}^{(1)}$. With this choice

$$C = \left(\overline{B(\xi; R)} \times [E - R, E + R] \right) \cap \mathcal{E}^{(1)} \tag{3.3}$$

is a compact subset of $\mathcal{E}^{(1)}$. For n large enough we have $(\xi_n, E_n) \in C$. By rotational symmetry we can assume that ξ and all the ξ_n's are collinear with a unit vector u. Write $\xi = ru$ and $\xi_n = r_n u$. There exist a sequence of momenta $s_n u$, with $s_n \in \mathcal{K}_C$ for n large enough, such that $((r_n - s_n)u, E_n - \omega(s_n u)) \in \Sigma_C \cap \mathcal{X}$, for n large enough. By compactness of $\mathcal{K}_C$ we can extract a convergent subsequence s_{n_j} converging to $s \in \mathbb{R}$. Then

$$\left(\xi - su, E - \omega(su) \right) = \lim_{j \to \infty} \left((r_{n_j} - s_{n_j})u, E_{n_j} - \omega(s_{n_j}u) \right) \in \Sigma_C \cap \mathcal{X},$$

since the set on the right-hand side is closed. Hence $(\xi, E) \in \mathcal{T}_{\parallel}^{(1)}$, which implies that $\mathcal{T}_{\parallel}^{(1)}$ is closed as a subset of $\mathcal{E}^{(1)}$.

We now turn to $\mathcal{T}_{\not\parallel}^{(1)}$. We again take a sequence $\{(\xi_n, E_n)\}_{n \in \mathbb{N}} \subset \mathcal{T}_{\not\parallel}^{(1)}$ converging to $(\xi, E) \in \mathcal{E}^{(1)}$. As above we can assume that there exists a unit vector u such that $\xi = ru$ and $\xi_n = r_n u$, with $r_n \to r$.

Since $(r_n u, E_n) \in \mathcal{T}_{\not\parallel}^{(1)}$ there must exist $k_n \in \mathbb{R}^\nu$ such that $(r_n u - k_n, E_n - \omega(k_n)) \in \mathcal{X}$ and $\nabla \omega(k_n) = 0$. By the, by now, standard argument, there exists a convergent subsequence $\{k_{n_j}\}$. Denote by k its limit.

We can now argue as for $\mathcal{T}_{\parallel}^{(1)}$ that

$$\left(\xi - k, E - \omega(k) \right) = \lim_{j \to \infty} \left(r_{n_j} u - k_{n_j}, E_{n_j} - \omega(k_{n_j}) \right) \in \mathcal{X}$$

and $\nabla \omega(k) = \lim_{j \to \infty} \nabla \omega(k_{n_j}) = 0$. Hence $(\xi, E) \in \mathcal{T}_{\not\parallel}^{(1)}$, which establishes the remaining part of (ii). $\qquad \square$

Proof of Theorem 1.5. Abbreviate for the purpose of this proof

$$\mathcal{U} = \left\{ (\xi, E) \in \mathcal{T}_{\mathcal{S}}^{(1)} \,\middle|\, (\xi, E) \notin \mathcal{T}_{\parallel}^{(1)} \cup \mathcal{T}_{\not\parallel}^{(1)} \right\}$$
$$\mathcal{U}(\xi) = \left\{ E \in \mathbb{R} \,\middle|\, (\xi, E) \in \mathcal{U} \right\}. \tag{3.4}$$

The sets $\mathcal{U}$ and $\mathcal{U}(\xi)$ are subsets of $\mathcal{T}_{\mathcal{S}}^{(1)}$ and $\mathcal{T}_{\mathcal{S}}^{(1)}(\xi)$ respectively.

Given Lemma 3.3 it remains to prove the following two statements:

$$\overline{\mathcal{U}} \cap \mathcal{E}^{(1)} \subset \mathcal{T}^{(1)} \tag{3.5}$$
$$\mathcal{U}(\xi) \cap \mathcal{E}^{(1)}(\xi) \quad \text{is locally finite.} \tag{3.6}$$

To prove (3.5), let $(\xi_n, E_n) \in \mathcal{U} \cap \mathcal{E}^{(1)}$ and assume $(\xi_n, E_n) \to (\xi, E)$ with $E < \Sigma_0^{(2)}(\xi)$. We need to argue that $(\xi, E) \in \mathcal{T}^{(1)}$.

Construct a compact set $C \subset \mathcal{E}^{(1)}$ containing (ξ, E) as in (3.3). For n large enough we have $(\xi_n, E_n) \in C$. For each (large) n we can find a $k_n \in \mathcal{K}_C$, a mass shell

$(\mathcal{A}_n, S_n) \in \mathcal{S}$, such that $\xi_n - k_n \in \mathcal{A}_n$, $E_n = S_n(\xi_n - k_n) + \omega(k_n)$ and $\nabla S_n(\xi_n - k_n) = \nabla\omega(k_n)$. Here we used that $(\xi_n, E_n) \in \mathcal{T}_{\mathcal{S}}^{(1)}$.

Since $\mathcal{K}_C$ is compact we can pass to a convergent subsequence $\{k_{n_\ell}\}$ with $k := \lim_{\ell \to \infty} k_{n_\ell} \in \mathcal{K}_C$. Abbreviate

$$p_\ell := \left(\xi_{n_\ell} - k_{n_\ell}, E_{n_\ell} - \omega(k_{n_\ell})\right) \in \Sigma_C. \tag{3.7}$$

Since Σ_C is closed we have

$$\lim_{\ell \to \infty} p_\ell = p := \left(\xi - k, E - \omega(k)\right) \in \Sigma_C \subset \Sigma_{\text{iso}}. \tag{3.8}$$

Recall that level crossings, as $S^{\nu-1}$-spheres inside $\mathcal{X}$, are isolated and only finitely many mass shells emanate from each crossing. Hence we can assume that there exists a distinguished mass shell $(\mathcal{A}, S) \in \mathcal{S}$ such that $p_\ell \in \mathcal{G}_S$, cf. (1.27), for all ℓ. We can furthermore assume that we are in one of the two following cases

$$\begin{array}{lll} \textbf{Case A} & \forall \ell: & \nabla\omega(k_{n_\ell}) \neq 0 \\ \textbf{Case B} & \forall \ell: & \nabla\omega(k_{n_\ell}) = 0. \end{array} \tag{3.9}$$

In Case A we must have for each ℓ an $s_\ell \in \mathbb{R}$ such that $k_{n_\ell} = s_\ell u$ and $\lim_{\ell \to \infty} s_\ell$ exists. If $k \to S^{(1)}(\xi; k) = S(\xi - k) + \omega(k)$ is not a constant function, the sequence s_ℓ must be eventually constant and hence $k \in \mathcal{A} + \xi$ and $p \in \mathcal{G}_S$. Here we used that $s \to S^{(1)}(\xi; su)$ continues analytically through level crossings. If on the other hand $k \to S^{(1)}(\xi; k)$ is a constant function, we can replace the k_{n_ℓ}'s by a constant $k \in \mathcal{A} + \xi$. Hence the new limit will satisfy $p \in \mathcal{G}_S$.

In Case B we have $\nabla S(\xi_{n_\ell} - k_{n_\ell}) = 0$, so we must have either S constant, or $|\xi_{n_\ell} - k_{n_\ell}|$ eventually constant and equal to $|\xi - k|$. In the latter case $k \in \mathcal{A} + \xi$ and $p \in \mathcal{G}_S$. We now assume that S is a constant function.

If ω is also constant we can redefine the k_{n_ℓ}'s as above and again arrive at $p \in \mathcal{G}_S$. If ω is not a constant, $|k_{n_\ell}|$ is eventually constant and equal to $r \geq 0$. First of all we observe that r is strictly smaller than the outer radius of $\mathcal{A}$. This is due to the choice of $(\xi_{n_\ell}, E_{n_\ell})$ away from $\mathcal{T}_{\#}^{(1)}$, cf. (1.33). We can thus replace the k_{n_ℓ}'s with possibly different k_{n_ℓ}'s in $rS^{\nu-1}$ such that the limit $k \in \mathcal{A} + \xi$.

Summing up, we have argued that either $p \in \mathcal{G}_S$, or we can make a different choice of sequence k_{n_ℓ} such that p ends up inside $\mathcal{G}_S$. Then, by continuity, we must have $E = S(\xi - k) + \omega(k)$ and $\nabla S(\xi - k) = \nabla\omega(k)$. Hence $(\xi, E) \in \mathcal{T}_{\mathcal{S}}^{(1)}$. This proves (3.5).

To verify (3.6), let $(\xi, E_n) \in \mathcal{U}(\xi)$, with $E_n < \Sigma_0^{(2)}(\xi)$, such that $E_n \to E \in \mathcal{U}(\xi)$, with $E < \Sigma_0^{(2)}(\xi)$. We have to prove that the sequence E_n is eventually constant. Assume towards a contradiction that it is not eventually constant. Hence we can assume, possibly passing to a subsequence, that it is strictly monotone.

Let $C \subset \mathcal{E}^{(1)}$ compact, be as in (3.2). For n sufficiently large we have $(\xi, E_n) \in C$. By the choice of E_n we can to each n identify a $k_n \in \mathcal{K}_C$ and a mass shell $(\mathcal{A}_n, S_n) \in \mathcal{S}$ such that $\xi - k_n \in \mathcal{A}_n$, $E_n = S_n(\xi - k_n) + \omega(k_n)$ and $\nabla S_n(\xi - k_n) = \nabla\omega(k_n)$.

As in the verification of (3.5) we can extract a subsequence $\{E_{n_\ell}\}$ together with a convergent sequence of momenta $\{k_{n_\ell}\}$, and a distinguished mass shell $(\mathcal{A}, S) \in \mathcal{S}$ such that $\xi - k, \xi - k_{n_\ell} \in \mathcal{A}$ and $p_\ell, p \in \mathcal{G}_S$, cf. (3.7) and (3.8). Here $k = \lim_{\ell \to \infty} k_{n_\ell}$.

We can furthermore assume that we are in either Case A or Case B, cf. (3.9). Here we used that $E \in \mathcal{U}(\xi)$, cf. (3.4), to rule out the possibility that $p \in \mathcal{X}$.

In Case A we reach a contradiction with E_{n_ℓ} being strictly monotone as follows. Write $\xi = ru$ for some unit vector u and $r \in \mathbb{R}$. If $\xi \neq 0$ the demand that $\nabla S(\xi - k_{n_\ell}) = \nabla \omega(k_{n_\ell})$, together with rotation invariance, forces all the k_{n_ℓ}'s to be collinear with u. If $\xi = 0$, we can again use rotation invariance and simply replace all the k_{n_ℓ}'s by $|k_{n_\ell}|u$ and thus arrive at the same situation. Hence the map $t \to S^{(1)}(\xi; tu) = S((r-t)u) + \omega(tu)$ is analytic and vanishes along a sequence with accumulation point inside its domain of analyticity. Hence it is constant, i.e. $E_{n_\ell} = S^{(1)}(\xi; k_{n_\ell})$ is constant.

In Case B we reach a contradiction as follows. Since $\nabla \omega(k_{n_\ell}) = 0 = \nabla S(\xi - k_{n_\ell})$ we can conclude that: Either ω is constant or $|k_{n_\ell}|$ is eventually constant. Furthermore, either S is constant or $|\xi - k_{n_\ell}|$ is eventually constant. Regardless of which of the 4 possible combinations we find ourselves in, we conclude again that $E_{n_\ell} = S(\xi - k_{n_\ell}) + \omega(k_{n_\ell})$ is eventually constant. $\qquad \square$

3.2 Some Geometric Considerations

The goal of this section is, given a

$$(\xi, E) \in \mathcal{E}^{(1)} \backslash \mathcal{T}^{(1)}, \tag{3.10}$$

to choose a compact interval $\mathcal{J} = [E - \delta, E + \delta]$ with

$$(\xi, E) \in \{\xi\} \times \mathcal{J} \subset \mathcal{E}^{(1)} \backslash \mathcal{T}^{(1)}, \tag{3.11}$$

such that states localized in $\mathcal{J}$ (at sharp total momentum ξ) can only break up into channels with non-zero breakup velocity.

Given $\mathcal{J} \subset \mathcal{E}^{(1)}(\xi)$ and $\mathcal{K} \subset \mathbb{R}^\nu$, we associate the sets

$$\mathcal{K}_\mathcal{J} := \mathcal{K}_{\{\xi\} \times \mathcal{J}}, \quad \Sigma_\mathcal{J} := \Sigma_{\{\xi\} \times \mathcal{J}} \quad \text{and} \quad \Sigma_\mathcal{J}(\mathcal{K}) := \Sigma_{\{\xi\} \times \mathcal{J}}(\mathcal{K}).$$

The set $\mathcal{K}_\mathcal{J}$ contains the momenta available for boson emission starting from a state localized with respect to energy in $\mathcal{J}$, whereas $\Sigma_\mathcal{J}$ labels the available interacting bound states the system can relax to. With the choice (3.11) of $\mathcal{J}$, some of the k's in $\mathcal{K}_\mathcal{J}$ may correspond to elements $(\xi - k, E - \omega(k))$ on level crossings, but only if ξ is non-zero, and then k is linearly independent of ξ. Here $E \in \mathcal{J}$. We introduce the notation

$$\mathcal{K}_\mathcal{J}^\mathcal{X} := \left\{ k \in \mathcal{K}_\mathcal{J} \,\middle|\, \exists \lambda \in \mathcal{J} : \left(\xi - k, \lambda - \omega(k) \right) \in \mathcal{X} \right\} \tag{3.12}$$

for the subset of $\mathcal{K}_\mathcal{J}$ corresponding to level crossings. In addition we write, for $\mathcal{K} \subset \mathbb{R}^\nu$,

$$\Sigma_\mathcal{J}^\mathcal{X} = \Sigma_\mathcal{J} \cap \mathcal{X} \quad \text{and} \quad \Sigma_\mathcal{J}^\mathcal{X}(\mathcal{K}) = \Sigma_\mathcal{J}(\mathcal{K}) \cap \mathcal{X}$$

for the reachable interacting bound states at level crossings.

For $\xi \neq 0$, we use the notation $\mathrm{O}_\nu(\xi)$ for the subgroup of the orthogonal group consisting of orthogonal matrices O satisfying $O\xi = \xi$. It is convenient, given $\xi \neq 0$, to introduce a change of coordinates. Let

$$\Pi_\xi : \mathbb{R}^{\nu-1} \to \{\eta \in \mathbb{R}^\nu \,|\, \xi \cdot \eta = 0\}$$

be a (linear) isometric isomorphism of $\mathbb{R}^{\nu-1}$ onto the orthogonal complement of ξ. We define a change of coordinates $\mathbf{k} \colon (0,\infty) \times [0,2\pi) \times S^{\nu-2} \to \mathbb{R}^\nu \backslash \{\xi\}$ by

$$\mathbf{k}(s,\vartheta,w) := \xi - s\cos(\vartheta)\frac{\xi}{|\xi|} + s\sin(\vartheta)\Pi_\xi(w). \tag{3.13}$$

Observe that Π_ξ induces a group isomorphism $\mathrm{O}_{\nu-1} \to \mathrm{O}_\nu(\xi)$ by mapping $O \in \mathrm{O}_{\nu-1}$ to $O_\xi \in \mathrm{O}_\nu(\xi)$, determined by the two relations $O_\xi\xi = \xi$ and $O_\xi\Pi_\xi = \Pi_\xi O$. For $O \in \mathrm{O}_{\nu-1}$ we have

$$O_\xi\mathbf{k}(r,\vartheta,w) = \mathbf{k}(r,\vartheta,Ow). \tag{3.14}$$

We will use the function $\mathbf{k}$ defining the change of coordinates also beyond angles confined to $[0,2\pi)$. Finally, note that

$$\mathbf{k}(r,2\pi-\vartheta,w) = \mathbf{k}(r,\vartheta,-w). \tag{3.15}$$

Lemma 3.5. *Assume Conditions 1.2 and 1.3, with $n_0 = 0$. Suppose $\nu \geq 2$. Let $(\xi,E) \in \mathcal{E}^{(1)} \backslash \mathcal{T}_\parallel^{(1)}$. There exists a finite number of radii $\{R_j\}_{j=1}^M$, with $R_i > 0$, and for each $i = 1,\ldots,M$, a finite set of angles $\{\vartheta_{i,j}\}_{j=1}^{M_i}$, with $\vartheta_{i,j} \in (0,\pi)$, such that*

$$\mathcal{K}_{\{E\}}^\mathcal{X} = \bigcup_{i=1}^M \bigcup_{j=1}^{M_i} \mathbf{k}(R_i,\vartheta_{i,j},S^{\nu-2}).$$

If $\xi = 0$ or ω is constant the set $\mathcal{K}_{\{E\}}^\mathcal{X}$ is empty.

Remark 3.6. 1) The set described above is a finite union of non-empty $S^{\nu-2}$-spheres, all centered along a line through the origin in the direction of ξ. They sit inside $S^{\nu-1}$-spheres of crossings centered at ξ with radius R_i.

2) In dimension $\nu = 1$ the set in question is empty by the choice of (ξ,E). In dimension 2 the set consists of finitely many points placed symmetrically around the line through the origin and ξ, with no points on the line through 0 and ξ. $\qquad\diamond$

Proof. First we observe that if $\xi = 0$ we have $\mathcal{K}_{\{E\}}^\mathcal{X} = \emptyset$. This is due to the assumption that ω is rotation invariant. For a similar reason, the set is also empty if ω is a constant function regardless of ξ. From now on we assume that $\xi \neq 0$ and that ω is not constant.

From Lemma 3.1 we know a priori that the sets $\mathcal{K}_{\{E\}}^\mathcal{X}$ and $\Sigma_{\{E\}}^\mathcal{X}$ are compact. In particular, there exist finitely many radii $R_1,\ldots,R_M$, and energies $\lambda_1,\ldots,\lambda_M$ such that

$$\Sigma_{\{E\}}^\mathcal{X} \subset \bigcup_{i=1}^M R_i S^{\nu-1} \times \{\lambda_i\} \subset \mathcal{X}. \tag{3.16}$$

The choice of (ξ,E) ensures that $R_i > 0$ for all $i = 1,\ldots,M$.

Clearly the set $\mathcal{K}_{\{E\}}^\mathcal{X}$ is invariant under rotations from the group $\mathrm{O}_\nu(\xi)$. Fix a unit vector v orthogonal to ξ. Take for example $v = \Pi_\xi(e_1)$. Put $u = \xi/|\xi|$.

What we need to show is that $\mathcal{K} = \mathcal{K}_{\{E\}}^\mathcal{X} \cap \mathrm{span}\{u,v\}$ is a finite set. The choice of (ξ,E) ensures that the intersection above does not contain any elements in $\mathbb{R}u = \mathbb{R}\xi$. The orbit under $\mathrm{O}_\nu(\xi)$ of $k \in \mathcal{K}_{\{E\}}^\mathcal{X} \cap \mathrm{span}\{u,v\}$ are exactly the $S^{\nu-2}$-spheres in the lemma, cf. (3.14).

Aiming for a contradiction we will assume that there exists a countably infinite sequence $\{k_n\}_{n \in \mathbb{N}} \subset \mathcal{K}$ consisting of distinct momenta. Observe that $(\xi - k_n, E - \omega(k_n)) \in \Sigma_{\{E\}}^{\mathcal{X}}$, and hence by (3.16) there must exist $1 \le i \le M$ and a subsequence $\{k_{n_j}\}$, with $|\xi - k_{n_j}| = R_i > 0$ and $E - \omega(k_{n_j}) = \lambda_i$ for all j. We can now write $k_{n_j} = \mathbf{k}(R_i, \vartheta_j, e_1)$, for a sequence of distinct angles $\vartheta_j \in [0, 2\pi)$.

Observe that $E - \omega(k_{n_j})$, and consequently $\omega(k_{n_j})$, is a constant sequence. Since the map $\mathbb{R} \ni \vartheta \to f(\vartheta) = \omega(\mathbf{k}(R_i, \vartheta, e_1))$ is a real analytic function, constant along a sequence ϑ_j that has a cluster point, we conclude that f must be a constant function. Since ω is not a constant function, this can only happen if $\mathbb{R} \ni \vartheta \to |\mathbf{k}(R_i, \vartheta, e_1)|$ is constant. But this is impossible because we assumed that $\xi \ne 0$, cf. (3.13). Hence $\mathcal{K}$ does not contain a countable sequence of distinct momenta and we conclude the lemma. Observe that (3.15) ensures that we can restrict the angles to $(0, \pi)$. $\qquad \square$

Let (ξ, E) be chosen as in (3.10). We construct torus neighborhoods $\mathbb{T}_{i,j}$, in the (r, ϑ, w) coordinate system, around the finitely many $S^{\nu-2}$-spheres in $\mathcal{K}_{\{E\}}^{\mathcal{X}}$ identified in Lemma 3.5. We can label these sets by radii and angles $(R_i, \vartheta_{i,j})$, $i = 1, \dots, M$ and $j = 1, \dots, M_i$, with $R_i > 0$ and $\vartheta_{i,j} \in (0, \pi)$. We define

$$\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r) := \Big\{ \mathbf{k}(r, \vartheta, w) \,\Big|\, |R_i - r| < \varepsilon_r, |\vartheta - \vartheta_{i,j}| < \varepsilon_\vartheta, w \in S^{\nu-2} \Big\},$$

where ε_ϑ measure the angular thickness of the torus, and ε_r the radial thickness.

In order to pick an appropriate angular and radial thickness for the tori we proceed in steps to ensure that a number of properties are satisfied. We first pick $0 < \varepsilon_r^{(4)}, \varepsilon_\vartheta^{(2)} \le 1$ such that

$$\begin{aligned} \varepsilon_r^{(4)} &< \min_{i=1,\dots,M} R_i \\ \varepsilon_\vartheta^{(2)} &< \min_{1 \le i \le M} \min_{1 \le j \le M_i} \{\vartheta_{i,j}, \pi - \vartheta_{i,j}\}. \end{aligned} \tag{3.17}$$

With this choice we have ensured that the tori will have their holes.

By the choice $E \notin \mathcal{T}_W^{(1)}(\xi)$ we know that $\nabla\omega(\mathbf{k}(R_i, \vartheta_{i,j}, w)) \ne 0$, for every $i = 1, \dots, M$, $j = 1, \dots, M_i$, and $w \in S^{\nu-2}$. In addition, by rotation invariance of ω, for any i and j the norm $|\nabla\omega(\mathbf{k}(R_i, \vartheta_{i,j}, w))|$ does not depend on w. By continuity of $\nabla\omega$, we can choose $0 < \varepsilon_\vartheta^{(1)} \le \varepsilon_\vartheta^{(2)}$ and $0 < \varepsilon_r^{(3)} \le \varepsilon_r^{(4)}$ such that

$$\forall i, j: \quad \inf_{k \in \mathbb{T}_{i,j}(\varepsilon_\vartheta^{(1)}, \varepsilon_r^{(3)})} |\nabla\omega(k)| > 0. \tag{3.18}$$

The choice of $\varepsilon_\vartheta^{(1)}$ and $\varepsilon_r^{(3)}$ implies that $\mathbb{T}_{i,j}(\varepsilon_\vartheta^{(1)}, \varepsilon_r^{(3)})$ are topological tori and they contain no k's parallel with ξ, nor are there k's with $\nabla\omega(k) = 0$.

Since $\nabla\omega$ does not vanish on the tori $\mathbb{T}_{i,j}(\varepsilon_\vartheta^{(1)}, \varepsilon_r^{(3)})$, and ω is rotation invariant, we find that $k \cdot \nabla\omega(k)$ does not vanish on the tori either. Recall that $k = 0$, being 'collinear' with ξ, is not in any of the tori. Hence $k \cdot \nabla\omega(k)$ has a sign, which we denote by $\sigma_{i,j} \in \{-1, +1\}$, for each $i = 1, \dots, M$ and $j = 1, \dots, M_i$. We note the identity

$$\forall k \in \mathbb{T}_{i,j}\big(\varepsilon_\vartheta^{(1)}, \varepsilon_r^{(3)}\big): \quad |\nabla\omega(k)| = \sigma_{i,j} \frac{k}{|k|} \cdot \nabla\omega(k). \tag{3.19}$$

Unfortunately the above choice of $\varepsilon_\vartheta^{(1)}$ and $\varepsilon_r^{(3)}$ does not quite suffice. At the center of the torus, i.e. for k's in the set $\mathbf{k}(R_i, \vartheta_{i,j}, S^{\nu-2})$, we know that $(\xi-k, E-\omega(k)) \in \mathcal{X}$. In fact for such k we always end at the same level crossing

$$\mathcal{X}_{i,j} := \left(R_i S^{\nu-1}\right) \times \left\{E - \omega(\mathbf{k}(R_i, \vartheta_{i,j}, e_1))\right\},$$

due to rotation invariance. (Different j's may a priori give rise to different crossings if there at different energies sit level crossings at the same radius in momentum space.) For other k's in the torus we need to be sure that $(\xi - k, E - \omega(k))$ does not land on a *different* crossing. That is, we have to identify $\varepsilon_\vartheta \leq \varepsilon_\vartheta^{(1)}$ and $\varepsilon_r^{(2)} \leq \varepsilon_r^{(3)}$ such that

$$\Sigma_{\{E\}}^{\mathcal{X}}\left(\overline{\mathbb{T}_{i,j}\left(\varepsilon_\vartheta, \varepsilon_r^{(2)}\right)}\right) \subset \mathcal{X}_{i,j}. \tag{3.20}$$

Here we can use that level crossings are isolated and that we only consider finitely many tori, to ensure that

$$d = d(\xi, E) := \min_{i,j} d(\mathcal{X}_{i,j}, \mathcal{X} \backslash \mathcal{X}_{i,j}) > 0. \tag{3.21}$$

For $k \in \overline{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r^{(2)})}$ we write first $k = \mathbf{k}(r, \vartheta, w)$ with $|r - R_i| \leq \varepsilon_r^{(2)}$, $|\vartheta - \vartheta_{i,j}| \leq \varepsilon_\vartheta$ and $w \in S^{\nu-2}$. Then we compute

$$\begin{pmatrix} \xi - k \\ E - \omega(k) \end{pmatrix} = \begin{pmatrix} \xi - \mathbf{k}(R_i, \vartheta_{i,j}, w) \\ E - \omega(\mathbf{k}(R_i, \vartheta_{i,j}, w)) \end{pmatrix} + \begin{pmatrix} \mathbf{k}(R_i, \vartheta_{i,j}, w) - \mathbf{k}(r, \vartheta, w) \\ \omega(\mathbf{k}(R_i, \vartheta_{i,j}, w)) - \omega(\mathbf{k}(r, \vartheta, w)) \end{pmatrix} \tag{3.22}$$

and estimate

$$\left| \begin{pmatrix} \mathbf{k}(R_i, \vartheta_{i,j}, w) - \mathbf{k}(r, \vartheta, w) \\ \omega(\mathbf{k}(R_i, \vartheta_{i,j}, w)) - \omega(\mathbf{k}(r, \vartheta, w)) \end{pmatrix} \right| \leq C_{i,j} \max\left\{\varepsilon_\vartheta, \varepsilon_r^{(2)}\right\},$$

using that $\nabla\omega$ is bounded to argue for the existence of the constant $C_{i,j}$. Put $C = \max_{i,j} C_{i,j}$. Since the first term on the right-hand side of (3.22) is an element of $\mathcal{X}_{i,j}$ we observe that if we choose $\varepsilon_\vartheta, \varepsilon_r^{(2)} \leq d/(2C)$ we can conclude that

$$\forall i, j \text{ and } k \in \overline{\mathbb{T}_{i,j}\left(\varepsilon_\vartheta, \varepsilon_r^{(2)}\right)} : \quad d((\xi - k, E - \omega(k)), \mathcal{X} \backslash \mathcal{X}_{i,j}) \geq \frac{d}{2}.$$

The constant d was defined in (3.21). We now make the choice

$$\varepsilon_\vartheta = \min\left\{\varepsilon_\vartheta^{(1)}, d/(2C)\right\}, \quad \varepsilon_r^{(2)} = \min\left\{\varepsilon_r^{(3)}, d/(2C)\right\},$$

and emphasize that with this choice the desired inclusion (3.20) holds true.

Our next task is to pick δ' small enough such that $\mathcal{K}_{[E-\delta', E+\delta']}^{\mathcal{X}}$ is contained inside the union over i and j of the tori $\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r^{(2)})$, and such that the inclusion (3.20) remains valid when $\{E\}$ is broadened to $\mathcal{J}'$.

Lemma 3.7. *Assume Conditions 1.2 and 1.3, with $n_0 = 0$. There exists $\delta' > 0$ such that $\mathcal{J}' = [E - \delta', E + \delta'] \subset \mathcal{E}^{(1)}(\xi)$ and*

$$\mathcal{K}_{\mathcal{J}'}^{\mathcal{X}} \subset \bigcup_{i=1}^M \bigcup_{j=1}^{M_i} \mathbb{T}_{i,j}\left(\varepsilon_\vartheta, \varepsilon_r^{(2)}\right) \tag{3.23}$$

$$\Sigma_{\mathcal{J}'}^{\mathcal{X}}\left(\overline{\mathbb{T}_{i,j}\left(\varepsilon_\vartheta, \varepsilon_r^{(2)}\right)}\right) \subset \mathcal{X}_{i,j}. \tag{3.24}$$

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Proof. Assume the inclusion (3.23) is false. Then there exists a sequence $\lambda_n \in [E - 1/n, E + 1/n] =: \mathcal{J}_n$ and $k_n \in \mathcal{K}_{\mathcal{J}_n}^{\mathcal{X}}$ with $(\xi - k_n, \lambda_n - \omega(k_n)) \in \mathcal{X}$ and $k_n \notin \cup_{i=1}^{M} \cup_{j=1}^{M_i} \mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r^{(2)})$.

By Lemma 3.1, we can extract a convergent subsequence $\{k_{n_j}\}$ converging to a momentum k. Since $\lambda_{n_j} \to E$ we must have $(\xi - k, E - \omega(k)) \in \mathcal{X}$, and hence $k \in \mathcal{K}_{\{E\}}^{\mathcal{X}}$. Since the tori are open we conclude furthermore that $k \notin \cup_{i=1}^{M} \cup_{j=1}^{M_i} \mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r^{(2)})$. But this contradicts Lemma 3.5 and we have thus established (3.23).

As for (3.24) we proceed in a similar fashion assuming that for any n there exists $\lambda_n \in [E - 1/n, E + 1/n]$ and $k_n \in \overline{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r^{(2)})}$ such that $(\xi - k_n, \lambda_n - \omega(k_n)) \in \mathcal{X} \backslash \mathcal{X}_{i,j}$.

Again, by Lemma 3.1, we must have a subsequence k_{n_j} converging to a momentum $k \in \overline{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r^{(2)})}$. For this k we must have $(\xi - k, E - \omega(k)) \in \mathcal{X} \backslash \mathcal{X}_{i,j}$, and hence $(\xi - k, E - \omega(k)) \in \Sigma_{\{E\}}^{\mathcal{X}}$. But this contradicts (3.20). $\qquad\square$

We identify the mass shells available for scattering channels, starting at momentum ξ and energy in $\mathcal{J}'$ to be

$$\mathcal{S}' := \left\{ (\mathcal{A}, S) \in \mathcal{S} \,\middle|\, \mathcal{G}_S \cap \Sigma_{\mathcal{J}'} \neq \emptyset \right\}. \tag{3.25}$$

By compactness of $\Sigma_{\mathcal{J}'}$ this set is finite. We list the radii of the spheres forming $\partial \mathcal{A}$, where $(\mathcal{A}, S) \in \mathcal{S}'$, as $R'_1, \ldots, R'_{M'}$. We *exclude* from the list of radii those already included in $R_1, R_2, \ldots, R_M$. With this choice we find that

$$\forall 1 \leq \ell \leq M' : \quad r'_\ell := d\left(\mathcal{K}_{\mathcal{J}'}, \partial B(\xi; R'_\ell) \right) > 0.$$

The next thing we need to do is to ensure that the set $\mathcal{K}_{\mathcal{J}'}$ approaches level crossings, or more precisely the spheres $\partial B(\xi; R_i)$, through the radial face of the tori. Define for $\varepsilon > 0$ the compact set

$$\mathcal{K}(\varepsilon) := \mathcal{K}_{\mathcal{J}'} \backslash \left(\bigcup_{i,j} \mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon) \right) \tag{3.26}$$

and for $i = 1, \ldots, M$ subsets

$$\mathcal{K}_i := \left\{ \mathbf{k}(r, \vartheta, w) \in \mathcal{K}(\varepsilon_r^{(2)}) \,\middle|\, |\vartheta - \vartheta_{i,j}| \geq \varepsilon_\vartheta, j = 1, \ldots, M_i \right\},$$

which again are compact sets. Let

$$r_i := d\left(\mathcal{K}_i, \partial B(\xi, R_i) \right) > 0,$$

where strict positivity follows from (3.23). We now pick an upper bound for the radial thickness ε_r to be

$$\varepsilon_r^{(1)} := \min \left\{ \varepsilon_r^{(2)}, \min_{1 \leq i \leq M} r_i, \min_{1 \leq \ell \leq M'} r'_\ell \right\}.$$

This choice ensures that for $\varepsilon \leq \varepsilon_r^{(1)}$ the set $\mathcal{K}(\varepsilon)$ approaches the spheres $\partial B(R_i, \xi)$, $i = 1, \ldots, M$, through the radial faces of the tori $\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon)$, not through their angular faces. In addition, k's in $\mathcal{K}(\varepsilon)$ stay at least a distance ε away from boundaries of annuli in which the relevant mass shells in $\mathcal{S}'$ are defined. To summarize: For all $0 < \varepsilon \leq \varepsilon_r^{(1)}$, $i \in \{1, \ldots, M\}$ and $\ell \in \{1, \ldots, M'\}$ we have

$$d\left(\mathcal{K}(\varepsilon), \partial B(\xi, R_i) \right) \geq \varepsilon \quad \text{and} \quad d\left(\mathcal{K}(\varepsilon), \partial B(\xi, R'_\ell) \right) \geq \varepsilon.$$

3.3 An Analytic Consideration

The next part of the construction is somewhat less obvious, in that it anticipates the proof of the Mourre estimate to follow. We need to construct a conjugate operator, i.e. a vector field, in a set like $\mathcal{K}_{\mathcal{J}'}$, but we proceed differently depending on whether we are inside or outside one of the tori introduced in the previous section. If we are inside a torus we want the conjugate operator to be a ϑ derivative. We now proceed to compute what turns out to be the relevant commutator and get something positive on the crossing. Then we pick $\varepsilon_r \leq \varepsilon_r^{(1)}$ small enough for the expression to remain positive in the torus.

We anticipate a conjugate operator of the form (2.2), cf. also (2.1). We require that the vector field v entering into the construction of the one-body conjugate operator a satisfies

$$\|v\|_\infty \leq 2 + \max_{1 \leq i \leq M} R_i. \tag{3.27}$$

We define, for $r > 0$, auxiliary Hamiltonians $G_\xi(r)$ on the Hilbert space $L^2(\mathbb{R} \times S^{\nu-2}; \mathcal{F})$ by the following direct integral construction

$$G_\xi(r) = \int_{\mathbb{R} \times S^{\nu-2}}^{\oplus} G_\xi(r, \vartheta, w)\, \mathrm{d}\vartheta \mathrm{d}w, \tag{3.28}$$

where

$$G_\xi(r, \vartheta, \omega) = H(\xi - \mathbf{k}(r, \vartheta, w)) + \omega(\mathbf{k}(r, \vartheta, w))\, \mathbb{1}_\mathcal{F}. \tag{3.29}$$

For a $\rho_{i,j} \in C_0^\infty(\mathbb{R}_\vartheta)$, we define a self-adjoint cutoff angular derivative

$$\tilde{a} = \sigma_{i,j} \tfrac{1}{2} \big\{ \mathrm{i}\partial_\vartheta \rho_{i,j} + \rho_{i,j} \mathrm{i}\partial_\vartheta \big\}.$$

We fix our choice of $\rho_{i,j}$ to be compactly supported in $(\vartheta_{i,j} - 2\varepsilon_\vartheta, \vartheta_{i,j} + 2\varepsilon_\vartheta)$, equal to 1 on

$$\Theta_{i,j} := (\vartheta_{i,j} - \varepsilon_\vartheta, \vartheta_{i,j} + \varepsilon_\vartheta) \tag{3.30}$$

and satisfying that $0 \leq \rho_{i,j} \leq 1$. Observe that $\tilde{a}$ only acts on the base space, not on the fiber $\mathcal{F}$. We now stitch A and $\tilde{a}$ together to get a conjugate operator on $\mathcal{F} \otimes L^2(\mathbb{R} \otimes S^{\nu-2})$

$$\widetilde{A}^{(1)} = A \otimes \mathbb{1}_{L^2(\mathbb{R} \times S^{\nu-2})} + \mathbb{1}_\mathcal{F} \otimes \tilde{a},$$

where we appeal to the identification $\mathcal{F} \otimes L^2(\mathbb{R} \times S^{\nu-2}) \simeq L^2(\mathbb{R} \times S^{\nu-2}; \mathcal{F})$. One can verify that $G_\xi(r)$ is of class $C^1(\widetilde{A}^{(1)})$ and

$$\mathrm{i}\big[G_\xi(r), \widetilde{A}^{(1)}\big]^\circ = \int_{\mathbb{R} \times S^{\nu-2}}^{\oplus} \mathrm{i}\big[G_\xi(r), \widetilde{A}^{(1)}\big]^\circ(\vartheta, w)\, \mathrm{d}\vartheta \mathrm{d}w,$$

where, as an identity on $\mathcal{F}$,

$$\mathrm{i}\big[G_\xi(r), \widetilde{A}^{(1)}\big]^\circ(\vartheta, w)$$
$$= \mathrm{i}\big[H(\xi - \mathbf{k}(r, \vartheta, w)), A\big]^\circ + \sigma_{i,j}\rho_{i,j}(\vartheta)\mathbf{v}_\xi(r, \vartheta, w) \cdot \nabla\omega(\mathbf{k}(r, \vartheta, w))\, \mathbb{1}_\mathcal{F},$$

and

$$\mathbf{v}_\xi(r, \vartheta, w) := \frac{\partial \mathbf{k}}{\partial \vartheta}(r, \vartheta, w) = r\Big(\cos(\vartheta)\Pi_\xi(w) + \sin(\vartheta)\frac{\xi}{|\xi|}\Big). \tag{3.31}$$

Let $k = \mathbf{k}(r, \vartheta, w) \neq 0$. By rotation invariance of $\nabla \omega$ we find that

$$\mathbf{v}_\xi(r, \vartheta, w) \cdot \nabla \omega(k) = (\mathbf{v}_\xi(r, \vartheta, w) \cdot k)(k \cdot \nabla \omega(k))/|k|^2,$$

which taken together with (3.19) and (3.13) enables us to establish that:

$$\forall k = \mathbf{k}(r, \vartheta, w) \in \mathbb{T}_{i,j}\left(\varepsilon_\vartheta, \varepsilon_r^{(1)}\right) : \quad \sigma_{i,j}\mathbf{v}_\xi(r, \vartheta, w) \cdot \nabla \omega(k) = r \sin(\vartheta)|\xi|\frac{|\nabla \omega(k)|}{|k|}.$$

This identity in conjunction with (3.18) implies, for all i and j, the crucial property

$$c_{i,j} := \inf_{\vartheta \in \Theta_{i,j}, w \in S^{\nu-2}} \sigma_{i,j}\mathbf{v}_\xi(R_i, \vartheta, w) \cdot \nabla \omega(\mathbf{k}(R_i, \vartheta, w)) > 0. \tag{3.32}$$

Note that $\sin(\vartheta) > 0$ for $\vartheta \in \overline{\Theta_{i,j}}$. The set $\Theta_{i,j}$ was defined in (3.30).

Now we pick and fix a $\chi'' \in C_0^\infty((E - \delta', E + \delta'))$. We choose χ'' such that $\chi'' = 1$ on $[E - 3\delta'/4, E + 3\delta'/4]$. Introduce bounded operators

$$B''(r) := \chi''(G_\xi(r))\mathrm{i}\Big[G_\xi(r), \widetilde{A}^{(1)}\Big]^\circ \chi''(G_\xi(r)). \tag{3.33}$$

We have

Lemma 3.8. *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. The maps $(0, \infty) \ni r \to \chi''(G_\xi(r))$ and $(0, \infty) \ni r \to B''(r)$ are locally Lipschitz, and furthermore: For any $0 < \bar{r} < \infty$ there exists $L > 0$ such that the following holds*

$$\forall r, r' \in (0, \bar{r}) : \quad \begin{cases} \|\chi''(G_\xi(r)) - \chi''(G_\xi(r'))\| \leq L|r - r'| \\ \|B''(r) - B''(r')\| \leq L|r - r'| \end{cases}$$

where L does not depend on v satisfying the constraint (3.27).

Proof. For the purpose of this proof we abbreviate

$$\mathbf{k} = \mathbf{k}(r, \vartheta, w) \quad \text{and} \quad \mathbf{k}' = \mathbf{k}(r', \vartheta, w).$$

We estimate first for $r, r' > 0$, using that $\nabla \omega$ is a bounded function,

$$|\omega(\mathbf{k}) - \omega(\mathbf{k}')| \leq C_1|\mathbf{k} - \mathbf{k}'|,$$

for some $C_1 > 0$. But

$$|\mathbf{k} - \mathbf{k}'| = |r - r'|\Big|\cos(\vartheta)\frac{\xi}{|\xi|} + \sin(\vartheta)\Pi_\xi(w)\Big| \leq 2|r - r'|, \tag{3.34}$$

such that, uniformly in ϑ and w,

$$|\omega(\mathbf{k}) - \omega(\mathbf{k}')| \leq 2C_1|r - r'|. \tag{3.35}$$

Next we compute as an identity between operators on $\mathcal{C}$

$$\Omega\big(\xi - \mathbf{k} - \mathrm{d}\Gamma(k)\big) - \Omega\big(\xi - \mathbf{k}' - \mathrm{d}\Gamma(k)\big) = \int_0^1 \nabla \Omega\big(\xi - s\mathbf{k} - (1-s)\mathbf{k}' - \mathrm{d}\Gamma(k)\big)\mathrm{d}s \, (\mathbf{k} - \mathbf{k}').$$

Appealing to (3.34) we arrive at

$$\left\|\Big(\Omega(\xi-\mathbf{k}-\mathrm{d}\Gamma(k))-\Omega(\xi-\mathbf{k}'-\mathrm{d}\Gamma(k))\Big)\Big(1+\Omega(\xi-\mathrm{d}\Gamma(k))\Big)^{-\frac{1}{2}}\right\|\leq C_2|r-r'|,\quad(3.36)$$

which holds uniformly in $0<r,r'\leq\bar{r}$, ϑ and w.

For $z\in\mathbb{C}$, with $\mathrm{Im}(z)\neq0$, we can now estimate using (3.35) and (3.36) to obtain

$$\left\|\Big((G_\xi(r)-z)^{-1}-(G_\xi(r')-z)^{-1}\Big)\psi\right\|\leq C|r-r'|\langle z\rangle|\mathrm{Im}(z)|^{-2}\left\|(H_0(\xi)+1)^{-\frac{1}{2}}\psi\right\|,$$

where $C>0$ does not depend on $0<r,r'\leq\bar{r}$, nor on z. Representing $\chi''(G_\xi(r))$ using an almost analytic extension of χ'' now yields the estimate

$$\left\|\Big(\chi''(G_\xi(r))-\chi''(G_\xi(r'))\Big)\psi\right\|\leq C|r-r'|\left\|(H_0(\xi)+1)^{-\frac{1}{2}}\psi\right\|.\qquad(3.37)$$

Here one should read $(H_0(\xi)+1)^{-\frac{1}{2}}$ as a (ϑ,w)-independent operator acting on each fiber $\mathcal{F}$ by the same operator. This in particular proves that the map $r\to\chi''(G_\xi(r))$ is locally Lipschitz and that the claimed bound holds.

We proceed to estimate the difference between the commutators

$$\begin{aligned}\mathrm{i}&\Big[G_\xi(r),\widetilde{A}^{(1)}\Big]^\circ-\mathrm{i}\Big[G_\xi(r'),\widetilde{A}^{(1)}\Big]^\circ\\&=\int_{\mathbb{R}\times S^{\nu-2}}^{\oplus}\mathrm{i}[H(\xi-\mathbf{k}),A]^\circ-\mathrm{i}[H(\xi-\mathbf{k}'),A]^\circ\\&\quad+\rho_{i,j}(\vartheta)\Big(\mathbf{v}_\xi(r,\vartheta,w)\cdot\nabla\omega(\mathbf{k})-\mathbf{v}_\xi(r',\vartheta,\rho)\cdot\nabla\omega(\mathbf{k}')\Big)\mathbb{1}_\mathcal{F}\,\mathrm{d}\vartheta\mathrm{d}w.\end{aligned}\qquad(3.38)$$

Appealing to (3.31) and (3.34) we find that

$$\left|\rho_{i,j}(\vartheta)\Big(\mathbf{v}_\xi(r,\vartheta,w)\cdot\nabla\omega(\mathbf{k})-\mathbf{v}_\xi(r',\vartheta,\rho)\cdot\nabla\omega(\mathbf{k}')\Big)\right|\leq C|r-r'|,\qquad(3.39)$$

for some $C>0$ independent of v, satisfying the constraint (3.27), and $0<r,r'\leq\bar{r}$. We then compute

$$\mathrm{i}[H(\xi-\mathbf{k}),A]^\circ-\mathrm{i}[H(\xi-\mathbf{k}'),A]^\circ=\mathrm{d}\Gamma(v)\cdot\Big\{\nabla\Omega(\xi-\mathbf{k}-\mathrm{d}\Gamma(k))-\nabla\Omega(\xi-\mathbf{k}'-\mathrm{d}\Gamma(k))\Big\}.\qquad(3.40)$$

Arguing as for (3.36) we get

$$\left\|(H_0(\xi)+1)^{-\frac{1}{2}}\Big(\mathrm{i}[H(\xi-\mathbf{k}),A]^\circ-\mathrm{i}[H(\xi-\mathbf{k}'),A]^\circ\Big)(H_0(\xi)+1)^{-1}\right\|\leq C|r-r'|\|v\|_\infty,\qquad(3.41)$$

where $C>0$ does not depend on $0<r,r'\leq\bar{r}$ or on v.

Putting together (3.38)–(3.41) we arrive at

$$\left\|(H_0(\xi)+1)^{-\frac{1}{2}}\Big(\mathrm{i}\Big[G_\xi(r),\widetilde{A}^{(1)}\Big]^\circ-\mathrm{i}\Big[G_\xi(r'),\widetilde{A}^{(1)}\Big]^\circ\Big)(H_0(\xi)+1)^{-1}\right\|\leq C|r-r'|,$$

The lemma now follows from the bound above together with (3.37). $\qquad\square$

Lemma 3.9. *Assume Conditions 1.2 and 1.3, with $n_0=1$. Let $i\in\{1,\dots,M\}$ and $j\in\{1,\dots,M_i\}$. We have*

$$\mathbb{1}_{\Theta_{i,j}}B''(R_i)=\mathbb{1}_{\Theta_{i,j}}\left(\int_{\mathbb{R}\times S^2}^{\oplus}\rho_{i,j}(\vartheta)\mathbf{v}_\xi(R_i,\vartheta,w)\cdot\nabla\omega(\mathbf{k}(R_i,\vartheta,w))\,\mathrm{d}\vartheta\mathrm{d}w\right)\chi''(G_\xi(R_i))^2.$$

Furthermore, we have

$$\mathbb{1}_{\Theta_{i,j}}B''(R_i)\geq c''\,\mathbb{1}_{\Theta_{i,j}}\chi''(G_\xi(R_i))^2,$$

for some $c''>0$, which does not depend on v satisfying the constraint (3.27).

Remark 3.10. The operator $\mathbb{1}_{\Theta_{i,j}}$ should be read as $\int_{\mathbb{R}\times S^2}^{\oplus} \mathbb{1}_{\Theta_{i,j}}(\vartheta)\, \mathbb{1}_{\mathcal{F}}\, \mathrm{d}\vartheta \mathrm{d}w$. A similar notation is used for $\rho_{i,j}$ in the proof below. $\diamond$

Proof. We begin by writing

$$\mathbb{1}_{\Theta_{i,j}}\, \chi''(G_\xi(R_i)) = \int_{\mathbb{R}\times S^2} \mathbb{1}_{\Theta_{i,j}}(\vartheta)\chi''(G_\xi(R_i,\vartheta,w))\, \mathrm{d}\vartheta \mathrm{d}w.$$

When ϑ is confined to the neighborhood $\Theta_{i,j}$ we have

$$\chi''\big(G_\xi(R_i,\vartheta,w)\big) = \chi''\big(H(\xi - \mathbf{k}(R_i,\vartheta,w)) + \omega(\mathbf{k}(R_i,\vartheta,w))\big)$$
$$= \mathbb{1}_{E-\omega(\mathbf{k}(R_i,\vartheta_{i,j},e_1))}\big(H(\xi - \mathbf{k}(R_i,\vartheta,w))\big).$$

This is due to the choice of δ', cf. (3.24), which ensures that we can at most land on one energy level, namely on the crossing $\mathcal{X}_{i,j}$ sitting at height E in energy-momentum space. By the virial theorem, cf. Theorem 2.10, this implies that

$$\int_{\mathbb{R}\times S^{\nu-2}}^{\oplus} \mathbb{1}_{\Theta_{i,j}}(\vartheta)\chi''\big(G_\xi(R_i,\vartheta,w)\big)\mathrm{i}\big[G_\xi(R_i),\widetilde{A}^{(1)}\big]^\circ(\vartheta,w)\chi''\big(G_\xi(R_i,\vartheta,w)\big)\, \mathrm{d}\vartheta \mathrm{d}w$$

$$= \int_{\mathbb{R}\times S^{\nu-2}}^{\oplus} \mathbb{1}_{\Theta_{i,j}}(\vartheta)\chi''\big(G_\xi(R_i,\vartheta,w)\big)$$
$$\times \mathrm{i}\Big[\big(E - \omega(\mathbf{k}(R_i,\vartheta_{i,j},e_1))\big)\mathbb{1}_{\mathcal{F}} + \omega(\mathbf{k}(R_i,\vartheta,w))\,\mathbb{1}_{\mathcal{F}},\widetilde{A}^{(1)}\Big]^\circ(\vartheta,w)$$
$$\times \chi''\big(G_\xi(R_i,\vartheta,w)\big)\, \mathrm{d}\vartheta \mathrm{d}w$$

$$= \int_{\mathbb{R}\times S^{\nu-2}}^{\oplus} \mathbb{1}_{\Theta_{i,j}}(\vartheta)\mathbf{v}_\xi(R_i,\vartheta,w)\cdot\nabla\omega(\mathbf{k}(R_i,\vartheta,w))\chi''\big(G_\xi(R_i,\vartheta,w)\big)^2\, \mathrm{d}\vartheta \mathrm{d}w.$$

This proves the first part. The second statement clearly follows from the first together with (3.32). One can choose

$$c'' := \inf_{|\vartheta_{i,j}-\vartheta|\leq\varepsilon_\vartheta} \sigma_{i,j}\mathbf{v}_\xi(R_i,\vartheta,w)\cdot\nabla\omega(\mathbf{k}(R_i,\vartheta,w)) > 0,$$

which is independent of w. $\square$

We now fix $\chi' \in C_0^\infty((E - 3\delta'/4, E + 3\delta'/4))$, with $\chi' = 1$ on $[E - \delta'/2, E + \delta'/2]$, and write

$$B'(r) = \chi'(G_\xi(r))\mathrm{i}\big[G_\xi(r),\widetilde{A}^{(1)}\big]^\circ\chi'(G_\xi(r)) = \chi'(G_\xi(r))B''(r)\chi'(G_\xi(r)). \tag{3.42}$$

The operator $B''(r)$ was defined in (3.33).

Proposition 3.11. *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. There exists $c' > 0$ and $0 < \varepsilon_r \leq \varepsilon_r^{(1)}$, independent of v satisfying (3.27), such that for all $i \in \{1,\ldots,M\}$ and $j \in \{1,\ldots,M_i\}$, we have*

$$\forall r \in [R_i - \varepsilon_r, R_i + \varepsilon_r]: \quad \mathbb{1}_{\Theta_{i,j}} B'(r) \geq c'\, \mathbb{1}_{\Theta_{i,j}}\, \chi'(G_\xi(r))^2.$$

Proof. Apply Lemmata 3.8 and 3.9. This yields the bounds

$$\begin{aligned}
\mathbb{1}_{\Theta_{i,j}} B''(r) &\geq \mathbb{1}_{\Theta_{i,j}} B''(R_i) - L|R_i - r| \\
&\geq c'' \mathbb{1}_{\Theta_{i,j}} \chi''(G_\xi(R_i))^2 - L|R_i - r| \\
&\geq c'' \mathbb{1}_{\Theta_{i,j}} \chi''(G_\xi(r))^2 - L(1 + c'')|R_i - r|.
\end{aligned}$$

Here L is the Lipschitz constant coming from Lemma 3.8 applied with

$$\bar{r} = \max_{1 \leq i \leq M} R_i + \varepsilon_r^{(1)}.$$

Choose $\varepsilon_r = \min\{\varepsilon_r^{(1)}, c''(2L(1 + c''))^{-1})\}$. Multiplying both sides first by $\chi'(G_\xi(r))$ from the left and the right, and subsequently by $\mathbb{1}_{\Theta_{i,j}}$, yields the result with $c' = c''/2$. Recall that the all the operators are fibered, i.e. they are functions of ϑ and w. $\square$

3.4   The Conjugate Operator

The task at hand in this section is the construction of the vector field $v_\xi \colon \mathbb{R}^\nu \to \mathbb{R}^\nu$, used to define the conjugate operators

$$a_\xi = \tfrac{1}{2}\big(v_\xi \cdot \mathrm{i}\nabla_k + \mathrm{i}\nabla_k \cdot v_\xi\big), \quad \text{and} \quad A_\xi = \mathrm{d}\Gamma(a_\xi). \tag{3.43}$$

These are operators on the form considered in Chapter 2, cf.(2.1) and (2.2). The vector field v_ξ will depend both on the total momentum ξ and on the energy localization we choose. Our first ingredient is a partition of unity in momentum space subordinate to an appropriately chosen open covering of $\mathcal{K}_{\mathcal{J}'}$. The first sets in the covering were constructed in the previous section, namely the disjoint open tori $\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)$.

Define a set of momenta

$$\mathcal{K} := \mathcal{K}(\varepsilon_r),$$

which is a compact subset of $\mathbb{R}^\nu$. Recall from (3.26) the definition of the set $\mathcal{K}(\varepsilon_r)$. We proceed to pick a $0 < \delta \leq \delta'$ with the property that the choice $\mathcal{J} = [E - \delta, E + \delta]$ ensures that $\Sigma_{\mathcal{J}}(\mathcal{K})$ is a graph, i.e. the projection onto momentum space $\mathbb{R}^\nu$ is injective. To do this we define the energy distance between mass shells $(\mathcal{A}, S) \in \mathcal{S}'$, away from $\varepsilon_r/2$ neighborhoods of their boundaries $\partial\mathcal{A}$: By compactness of $\Sigma_{\mathcal{J}'} \subset \Sigma_{\mathrm{iso}}$, there exist $P > 0$ and $\sigma > 0$ such that

$$\Sigma_{\mathcal{J}'}(\mathcal{K}) \subset \Sigma_{\mathcal{J}'} \subset \big\{(p, \lambda) \,\big|\, |p| \leq P \text{ and } \lambda \leq \Sigma_{\mathrm{ess}}(p) - \sigma\big\}.$$

Hence we define for $(\mathcal{A}, S) \in \mathcal{S}'$

$$\delta_{(\mathcal{A}, S)} = \inf\big\{d(S(p), \sigma_{\mathrm{pp}}(H(p))\backslash\{S(p)\}) \,\big|\, p \in \mathcal{A}_{\varepsilon_r, P}, S(p) \leq \Sigma_{\mathrm{ess}}(p) - \sigma\big\},$$

$$\text{where} \quad \mathcal{A}_{\varepsilon_r, P} = \big\{p \in \mathcal{A} \,\big|\, d(p, \partial\mathcal{A}) \geq \varepsilon_r/2\big\} \cap \overline{B_P(0)}.$$

Again, by compactness, $\delta = \min_{(\mathcal{A}, S) \in \mathcal{S}'} \delta_{(\mathcal{A}, S)} > 0$. Recall that $\mathcal{S}'$ denotes the finite collection of mass shells available for scattering, cf. (3.25).

We split $\mathcal{K}$ into compact components pertaining to shells $(\mathcal{A}, S) \in \mathcal{S}'$

$$\mathcal{K}_{(\mathcal{A}, S)} = \mathcal{K} \cap (\mathcal{A} + \xi) = \big\{k \in \mathcal{K} \,\big|\, \xi - k \in \mathcal{A}\big\}.$$

We pick open neighborhoods

$$\mathcal{V}_{(\mathcal{A},S)} = \left\{ k \in \mathbb{R}^\nu \,\middle|\, d(k, \mathcal{K}_{(\mathcal{A},S)}) < \varepsilon_r/2 \right\} \subset \mathcal{A} + \xi$$

which inherit the property of $\mathcal{K}$ that the distance from $S(\xi - k)$ to the nearest eigenvalue in $\sigma_{\mathrm{pp}}(H(\xi - k)) \backslash \{S(\xi - k)\}$ is at least a distance δ away. In addition we remark that the sets $\mathcal{V}_{(\mathcal{A},S)}$ are pairwise disjoint but overlap with possibly existing tori $\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)$, provided $\partial B(0, R_i)$ is a boundary of $\mathcal{A}$.

To make a partition of unity we choose first $\varphi_{i,j} \in C_0^\infty(\mathbb{T}_{i,j}(2\varepsilon_\vartheta, 2\varepsilon_r))$ such that $\varphi_{i,j} = 1$ on $\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)$. It will be convenient to use a product construction such that

$$\varphi_{i,j}(\mathbf{k}(r, \vartheta, w)) = \rho_{i,j}(\vartheta)\widetilde{\rho}_i(r),$$

where $\rho_{i,j}$ was introduced in the previous section and $\widetilde{\rho}_i \in C_0^\infty((-2\varepsilon_r, 2\varepsilon_r))$, with $\widetilde{\rho}_i = 1$ on $[R_i - \varepsilon_r, R_i + \varepsilon_r]$ and satisfying that $0 \leq \widetilde{\rho}_i \leq 1$.

Using the smooth Urysohn lemma on the pairs $\mathcal{K}_{(\mathcal{A},S)} \subset V_{(\mathcal{A},S)}$, with $(\mathcal{A}, S) \in \mathcal{S}'$, yields smooth functions $\widetilde{\varphi}_{(\mathcal{A},S)}$ with compact support in $V_{(\mathcal{A},S)}$ and equal to 1 on $\mathcal{K}_{(\mathcal{A},S)}$. By a standard construction we can replace these by (possibly) smaller functions $\varphi_{(\mathcal{A},S)}$ with the same two properties and the additional property that

$$\forall k \in \mathcal{K}_{\mathcal{J}'}: \quad \sum_{i,j} \varphi_{i,j}(k) + \sum_{(\mathcal{A},S) \in \mathcal{S}'} \varphi_{(\mathcal{A},S)}(k) = 1. \tag{3.44}$$

We can now construct our vector field

$$v_\xi = \sum_{i,j} v_{i,j}^\xi + \sum_{(\mathcal{A},S) \in \mathcal{S}'} v_{(\mathcal{A},S)}^\xi, \tag{3.45}$$

where

$$v_{i,j}^\xi(k) = \sigma_{i,j}\varphi_{i,j}(k)\mathbf{v}_\xi(k) \quad \text{and} \quad v_{(\mathcal{A},S)}^\xi(k) = \varphi_{(\mathcal{A},S)}(k)\frac{\nabla S^{(1)}(\xi; k)}{|\nabla S^{(1)}(\xi; k)|}. \tag{3.46}$$

We have thus finished the construction of the conjugate operator A_ξ, cf. (3.43). We remark that the construction of v_ξ is consistent with the constraint (3.27), cf. (3.31). The signs $\sigma_{i,j}$ and the vector field $\mathbf{v}_\xi$ were introduced in (3.19) and (3.31) respectively.

In the following we make use of the notation

$$A_\xi^{\mathrm{x}} = \mathrm{d}\Gamma^{\mathrm{x}}(a_\xi) = A_\xi \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{F} \otimes A_\xi \qquad \text{on } \mathcal{F}^{\mathrm{x}} \tag{3.47}$$

and observe the direct sum decomposition

$$A_\xi^{\mathrm{x}} = \oplus_{\ell=0}^\infty A_\xi^{(\ell)}, \qquad \text{where } A_\xi^{(\ell)} = A_\xi \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} + \mathbb{1}_\mathcal{F} \otimes \mathrm{d}\Gamma(a_\xi)_{|\mathcal{F}^{(\ell)}} \text{ on } \mathcal{F} \otimes \mathcal{F}^{(\ell)}. \tag{3.48}$$

In particular, $A_\xi^{(0)} = A_\xi$. See also Section 1.2 for notation and constructions pertaining to extended objects.

The following proposition is a Mourre estimate for $H^{(1)}(\xi)$, with conjugate operator $A_\xi^{(1)}$, stating that a composite system consisting of a dressed matter particle and a free boson at total momentum ξ, localized in energy in the interval $\mathcal{J}$, has non-zero breakup velocity. This is the source of positivity in the Mourre estimate for H we prove in the following section.

Proposition 3.12. *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. Let $\chi \in C_0^\infty(\mathcal{J})$, with $\chi \geq 0$. Then there exists $c > 0$ such that*

$$\chi\left(H^{(1)}(\xi)\right)\mathrm{i}\left[H^{(1)}(\xi), A_\xi^{(1)}\right]^\circ \chi\left(H^{(1)}(\xi)\right) \geq c\chi\left(H^{(1)}(\xi)\right)^2.$$

Proof. Let

$$B(k) = \chi\left(H^{(1)}(\xi; k)\right)\mathrm{i}\left[H^{(1)}(\xi), A_\xi^{(1)}\right]^\circ(k)\chi\left(H^{(1)}(\xi; k)\right)$$

and write

$$\begin{aligned}
B &:= \chi\left(H^{(1)}(\xi)\right)\mathrm{i}\left[H^{(1)}(\xi), A_\xi^{(1)}\right]^\circ \chi\left(H^{(1)}(\xi)\right) \\
&= \int_{\mathbb{R}^\nu}^\oplus B(k)\,\mathrm{d}k \\
&= \sum_{i,j} \int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k)B(k)\,\mathrm{d}k + \sum_{(\mathcal{A},S)\in\mathcal{S}'} \int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k)B(k)\,\mathrm{d}k,
\end{aligned}$$

where the summation over i and j is understood to be over $i = 1, \ldots, M$ and $j = 1, \ldots, M_i$.

We split the operator $A_\xi^{(1)}$ into the sum

$$A_\xi^{(1)} = A_\xi \otimes \mathbb{1}_\mathfrak{h} + \sum_{i,j} \mathbb{1}_\mathcal{F} \otimes a_{i,j} + \sum_{(\mathcal{A},S)\in\mathcal{S}'} \mathbb{1}_\mathcal{F} \otimes a_{(\mathcal{A},S)},$$

corresponding to the construction of v_ξ, cf. (3.45) and (3.48). This induces a decomposition of $B(k)$ using Lemma 2.6 and Proposition 2.8:

$$B(k) = B_0(k) + \sum_{i,j} B_{i,j}(k) + \sum_{(\mathcal{A},S)\in\mathcal{S}'} B_{(\mathcal{A},S)}(k),$$

where

$$\begin{aligned}
B_0(k) &= \chi\left(H^{(1)}(\xi; k)\right)\mathrm{i}\left[H^{(1)}(\xi), A_\xi \otimes \mathbb{1}_\mathfrak{h}\right]^\circ(k)\chi\left(H^{(1)}(\xi; k)\right), \\
B_{i,j}(k) &= \chi\left(H^{(1)}(\xi; k)\right)\mathrm{i}\left[H^{(1)}(\xi), \mathbb{1}_\mathcal{F} \otimes a_{i,j}\right]^\circ(k)\chi\left(H^{(1)}(\xi; k)\right), \\
B_{(\mathcal{A},S)}(k) &= \chi\left(H^{(1)}(\xi; k)\right)\mathrm{i}\left[H^{(1)}(\xi), \mathbb{1}_\mathcal{F} \otimes a_{(\mathcal{A},S)}\right]^\circ(k)\chi\left(H^{(1)}(\xi; k)\right).
\end{aligned}$$

Observe that that $k \to B_{i,j}(k)$ has support in the torus $\mathbb{T}_{i,j}(2\varepsilon_\vartheta, 2\varepsilon_r)$ and $k \to B_{(\mathcal{A},S)}(k)$ has support in $V_{(\mathcal{A},S)}$. Using these support properties we compute

$$\int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k)B(k)\,\mathrm{d}k = \int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k)\left(B_0(k) + B_{i,j}(k)\right)\mathrm{d}k.$$

Write $\widetilde{B}_{i,j}(k) = B_0(k) + B_{i,j}(k)$ and observe that

$$\int_{\mathbb{R}\times S^{\nu-2}}^\oplus \widetilde{B}_{i,j}(\mathbf{k}(r,\vartheta,w))\mathrm{d}\vartheta\mathrm{d}w = \chi(G_\xi(r))B'(r)\chi(G_\xi(r)),$$

where $B'(r)$ was introduced in (3.42) and $G_\xi(r)$ in (3.28) and (3.29). Here we used that $\chi\chi' = \chi$. We have obtained the identity

$$\int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k) B(k) \, \mathrm{d}k$$
$$= \int_{[0,\infty)}^\oplus \mathbb{1}_{(R_i - \varepsilon_r, R_i + \varepsilon_r)}(r) \left\{ \int_{\mathbb{R} \times S^{\nu-2}}^\oplus \mathbb{1}_{\Theta_{i,j}}(\vartheta) \widetilde{B}_{i,j}(\mathbf{k}(r, \vartheta, w)) \, \mathrm{d}\vartheta \mathrm{d}w \right\} \mathrm{d}r$$
$$= \int_{[0,\infty)}^\oplus \mathbb{1}_{(R_i - \varepsilon_r, R_i + \varepsilon_r)}(r) \, \mathbb{1}_{\Theta_{i,j}} \, \chi(G_\xi(r)) B'(r) \chi(G_\xi(r)) \, \mathrm{d}r.$$

From Proposition 3.11 we thus find a $c' > 0$, independent of i and j, such that

$$\int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k) B(k) \, \mathrm{d}k \geq c' \int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k) \chi\big(H^{(1)}(\xi; k)\big)^2 \mathrm{d}k. \qquad (3.49)$$

To deal with the remaining contributions we compute, using the support properties of $B_{(\mathcal{A},S)}$,

$$\mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) B(k) = \mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) \Big\{ B_0(k) + B_{(\mathcal{A},S)}(k) + \sum_{i,j} B_{i,j}(k) \Big\}.$$

Since $\mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) \chi(H^{(1)}(\xi; k)) = \mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) \, \mathbb{1}_{\{S^{(1)}(\xi;k)\}}(H^{(1)}(\xi; k))$ we can apply the virial theorem, cf. Theorem 2.10, to compute for a.e. $k = \mathbf{k}(r, \vartheta, w) \in \mathcal{K}_{(\mathcal{A},S)}$

$$B_0(k) = 0,$$
$$B_{(\mathcal{A},S)}(k) = \varphi_{(\mathcal{A},S)}(k) |\nabla_k S^{(1)}(\xi; k)| \geq c'_{(\mathcal{A},S)} \varphi_{(\mathcal{A},S)}(k),$$
$$B_{i,j}(k) = \sigma_{i,j} \varphi_{i,j}(k) \mathbf{v}_\xi(r, \vartheta, w) \cdot \nabla\omega(\mathbf{k}(r, \vartheta, w)) \geq c_{i,j} \varphi_{i,j}(k).$$

Here the constants $c_{i,j}$ are defined in (3.32),

$$c'_{(\mathcal{A},S)} := \inf_{k \in \mathcal{K}_{(\mathcal{A},S)}} |\nabla_k S^{(1)}(\xi; k)| > 0$$

and positivity follows from $\mathcal{K}_{(\mathcal{A},S)}$ being compact and $\mathcal{J}$ being chosen to not contain threshold energies. Summing up, cf. (3.44) and recalling that for distinct shells $\mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}} \varphi_{(\mathcal{A}',S')} = 0$, we get

$$\int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) B(k) \, \mathrm{d}k \geq c_{(\mathcal{A},S)} \int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) \chi\big(H^{(1)}(\xi; k)\big)^2 \mathrm{d}k, \qquad (3.50)$$

with $c_{(\mathcal{A},S)} = \min\{c'_{(\mathcal{A},S)}, \min_{i,j} c_{i,j}\}$.

Combining (3.49) and (3.50) we finally get

$$B \geq c \int_{\mathbb{R}^\nu}^\oplus \Big\{ \sum_{i,j} \mathbb{1}_{\mathbb{T}_{i,j}(\varepsilon_\vartheta, \varepsilon_r)}(k) + \sum_{(\mathcal{A},S) \in \mathcal{S}'} \mathbb{1}_{\mathcal{K}_{(\mathcal{A},S)}}(k) \Big\} \chi\big(H^{(1)}(\xi; k)\big)^2 \mathrm{d}k$$
$$= c\chi\big(H^{(1)}(\xi)\big)^2,$$

with $c = \min\{c', \min_{(\mathcal{A},S) \in \mathcal{S}'} c_{(\mathcal{A},S)}\}$. $\qquad \square$

3.5 The Mourre Estimate

We will make use of a geometric partition of unity, introduced in [11], and since used frequently, cf, [2, 12, 16, 22, 38], to perform localization arguments in non-relativistic QFT.

The input is a pair of smooth functions $j_0, j_\infty \colon \mathbb{R}^\nu \to [0,1]$, with the properties $j_0^2 + j_\infty^2 = 1$ and $j_0 = 1$ on $\{x \in \mathbb{R}^\nu \,|\, |x| \leq 1\}$ and $j_\infty = 1$ on $\{x \in \mathbb{R}^\nu \,|\, |x| \geq 2\}$. We scale these functions and define for $R > 0$ localizations $j_\#^R(x) = j_\#(x/R)$ inside and outside of balls with a radius scaling like R. Reading now $x = \mathrm{i}\nabla_k$ these operators become bounded self-adjoint operators on $\mathfrak{h}$ and we form the vector operator $j^R = (j_0^R, j_\infty^R) \colon \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$. It satisfies $(j^R)^* j^R = \mathbb{1}_\mathfrak{h}$. The operator $\Gamma(j^R)$ now maps $\mathcal{F} = \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h} \oplus \mathfrak{h})$, and composing with the canonical identification operator $U \colon \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) = \mathcal{F}^\mathrm{x}$ we get

$$\check{\Gamma}(j^R) := U\Gamma(j^R) \colon \mathcal{F} \to \mathcal{F}^\mathrm{x}.$$

The operator is a 'partition of unity' in that it is an isometry, i.e. $\check{\Gamma}(j^R)^* \check{\Gamma}(j^R) = \mathbb{1}_\mathcal{F}$.

Lemma 3.13. *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. Let $f \in C_0^\infty(\mathbb{R})$. Then*

(i) $\check{\Gamma}(j^R) f(H(\xi)) = f(H^\mathrm{x}(\xi))\check{\Gamma}(j^R) + o_R(1)$

(ii) $\check{\Gamma}(j^R) f(H(\xi))\mathrm{i}[H(\xi), A_\xi]^\circ f(H(\xi))$

$\quad = f(H^\mathrm{x}(\xi))\mathrm{i}[H^\mathrm{x}(\xi), A_\xi^\mathrm{x}]^\circ f(H^\mathrm{x}(\xi))\check{\Gamma}(j^R) + o_R(1)$

Remark 3.14. We note that (i) was already proved in [38] in the case $s_\Omega \in \{0, 1, 2\}$. As the assumption of s_Ω being integer is only used in the proof of this result in [38], this new proof now secures the validity of the results in [38] for non-integer values of s_Ω. ◇

Proof. In the following we fix a $\lambda < \Sigma_0$, cf. (1.15). We will start by proving the following statements: For $p \in \{1, \ldots, \nu\}$ and $w \in C_0^\infty(\mathbb{R}^\nu)$ we claim that

(a) $\check{\Gamma}(j^R) f(H(\xi)) \colon \mathcal{F} \to \mathcal{D}_{1/2}^\mathrm{x}$ and $f(H^\mathrm{x}(\xi))\check{\Gamma}(j^R) \colon \mathcal{D}_{1/2}^* \to \mathcal{F}^\mathrm{x}$ for any $R > 1$ and,

$$(H^\mathrm{x}(\xi) - \lambda)^{-\frac{1}{2}}\big(\check{\Gamma}(j^R) H(\xi) - H^\mathrm{x}(\xi)\check{\Gamma}(j^R)\big) f(H(\xi)) = o_R(1),$$
$$f(H^\mathrm{x}(\xi))\big(\check{\Gamma}(j^R) H(\xi) - H^\mathrm{x}(\xi)\check{\Gamma}(j^R)\big)(H(\xi) - \lambda)^{-\frac{1}{2}} = o_R(1).$$

(b) $\big(\check{\Gamma}(j^R)\partial_p\Omega(\xi - \mathrm{d}\Gamma(k)) - \partial_p\Omega(\xi - \mathrm{d}\Gamma^\mathrm{x}(k))\check{\Gamma}(j^R)\big) f(H(\xi)) = o_R(1).$

(c) $f(H^\mathrm{x}(\xi))\big(\check{\Gamma}(j^R)\mathrm{d}\Gamma(w) - \mathrm{d}\Gamma^\mathrm{x}(w)\check{\Gamma}(j^R)\big)(H_0(\xi) - \lambda)^{-\frac{1}{2}} = o_R(1).$

(d) $f(H^\mathrm{x}(\xi))\big(\check{\Gamma}(j^R)\phi(\mathrm{i}a_\xi g) - \phi(\mathrm{i}a_\xi g) \otimes \mathbb{1}_\mathcal{F}\,\check{\Gamma}(j^R)\big) f(H(\xi)) = o_R(1).$

In the rest of the proof we abbreviate $H_0 = H_0(\xi)$, $H = H(\xi)$, $A = A_\xi$, $H^\mathrm{x} = H^\mathrm{x}(\xi)$, $A^\mathrm{x} = A_\xi^\mathrm{x}$, $\phi^\mathrm{x}(w) = \phi(w) \otimes \mathbb{1}_\mathcal{F}$, and $\check{\Gamma} = \check{\Gamma}(j^R)$. For notational convenience, we write $M \overset{o}{=} N$ if $M = N + o_R(1)$.

Ad (a). We only prove half of the statement as the other half follows by a symmetric argument. Note that $(H^{\mathrm{x}}-\lambda)^{-1/2}(\check{\Gamma}(H-\Omega(\xi-\mathrm{d}\Gamma(k)))-(H^{\mathrm{x}}-\Omega(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k)))\check{\Gamma})f(H) = o_R(1)$ by (the proof of) [38, Lemma 3.2]. Hence, to prove the statement, we need only show that

$$(H^{\mathrm{x}}-\lambda)^{-\frac{1}{2}}\big(\check{\Gamma}\Omega(\xi-\mathrm{d}\Gamma(k))-\Omega(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k))\check{\Gamma}\big)f(H) = o_R(1). \tag{3.51}$$

In order to use a commutator expansion formula, we find it useful to cast the statement differently by extending $\check{\Gamma}$ and its adjoint to operators on $\mathcal{F}^{\mathrm{x}}$. Let, for the purpose of this proof only, $P\colon \mathcal{F}^{\mathrm{x}} \to \mathcal{F}$ be the projection

$$\mathcal{F}^{\mathrm{x}} = \mathcal{F} \oplus \Big(\bigoplus_{\ell=1}^{\infty} \mathcal{F} \otimes \mathcal{F}^{(\ell)}\Big) \ni (u,v) \mapsto u \in \mathcal{F}$$

and $I = P^*\colon \mathcal{F} \to \mathcal{F}^{\mathrm{x}}$ the injection

$$\mathcal{F} \ni u \mapsto Iu = (u,0) \in \mathcal{F} \oplus \Big(\bigoplus_{\ell=1}^{\infty} \mathcal{F} \otimes \mathcal{F}^{(\ell)}\Big).$$

Define $\check{\Gamma}^{\mathrm{x}}\colon \mathcal{F}^{\mathrm{x}} \to \mathcal{F}^{\mathrm{x}}$ by $\check{\Gamma}^{\mathrm{x}} = \check{\Gamma}P$. Note that PI is the identity on $\mathcal{F}$ and that

$$\check{\Gamma}^{\mathrm{x}}I = \check{\Gamma}, \quad H^{\mathrm{x}}I = IH, \quad A^{\mathrm{x}}I = IA \quad \text{and} \quad \phi^{\mathrm{x}}(g)I = I\phi(g).$$

We write, using [38, Lemma 3.6],

$$\begin{aligned}
(H^{\mathrm{x}}-\lambda)^{-\frac{1}{2}}&\big[\check{\Gamma}^{\mathrm{x}},\Omega(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k))\big]f(H^{\mathrm{x}}) \\
&= \Big\{(H^{\mathrm{x}}-\lambda)^{-\frac{1}{2}}\big[\check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2},\Omega(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k))\big]\Big\}(N^{\mathrm{x}}+1)^2 f(H^{\mathrm{x}}).
\end{aligned} \tag{3.52}$$

The estimate (3.51) follows if the term in the brackets above is $o_R(1)$. The commutator $[\check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2}, \Omega(\xi-d\Gamma^{\mathrm{x}}(k))]$ satisfies the assumptions of [45, Theorem 3] with

$$B = \check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2}, \quad A = \xi - \mathrm{d}\Gamma^{\mathrm{x}}(k), \quad f_\lambda = \Omega, \quad s = s_\Omega, \quad n_0 = 2 \quad \text{and} \quad n = 1.$$

Hence, by [45, Theorem 3] we obtain the commutator expansion

$$\begin{aligned}
\big[\check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2},\Omega(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k))\big] &= \sum_{p=1}^{\nu} \partial_p\Omega(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k))\,\mathrm{ad}_{\xi_p-\mathrm{d}\Gamma^{\mathrm{x}}(k_p)}\big(\check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2}\big) \\
&\quad + R_1\big(\xi-\mathrm{d}\Gamma^{\mathrm{x}}(k),\check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2}\big).
\end{aligned} \tag{3.53}$$

The remainder $R_1(A,B)$ satisfies for some $C > 0$ the bound

$$\|R_1(A,B)\| \leq C \sum_{|\alpha|=2} \|\mathrm{ad}_A^\alpha(B)\|. \tag{3.54}$$

Let $\psi \in \mathcal{F}^{(m)}$ and $\varphi \in \mathcal{F} \otimes \mathcal{F}^{(\ell)}$, with $m \geq 0$ and $\ell \geq 1$. Note that $P(\psi\otimes|0\rangle, \varphi) = \psi$. One can readily verify that

$$\mathrm{ad}_{\mathrm{d}\Gamma^{\mathrm{x}}(k)}^\alpha(\check{\Gamma}^{\mathrm{x}})\begin{pmatrix}\psi\otimes|0\rangle\\\varphi\end{pmatrix} = U\sum_{\sum\alpha^{(\ell)}=\alpha}\frac{\alpha!}{\prod_{\ell=1}^m \alpha^{(\ell)}!}\bigotimes_{\ell=1}^m\begin{pmatrix}\mathrm{ad}_k^{\alpha^{(\ell)}}(j_0^R)\\\mathrm{ad}_k^{\alpha^{(\ell)}}(j_\infty^R)\end{pmatrix}\psi, \tag{3.55}$$

and

$$\mathrm{ad}_k^{\alpha^{(\ell)}}\left(j_\#^R\right) = i^{|\alpha^{(\ell)}|}R^{-|\alpha^{(\ell)}|}\left(\partial^{\alpha^{(\ell)}}j_\#\right)\left(\frac{x}{R}\right) = O\left(R^{-|\alpha^{(\ell)}|}\right),$$

where the sums are over all ordered sets of multi-indicies $\{\alpha^{(\ell)}\}_{\ell=1}^m$ in such a manner that $\sum_{\ell=1}^m \alpha^{(\ell)} = \alpha$. The identity (3.55) extends by linearity to an identity between bounded operators from $\mathcal{F}^{(m)} \otimes \mathcal{F}$ to $\mathbb{1}[N^{\mathrm{x}} = m]\mathcal{F}^{\mathrm{x}}$, where $N^{\mathrm{x}} = N \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes N$ is the extended number operator. Since

$$\sum_{\sum \alpha^{(\ell)}=\alpha} \frac{\alpha!}{\prod_\ell \alpha^{(\ell)}!} = m^{|\alpha|},$$

it follows that $\mathrm{ad}_{\mathrm{d}\Gamma^{\mathrm{x}}(k)}^\alpha(\check{\Gamma}^{\mathrm{x}})(N^{\mathrm{x}}+1)^{-|\alpha|} = O(R^{-|\alpha|})$, with respect to the norm on $\mathcal{B}(\mathcal{F}^{\mathrm{x}})$, and hence that

$$\sum_{p=1}^\nu \left\| \mathrm{ad}_{\xi_p - \mathrm{d}\Gamma^{\mathrm{x}}(k_p)}\left(\check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2}\right)\right\| + \left\|R_1\left(\xi - \mathrm{d}\Gamma^{\mathrm{x}}(k), \check{\Gamma}^{\mathrm{x}}(N^{\mathrm{x}}+1)^{-2}\right)\right\| = O(R^{-1}).$$

As $s_\Omega \leq 2$, $(H - \lambda)^{-1/2}\partial_p\Omega(\xi - \mathrm{d}\Gamma^{\mathrm{x}}(k))$ is bounded. These two observations together with (3.52), (3.53) and (3.54) imply the claim (a).

By an analogous argument we get (b). The proof of (c) and (d) can be found in the proof of [38, Lemma 3.2].

We can now prove (i). Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $f = f\chi$. We pull the energy localization through $\check{\Gamma}$ in two steps, using both bounds in (a) along the way,

$$
\begin{aligned}
f(H^{\mathrm{x}})\check{\Gamma} &= f(H^{\mathrm{x}})\chi(H^{\mathrm{x}})\check{\Gamma} \\
&= f(H^{\mathrm{x}})\check{\Gamma}\chi(H) + \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\chi}(z)(H^{\mathrm{x}} - z)^{-1}f(H^{\mathrm{x}})\left(\check{\Gamma}H - H^{\mathrm{x}}\check{\Gamma}\right)(H - z)^{-1}\mathrm{d}z \\
&\stackrel{o}{=} f(H^{\mathrm{x}})\check{\Gamma}\chi(H) \\
&= \check{\Gamma}f(H) + \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(H^{\mathrm{x}} - z)^{-1}\left(\check{\Gamma}H - H^{\mathrm{x}}\check{\Gamma}\right)\chi(H)(H - z)^{-1}\mathrm{d}z \\
&\stackrel{o}{=} \check{\Gamma}f(H).
\end{aligned}
$$

This computation establishes (i).

Finally we argue for the validity of (ii). Let $\chi \in C_0^\infty(\mathbb{R})$ be as in the proof of (i). By (i) and (a) we see that

$$
\begin{aligned}
f(H^{\mathrm{x}})[H^{\mathrm{x}}, A^{\mathrm{x}}]^\circ f(H^{\mathrm{x}})\check{\Gamma} & \\
&= f(H^{\mathrm{x}})[H^{\mathrm{x}}, A^{\mathrm{x}}]^\circ f(H^{\mathrm{x}})\check{\Gamma}\chi(H) + f(H^{\mathrm{x}})[H^{\mathrm{x}}, A^{\mathrm{x}}]^\circ f(H^{\mathrm{x}})o_R(1) \\
&\stackrel{o}{=} f(H)[H, A]^\circ\check{\Gamma}f(H) \\
&\quad + \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)f(H)[H^{\mathrm{x}}, A^{\mathrm{x}}]^\circ(H^{\mathrm{x}} - z)^{-1}\left(\check{\Gamma}H - H^{\mathrm{x}}\Gamma^{\mathrm{x}}\right)\chi(H)(H - z)^{-1}\mathrm{d}z \\
&\stackrel{o}{=} f(H^{\mathrm{x}})[H^{\mathrm{x}}, A^{\mathrm{x}}]^\circ\check{\Gamma}f(H),
\end{aligned}
$$

(3.56)

Here we used that $f(H^{\mathrm{x}})[H^{\mathrm{x}}, A^{\mathrm{x}}]^\circ(H^{\mathrm{x}} - \lambda)^{-1/2}$ is bounded. The right-hand side of

(3.56) splits into three terms

$$f(H^{\mathrm{x}})[H^{\mathrm{x}}, A^{\mathrm{x}}]^{\circ}\check{\Gamma}f(H) = f(H^{\mathrm{x}})\mathrm{d}\Gamma^{\mathrm{x}}(v_\xi \cdot \nabla\omega)\check{\Gamma}f(H) \tag{3.57a}$$

$$- f(H^{\mathrm{x}})\mathrm{d}\Gamma^{\mathrm{x}}(v_\xi) \cdot \nabla\Omega(\xi - \mathrm{d}\Gamma^{\mathrm{x}}(k))\check{\Gamma}f(H) \tag{3.57b}$$

$$- f(H^{\mathrm{x}})\phi^{\mathrm{x}}(\mathrm{i}a_\xi g)\check{\Gamma}f(H). \tag{3.57c}$$

Now by (c) we find $(3.57a) \stackrel{o}{=} f(H^{\mathrm{x}})\check{\Gamma}\mathrm{d}\Gamma(v_\xi \cdot \nabla\omega)f(H)$, by (b), (c) and (a)

$$(3.57b) \stackrel{o}{=} -f(H^{\mathrm{x}})\sum_{p=1}^{\nu}\mathrm{d}\Gamma^{\mathrm{x}}(v_{\xi;p})\check{\Gamma}\partial_p\Omega(\xi - \mathrm{d}\Gamma(k))f(H)$$

$$= -f(H^{\mathrm{x}})\check{\Gamma}\mathrm{d}\Gamma(v_\xi) \cdot \nabla\Omega(\xi - \mathrm{d}\Gamma(k))f(H)$$

$$- f(H)\sum_{p=1}^{\nu}\Big(\mathrm{d}\Gamma^{\mathrm{x}}(v_{\xi;p})\check{\Gamma} - \check{\Gamma}\mathrm{d}\Gamma(v_{\xi;p})\Big)(H_0 - \lambda)^{-\frac{1}{2}}$$

$$\cdot \partial_p\Omega(\xi - \mathrm{d}\Gamma(k))(H_0 - \lambda)^{\frac{1}{2}}f(H)$$

$$\stackrel{o}{=} -f(H^{\mathrm{x}})\check{\Gamma}\mathrm{d}\Gamma(v_\xi) \cdot \nabla\Omega(\xi - \mathrm{d}\Gamma(k))f(H),$$

and by (d) we conclude that $(3.57c) \stackrel{o}{=} -f(H^{\mathrm{x}})\check{\Gamma}\phi(\mathrm{i}a_\xi g)f(H)$. Putting this together – and again using (a) and (i) – we see that

$$(3.57) \stackrel{o}{=} f(H^{\mathrm{x}})\check{\Gamma}[H, A]^{\circ}f(H)$$

$$= \chi(H^{\mathrm{x}})\check{\Gamma}f(H)[H, A]^{\circ}f(H)$$

$$+ \frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}(z)(H^{\mathrm{x}} - z)^{-1}\chi(H^{\mathrm{x}})\Big(\check{\Gamma}H - H^{\mathrm{x}}\check{\Gamma}\Big)(H - z)^{-1}[H, A]^{\circ}f(H)\mathrm{d}z$$

$$\stackrel{o}{=} \check{\Gamma}f(H)[H, A]^{\circ}f(H) + o_R(1)f(H)[H, A]^{\circ}f(H),$$

as desired. $\qquad\square$

We will make use of another partition of unity, this time in momentum space. It has proved useful for the type of models studied here [22, 40]. We take, for $r > 0$, sharp localizations $\mathbb{1}_0^r$ and $\mathbb{1}_\infty^r$ onto sets $\Lambda_r = \{k \in \mathbb{R}^\nu \,|\, |k| \le r\}$ and Λ_r^{c} respectively. As multiplication operators they are projections and this allows us to view the vector operators $\mathbb{1}^r = (\mathbb{1}_0^r, \mathbb{1}_\infty^r)$ as operators from $\mathfrak{h}$ to $\mathfrak{h}_<^r \oplus \mathfrak{h}_>^r$, with $\mathfrak{h}_<^r = L^2(\Lambda_r)$ and $\mathfrak{h}_>^r = L^2(\Lambda_r^{\mathrm{c}})$. We can now lift this vector operator to a unitary operator

$$\check{\Gamma}(\mathbb{1}^r) = U\Gamma(\mathbb{1}^r)\colon \mathcal{F} \to \mathcal{F}_<^r \otimes \mathcal{F}_>^r,$$

where $\mathcal{F}_<^r = \Gamma(\mathfrak{h}_<^r)$ and $\mathcal{F}_>^r = \Gamma(\mathfrak{h}_>^r)$. Here U is again the canonical identification operator and we have abused notation by using the same notation $\check{\Gamma}$, although these operators map into a smaller space than $\mathcal{F}^{\mathrm{x}}$ and they are unitary, not merely isometric.

Since the Hamiltonian does not involve k-derivatives, we will not pick up localization errors, when applying a partition of unity in momentum space. We introduce

notation for the fiber Hamiltonians $H_r(\xi)$ restricted to $\mathcal{F}^r_<$, the new 'extended' Hamiltonian $H^x_r(\xi)$ and its building blocks $H^{(\ell)}_r(\xi)$ and their fiber operators $H^{(\ell)}_r(\xi; \underline{k})$:

$$H_r(\xi) = \mathrm{d}\Gamma\big(\omega_{|\mathfrak{h}^r_<}\big) + \Omega\big(\xi - \mathrm{d}\Gamma(k_{|\mathfrak{h}^r_<})\big) + \phi\big(\mathbb{1}^r_0\, g\big)$$
$$H^{(\ell)}_r(\xi; \underline{k}) = H\big(\xi - \textstyle\sum_{j=1}^\ell k_j\big) + \big(\textstyle\sum_{j=1}^\ell \omega(k_j)\big) \mathbb{1}_{\mathcal{F}^r_<}$$
$$H^{(\ell)}_r(\xi) = \int^\oplus_{(\Lambda^c_r)^\ell} H^{(\ell)}_r(\xi; \underline{k})\, \mathrm{d}k \tag{3.58}$$
$$H^x_r(\xi) = H_r(\xi) \oplus \Big(\bigoplus_{\ell=1}^\infty H^{(\ell)}_r(\xi)\Big).$$

The direct sum above is with respect to the splitting

$$\mathcal{F}^r_< \otimes \mathcal{F}^r_> = \mathcal{F}^r_< \oplus \Big(\bigoplus_{\ell=1}^\infty \mathcal{F}^r_< \otimes \Gamma^{(\ell)}\big(\mathfrak{h}^r_>\big)\Big), \tag{3.59}$$

where we identify $\mathcal{F}^r_< \otimes \Gamma^{(\ell)}(\mathfrak{h}^r_>)$ with $L^2_{\mathrm{sym}}((\Lambda^c_r)^\ell; \mathcal{F}^r_<)$.

Lemma 3.15. *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. Then*

$$\check{\Gamma}(\mathbb{1}^r)f(H(\xi)) = f(H^x_r(\xi))\check{\Gamma}(\mathbb{1}^r) + o_r(1).$$

Proof. Note that $H(\xi) = \check{\Gamma}(\mathbb{1}^r)^* H^x_r(\xi)\check{\Gamma}(\mathbb{1}^r) + \phi(\mathbb{1}^r_\infty\, g)$. Composing with the unitary operator $\check{\Gamma}(\mathbb{1}^r)$ from the left on both sides yields

$$\check{\Gamma}(\mathbb{1}^r)H(\xi) = H^x_r(\xi)\check{\Gamma}(\mathbb{1}^r) + \check{\Gamma}(\mathbb{1}^r)\phi(\mathbb{1}^r_\infty\, g).$$

Subtracting $z\check{\Gamma}(\mathbb{1}^r)$ on both sides and multiplying with $(H^x_r(\xi) - z)^{-1}$ and $(H(\xi) - z)^{-1}$ from the left and the right respectively, we get

$$(H^x_r(\xi) - z)^{-1}\check{\Gamma}(\mathbb{1}^r)$$
$$= \check{\Gamma}(\mathbb{1}^r)(H(\xi) - z)^{-1} + (H^x_r(\xi) - z)^{-1}\check{\Gamma}(\mathbb{1}^r)\phi(\mathbb{1}^r_\infty\, g)(H(\xi) - z)^{-1},$$

where the term involving $\phi(\mathbb{1}^r_\infty\, g)$ is of order $\langle z \rangle^{1/2}|\mathrm{Im}z|^{-2}o_r(1)$. The result is now obtained using the calculus of almost analytic extensions. $\square$

Let $\kappa > 0$ and $\lambda \in \mathbb{R}$. Denote by $E_{0,1}\colon \mathbb{R} \to \mathbb{R}$ the indicator function for the set $[-1, 1]$. Abbreviate $E_{\lambda,\kappa}(t) = E_{0,1}((t - \lambda)/\kappa)$, the indicator function for the set $[\lambda - \kappa, \lambda + \kappa]$.

Theorem 3.16 (Mourre Estimate). *Assume Conditions 1.2 and 1.3, with $n_0 = 1$. Let $(\xi, \lambda) \in \mathcal{E}^{(1)}\backslash(\mathcal{T}^{(1)}\cup\mathrm{Exc})$. Then there exist $\kappa > 0$, $c > 0$ and a compact self-adjoint operator K such that*

$$E_{\lambda,\kappa}(H(\xi))\mathrm{i}[H(\xi), A_\xi]^\circ E_{\lambda,\kappa}(H(\xi)) \geq cE_{\lambda,\kappa}(H(\xi)) + K. \tag{3.60}$$

Proof. For a pair $(\xi, \lambda) \in \mathcal{E}^{(1)} \backslash (\mathcal{T}^{(1)} \cup \mathrm{Exc})$ we have in (3.45) constructed a vector field giving rise to a conjugate operator A_ξ, cf. (2.2), and hence extended conjugate operators $A_\xi^{(\ell)}$ and A_ξ^{x}, cf. Section 2.3. We recall from Proposition 2.5 and 2.8 that $H(\xi)$ is of class $C^1(A_\xi)$, $H^{(\ell)}(\xi)$ is of class $C^1(A_\xi^{(\ell)})$, and $H^{\mathsf{x}}(\xi)$ is of class $C^1(A_\xi^{\mathsf{x}})$.

Let $f \in C_0^\infty\big(\big(\Sigma_0^{(1)}(\xi), \Sigma_0^{(2)}(\xi)\big)\big)$. Calculate using Corollary 3.13

$$
\begin{aligned}
&f(H(\xi)) \, \mathrm{i}[H(\xi), A_\xi]^\circ \, f(H(\xi)) \\
&= \check{\Gamma}(j^R)^* \, \check{\Gamma}(j^R) \, f(H(\xi)) \, \mathrm{i}[H(\xi), A_\xi]^\circ \, f(H(\xi)) \\
&= \check{\Gamma}(j^R)^* \, f(H^{\mathsf{x}}(\xi)) \, \mathrm{i}[H^{\mathsf{x}}(\xi), A_\xi^{\mathsf{x}}]^\circ \, f(H^{\mathsf{x}}(\xi)) \, \check{\Gamma}(j^R) + o_R(1)
\end{aligned}
\tag{3.61}
$$

From Proposition 2.8 we know that

$$
f(H^{\mathsf{x}}(\xi)) \, \mathrm{i}[H^{\mathsf{x}}(\xi), A_\xi^{\mathsf{x}}]^\circ \, f(H^{\mathsf{x}}(\xi)) = \bigoplus_{\ell=0}^\infty f\big(H^{(\ell)}(\xi)\big) \, \mathrm{i}\big[H^{(\ell)}(\xi), A_\xi^{(\ell)}\big]^\circ f\big(H^{(\ell)}(\xi)\big),
\tag{3.62}
$$

where $H^{(0)}(\xi) := H(\xi)$ and $A_\xi^{(0)} := A_\xi$. Recalling (1.17), (1.19) and (1.23) we find that for $\ell \geq 2$ we have $H^{(\ell)}(\xi) \geq \Sigma_0^{(\ell)}(\xi) \, \mathbb{1}_{\mathcal{H}^{(\ell)}} \geq \Sigma_0^{(2)}(\xi) \, \mathbb{1}_{\mathcal{H}^{(\ell)}}$. It follows that

$$
\forall \ell \geq 2 : \quad f\big(H^{(\ell)}(\xi)\big) = 0.
\tag{3.63}
$$

This takes care of the contributions to (3.62) with $\ell \geq 2$, where we can simply write $f(H^{(\ell)}(\xi))\mathrm{i}[H^{(\ell)}(\xi), A_\xi^{(\ell)}]^\circ f(H^{(\ell)}(\xi)) = f(H^{(\ell)}(\xi))^2$, both sides being elaborate zeroes.

If we insert (3.62) into (3.61) and look at the $\ell = 0$ contribution, we get

$$
\begin{aligned}
&\Gamma(j_0^R)^* \, f(H(\xi)) \, \mathrm{i}[H(\xi), A_\xi]^\circ \, f(H(\xi)) \, \Gamma(j_0^R) \\
&= \Gamma(j_0^R)^* \, f(H(\xi)) \, \mathrm{i}[H(\xi), A_\xi]^\circ \, h(H(\xi)) \, f(H(\xi)) \, \Gamma(j_0^R) = BK,
\end{aligned}
\tag{3.64}
$$

where

$$
B = \Gamma(j_0^R)^* \, f(H(\xi)) \, \mathrm{i}[H(\xi), A_\xi]^\circ \, h(H(\xi)) \quad \text{and} \quad K = f(H(\xi)) \, \Gamma(j_0^R).
$$

Here $h \in C_0^\infty(\mathbb{R})$ equals 1 on the support of f. Note that B is bounded, so to see that BK is compact, it is enough to prove that K is compact. Now by Lemma 3.15

$$
K = \check{\Gamma}(\mathbb{1}^r)^* f(H_r^{\mathsf{x}}(\xi)) \check{\Gamma}(\mathbb{1}^r) \Gamma(j_0^R) + o_r(1).
\tag{3.65}
$$

Like before, we split with respect to the direct sum decomposition (3.59), cf. also (3.58), and find

$$
\begin{aligned}
&\check{\Gamma}(\mathbb{1}^r)^* f(H_r^{\mathsf{x}}(\xi)) \check{\Gamma}(\mathbb{1}^r) \Gamma(j_0^R) \\
&= \check{\Gamma}(\mathbb{1}^r)^* \Big\{ f(H_r(\xi)) \oplus \Big(\bigoplus_{\ell=1}^\infty f(H_r^{(\ell)}(\xi)) \Big) \Big\} \check{\Gamma}(\mathbb{1}^r) \Gamma(j_0^R).
\end{aligned}
\tag{3.66}
$$

Observe now that by a variational argument we have the bound from below $\inf \sigma(H_r(\xi)) \geq \inf \sigma(H_0(\xi) + \phi(\mathbb{1}_0^r g))$, and furthermore by monotonicity of $\Sigma_0(\xi)$ as a function of the coupling g, cf. [40, Corollary 2.5 (i)], we get $\inf \sigma(H_0(\xi) + \phi(\mathbb{1}_0^r g)) \geq \Sigma_0(\xi)$. Hence

$$
H_r^{(\ell)}(\xi; \underline{k}) \geq \Sigma_0^{(\ell)}(\xi; \underline{k}) \, \mathbb{1}_{\mathcal{F}_<^r} \quad \text{and} \quad H_r^{(\ell)}(\xi) \geq \Sigma_0^{(\ell)}(\xi) \, \mathbb{1}_{\mathcal{F}_<^r}.
$$

This ensures that $f(H_r^{(\ell)}(\xi)) = 0$ for $\ell \geq 2$.

Choose and $\varepsilon > 0$ with $\varepsilon < d(\operatorname{supp}(f), \Sigma_0(\xi))$. As for the term with $\ell = 1$ we get similarly that $H_r^{(1)}(\xi; k) \geq \Sigma_0^{(1)}(\xi; k) \mathbb{1}_{\mathcal{F}_<^r}$ and by (3.1) (if ω is bounded) r can thus be chosen so large that $\Sigma_0^{(1)}(\xi; k) = \Sigma_0(\xi - k) + \omega(k) \geq \Sigma_0^{(2)}(\xi) - \varepsilon$, for any $|k| \geq r$. Hence, for r large enough and $|k| \geq r$ we have $f(H_r^{(1)}(\xi; k)) = 0$.

The only non-zero contribution to (3.66) for large r is thus the remaining $\ell = 0$ term $\Gamma(\mathbb{1}_0^r) f(H_r(\xi)) \Gamma(\mathbb{1}_0^r) \Gamma(j_0^R)$, which clearly is compact. Hence we see by letting $r \to \infty$ in (3.65) that K is compact.

By (3.63) we only get one non-compact contribution when inserting (3.62) into (3.61), namely one coming from the term $\ell = 1$, which is

$$f\big(H^{(1)}(\xi)\big) \, \mathrm{i} \big[H^{(1)}(\xi), A_\xi^{(1)}\big]^\circ f\big(H^{(1)}(\xi)\big).$$

We can now apply Proposition 3.12. Let $\mathcal{J} = (\lambda - \delta, \lambda + \delta)$ from Proposition 3.12, which we apply with an f chosen to be equal to one on $[\lambda - \delta/2, \lambda + \delta/2]$. We thus get

$$f(H(\xi)) \, \mathrm{i}[H(\xi), A_\xi]^\circ \, f(H(\xi)) \geq c f(H(\xi))^2 - c\,\Gamma(j_0^R) f(H(\xi))^2 \Gamma(j_0^R) + BK,$$

which implies the theorem, with $\kappa = \delta/2$, since $\Gamma(j_0^R) f(H(\xi))^2 \Gamma(j_0^R)$ was demonstrated to be compact above. $\qquad\square$

Proof of Theorem 1.6. Items (i) and (ii) are standard consequences of $H(\xi)$ being of class $C^1(A)$, cf. Proposition 2.5 '$k = 1$', the virial theorem [20], and the Mourre estimate Theorem 3.16. See e.g. [10, Chapter 4.3] and [31, Section VI]. The last statement (iii) follows from Theorem 1.5 once we have observed that the Mourre estimate is continuous in ξ and E: Let $(\xi_0, E_0) \in \mathcal{E}^{(1)} \backslash (\Sigma_{\mathrm{pp}} \cup \mathcal{T}^{(1)} \cup \mathrm{Exc})$ be given. Then the Mourre estimate, cf. Theorem 3.16,

$$f_{E,\kappa}(H(\xi)) \mathrm{i}[H(\xi), A_{\xi_0}] f_{E,\kappa}(H(\xi)) \geq c f_{E,\kappa}(H(\xi))^2$$

holds true at $(\xi, E) = (\xi_0, E_0)$ for some $\kappa, c > 0$. Here $f \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp} f \subset [-1, 1]$, $0 \leq f \leq 1$ and $f(t) = 1$ for $|t| < 1/2$. Finally, $f_{\lambda,\kappa}(t) = f((t - \lambda)/\kappa)$. Recall that, being away from the point spectrum, one can squeeze away the compact error in the Mourre estimate, by passing to a smaller κ. We leave it to the reader to argue that both sides of the estimate above are jointly continuous in ξ and E, hence an estimate of the same form, possibly with a smaller κ, will hold in a small neighborhood of (ξ_0, E_0). Hence, by the standard virial theorem [20], there can be no point spectrum in a small neighborhood of (ξ_0, E_0). Taken together with relative closedness of $(\mathcal{T}^{(1)} \cup \mathrm{Exc}) \cap \mathcal{E}^{(1)}$ in $\mathcal{E}^{(1)}$ we are done. $\qquad\square$

Proof of Theorem 1.7. Under the assumptions of the theorem we have $H(\xi)$ of class $C^2(A)$, cf. Proposition 2.5 '$k = 2$'. Hence we can conclude from Theorem 3.16 the limiting absorption principle

$$\sup_{\substack{z \in \mathbb{C}, \, \mathrm{Im}\, z \neq 0 \\ \mathrm{Re}\, z \in \mathcal{J}}} \left\| \langle A \rangle^{-s} (H(\xi) - z)^{-1} \langle A \rangle^{-s} \right\| < \infty, \tag{3.67}$$

where $s > 1/2$ and $\mathcal{J} \subset \mathcal{E}^{(1)}(\xi) \backslash (\mathcal{T}^{(1)}(\xi) \cup \mathrm{Exc}(\xi) \cup \sigma_{\mathrm{pp}}(H(\xi)))$ is a compact interval. For a proof of this estimate, we refer the reader to [3, 23]. It is a well-known consequence of (3.67), together with Theorem 1.6, that the singular continuous part of $\sigma(H(\xi)) \cap \mathcal{E}^{(1)}(\xi)$ is empty. See e.g. [48, Theorem XIII.20]. $\qquad\square$

A   Fibered Operators

Let $\mathcal{F}$ be a separable Hilbert space and (X, S) a measurable space, i.e. S is a σ-algebra of subsets of X.

Let $H(x)$, for $x \in X$, be a family of self-adjoint operators on $\mathcal{F}$ with domain $\mathcal{D}(x)$. The family is said to be *weakly resolvent measurable* if the map $x \to (H(x) + \mathrm{i})^{-1}$ is weakly - hence strongly - measurable. Henceforth we simply write measurable. This implies the same property for $x \to (H(x) - z)^{-1}$ for any $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$. We remark that if $X = \mathbb{R}$ and S is the Lebesgue measurable subsets of $\mathbb{R}$, then being weakly resolvent measurable is equivalent to being measurable in the sense introduced by Nussbaum in [43], a property called 'N-measurable' in [28]. For equivalence of weak resolvent measurability and N-measurability for self-adjoint families of operators see [28, Theorem 4.11].

By Stone-Weierstrass we can conclude that for any $f \in C_0(\mathbb{R})$, the map $x \to f(H(x))$ is measurable. Choosing a sequence $f_n \in C_0(\mathbb{R})$ with $f_n(t) \to 0$ for $t \neq \lambda$ and $f_n(\lambda) = 1$ yields measurability of eigenprojections $x \to \mathbb{1}_{\{\lambda\}}(H(x))$. Stone's formula now gives measurability of $x \to \mathbb{1}_I(H(x))$ for any interval I. Since the collection of Borel sets E for which $x \to \mathbb{1}_E(H(x))$ is measurable form a σ-algebra, we can conclude that the property must hold true for all Borel sets.

Equip the Cartesian product $X \times \mathbb{R}$ with the product σ-algebra $S \times \mathrm{Borel}(\mathbb{R})$. Let $F \subset \{\psi \in \mathcal{F} \,|\, \|\psi\| = 1\}$ be a countable dense subset of the unit ball. For $\psi \in F$ put

$$f_\psi(x, \lambda) = \|(H(x) - \lambda)(H(x) + \mathrm{i})^{-1}\psi\| / \|(H(x) + \mathrm{i})^{-1}\psi\|.$$

Then $(x, \lambda) \to f_\psi(x, \lambda)$ is measurable. Put $\Sigma_n = \cup_{\psi \in F} f_\psi^{-1}((-\infty, 1/n))$. Since the joint spectrum $\Sigma = \{(x, \lambda) \,|\, \lambda \in \sigma(H(x))\} \subset X \times \mathbb{R}$ can be written as $\cap_{n=1}^\infty \Sigma_n$ we conclude that Σ is measurable.

Similarly, for $\psi \in F$ and $n \in \mathbb{N}$, we can define $S \times \mathrm{Borel}(\mathbb{R})$ measurable functions

$$f_\psi^{(n)}(x, \lambda) = \|(n(H(x) - \lambda) + \mathrm{i})^{-1}\psi\|.$$

By the spectral theorem together with the dominated convergence theorem, we find that $f_\psi^{(n)}(x, \lambda) \to \|\mathbb{1}_{\{\lambda\}}(H(x))\psi\|$, which is thus a measurable function of x and λ. Taking supremum over $\psi \in F$, we conclude that $(x, \lambda) \to \|\mathbb{1}_{\{\lambda\}}(H(x))\|$ is measurable and hence the joint point spectrum $\Sigma_{\mathrm{pp}} = \{(x, \lambda) \,|\, \lambda \in \sigma_{\mathrm{pp}}(H(x))\}$ is an $S \times \mathrm{Borel}(\mathbb{R})$ measurable set.

Let now μ be a positive measure defined on the σ-algebra S. Denote by $\mathcal{H}$ the Hilbert space $L^2(X; \mathcal{F}) \simeq \mathcal{F} \otimes L^2(X)$, consisting of all measurable functions $X \ni x \to \psi(x)$ with $\int_X \|\psi(x)\|_{\mathcal{F}}^2 \mathrm{d}\mu(x) < \infty$. The construction

$$R(z) = \int_X^\oplus (H(x) - z)^{-1} \, \mathrm{d}\mu(x)$$

yields a family of bounded operators on $\mathcal{H}$ satisfying the first resolvent formula. The operator $R(z)$ satisfies $R(z)^* = R(\bar{z})$ and has kernel $\{0\}$. Hence, it is the resolvent family of a self-adjoint operator H densely defined on $\mathcal{H}$. Its domain is $\mathcal{D}(H) = R(z)\mathcal{H}$, which is independent of $z \in \{z \in \mathbb{C} \,|\, \mathrm{Im} z \neq 0\}$.

Define a different domain by

$$\widetilde{\mathcal{D}}(H) = \left\{\psi \in \mathcal{H} \,\middle|\, \psi(x) \in \mathcal{D}(x) \text{ a.e. and } \int_X \|H(x)\psi(x)\|_{\mathcal{F}}^2 \, \mathrm{d}\mu(x) < \infty\right\}.$$

We remark that for $\psi \in \mathcal{H}$ and $\mathrm{Im}\, z \neq 0$ we have $(R(z)\psi)(x) = (H(x) - z)^{-1}\psi(x)$ a.e. and hence $\mathcal{D}(H) \subset \tilde{\mathcal{D}}(H)$. Furthermore we can on $\tilde{\mathcal{D}}(H)$ define a symmetric operator $\widetilde{H}$ by $(\widetilde{H}\psi)(x) = H(x)\psi(x)$. It is easy to see that $H \subset \widetilde{H}$ and since H was self-adjoint we must have $H = \widetilde{H}$ and in particular $\mathcal{D}(H) = \tilde{\mathcal{D}}(H)$. We remark that even with only weak measurability of $x \to H(x)$ one can always construct $\widetilde{H}$ as a closed operator, but without assumptions beyond weak measurability one may not arrive at a densely defined operator, cf. [28, Remark 4.7].

The spectral resolution $\mathbb{1}_E(H)$, with $E \subset \mathbb{R}$ Borel, can be explicitly computed to be

$$\mathbb{1}_E(H) = \int_X^\oplus \mathbb{1}_E(H(x))\, \mathrm{d}\mu(x).$$

If $\mathcal{D}_0 \subset \mathcal{D}(x)$ for (almost) every x is a common core for $H(x)$, then one can construct an essentially self-adjoint operator on the dense set of $\psi \in \mathcal{H}$, with a.e. $\psi(x) \in \mathcal{D}_0$. The closure coincides with H and is the situation we find ourselves in with the fibered Nelson Hamiltonian, where $X = \mathbb{R}^\nu$ equipped with the Borel σ-algebra and Lebesgue measure.

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