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RESTRICTION OF COMPLEMENTARY SERIES REPRESENTATIONS OF $O(1,N)$ TO SYMMETRIC SUBGROUPS

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Restriction of complementary series representations of $O(1, N)$ to symmetric subgroups

Jan Möllers, Yoshiki Oshima

Abstract

We find the complete branching law for the restriction of complementary series representations of $O(1, n+1)$ to the symmetric subgroup $O(1, m+1) \times O(n-m)$, $0 \leq m < n$. The decomposition consists of a continuous part and a discrete part which is trivial for some parameters. The continuous part is given by a direct integral of principal series representations whereas the discrete part consists of finitely many complementary series representations. The explicit Plancherel formula is computed on the Fourier transformed side of the non-compact realization of the complementary series by using the spectral decomposition of a certain hypergeometric type ordinary differential operator. The main tool connecting this differential operator with the representations are second order Bessel operators which describe the Lie algebra action in this realization.

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Contents

Introduction

In the unitary representation theory of reductive Lie groups one is mainly concerned with the following two problems as was advocated in [8]:

(1) Classify all irreducible unitary representations of a given reductive Lie group,

(2) Decompose a given unitary representation into irreducible ones.

While problem (1) is a long-standing problem in general, there is a classification of all irreducible unitary representations for certain subclasses of groups, among them semisimple Lie groups of rank one (see [1, 7]). Our focus is on the rank one group $G = O(1, n+1)$, $n \in \mathbb{N}$, for which we study problem (2).

All irreducible unitary representations of G are obtained as subrepresentations of representations induced from a parabolic subgroup $P = MAN$ on the level of $(\mathfrak{g}, K)$-modules. Up to conjugation P is unique and $M \cong O(n) \times (\mathbb{Z}/2\mathbb{Z})$, $A \cong \mathbb{R}_+$ and $N \cong \mathbb{R}^n$. We restrict our attention to representations induced from characters of P. Denote by $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ the representation of G, which is induced from the character of P given by the character $\sigma \in \mathbb{C}$ of A and the character $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ of the second factor of $M \cong O(n) \times (\mathbb{Z}/2\mathbb{Z})$ (normalized parabolic induction). In this parameterization $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ is irreducible and unitarizable if and only if $\sigma \in i\mathbb{R} \cup (-n, n)$. By abuse of notation we denote by $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ also the corresponding irreducible unitary representations. For $\sigma \in i\mathbb{R}$ the representations $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ are called unitary principal series representations and for $\sigma \in (-n, 0) \cup (0, n)$ they are called complementary series representations. We have natural isomorphisms $\pi_{-\sigma,\varepsilon}^{O(1,n+1)} \cong \pi_{\sigma,\varepsilon}^{O(1,n+1)}$ for $\sigma \in i\mathbb{R} \cup (-n, n)$.

In this paper we solve problem (2) for the restriction of $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$, $\sigma \in i\mathbb{R} \cup (-n, n)$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, to any symmetric subgroup of G. By Berger's list [2] any non-trivial symmetric subgroup of G is either conjugate to

$$\begin{aligned} &K = O(1) \times O(n+1) && \text{or} \\ &H = O(1, m+1) \times O(n-m), && 0 \le m < n. \end{aligned}$$

Since K is a maximal compact subgroup of G the branching law for the restriction of $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ to K is simply the K-type decomposition (1.2) which is well-known. The branching to H is the main topic of this paper. In the formulation of the branching law we use the convention $[0, \alpha) = \emptyset$ for $\alpha \le 0$.

Theorem (see Theorem 4.7). *For $\sigma \in i\mathbb{R} \cup (-n, n)$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ the representation $\pi_{\sigma,\varepsilon}^G$ of $G = O(1, n+1)$ decomposes into irreducible representations of $H = O(1, m+1) \times O(n-m)$, $0 \le m < n$, as follows:*

$$\pi_{\sigma,\varepsilon}^G\big|_H \cong \sum_{k=0}^{\infty}{}^{\oplus} \left(\int_{i\mathbb{R}_+}^{\oplus} \pi_{\tau,\varepsilon+k}^{O(1,m+1)} \, \mathrm{d}\tau \oplus \bigoplus_{j \in \mathbb{Z} \cap \left[0, \frac{|\operatorname{Re}\sigma| - n + m - 2k}{4}\right)} \pi_{|\operatorname{Re}\sigma| - n + m - 2k - 4j, \varepsilon + k}^{O(1,m+1)} \right) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}),$$

where $\mathcal{H}^k(\mathbb{R}^{n-m})$ denotes the irreducible representation of $O(n-m)$ on the space of solid spherical harmonics of degree k on $\mathbb{R}^{n-m}$.

The explicit Plancherel formula is given in Theorem 4.1. First of all, the restriction $\pi^G_{\sigma,\varepsilon}|_H$ is decomposed with respect to the action of $O(n-m)$, the second factor of H. Then the decomposition of each $\mathcal{H}^k(\mathbb{R}^{n-m})$-isotypic component into irreducible representations of $O(1, m+1)$ contains continuous and discrete spectrum in general. The continuous part is a direct integral of unitary principal series representations $\pi^{O(1,m+1)}_{\tau,\varepsilon+k}$. The discrete part appears if and only if $k < \frac{|\mathrm{Re}\,\sigma|-n+m}{2}$ and is a finite direct sum of complementary series representations. Therefore the whole branching law of $\pi^G_{\sigma,\varepsilon}|_H$ contains only finitely many discrete components and the discrete part is non-trivial if and only if $|\mathrm{Re}\,\sigma| > n-m$. In particular for $m > 0$ there is always at least one discrete component if σ is sufficiently close to the first reduction point n or $-n$.

For $\sigma \in i\mathbb{R}$ the decomposition is purely continuous. In this case the branching law is actually equivalent to the Plancherel formula for the Riemannian symmetric space $O(1, m+1)/(O(1)\times O(m+1))$ (see Appendix A) and therefore easy to derive. We remark that a similar method was used in [9] for the branching laws of the most degenerate principal series representations of $GL(n,\mathbb{R})$ with respect to symmetric pairs. However, for the complementary series representations, i.e. $\sigma \in (-n, 0)\cup(0, n)$, the decomposition cannot be obtained in the same way.

The proof of the Plancherel formula we present works uniformly for $\sigma \in i\mathbb{R} \cup (-n, n)$. It uses the "Fourier transformed realization" of $\pi^G_{\sigma,\varepsilon}$ on $L^2(\mathbb{R}^n, |x|^{-\mathrm{Re}\,\sigma}\,\mathrm{d}x)$. For this consider first the non-compact realization on the nilradical $\overline{N}$ of the parabolic subgroup $\overline{P}$ opposite to P. We then take the Euclidean Fourier transform on $\overline{N} \cong \mathbb{R}^n$ to obtain a realization of $\pi^G_{\sigma,\varepsilon}$ on $L^2(\mathbb{R}^n, |x|^{-\mathrm{Re}\,\sigma}\,\mathrm{d}x)$. The advantage of this realization is that the invariant form is simply the L^2-inner product. The Lie algebra action in the Fourier transformed picture is given by differential operators up to order two, the crucial operators being the second order Bessel operators studied in [6, 12]. Using these operators we reduce the branching law to the spectral decomposition of an ordinary differential operator of hypergeometric type on $L^2(\mathbb{R}_+)$ (see Section 2). The spectral decomposition of this operator is derived in Section 3 and used in Section 4 to obtain the branching law and the explicit Plancherel formula. An interesting formula for the intertwining operators realizing the branching law in the non-compact picture on $\overline{N}$ is computed in Section 5. These intertwining operators will be subject of a subsequent paper.

Up to now only partial results regarding the branching of $\pi^G_{\sigma,\varepsilon}$, $\sigma \in (-n, n)$, to H were known:

- For $n = 2$ and $m = 1$ the full decomposition was given by Mukunda [13] using the non-compact picture. This case corresponds to the branching law $\mathrm{SL}(2,\mathbb{C}) \searrow \mathrm{SL}(2,\mathbb{R})$.

- For $n \geq 2$ and $m = n-1$ Speh–Venkataramana [14, Theorem 1] proved the existence of the discrete component $\pi^{O(1,n)}_{\sigma-1}$ in $\pi^{O(1,n+1)}_{\sigma}$ for $\sigma \in (1, n)$ (special case $j = k = 0$ in our Theorem). They also use the Fourier transformed picture for their proof. This is a special case of their more general result for complementary series representations of G on differential forms, i.e. induced from more general (possibly non-scalar) P-representations.

- The same special case was obtained by Zhang [19, Theorem 3.6]. He actually proved that for all rank one groups $G = \mathrm{SU}(1, n+1; \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, resp. $G = F_{4(-20)}$ certain complementary series representations of $H = \mathrm{SU}(1, n; \mathbb{F})$ resp. $H = \mathrm{Spin}(8, 1)$ occur discretely in some spherical complementary series representations of G. His proof uses the compact picture and explicit estimates for the restriction of K-finite vectors.

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Notation. $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

1 L^2-realization of the complementary series of $O(1, n+1)$

In this section we recall the necessary geometry of the group $G = O(1, n+1)$ and some of its representation theory.

1.1 Subgroups and decompositions

Let $G = O(1, n+1)$, $n \geq 1$, realized as the subgroup of $\mathrm{GL}(n+2, \mathbb{R})$ leaving the quadratic form

$$\mathbb{R}^{n+2} \to \mathbb{R}, \quad x = (x_1, \ldots, x_{n+2})^t \mapsto x_1^2 - (x_2^2 + \cdots + x_{n+2}^2),$$

invariant. We fix the Cartan involution θ of G given by $\theta(g) = g^{-t} = (g^t)^{-1}$, $g \in G$, which corresponds to the maximal compact subgroup $K := G^\theta = O(1) \times O(n+1)$. On the Lie algebra level the Lie algebra $\mathfrak{g}$ of G has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the ± 1 eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ of θ where $\mathfrak{k}$ is the Lie algebra of K. Choose the maximal abelian subalgebra $\mathfrak{a} := \mathbb{R}H \subseteq \mathfrak{p}$ spanned by the element

$$H := 2(E_{1,n+2} + E_{n+2,1}),$$

where E_{ij} denotes the $(n+2) \times (n+2)$ matrix with 1 in the (i,j)-entry and 0 elsewhere. The root system of the pair $(\mathfrak{g}, \mathfrak{a})$ consists only of the roots $\pm 2\gamma$ where $\gamma \in \mathfrak{a}_\mathbb{C}^*$ is defined by $\gamma(H) := 1$. Put

$$\mathfrak{n} := \mathfrak{g}_{2\gamma}, \qquad \overline{\mathfrak{n}} := \mathfrak{g}_{-2\gamma} = \theta\mathfrak{n}$$

and let

$$N := \exp_G(\mathfrak{n}), \quad \overline{N} := \exp_G(\overline{\mathfrak{n}}) = \theta N$$

be the corresponding analytic subgroups of G. Since $\dim(\mathfrak{n})=\dim(\overline{\mathfrak{n}})=n$ the half sum of all positive roots is given by $\rho=n\gamma$. We introduce the following coordinates on N and $\overline{N}$: For $1\leq j\leq n$ let

$$\begin{aligned} N_j &:= E_{1,j+1}+E_{j+1,1}-E_{j+1,n+2}+E_{n+2,j+1}, \\ \overline{N}_j &:= E_{1,j+1}+E_{j+1,1}+E_{j+1,n+2}-E_{n+2,j+1}. \end{aligned}$$

For $x\in\mathbb{R}^n$ let

$$n_x := \exp\Big(\sum_{j=1}^n x_jN_j\Big)\in N, \qquad \overline{n}_x := \exp\Big(\sum_{j=1}^n x_j\overline{N}_j\Big)\in \overline{N}.$$

Further put $M:=Z_K(\mathfrak{a})$ and $A:=\exp(\mathfrak{a})$ and denote by $\mathfrak{m}$ the Lie algebra of M. We write $M=M^+\cup m_0M^+$ where

$$\begin{aligned} M^+ &:= \{\operatorname{diag}(1,k,1):k\in O(n)\}\cong O(n) \qquad \text{and} \\ m_0 &:= \operatorname{diag}(-1,1,\ldots,1,-1). \end{aligned}$$

Via conjugation the element m_0 acts on N and $\overline{N}$ by

$$m_0n_xm_0^{-1}=n_{-x} \quad \text{and} \quad m_0\overline{n}_xm_0^{-1}=\overline{n}_{-x}$$

and the action of $m\in M^+\cong O(n)$ on N and $\overline{N}$ by conjugation is given by

$$mn_xm^{-1}=n_{mx} \quad \text{and} \quad m\overline{n}_xm^{-1}=\overline{n}_{mx}$$

for $x\in\mathbb{R}^n$, where mx is the usual action of $O(n)$ on $\mathbb{R}^n$. Further A acts on N and $\overline{N}$ by

$$e^{tH}n_xe^{-tH}=n_{e^{2t}x} \quad \text{and} \quad e^{tH}\overline{n}_xe^{-tH}=\overline{n}_{e^{-2t}x}$$

for $x\in\mathbb{R}^n$, $t\in\mathbb{R}$. The following decomposition holds

$$\mathfrak{g}=\overline{\mathfrak{n}}\oplus\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n} \qquad \text{(Gelfand–Naimark decomposition)}.$$

The groups

$$P:=MAN \quad \text{and} \quad \overline{P}:=MA\overline{N}=\theta(P)$$

are opposite parabolic subgroups in G and $\overline{N}P\subseteq G$ is an open dense subset. Let $W:=N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group corresponding to $\mathfrak{a}$. Then $W=\{\mathbf{1},[w_0]\}$ where the non-trivial element is represented by the matrix

$$w_0=\operatorname{diag}(-1,1,\ldots,1)\in K.$$

The element w_0 has the property that $w_0Nw_0^{-1}=\overline{N}$ and hence $w_0Pw_0^{-1}=\overline{P}$. More precisely,

$$w_0n_xw_0^{-1}=\overline{n}_{-x} \quad \text{and} \quad w_0e^{tH}w_0^{-1}=e^{-tH}.$$

We have the disjoint union

$$G=\overline{P}\cup\overline{P}w_0\overline{P} \qquad \text{(Bruhat decomposition)}.$$

The following lemma is a straightforward calculation:

Lemma 1.1. *For $x \in \mathbb{R}^n$, $x \neq 0$, we have $w_0^{-1}\overline{n}_x = \overline{n}_y m e^{tH} n_z \in \overline{N}P$ with*

$$\begin{aligned} y &= -|x|^{-2}x, \\ z &= |x|^{-2}x, \\ t &= \log|x|, \end{aligned} \qquad m = \begin{pmatrix} -1 & & \\ & \mathbf{1}_n - 2|x|^{-2}xx^t & \\ & & -1 \end{pmatrix}.$$

Let τ be the involution of G given by conjugation with the matrix

$$\operatorname{diag}(\mathbf{1}_m, -\mathbf{1}_{n-m}, 1).$$

Then the symmetric subgroup $H := G^\tau$ is isomorphic to $O(1, m+1) \times O(n-m)$. The subgroup H is generated by the subgroups N_H, $\overline{N}_H$, M_H and A, where (viewing $\mathbb{R}^m$ as the subspace $\mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$)

$$N_H := \{n_x : x \in \mathbb{R}^m\} \quad \text{and} \quad \overline{N}_H := \{\overline{n}_x : x \in \mathbb{R}^m\}$$

and $M_H := M_H^+ \cup m_0 M_H^+$ with

$$M_H^+ := \{\operatorname{diag}(1, k_1, k_2, 1) : k_1 \in O(m), k_2 \in O(n-m)\} \cong O(m) \times O(n-m).$$

Also denote by

$$P_H := M_H A N_H \quad \text{and} \quad \overline{P}_H := M_H A \overline{N}_H$$

the corresponding parabolic subgroups. We write $\mathfrak{h}$ for the Lie algebra of H.

1.2 Principal series representations – non-compact picture and standard intertwining operators

We identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ by $\lambda \mapsto \lambda(H)$, i.e. $\sigma \in \mathbb{C}$ corresponds to $\sigma\gamma \in \mathfrak{a}_\mathbb{C}^*$. Under this identification ρ corresponds to n. For $\sigma \in \mathbb{C}$ let e^σ be the character of A given by $e^\sigma(e^{tH}) = e^{\sigma t}$, $t \in \mathbb{R}$. Further, for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ denote by ξ_ε the character of $M = M^+ \cup m_0 M^+$ with $\xi_\varepsilon(m_0) = (-1)^\varepsilon$ and $\xi_\varepsilon(m) = 1$ for $m \in M^+$. For $\sigma \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ we consider the character $\chi_{\sigma,\varepsilon} := \xi_\varepsilon \otimes e^\sigma \otimes \mathbf{1}$ on $P = MAN$ and induce it to a representation of G:

$$\begin{aligned} \widetilde{I}^G_{\sigma,\varepsilon} &:= \operatorname{Ind}_P^G(\chi_{\sigma,\varepsilon}) \\ &= \{f \in C^\infty(G) : f(gman) = \xi_\varepsilon(m)^{-1} a^{-\sigma-\rho} f(g) \,\forall\, g \in G, man \in P = MAN\}. \end{aligned}$$

The group G acts on $\widetilde{I}^G_{\sigma,\varepsilon}$ by left-translations and this action will be denoted by $\widetilde{\pi}^G_{\sigma,\varepsilon}$. Since $\overline{N}P \subseteq G$ is dense, a function in $\widetilde{I}^G_{\sigma,\varepsilon}$ is already uniquely determined by its values on $\overline{N}$ and for $f \in \widetilde{I}^G_{\sigma,\varepsilon}$ we put

$$f_{\overline{N}}(x) := f(\overline{n}_x), \qquad x \in \mathbb{R}^n.$$

Let $I^G_{\sigma,\varepsilon} := \{f_{\overline{N}} : f \in \widetilde{I}^G_{\sigma,\varepsilon}\}$ and denote by $\pi^G_{\sigma,\varepsilon}$ the corresponding induced action, i.e.

$$\pi^G_{\sigma,\varepsilon}(g) f_{\overline{N}} := (\widetilde{\pi}^G_{\sigma,\varepsilon}(g) f)_{\overline{N}}, \qquad f \in \widetilde{I}^G_{\sigma,\varepsilon}.$$

In view of the Bruhat decomposition $G = \overline{P} \cup \overline{P} w_0 \overline{P}$ this action can be completely described by the action of $\overline{P}$ and w_0. Using Lemma 1.1 we find

$$\begin{aligned}
\pi^G_{\sigma,\varepsilon}(\overline{n}_a)f(x) &= f(x-a), & \overline{n}_a &\in \overline{N},\\
\pi^G_{\sigma,\varepsilon}(m)f(x) &= f(m^{-1}x), & m &\in M^+ \cong O(n),\\
\pi^G_{\sigma,\varepsilon}(m_0)f(x) &= (-1)^\varepsilon f(-x), & &\\
\pi^G_{\sigma,\varepsilon}(e^{tH})f(x) &= e^{(\sigma+n)t} f(e^{2t}x), & e^{tH} &\in A,\\
\pi^G_{\sigma,\varepsilon}(w_0)f(x) &= (-1)^\varepsilon |x|^{-\sigma-n} f(-|x|^{-2}x). & &
\end{aligned}$$

This also gives the following expressions for the differential action $\mathrm{d}\pi^G_\sigma = \mathrm{d}\pi^G_{\sigma,\varepsilon}$ of the Lie algebra $\mathfrak{g}$, which is independent of ε:

$$\begin{aligned}
\mathrm{d}\pi^G_\sigma(\overline{N}_j)f(x) &= -\frac{\partial f}{\partial x_j}(x), & j &= 1,\ldots,n,\\
\mathrm{d}\pi^G_\sigma(T)f(x) &= -D_{Tx}f(x), & T &\in \mathfrak{m} \cong \mathfrak{so}(n),\\
\mathrm{d}\pi^G_\sigma(H)f(x) &= (2E+\sigma+n)\,f(x), & &\\
\mathrm{d}\pi^G_\sigma(N_j)f(x) &= -|x|^2\frac{\partial f}{\partial x_j}(x) + x_j\,(2E+\sigma+n)\,f(x), & j &= 1,\ldots,n,
\end{aligned}$$

where D_a denotes the directional derivative in direction $a \in \mathbb{R}^n$ and $E = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$ is the Euler operator on $\mathbb{R}^n$. For the action of $\mathfrak{n}$ we have used the identity $\mathrm{d}\pi^G_\sigma(N_a) = \pi^G_{\sigma,\varepsilon}(w_0)\mathrm{d}\pi^G_\sigma(\overline{N}_{-a})\pi^G_{\sigma,\varepsilon}(w_0^{-1})$.

Now suppose $\sigma \in (0,n)$ and consider the Knapp–Stein intertwining operator $\widetilde{J}(\sigma,\varepsilon) : \widetilde{I}^G_{\sigma,\varepsilon} \to \widetilde{I}^G_{-\sigma,\varepsilon}$ given by

$$\widetilde{J}(\sigma,\varepsilon)f(g) := \int_{\overline{N}} f(gw_0\overline{n})\,\mathrm{d}\overline{n}, \qquad g \in G,\ f \in \widetilde{I}^G_{\sigma,\varepsilon},$$

where $\mathrm{d}\overline{n}$ is the Haar measure on $\overline{N}$ given by the push-forward of the Lebesgue measure on $\mathbb{R}^n$ by the map $\mathbb{R}^n \to \overline{N}$, $x \mapsto \overline{n}_x$. This intertwining operator induces an intertwining operator $J(\sigma,\varepsilon) : I^G_{\sigma,\varepsilon} \to I^G_{-\sigma,\varepsilon}$ by $J(\sigma,\varepsilon)f_{\overline{N}} := (\widetilde{J}(\sigma,\varepsilon)f)_{\overline{N}}$, $f \in \widetilde{I}^G_{\sigma,\varepsilon}$. Using Lemma 1.1 we obtain

$$\begin{aligned}
J(\sigma,\varepsilon)f_{\overline{N}}(x) &= \int_{\mathbb{R}^n} f(\overline{n}_x w_0 \overline{n}_z)\,\mathrm{d}z\\
&= (-1)^\varepsilon \int_{\mathbb{R}^n} |z|^{-\sigma-n} f_{\overline{N}}(x - |z|^{-2}z)\,\mathrm{d}z.
\end{aligned}$$

Consider the coordinate change $y := x - |z|^{-2}z$. Its Jacobian $|\det(\frac{\partial y}{\partial z})|$ is homogeneous of degree $-2n$, $O(n)$-invariant and has value 1 for $z = e_1$. Hence it is equal to $|z|^{-2n}$. This finally gives

$$\begin{aligned}
J(\sigma,\varepsilon)f(x) &= (-1)^\varepsilon \int_{\mathbb{R}^n} |x-y|^{\sigma-n} f(y)\,\mathrm{d}y\\
&= (-1)^\varepsilon (|-|^{\sigma-n} * f)(x),
\end{aligned}$$

so $J(\sigma,\varepsilon)$ is up to sign given by convolution with the function $|-|^{\sigma-n}$. We define a G-invariant Hermitian form $(-\,|\,-)_{\sigma,\varepsilon}$ on $I^G_{\sigma,\varepsilon}$ by

$$(f\,|\,g)_{\sigma,\varepsilon} := (-1)^\varepsilon (f\,|\,J(\sigma,\varepsilon)g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\sigma-n} f(x)\overline{g(y)}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.1}$$

For $\sigma\in(0,n)$ this form is in fact positive definite and in this case the closure $\mathcal{H}^G_{\sigma,\varepsilon}$ of $I^G_{\sigma,\varepsilon}$ with respect to the inner product $(-\,|\,-)_{\sigma,\varepsilon}$ gives an irreducible unitary representation $(\mathcal{H}^G_{\sigma,\varepsilon},\pi^G_{\sigma,\varepsilon})$ of G. Using the intertwining operator $J(\sigma,\varepsilon): I^G_{\sigma,\varepsilon}\to I^G_{-\sigma,\varepsilon}$ one also obtains a unitarization $(\mathcal{H}^G_{-\sigma,\varepsilon},\pi^G_{-\sigma,\varepsilon})\cong(\mathcal{H}^G_{\sigma,\varepsilon},\pi^G_{\sigma,\varepsilon})$ of $I^G_{-\sigma,\varepsilon}$.

For $\sigma\in i\mathbb{R}$ the usual L^2-inner product provides unitarizations $(\mathcal{H}^G_{\sigma,\varepsilon},\pi^G_{\sigma,\varepsilon})$ on $\mathcal{H}^G_{\sigma,\varepsilon}=L^2(\mathbb{R}^n)$ and these representations form the unitary principal series. For $\sigma\in(-n,0)\cup(0,n)$ they comprise the complementary series for G. Note that for any $\sigma\in(-n,n)\cup i\mathbb{R}$ the analytic continuation of the operator $J(\sigma,\varepsilon)$ provides an intertwining operator between the irreducible unitary representations $(\mathcal{H}^G_{\sigma,\varepsilon},\pi^G_{\sigma,\varepsilon})$ and $(\mathcal{H}^G_{-\sigma,\varepsilon},\pi^G_{-\sigma,\varepsilon})$.

From the compact picture it is easy to see that the K-type decomposition of the representations $\pi_{\sigma,\varepsilon}$ is given by

$$\pi^G_{\sigma,\varepsilon}\big|_K \cong \sum_{k=0}^{\infty}{}^{\oplus}\ \mathcal{H}^k(\mathbb{R}^{n+1}), \tag{1.2}$$

where $O(n+1)$ acts as usual on $\mathcal{H}^k(\mathbb{R}^{n+1})$ and $O(1)$ acts by $(-1)^{\varepsilon+k}$, giving combined the action of $K\cong O(1)\times O(n+1)$.

1.3 The Fourier transformed picture

Consider the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}:\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ given by

$$\mathcal{F}_{\mathbb{R}^n}u(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i(x\,|\,y)}u(y)\,\mathrm{d}y. \tag{1.3}$$

For $\sigma\in(-n,n)\cup i\mathbb{R}$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ we define a representation $\rho^G_{\sigma,\varepsilon}$ of G on $\mathcal{F}^{-1}_{\mathbb{R}^n}\mathcal{H}^G_{\sigma,\varepsilon}$ by

$$\pi^G_{\sigma,\varepsilon}(g)\circ\mathcal{F}_{\mathbb{R}^n} = \mathcal{F}_{\mathbb{R}^n}\circ\rho^G_{\sigma,\varepsilon}(g), \qquad g\in G.$$

It is easy to calculate the group action of $\overline{P}=MA\overline{N}$:

$$\rho_{\sigma,\varepsilon}(\overline{n}_a)f(x) = e^{i(x\,|\,a)}f(x), \qquad \overline{n}_a\in\overline{N}, \tag{1.4}$$
$$\rho_{\sigma,\varepsilon}(m)f(x) = f(m^{-1}x), \qquad m\in M^+\cong O(n), \tag{1.5}$$
$$\rho_{\sigma,\varepsilon}(m_0)f(x) = (-1)^\varepsilon f(-x), \tag{1.6}$$
$$\rho_{\sigma,\varepsilon}(e^{tH})f(x) = e^{(\sigma-n)t}f(e^{-2t}x), \qquad t\in\mathbb{R}. \tag{1.7}$$

The action of w_0 in the Fourier transformed picture is more involved (see e.g. [17, Proposition 2.3]). Note that by these formulas the restriction $\rho_{\sigma,\varepsilon}|_{\overline{P}}$ also acts on $C^\infty(\mathbb{R}^m\setminus\{0\})$. Using the classical intertwining relations

$$x_j\circ\mathcal{F}_{\mathbb{R}^n} = \mathcal{F}_{\mathbb{R}^n}\circ\left(-i\tfrac{\partial}{\partial x_j}\right),$$
$$\tfrac{\partial}{\partial x_j}\circ\mathcal{F}_{\mathbb{R}^n} = \mathcal{F}_{\mathbb{R}^n}\circ(-ix_j)$$

it is easy to compute the differential action $\mathrm{d}\rho^G_\sigma$ of $\rho^G_{\sigma,\varepsilon}$:

$$\mathrm{d}\rho^G_\sigma(\overline{N}_j)f(x) = ix_jf(x), \qquad j=1,\dots,n, \tag{1.8}$$
$$\mathrm{d}\rho^G_\sigma(T)f(x) = -D_{Tx}f(x), \qquad T\in\mathfrak{m}\cong\mathfrak{so}(n), \tag{1.9}$$
$$\mathrm{d}\rho^G_\sigma(H)f(x) = -\left(2E-\sigma+n\right)f(x), \tag{1.10}$$
$$\mathrm{d}\rho^G_\sigma(N_j)f(x) = -i\mathcal{B}^{n,\sigma}_jf(x), \qquad j=1,\dots,n, \tag{1.11}$$

where we abbreviate

$$\mathcal{B}^{n,\sigma}_j := x_j\Delta - \left(2E-\sigma+n\right)\frac{\partial}{\partial x_j}.$$

The operators $\mathcal{B}^{n,\sigma}_j$ are called *Bessel operators* and were for $G=O(1,n+1)$ first studied in [6]. They are polynomial differential operators on $\mathbb{R}^n$ and hence the action $\mathrm{d}\rho^G_\sigma$ defines a representation of $\mathfrak{g}$ on $C^\infty(\Omega)$ for every open subset $\Omega\subseteq\mathbb{R}^n$.

To describe the representation spaces $\mathcal{F}^{-1}_{\mathbb{R}^n}\mathcal{H}^G_{\sigma,\varepsilon}$ in the Fourier transformed picture we recall that the Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ intertwines convolution and multiplication operators. Further, the Riesz distributions $R_\lambda\in\mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle R_\lambda,\varphi\rangle = \frac{2^{-\frac{\lambda}{2}}}{\Gamma(\frac{\lambda+n}{2})}\int_{\mathbb{R}^n}\varphi(x)|x|^\lambda\,\mathrm{d}x, \qquad \varphi\in\mathcal{S}(\mathbb{R}^n),$$

for $\operatorname{Re}\lambda\gg0$ and extended analytically to $\lambda\in\mathbb{C}$ satisfy the following classical functional equation (see [4, equation (2') in II.3.3])

$$\mathcal{F}_{\mathbb{R}^n}R_\lambda = R_{-\lambda-n}.$$

With this and (1.1) we see that in the Fourier transformed picture the representations $\rho^G_{\sigma,\varepsilon}$ are realized on $\mathcal{F}^{-1}_{\mathbb{R}^n}\mathcal{H}^G_{\sigma,\varepsilon} = L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$, $\sigma\in i\mathbb{R}\cup(-n,n)$, and the Fourier transform is a unitary (up to scalar multiples) isomorphism $\mathcal{F}_{\mathbb{R}^n}: L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)\to\mathcal{H}^G_{\sigma,\varepsilon}$ intertwining the representations $\rho^G_{\sigma,\varepsilon}$ and $\pi^G_{\sigma,\varepsilon}$. The standard intertwining operators $J(\sigma,\varepsilon)$ are in this picture given by multiplication

$$L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)\to L^2(\mathbb{R}^n,|x|^{\operatorname{Re}\sigma}\,\mathrm{d}x), \qquad f(x)\mapsto|x|^{-\sigma}f(x).$$

The K-type decomposition (1.2) is difficult to see in the Fourier transformed picture. However, one can still explicitly describe the space of K-finite vectors. For this recall the renormalized K-Bessel function $\widetilde{K}_\alpha(z)$ from Appendix B.1. It is easy to see that the vector

$$\psi^G_\sigma(x) := \widetilde{K}_{-\frac{\sigma}{2}}(|x|), \qquad x\in\mathbb{R}^n\setminus\{0\}, \tag{1.12}$$

is $\mathfrak{k}$-fixed and constitutes the minimal $\mathfrak{k}$-type. Note that as K-representation the minimal K-type is for $\varepsilon\neq0$ not the trivial representation since $m_0\in K$ acts by $(-1)^\varepsilon$. To describe the underlying $(\mathfrak{g},K)$-module we denote for $f\in C^\infty(\mathbb{R}_+)$ and $k\in\mathbb{N}_0$ by $f\otimes|x|^{2k}$ the function $f(|x|)|x|^{2k}$ and by $f\otimes|x|^{2k}\mathbb{C}[x_1,\dots,x_n]$ the space of all functions of the form $f(|x|)|x|^{2k}p(x)$ for some polynomial $p\in\mathbb{C}[x_1,\dots,x_n]$.

Lemma 1.2. *The underlying $(\mathfrak{g},K)$-module of $(\rho_{\sigma,\varepsilon},L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x))$ is given by*

$$L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)_K = \bigoplus_{k=0}^\infty\widetilde{K}_{-\frac{\sigma}{2}+k}\otimes|x|^{2k}\mathbb{C}[x_1,\dots,x_n]. \tag{1.13}$$

Proof. Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \overline{\mathfrak{n}}$ the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ decomposes by the Poincaré–Birkhoff–Witt Theorem into $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\overline{\mathfrak{n}})\mathcal{U}(\mathfrak{a})\mathcal{U}(\mathfrak{k})$. The $(\mathfrak{g}, K)$-module $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)_K$ is generated by the $\mathfrak{k}$-fixed vector ψ_σ^G and hence

$$L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)_K = \mathcal{U}(\mathfrak{g})\psi_\sigma^G = \mathcal{U}(\overline{\mathfrak{n}})\mathcal{U}(\mathfrak{a})\psi_\sigma^G.$$

By (1.8) and (1.10) we have $\mathcal{U}(\overline{\mathfrak{n}}) = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathcal{U}(\mathfrak{a}) = \mathbb{C}[E]$. Using (B.3) we further find that the Euler operator E acts on functions of the form $\widetilde{K}_\alpha(|x|)|x|^{2k}$, $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_0$, by

$$E\left(\widetilde{K}_\alpha(|x|)|x|^{2k}\right) = -\tfrac{1}{2}\widetilde{K}_{\alpha+1}(|x|)|x|^{2k+2} + 2k\widetilde{K}_\alpha(|x|)|x|^{2k}$$

Hence

$$\begin{aligned}\mathcal{U}(\overline{\mathfrak{n}})\mathcal{U}(\mathfrak{a})\psi_\sigma^G &= \mathcal{U}(\overline{\mathfrak{n}}) \bigoplus_{k=0}^\infty \mathbb{C}\left(\widetilde{K}_{-\frac{\sigma}{2}+k} \otimes |x|^{2k}\right) \\ &= \bigoplus_{k=0}^\infty \widetilde{K}_{-\frac{\sigma}{2}+k} \otimes |x|^{2k}\mathbb{C}[x_1, \ldots, x_n]\end{aligned}$$

and the claim follows. □

2 Reduction to an ordinary differential operator

This section deals with the reduction of the branching problem for $\rho_{\sigma,\varepsilon}^G|_H$ to an ordinary differential equation on $\mathbb{R}_+$.

Consider the L^2-realization $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$ of the representation $\rho_{\sigma,\varepsilon}^G$ where we split variables $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. We realize unitary principal series and complementary series representations $\rho_{\tau,\delta}^{O(1,m+1)}$ of the first factor $O(1, m+1)$ of $H = O(1, m+1) \times O(n-m)$ in the same way on $L^2(\mathbb{R}^m, |x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)$. For the second factor $O(n-m)$ denote by $\mathcal{H}^k(\mathbb{R}^{n-m})$ its representation on solid spherical harmonics on $\mathbb{R}^{n-m}$ of degree $k \in \mathbb{N}_0$ by left-translation.

Proposition 2.1. *Let $\sigma \in (-n, n) \cup i\mathbb{R}$ and $\tau \in (-m, m) \cup i\mathbb{R}$. For every solution $F \in C^\infty(\mathbb{R}_+)$ of the second-order ordinary differential equation*

$$\begin{aligned}t(1+t)u''(t) + \left(\tfrac{-\sigma+2k+n-m+2}{2}t + \tfrac{2k+n-m}{2}\right)u'(t)& \\ + \tfrac{1}{4}\left(\left(\tfrac{-\sigma+2k+n-m}{2}\right)^2 - \left(\tfrac{\tau}{2}\right)^2\right)u(t) &= 0\end{aligned}$$

which is regular at $t = 0$ the map

$$\begin{aligned}\Psi : C^\infty(\mathbb{R}^m \setminus \{0\}) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) &\to C^\infty(\mathbb{R}^n \setminus \{x = 0\}), \\ \Psi(f \otimes \phi)(x, y) &:= |x|^{\frac{\sigma-\tau-2k-n+m}{2}} F\left(\tfrac{|y|^2}{|x|^2}\right) f(x)\phi(y),\end{aligned}$$

is $\overline{P}_H$- and $\mathfrak{h}$-equivariant if $C^\infty(\mathbb{R}^m \setminus \{0\})$ carries the representation $\rho_{\tau,\varepsilon+k}^{O(1,m+1)}|_{\overline{P}_H}$ (resp. $\mathrm{d}\rho_\tau^{O(1,m+1)}$) and $C^\infty(\mathbb{R}^n \setminus \{x = 0\})$ the representation $\rho_{\sigma,\varepsilon}^G|_{\overline{P}_H}$ (resp. $\mathrm{d}\rho_\sigma^G|_{\mathfrak{h}}$).

Proof. Put $\mu := 2k+n-m$ and $\alpha := \frac{\sigma-\tau-\mu}{2}$ so that

$$\Psi(f\otimes\phi)(x,y) = |x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y).$$

Since $\mathfrak{h} = \mathfrak{n}_H + \mathfrak{m}_H + \mathfrak{a} + \overline{\mathfrak{n}}_H$ it suffices to check the intertwining property for $\overline{N}_H$, M_H, A and $\mathfrak{n}_H$.

(i) For $\overline{n}_a \in \overline{N}_H$ both $\rho^G_{\sigma,\varepsilon}(\overline{n}_a)$ and $\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(\overline{n}_a)$ are by (1.4) the multiplication operators $e^{i(x\,|\,a)}$ and hence the intertwining property is clear.

(ii) Let $m = \mathrm{diag}(1,k_1,k_2,1) \in M_H^+$, $k_1 \in O(m)$, $k_2 \in O(n-m)$. Then with $m' = \mathrm{diag}(1,k_1,\mathbf{1}_{n-m+1})$ we have by (1.5)

$$\begin{aligned}\rho^G_{\sigma,\varepsilon}(m)\Psi(f\otimes\phi)(x,y) &= \Psi(f\otimes\phi)(k_1^{-1}x,k_2^{-1}y)\\ &= |k_1^{-1}x|^\alpha F(\tfrac{|k_2^{-1}y|^2}{|k_1^{-1}x|^2})f(k_1^{-1}x)\phi(k_2^{-1}y)\\ &= |x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(k_1^{-1}x)\phi(k_2^{-1}y)\\ &= \Psi(\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(m')f\otimes(k_2\cdot\phi))(x,y).\end{aligned}$$

Further, for m_0 we have with (1.6)

$$\begin{aligned}\rho^G_{\sigma,\varepsilon}(m_0)\Psi(f\otimes\phi)(x,y) &= (-1)^\varepsilon\Psi(f\otimes\phi)(-x,-y)\\ &= (-1)^\varepsilon|(-x)|^\alpha F(\tfrac{|(-y)|^2}{|(-x)|^2})f(-x)\phi(-y)\\ &= (-1)^{\varepsilon+k}|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(-x)\phi(y)\\ &= \Psi(\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(m_0)f\otimes\phi)(x,y).\end{aligned}$$

(iii) For $a = e^{tH} \in A$ we obtain with (1.7)

$$\begin{aligned}\rho^G_{\sigma,\varepsilon}(a)\Psi(f\otimes\phi)(x,y) &= e^{(\sigma-n)t}\Psi(f\otimes\phi)(e^{-2t}x,e^{-2t}y)\\ &= e^{(\sigma-n)t}|e^{-2t}x|^\alpha F(\tfrac{|e^{-2t}y|^2}{|e^{-2t}x|^2})f(e^{-2t}x)\phi(e^{-2t}y)\\ &= e^{(\sigma-n-2\alpha-2k)t}|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(e^{-2t}x)\phi(y)\\ &= e^{(\tau-m)t}|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(e^{-2t}x)\phi(y)\\ &= \Psi(\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(a)f\otimes\phi)(x,y).\end{aligned}$$

(iv) To show the intertwining property for $\mathfrak{n}_H$ it suffices by (1.11) to show the identity

$$\mathcal{B}^{n,\sigma}_j\Psi(f\otimes\phi) = \Psi(\mathcal{B}^{m,\tau}_j f\otimes\phi)$$

for $j=1,\ldots,m$ which follows from the next lemma. □

For $\sigma,\mu \in \mathbb{C}$ we introduce the ordinary differential operator

$$\mathcal{D}_{\sigma,\mu} := t(1+t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{\mu-\sigma+2}{2}t + \frac{\mu}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}. \tag{2.1}$$

Lemma 2.2. *Let $\sigma, \tau, \alpha \in \mathbb{C}$, $k \in \mathbb{N}_0$, $F \in C^\infty([0,\infty))$, $f \in C^\infty(\mathbb{R}^m \setminus \{0\})$ and $\phi \in \mathcal{H}^k(\mathbb{R}^{n-m})$. Then for every $j = 1, \ldots, m$ we have*

$$\begin{aligned}
&\mathcal{B}_j^{n,\sigma}\left[|x|^\alpha F(\tfrac{|y|^2}{|x|^2}) f(x)\phi(y)\right] \\
&\qquad = |x|^\alpha F(\tfrac{|y|^2}{|x|^2}) \mathcal{B}_j^{m,\tau} f(x)\phi(y) + x_j|x|^{\alpha-2} f(x)\phi(y)\left(4\mathcal{D}_{\sigma,\mu} + \alpha(\sigma-\mu-\alpha)\right) F(\tfrac{|y|^2}{|x|^2}).
\end{aligned}$$

Proof. We first note the following basic identities, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the gradients in $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$ respectively, and Δ_x and Δ_y the Laplacians on $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively:

$$\begin{aligned}
\frac{\partial}{\partial x}|x|^\alpha &= \alpha|x|^{\alpha-2}x, & \Delta_x|x|^\alpha &= \alpha(\alpha+m-2)|x|^{\alpha-2}, \\
\frac{\partial}{\partial x}F(\tfrac{|y|^2}{|x|^2}) &= -\frac{2|y|^2}{|x|^4}F'(\tfrac{|y|^2}{|x|^2})x, & \Delta_x F(\tfrac{|y|^2}{|x|^2}) &= 4\frac{|y|^4}{|x|^6}F''(\tfrac{|y|^2}{|x|^2}) - 2(m-4)\frac{|y|^2}{|x|^4}F'(\tfrac{|y|^2}{|x|^2}), \\
\frac{\partial}{\partial y}F(\tfrac{|y|^2}{|x|^2}) &= \frac{2}{|x|^2}F'(\tfrac{|y|^2}{|x|^2})y, & \Delta_y F(\tfrac{|y|^2}{|x|^2}) &= \frac{4|y|^2}{|x|^4}F''(\tfrac{|y|^2}{|x|^2}) + \frac{2(n-m)}{|x|^2}F'(\tfrac{|y|^2}{|x|^2}).
\end{aligned}$$

The calculation is split into several parts. In what follows we abbreviate $t := \frac{|y|^2}{|x|^2}$.

(i) We begin with calculating $x_j\Delta_x\Psi(f\otimes\phi)$:

$$\begin{aligned}
&x_j\Delta_x\Psi(f\otimes\phi)(x,y) \\
&\quad = \Psi(x_j\Delta_x f\otimes\phi)(x,y) + x_j\Delta_x|x|^\alpha \cdot F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y) \\
&\qquad + x_j\Delta_x F(\tfrac{|y|^2}{|x|^2}) \cdot |x|^\alpha f(x)\phi(y) + 2x_j\frac{\partial|x|^\alpha}{\partial x}\cdot\frac{\partial f}{\partial x}(x)\cdot F(\tfrac{|y|^2}{|x|^2})\phi(y) \\
&\qquad + 2x_j\frac{\partial|x|^\alpha}{\partial x}\cdot\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial x}\cdot f(x)\phi(y) + 2x_j\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial x}\cdot\frac{\partial f}{\partial x}(x)\cdot|x|^\alpha\phi(y) \\
&\quad = \Psi(x_j\Delta_x f\otimes\phi)(x,y) + x_j|x|^{\alpha-2}Ef(x)\phi(y)\left(-4tF'(t) + 2\alpha F(t)\right) \\
&\qquad + x_j|x|^{\alpha-2}f(x)\phi(y)\left(4t^2F''(t) - 2(2\alpha+m-4)tF'(t) + \alpha(\alpha+m-2)F(t)\right).
\end{aligned}$$

(ii) Next we calculate $x_j\Delta_y\Psi(f\otimes\phi)$:

$$\begin{aligned}
&x_j\Delta_y\Psi(f\otimes\phi)(x,y) \\
&\quad = x_j\Delta_y F(\tfrac{|y|^2}{|x|^2})\cdot|x|^\alpha f(x)\phi(y) + x_j\Delta_y\phi(y)\cdot|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(x) \\
&\qquad + 2x_j\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial y}\cdot\frac{\partial\phi}{\partial y}\cdot|x|^\alpha f(x) \\
&\quad = x_j|x|^{\alpha-2}f(x)\phi(y)\left(4tF''(t) + 2(2k+n-m)F'(t)\right)
\end{aligned}$$

since $E\phi = k\phi$ and $\Delta_y\phi = 0$.

(iii) We now calculate $\frac{\partial}{\partial x_j}\Psi(f\otimes\phi)$:

$$\begin{aligned}&\frac{\partial}{\partial x_j}\Psi(f\otimes\phi)(x,y)\\&\quad=\frac{\partial|x|^\alpha}{\partial x_j}\cdot F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y)+\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial x_j}\cdot|x|^\alpha f(x)\phi(y)\\&\qquad+\frac{\partial f}{\partial x_j}(x)\cdot|x|^\alpha F(\tfrac{|y|^2}{|x|^2})\phi(y)\\&\quad=\frac{\partial f}{\partial x_j}(x)\cdot|x|^\alpha F(\tfrac{|y|^2}{|x|^2})\phi(y)+x_j|x|^{\alpha-2}f(x)\phi(y)\left(-2tF'(t)+\alpha F(t)\right).\end{aligned}$$

(iv) Next we find $(2E-\sigma+n)\frac{\partial}{\partial x_j}\Psi(f\otimes\phi)$ by using (iii):

$$\begin{aligned}&(2E-\sigma+n)\frac{\partial}{\partial x_j}\Psi(f\otimes\phi)(x,y)\\&\quad=(2E-\sigma+n+2(\alpha+k))\frac{\partial f}{\partial x_j}(x)\cdot|x|^\alpha F(\tfrac{|y|^2}{|x|^2})\phi(y)\\&\qquad+2x_j|x|^{\alpha-2}Ef(x)\phi(y)\left(-2tF'(t)+\alpha F(t)\right)\\&\qquad+(2(\alpha+k-1)-\sigma+n)x_j|x|^{\alpha-2}f(x)\phi(y)\left(-2tF'(t)+\alpha F(t)\right)\end{aligned}$$

since $E|x|^\beta=\beta|x|^\beta$, $EF(\frac{|y|^2}{|x|^2})=0$ and $E\phi=k\phi$.

Now, putting (i), (ii) and (iv) together gives the claimed identity. □

3 Spectral decomposition of a self-adjoint second-order differential operator on $\mathbb{R}_+$

In this section we find the spectral decomposition of the second-order differential operator $\mathcal{D}_{\sigma,\mu}$ on $L^2(\mathbb{R}_+,t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}t)$ using the theory developed by Weyl–Titchmarsh–Kodaira (see [11, 16]).

We fix $\sigma\in i\mathbb{R}\cup(0,n)$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$. (In the case $\sigma\in(-n,0)$ only the derivation of the discrete spectrum in Section 3.5 is slightly different. However, since $\pi_{\sigma,\varepsilon}\cong\pi_{-\sigma,\varepsilon}$ the decomposition of the representations is again the same and it suffices to consider $\sigma\in i\mathbb{R}\cup(0,n)$.) Further fix $k\in\mathbb{N}_0$ and put $\mu:=2k+n-m$. We assume that $m<n$ so that $\mu>0$. Proposition 2.1 suggests that the decomposition of the $O(n-m)$-isotypic component of $\mathcal{H}^k(\mathbb{R}^{n-m})$ in $\rho^G_{\sigma,\varepsilon}$ into irreducible $O(1,m+1)$-representations is given by the spectral decomposition of the differential operator $\mathcal{D}_{\sigma,\mu}$ defined in (2.1). Writing

$$\mathcal{D}_{\sigma,\mu}=t(1+t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}+((a+b+1)t+c)\frac{\mathrm{d}}{\mathrm{d}t}$$

with

$$a=-\frac{\sigma-\mu}{4}+\frac{\tau}{4},\qquad b=-\frac{\sigma-\mu}{4}-\frac{\tau}{4},\qquad c=\frac{\mu}{2},$$

it is easy to see from (B.4) that the hypergeometric function

$$F(t,\tau) := {}_2F_1\left(a,b;c;-t\right) \tag{3.1}$$

solves the equation

$$\mathcal{D}_{\sigma,\mu}u + \lambda^* u = 0, \qquad \lambda^* = ab = \left(\frac{\sigma-\mu}{4}\right)^2 - \left(\frac{\tau}{4}\right)^2.$$

We find a spectral decomposition of $\mathcal{D}_{\sigma,\mu}$ in terms of $F(t,\tau)$.

3.1 Simplifications

Following [16, Examples 4.17 & 4.18] we first make the transformation $t = \sinh^2(\frac{x}{2})$. Using $t\frac{\mathrm{d}}{\mathrm{d}t} = \tanh(\frac{x}{2})\frac{\mathrm{d}}{\mathrm{d}x}$ we write the operator $\mathcal{D}_{\sigma,\mu}$ as

$$\begin{aligned}\mathcal{D}_{\sigma,\mu} &= \frac{1}{t}\left((1+t)\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + \left(\frac{\mu-\sigma}{2}t + \frac{\mu-2}{2}\right)t\frac{\mathrm{d}}{\mathrm{d}t}\right)\\ &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x}\end{aligned}$$

with

$$\beta(x) = \frac{\mu-1}{2}\tanh\left(\frac{x}{2}\right)^{-1} - \frac{\sigma-1}{2}\tanh\left(\frac{x}{2}\right).$$

Putting

$$y(x) = r(x)^{-1}u\left(\sinh^2\left(\frac{x}{2}\right)\right) \quad \text{with} \quad r(x) = \sinh\left(\frac{x}{2}\right)^{-\frac{\mu-1}{2}}\cosh\left(\frac{x}{2}\right)^{\frac{\sigma-1}{2}}$$

we finally see that the differential equation $\mathcal{D}_{\sigma,\mu}u + \lambda^* u = 0$ is equivalent to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (\lambda^* - q^*(x))y = 0$$

with

$$\begin{aligned}q^*(x) &= \tfrac{1}{4}\beta(x)^2 + \tfrac{1}{2}\beta'(x)\\ &= \frac{(\mu-1)(\mu-3)}{16}\tanh\left(\frac{x}{2}\right)^{-2} - \frac{\mu(\sigma-2)+1}{8} + \frac{(\sigma+1)(\sigma-1)}{16}\tanh\left(\frac{x}{2}\right)^2.\end{aligned}$$

To stay in line with [16, Examples 4.17 & 4.18] we shift the eigenvalues by putting $q(x) := q^*(x) - \left(\frac{\sigma-\mu}{4}\right)^2$ and $\lambda := \lambda^* - \left(\frac{\sigma-\mu}{4}\right)^2$ and obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (\lambda - q(x))y = 0. \tag{3.2}$$

Note that $q(x)$ is real-valued for $\sigma \in i\mathbb{R}\cup\mathbb{R}$ and hence the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2} - q(x)$ is formally self-adjoint on $L^2(\mathbb{R}_+)$.

3.2 Singularities and the boundary condition

The differential equation (3.2) has regular singular points at $x = 0$ and $x = \infty$. The corresponding asymptotic behaviour of solutions at $x = 0$ is given by $x^{\frac{\mu-1}{2}}$ and $x^{-\frac{\mu-3}{2}}$ for $\mu \neq 2$ and by $x^{\frac{1}{2}}$ and $\log(x)x^{\frac{1}{2}}$ for $\mu = 2$. Hence $x = 0$ is of limit point type (LPT) if $\mu \geq 4$ and of limit circle type (LCT) if $\mu = 1, 2, 3$. The solution

$$\eta_1(x,\lambda) = r(x)^{-1} {}_2F_1(a, b; c; -\sinh^2(\tfrac{x}{2}))$$

has asymptotic behaviour $x^{\frac{\mu-1}{2}}$ near $x = 0$, where

$$a = -\frac{\sigma-\mu}{4} + i\sqrt{\lambda}, \qquad b = -\frac{\sigma-\mu}{4} - i\sqrt{\lambda}, \qquad c = \frac{\mu}{2}.$$

Note that $\eta_1(x,\lambda)$ is holomorphic in $\lambda \in \mathbb{C}$ and real-valued if $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R} \cup i\mathbb{R}$. Indeed we can easily see $\overline{\eta_1(x,\lambda)} = \eta_1(x,\lambda)$ for $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R} \cup i\mathbb{R}$ using Kummer's transformation formula (B.7). In the case of (LCT) at $x = 0$ we impose an additional boundary condition (which is automatic in the case of (LPT)). For this we use the solution $\eta_1(x,\lambda)$ for a fixed λ_0 and impose

$$\lim_{x\to 0} W(\eta_1(-,\lambda_0), u)(x) = 0, \tag{BC}$$

where $W(u, v) = u'v - uv'$ denotes the Wronskian. Then in both (LPT) and (LCT) cases $\eta_1(x,\lambda)$ is the unique solution of (3.2) which is L^2 near $x = 0$ and satisfies the boundary condition (BC). Near $x = \infty$ we consider the solution

$$\eta_2(x,\lambda) = r(x)^{-1} \sinh^{-2b}(\tfrac{x}{2}) {}_2F_1(b, b-c+1; b-a+1; -\sinh^{-2}(\tfrac{x}{2})),$$

which has the asymptotic behaviour $e^{ix\sqrt{\lambda}}$ near $x = \infty$ and hence is L^2 near $x = \infty$ for $0 < \arg(\sqrt{\lambda}) < \pi$. The other solution is obtained by interchanging a and b and has asymptotics $e^{-ix\sqrt{\lambda}}$ whence $x = \infty$ is always of (LPT). Altogether the operator in (3.2) extends to a self-adjoint operator on $L^2(\mathbb{R}_+)$ under the boundary condition (BC).

3.3 Titchmarsh–Kodaira's spectral theorem

We calculate the Wronskian

$$\begin{aligned}
&W(\eta_1, \eta_2)(\lambda) \\
&\quad = r(x)^{-2} W({}_2F_1(a, b; c; -\sinh^2(\tfrac{-}{2})), \\
&\qquad\qquad \sinh^{-2b}(\tfrac{-}{2}) {}_2F_1(b, b-c+1; b-a+1; -\sinh^{-2}(\tfrac{-}{2})))(x) \\
&\quad = r(x)^{-2} \sinh(\tfrac{x}{2}) \cosh(\tfrac{x}{2}) W({}_2F_1(a, b; c; -z), \\
&\qquad\qquad z^{-b} {}_2F_1(b, b-c+1; b-a+1; -\tfrac{1}{z}))(\sinh^2(\tfrac{x}{2})) \\
&\quad = r(x)^{-2} \sinh(\tfrac{x}{2}) \cosh(\tfrac{x}{2}) \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} \\
&\qquad \times W(z^{-a} {}_2F_1(a, a-c+1; a-b+1; -\tfrac{1}{z}), \\
&\qquad\qquad z^{-b} {}_2F_1(b, b-c+1; b-a+1; -\tfrac{1}{z}))(\sinh^2(\tfrac{x}{2})) \\
&\quad = (a-b) \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} = \frac{2i\sqrt{\lambda}\,\Gamma(-2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}{\Gamma(-\frac{\sigma-\mu}{4} - i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4} - i\sqrt{\lambda})},
\end{aligned}$$

which depends only on λ. Now for $f \in L^2(\mathbb{R}_+)$ real-valued define

$$\Phi(x,\lambda) := \frac{1}{W(\eta_1,\eta_2)(\lambda)}\Big[\eta_2(x,\lambda)\int_0^x \eta_1(y,\lambda)f(y)\,\mathrm{d}y + \eta_1(x,\lambda)\int_x^\infty \eta_2(y,\lambda)f(y)\,\mathrm{d}y\Big].$$

Then by [16, Section 3.1] we have

$$f(x) = \lim_{\delta\to 0} -\frac{1}{i\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \Phi(x,\lambda)\,\mathrm{d}\lambda = \lim_{\delta\to 0} -\frac{1}{\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \operatorname{Im}\Phi(x,\lambda)\,\mathrm{d}\lambda.$$

3.4 The continuous spectrum

We first treat the integration over the interval $(0,\infty)$. In this case $\sqrt{\lambda} \in \mathbb{R}_+$. We study the cases $\sigma \in (0,n)$ and $\sigma \in i\mathbb{R}$ separately. Recall that in both cases $\eta_1(x,\lambda) \in \mathbb{R}$.

(i) $\sigma \in (0,n)$. In this case $\overline{a} = b$ and $c \in \mathbb{R}$. Using $\overline{\Gamma(z)} = \Gamma(\overline{z})$ we obtain

$$\begin{aligned}\operatorname{Im}\left(\frac{\eta_2(x,\lambda)}{W(\eta_1,\eta_2)(\lambda)}\right) &= \frac{1}{2ir(x)}\Big[\frac{\Gamma(b)\Gamma(c-a)}{(a-b)\Gamma(b-a)\Gamma(c)} \\ &\quad\times \sinh^{-2b}(\tfrac{x}{2})\,{}_2F_1\big(b,b-c+1;b-a+1;-\sinh^{-2}(\tfrac{x}{2})\big) \\ &\quad- \frac{\Gamma(a)\Gamma(c-b)}{(b-a)\Gamma(a-b)\Gamma(c)}\sinh^{-2a}(\tfrac{x}{2})\,{}_2F_1\big(a,a-c+1;a-b+1;-\sinh^{-2}(\tfrac{x}{2})\big)\Big] \\ &= \frac{1}{2i(a-b)r(x)}\frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(a-b)\Gamma(b-a)\Gamma(c)^2}\,{}_2F_1(a,b;c;-\sinh^2(\tfrac{x}{2})) \\ &= \frac{1}{2i(a-b)}\left|\frac{\Gamma(a)\Gamma(c-b)}{\Gamma(a-b)\Gamma(c)}\right|^2\eta_1(x,\lambda) \\ &= -\frac{1}{4\sqrt{\lambda}}\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2\eta_1(x,\lambda)\end{aligned}$$

whence

$$\operatorname{Im}\Phi(x,\lambda) = -\frac{1}{4\sqrt{\lambda}}\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2\eta_1(x,\lambda)\int_0^\infty \eta_1(y,\lambda)f(y)\,\mathrm{d}y. \tag{3.3}$$

(ii) $\sigma \in i\mathbb{R}$. Note that $\overline{a} = c-a$, $\overline{b} = c-b$ and $c \in \mathbb{R}$. Hence we find by Kummer's transformation formula (B.7)

$$\operatorname{Im}\left(\frac{\eta_2(x,\lambda)}{W(\eta_1,\eta_2)(\lambda)}\right)=\frac{1}{2ir(x)}\Big[\frac{\Gamma(b)\Gamma(c-a)}{(a-b)\Gamma(b-a)\Gamma(c)}$$

$$\times\sinh^{-2b}(\tfrac{x}{2})_2F_1\big(b,b-c+1;b-a+1;-\sinh^{-2}(\tfrac{x}{2})\big)$$

$$-\cosh^{\sigma}(\tfrac{x}{2})\frac{\Gamma(c-b)\Gamma(a)}{(b-a)\Gamma(a-b)\Gamma(c)}$$

$$\times\sinh^{-2(c-b)}(\tfrac{x}{2})_2F_1\big(c-b,1-b;a-b+1;-\sinh^{-2}(\tfrac{x}{2})\big)\Big]$$

$$=\frac{1}{2ir(x)}\Big[\frac{\Gamma(b)\Gamma(c-a)}{(a-b)\Gamma(b-a)\Gamma(c)}$$

$$\times\sinh^{-2b}(\tfrac{x}{2})_2F_1\big(b,b-c+1;b-a+1;-\sinh^{-2}(\tfrac{x}{2})\big)$$

$$-\frac{\Gamma(c-b)\Gamma(a)}{(b-a)\Gamma(a-b)\Gamma(c)}\sinh^{-2a}(\tfrac{x}{2})_2F_1\big(a,a-c+1;a-b+1;-\sinh^{-2}(\tfrac{x}{2})\big)\Big]$$

$$=-\frac{1}{4\sqrt{\lambda}}\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2\eta_1(x,\lambda)$$

where we have used the same calculation as above. This shows that (3.3) also holds in the case $\sigma\in i\mathbb{R}$.

In both cases we collect the continuous spectrum

$$-\frac{1}{\pi}\int_0^\infty \operatorname{Im}\Phi\,(x,\lambda)\,\mathrm{d}\lambda$$

$$=\frac{1}{4\pi}\int_0^\infty\eta_1(x,\lambda)\left(\int_0^\infty\eta_1(y,\lambda)f(y)\,\mathrm{d}y\right)\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2\frac{\mathrm{d}\lambda}{\sqrt{\lambda}}.$$

3.5 The discrete spectrum

Next we consider the integration over the interval $(-\infty,0)$. Here $\sqrt{\lambda}\in i\mathbb{R}_+$. Again we treat the cases $\sigma\in(0,n)$ and $\sigma\in i\mathbb{R}$ separately. Recall that in both cases $\eta_1(x,\lambda)\in\mathbb{R}$ for all $x\in\mathbb{R}_+$.

(i) $\sigma\in(0,n)$. We have $a,b,c\in\mathbb{R}$ which gives $\eta_2(x,\lambda)\in\mathbb{R}$ for all $x\in\mathbb{R}_+$. The poles of the function $\Phi(x,\lambda)$ as a function of λ are the zeros of the Wronskian. Since $\operatorname{Re}(\frac{\sigma+\mu}{4}-i\sqrt{\lambda})>\frac{\mu}{4}>0$ the poles of $\Phi(x,\lambda)$ are all simple and exactly at the points where $-\frac{\sigma-\mu}{4}-i\sqrt{\lambda}\in-\mathbb{N}_0$. This gives $i\sqrt{\lambda}=-\frac{\sigma-\mu}{4}+j$ and $\lambda=-\big(\frac{\sigma-\mu}{4}-j\big)^2$ for $j\in\mathbb{N}_0$ with $j<\frac{\sigma-\mu}{4}$. Consequently we have $b=-j$ and therefore, by (B.5)

$$\eta_1(x,\lambda)=\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\eta_2(x,\lambda)$$

$$=\frac{\Gamma(-\frac{\sigma-\mu}{2}+2j)\Gamma(\frac{\mu}{2})}{\Gamma(-\frac{\sigma-\mu}{2}+j)\Gamma(j+\frac{\mu}{2})}\eta_2(x,\lambda).$$

Using $\operatorname{res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$ we find

$$\begin{aligned}
&\operatorname{res}_{\lambda=-\left(\frac{\sigma-\mu}{4}-j\right)^2} \frac{1}{W(\eta_1,\eta_2)(\lambda)} \\
&\quad = \frac{\Gamma(\frac{\sigma}{2}-j)}{(2j-\frac{\sigma-\mu}{2})\Gamma(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\mu}{2})} \operatorname{res}_{\lambda=-\left(\frac{\sigma-\mu}{4}-j\right)^2} \Gamma\left(-\frac{\sigma-\mu}{4}-i\sqrt{\lambda}\right) \\
&\quad = \frac{\Gamma(\frac{\sigma}{2}-j)}{(2j-\frac{\sigma-\mu}{2})\Gamma(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\mu}{2})} \left(2j-\frac{\sigma-\mu}{2}\right)\frac{(-1)^j}{j!} \\
&\quad = \frac{(-1)^j\Gamma(\frac{\sigma}{2}-j)}{j!\Gamma(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\mu}{2})}.
\end{aligned}$$

Since $\operatorname{Im}\Phi(x,\lambda)=0$ for $\lambda\in(-\infty,0)$ not a pole we obtain by the residue theorem the discrete spectrum

$$\begin{aligned}
&-\frac{1}{\pi}\lim_{\delta\to 0}\int_{-\infty+i\delta}^{i\delta} \operatorname{Im}\Phi\left(x,\lambda\right)\,\mathrm{d}\lambda \\
&\quad = \sum_{j\in[0,\frac{\sigma-\mu}{4})\cap\mathbb{Z}} \operatorname{res}_{\lambda=-\left(\frac{\sigma-\mu}{4}-j\right)^2}\Phi(x,\lambda) \\
&\quad = \sum_{j\in[0,\frac{\sigma-\mu}{4})\cap\mathbb{Z}} \frac{(-1)^j\Gamma(\frac{\sigma}{2}-j)\Gamma(-\frac{\sigma-\mu}{2}+j)\Gamma(j+\frac{\mu}{2})}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-2j)\Gamma(-\frac{\sigma-\mu}{2}+2j)}\eta_1(x,-\left(\tfrac{\sigma-\mu}{4}-j\right)^2) \\
&\qquad \times\int_0^\infty \eta_1(y,-\left(\tfrac{\sigma-\mu}{4}-j\right)^2)f(y)\,\mathrm{d}y.
\end{aligned}$$

(ii) $\sigma\in i\mathbb{R}$. We have $\overline{a}=c-b$, $\overline{b}=c-a$ and $c\in\mathbb{R}$. By Kummer's transformation formula (B.7)

$$\begin{aligned}
\overline{\eta_2(x,\lambda)} &= r(x)^{-1}\cosh^\sigma(\tfrac{x}{2})\sinh^{-2(c-a)}(\tfrac{x}{2})\,{}_2F_1(c-a,1-a;b-a+1;-\sinh^{-2}(\tfrac{x}{2})) \\
&= r(x)^{-1}\sinh^{-2b}(\tfrac{x}{2})\,{}_2F_1(b-c+1,b;b-a+1;-\sinh^{-2}(\tfrac{x}{2})) \\
&= \eta_2(x,\lambda).
\end{aligned}$$

Further since $\overline{\Gamma(z)}=\Gamma(\overline{z})$ we also find that the Wronskian $W(\eta_1,\eta_2)$ is real-valued. Hence $\operatorname{Im}\Phi(x,\lambda)=0$ for $\lambda\in(-\infty,0)$. Since $W(\eta_1,\eta_2)$ has no poles in $\lambda\in(-\infty,0)$ for $\sigma\in i\mathbb{R}$ there is no discrete spectrum in this case.

3.6 The spectral theorem for $\mathcal{D}_{\sigma,\mu}$

Together this gives the spectral decomposition of $L^2(\mathbb{R}_+)$ into the eigenfunctions $\eta_1(x,\lambda)$. For the precise statement let

$$S(\sigma,\mu) := (0,\infty)\cup\bigcup_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}}\left\{-\left(\frac{\sigma-\mu}{4}-j\right)^2\right\}.$$

Note that $S(\sigma,\mu)=(0,\infty)$ for $\sigma\in i\mathbb{R}$. On $S(\sigma,\mu)$ we define a measure $\mathrm{d}\nu_{\sigma,\mu}$ by

$$\int_{S(\sigma,\mu)} g(\lambda)\,\mathrm{d}\nu_{\sigma,\mu}(\lambda) := \frac{1}{4\pi}\int_0^\infty g(\lambda)\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2\frac{\mathrm{d}\lambda}{\sqrt{\lambda}}$$
$$+\sum_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}}\frac{(-1)^j\Gamma(\frac{\sigma}{2}-j)\Gamma(-\frac{\sigma-\mu}{2}+j)\Gamma(j+\frac{\mu}{2})}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-2j)\Gamma(-\frac{\sigma-\mu}{2}+2j)}g(-\left(\tfrac{\sigma-\mu}{4}-j\right)^2).$$

Then by [16, Sections 3.1 & 3.7] we have:

Theorem 3.1. *For $\sigma\in i\mathbb{R}\cup(0,n)$ and $\mu\in\mathbb{N}$ the map*

$$L^2(\mathbb{R}_+)\xrightarrow{\sim}L^2(S(\sigma,\mu),\,\mathrm{d}\nu_{\sigma,\mu}),\qquad f\mapsto g(\lambda)=\int_0^\infty \eta_1(x,\lambda)f(x)\,\mathrm{d}x,$$

is a unitary isomorphism with inverse

$$L^2(S(\sigma,\mu),\,\mathrm{d}\nu_{\sigma,\mu})\xrightarrow{\sim}L^2(\mathbb{R}_+),\qquad g\mapsto f(x)=\int_{S(\sigma,\mu)}\eta_1(x,\lambda)g(\lambda)\,\mathrm{d}\nu_{\sigma,\mu}(\lambda).$$

For our application we need the spectral decomposition of the operator $\mathcal{D}_{\sigma,\mu}$ which follows from Theorem 3.1 by the transformation $u(t)\mapsto r(x)^{-1}u(\sinh^2(\frac{x}{2}))$. To state this put

$$T(\sigma,\mu):=i\mathbb{R}_+\cup\bigcup_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}}\{\sigma-\mu-4j\}$$

and define a measure $\mathrm{d}m_{\sigma,\mu}$ on $T(\sigma,\mu)$ by

$$\int_{T(\sigma,\mu)} g(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau) := \frac{1}{8\pi i}\int_{i\mathbb{R}_+} g(\tau)\left|\frac{\Gamma(\frac{-\sigma+\mu+\tau}{4})\Gamma(\frac{\sigma+\mu+\tau}{4})}{\Gamma(\frac{\tau}{2})\Gamma(\frac{\mu}{2})}\right|^2\mathrm{d}\tau$$
$$+\sum_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}}\frac{(-1)^j\Gamma(\frac{\sigma}{2}-j)\Gamma(-\frac{\sigma-\mu}{2}+j)\Gamma(j+\frac{\mu}{2})}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-2j)\Gamma(-\frac{\sigma-\mu}{2}+2j)}g(\sigma-\mu-4j). \tag{3.4}$$

Corollary 3.2. *For $\sigma\in i\mathbb{R}\cup(0,n)$ and $\mu\in\mathbb{N}$ the map*

$$L^2(\mathbb{R}_+,t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}t)\xrightarrow{\sim}L^2(T(\sigma,\mu),\,\mathrm{d}m_{\sigma,\mu}),$$
$$f\mapsto g(\tau)=\int_0^\infty F(t,\tau)f(t)t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\sigma}{2}}\,\mathrm{d}t$$

is a unitary isomorphism with inverse

$$L^2(T(\sigma,\mu),\,\mathrm{d}m_{\sigma,\mu})\xrightarrow{\sim}L^2(\mathbb{R}_+,t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}t),$$
$$g\mapsto f(t)=\int_{T(\sigma,\mu)}F(t,\tau)g(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau).$$

Remark 3.3. For the discrete part, namely for $\tau = \sigma - \mu - 4j$, $j \in \mathbb{N}_0$, the Gauß hypergeometric function $F(t,\tau)$ degenerates to a polynomial in t of degree j. More precisely, we have (see (B.8))

$$F(t, \sigma - \mu - 4j) = \frac{j!}{(\frac{\mu}{2})_n} P_j^{(\frac{\mu-2}{2}, -\frac{\sigma}{2})}(1+2t),$$

where $P_n^{(\alpha,\beta)}(z)$ denote the Jacobi polynomials.

Remark 3.4. For $\sigma \in (0,n)$ the results of Corollary 3.2 can also be found in [3, formula (A.11)] where the hypergeometric transform appears (essentially) as the radial part of the spherical Fourier transform on $\mathrm{SU}(1,n)/\mathrm{SU}(n)$. Since our approach provides a unified treatment of both complementary series and principal series, including the case $\sigma \in i\mathbb{R}$, we gave a detailed proof in this section for convenience.

4 Decomposition of representations and the Plancherel formula

Using the spectral decomposition of $\mathcal{D}_{\sigma,\mu}$ obtained in Corollary 3.2 we find in this section the explicit Plancherel formula for the decomposition of $\rho^G_{\sigma,\varepsilon}|_H$.

Let us first consider the action of $O(n-m)$ on $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$ which gives the following decomposition as $O(n-m)$-representations:

$$\begin{aligned} &L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y) \\ &\quad = \sum_{k=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}^m \times \mathbb{R}_+, (|x|^2 + r^2)^{-\frac{\operatorname{Re}\sigma}{2}} r^{2k+n-m-1}\,\mathrm{d}x\,\mathrm{d}r) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}), \end{aligned} \qquad (4.1)$$

where $r = |y|$. We fix a summand for some $k \in \mathbb{N}_0$ and put again $\mu = 2k + n - m$. The coordinate change $t := \frac{r^2}{|x|^2}$ gives

$$\begin{aligned} &L^2(\mathbb{R}^m \times \mathbb{R}_+, (|x|^2 + r^2)^{-\frac{\operatorname{Re}\sigma}{2}} r^{\mu-1}\,\mathrm{d}x\,\mathrm{d}r) \\ &\qquad = L^2(\mathbb{R}^m \times \mathbb{R}_+, \tfrac{1}{2}|x|^{-\operatorname{Re}\sigma+\mu} t^{\frac{\mu-2}{2}} (1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}x\,\mathrm{d}t). \end{aligned}$$

Since

$$\begin{aligned} &L^2(\mathbb{R}^m \times \mathbb{R}_+, \tfrac{1}{2}|x|^{-\operatorname{Re}\sigma+\mu} t^{\frac{\mu-2}{2}} (1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}x\,\mathrm{d}t) \\ &\qquad \cong L^2(\mathbb{R}^m, \tfrac{1}{2}|x|^{-\operatorname{Re}\sigma+\mu}\,\mathrm{d}x) \widehat{\otimes} L^2(\mathbb{R}_+, t^{\frac{\mu-2}{2}} (1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}t) \end{aligned}$$

we can apply Theorem 3.1 to find that the map

$$\begin{aligned} &L^2(\mathbb{R}^m \times \mathbb{R}_+, \tfrac{1}{2}|x|^{-\operatorname{Re}\sigma+\mu} t^{\frac{\mu-2}{2}} (1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}x\,\mathrm{d}t) \\ &\qquad \to \int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau) \end{aligned}$$

given by

$$f(x,t) \mapsto \hat{f}(x,\tau) := |x|^{-\frac{\sigma-\tau-\mu}{2}} \int_0^\infty F(t,\tau) f(x,t) t^{\frac{\mu-2}{2}} (1+t)^{-\frac{\sigma}{2}} \, \mathrm{d}t$$

is a unitary isomorphism, where $F(t,\tau)$ is defined by (3.1) and the measure $\mathrm{d}m_{\sigma,\mu}$ is given by (3.4). Its inverse is given by

$$g(x,\tau) \mapsto \check{g}(x,t) := \int_{T(\sigma,\mu)} |x|^{\frac{\sigma-\tau-\mu}{2}} F(t,\tau) g(x,\tau) \, \mathrm{d}m_{\sigma,\mu}(\tau).$$

Now we put these things together. For $\sigma \in i\mathbb{R} \cup (0,n)$ and $k \in \mathbb{N}_0$ we put $\mu := 2k+n-m$ and define an operator

$$\begin{aligned}\Psi(\sigma,k) : \left(\int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}|x|^{-\operatorname{Re}\tau} \, \mathrm{d}x) \, \mathrm{d}m_{\sigma,\mu}(\tau) \right) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) \\ \to L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma} \, \mathrm{d}x \, \mathrm{d}y)\end{aligned}$$

by

$$\begin{aligned}&\Psi(\sigma,k)\,(f \otimes \phi)\,(x,y) \\ &:= |x|^{\frac{\sigma-\tau-\mu}{2}} \phi(y) \int_{T(\sigma,\mu)} {}_2F_1\left(\tfrac{\mu-\sigma+\tau}{4}, \tfrac{\mu-\sigma-\tau}{4}; \tfrac{\mu}{2}; -\tfrac{|y|^2}{|x|^2} \right) f(x,\tau) \, \mathrm{d}m_{\sigma,\mu}(\tau).\end{aligned}$$

Theorem 4.1. *For $\sigma \in i\mathbb{R} \cup (0,n)$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ the map $\Psi(\sigma,k)$ is H-equivariant between the representations*

$$\int_{T(\sigma,\mu)}^{\oplus} \rho_{\tau,\varepsilon+k}^{O(1,m+1)} \, \mathrm{d}m_{\sigma,\mu}(\tau) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) \to \rho_{\sigma,\varepsilon}^G\big|_H$$

and constructs the $\mathcal{H}^k(\mathbb{R}^{n-m})$-isotypic component in $\rho_{\sigma,\varepsilon}^G|_H$. The following Plancherel formula holds:

$$\begin{aligned}&\|\Psi(\sigma,k)(f \otimes \phi)\|^2_{L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma} \, \mathrm{d}x \, \mathrm{d}y)} \\ &\quad = \int_{T(\sigma,\mu)} \|f(-,\tau)\|^2_{L^2(\mathbb{R}^m, \frac{1}{2}|x|^{-\operatorname{Re}\tau} \, \mathrm{d}x)} \, \mathrm{d}m_{\sigma,\mu}(\tau) \cdot \|\phi\|^2_{L^2(S^{n-m-1})}.\end{aligned}$$

Proof. We have already seen that $\Psi(\sigma,k)$ gives a unitary isomorphism so that the Plancherel formula above holds. Further, by Proposition 2.1 the map $\Psi(\sigma,k)$ intertwines the actions of $M_H A \overline{N}_H$ on smooth vectors and hence on the Hilbert spaces. Since H is generated by $M_H A \overline{N}_H$ and N_H it remains to prove the intertwining property for N_H. For this we use the Lie algebra action.

Lemma 4.2. *Let L be a connected Lie group with Lie algebra $\mathfrak{l}$ and let $(\rho_1, \mathcal{H}_1)$ and $(\rho_2, \mathcal{H}_2)$ be unitary representations of L. Suppose that a continuous linear map $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ is given and there exist subspaces $V_1 \subset \mathcal{H}_1$ and $V_2 \subset \mathcal{H}_2$ such that*

(i) V_i is dense in $\mathcal{H}_i$ for $i = 1,2$,

(ii) V_i is contained in the space of analytic vectors $\mathcal{H}_i^\omega$ for $i = 1,2$,

(iii) V_i *is* $\mathrm{d}\rho_i$*-stable for* $i = 1, 2$*,*

(iv) $(\varphi(\mathrm{d}\rho_1(X)v_1) \mid v_2)_{\mathcal{H}_2} = -(\varphi(v_1) \mid \mathrm{d}\rho_2(X)v_2)_{\mathcal{H}_2}$ *for* $v_1 \in V_1$*,* $v_2 \in V_2$ *and* $X \in \mathfrak{l}$*.*

Then φ *is* L*-equivariant.*

Proof. For $v_1 \in V_1$ and $v_2 \in V_2$ we put

$$f_{v_1,v_2}(g) := (\varphi(\rho_1(g)v_1) \mid v_2)_{\mathcal{H}_2}, \qquad g \in L,$$
$$h_{v_1,v_2}(g) := (\rho_2(g)\varphi(v_1) \mid v_2)_{\mathcal{H}_2} = (\varphi(v_1) \mid \rho_2(g^{-1})v_2)_{\mathcal{H}_2}, \qquad g \in L,$$

which are analytic functions on L by (ii). For a smooth function f on L and $X \in \mathfrak{l}$ we define derivatives by

$$(R(X)f)(g) := \lim_{t\to 0} \frac{f(ge^{tX}) - f(g)}{t}, \qquad (L(X)f)(g) := \lim_{t\to 0} \frac{f(e^{-tX}g) - f(g)}{t}.$$

We have $R(X)f(e) = -L(X)f(e)$ for the identity element $e \in L$ and $R(X)$ commutes with $L(X')$ for any $X, X' \in \mathfrak{l}$. Hence

$$\begin{aligned} R(X_1)R(X_2)\cdots R(X_k)f(e) &= -L(X_1)R(X_2)\cdots R(X_k)f(e) \\ &= -R(X_2)\cdots R(X_k)L(X_1)f(e) \\ &\;\;\vdots \\ &= (-1)^k L(X_k)\cdots L(X_2)L(X_1)f(e) \end{aligned}$$

for $X_1, \ldots, X_k \in \mathfrak{l}$. Then (iv) implies

$$\begin{aligned} R(X_1)\cdots R(X_k)f_{v_1,v_2}(e) &= f_{\mathrm{d}\rho_1(X_1)\cdots \mathrm{d}\rho_1(X_k)v_1,v_2}(e) \\ &= (-1)^k h_{v_1,\mathrm{d}\rho_2(X_k)\cdots \mathrm{d}\rho_2(X_1)v_2}(e) \\ &= (-1)^k L(X_k)\cdots L(X_1)h_{v_1,v_2}(e) \\ &= R(X_1)\cdots R(X_k)h_{v_1,v_2}(e). \end{aligned}$$

Since f_{v_1,v_2} and h_{v_1,v_2} are analytic functions, they coincide. Therefore $\varphi(\rho_1(g)v_1) = \rho_2(g)\varphi(v_1)$ for $v_1 \in V_1$ and hence $\varphi(\rho_1(g)v) = \rho_2(g)\varphi(v)$ for any $v \in \mathcal{H}_1$ by (i). □

We apply the lemma to the map $\varphi = \Psi(\sigma, k) : \mathcal{H}_1 \to \mathcal{H}_2$ where

$$\mathcal{H}_1 := \left(\int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau)\right) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}),$$
$$\mathcal{H}_2 := L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y).$$

So let ρ_1 and ρ_2 be the restrictions of

$$\left(\int_{T(\sigma,\mu)}^{\oplus} \rho^{O(1,m+1)}_{\tau,\varepsilon+k}\,\mathrm{d}m_{\sigma,\mu}(\tau)\right) \boxtimes \mathbf{1} \quad \text{and} \quad \rho^G_{\sigma,\varepsilon}$$

to $L = N_H$, respectively. We regard an element

$$f \in \int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau)$$

as a function $f(x,\tau)$ on $(\mathbb{R}^m\setminus\{0\})\times T(\sigma,\mu)$ and let V_1 be the space consisting of linear combinations of the functions on $(\mathbb{R}^m\setminus\{0\})\times T(\sigma,\mu)\times\mathbb{R}^{n-m}$ of the form

$$(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)\phi(y)\chi(\tau),$$

where $X\in\mathcal{U}(\mathfrak{h})$, $\psi_\tau^{O(1,m+1)}$ is the spherical vector of $\rho_{\tau,\varepsilon+k}^{O(1,m+1)}$ as defined in (1.12), $\phi\in\mathcal{H}^k(\mathbb{R}^{n-m})$ and $\chi\in C_c(T(\sigma,\mu))$, i.e. χ is a continuous function on $T(\sigma,\mu)$ with compact support. Further let V_2 be the space of all K-finite vectors in $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$. We now check conditions (i)–(iv):

(i) V_1 is dense in $\mathcal{H}_1$ since $C_c(T(\sigma,\mu))$ is dense in $L^2(T(\sigma,\mu),\mathrm{d}m_{\sigma,\mu})$ and the space of $(K\cap O(1,m+1))$-finite vectors for $\rho_\tau^{O(1,m+1)}$ is generated by $\psi_\tau^{O(1,m+1)}(x)$ and dense in $L^2(\mathbb{R}^m,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)$. The space V_2 is dense in $\mathcal{H}_2$ since it is the space of K-finite vectors for $\rho_{\sigma,\varepsilon}^{O(1,n+1)}$.

(ii) K-finite vectors are analytic vectors for G and in particular for $N_H\subseteq G$, hence $V_2\subseteq\mathcal{H}_2^\omega$. The inclusion $V_1\subseteq\mathcal{H}_1^\omega$ follows from the lemma below.

(iii) It is clear that V_2 is $\mathrm{d}\rho_2$-stable since the space of K-finite vectors is $\mathrm{d}\rho_\sigma^{O(1,n+1)}$-stable. That V_1 is $\mathrm{d}\rho_1$-stable follows from the definition of V_1.

Lemma 4.3. *Let*

$$(\rho_1',\mathcal{H}_1') := \left(\int_{T(\sigma,\mu)}^\oplus \rho_{\tau,\varepsilon+k}^{O(1,m+1)}\,\mathrm{d}m_{\sigma,\mu}(\tau), \int_{T(\sigma,\mu)}^\oplus L^2(\mathbb{R}^m,\tfrac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau)\right).$$

A function $f(x,\tau)$ on $(\mathbb{R}^m\setminus\{0\})\times T(\sigma,\mu)$ of the form

$$f(x,\tau):=(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)\chi(\tau)$$

for $X\in\mathcal{U}(\mathfrak{h})$ and $\chi\in C_c(T(\sigma,\mu))$ is an analytic vector of ρ_1'.

Proof. Let $\chi=\chi_c+\chi_d$ be the decomposition into continuous part and discrete part so that $\chi_c\in C_c(i\mathbb{R}_+)$ and $\chi_d\in C_c(T(\sigma,\mu)\cap(0,m))$. Since

$$(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)$$

is an analytic vector of $\rho_{\tau,\varepsilon+k}^{O(1,m+1)}$, the discrete part $(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)\chi_d(\tau)$ is an analytic vector of ρ_1'. Therefore we may and do assume $\chi\in C_c(i\mathbb{R}_+)$. It is enough to prove that for any $g_0\in O(1,m+1)$ there exists a neighborhood $0\in U\subset\mathfrak{so}(1,m+1)$ such that

$$a_N:=\left\|\rho_1'(\exp Y)\rho_1'(g_0)f(x,\tau)-\sum_{l=0}^N\frac{1}{l!}\mathrm{d}\rho_1'(Y)^l\rho_1'(g_0)f(x,\tau)\right\|_{\mathcal{H}_1'}^2\to 0$$

as $N\to 0$ for $Y\in U$. Consider the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^m}$ with respect to the variable x (see (1.3)) which gives a unitary equivalence between

$$\rho_1'=\int_{T(\sigma,\mu)}^\oplus\rho_{\tau,\varepsilon+k}^{O(1,m+1)}\,\mathrm{d}m_{\sigma,\mu}(\tau)\quad\text{and}\quad\pi_1:=\int_{T(\sigma,\mu)}^\oplus\pi_{\tau,\varepsilon+k}^{O(1,m+1)}\,\mathrm{d}m_{\sigma,\mu}(\tau).$$

Put $h(x,\tau) := \mathcal{F}_{\mathbb{R}^m}(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)$ then

$$a_N = \int_{i\mathbb{R}_+} \Big\| \pi_1(\exp Y)\pi_1(g_0)h(x,\tau) - \sum_{l=0}^{N} \frac{1}{l!}\mathrm{d}\pi_1(Y)^l\pi_1(g_0)h(x,\tau)\Big\|^2_{L^2(\mathbb{R}^m,\frac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)} |\chi(\tau)|^2\,\mathrm{d}m_{\sigma,\mu}(\tau).$$

As in Section 1.2 the function $h(x,\tau)$ corresponds to a function $\tilde{h}(g,\tau)$ on $O(1,m+1)\times i\mathbb{R}_+$ satisfying $\tilde{h}(gman,\tau) = \xi_{\varepsilon+k}(m)^{-1}a^{-\tau-\rho}\tilde{h}(g,\tau)$ for $m \in O(1,m+1)\cap M$, $a \in A$ and $n \in N_H$. Consequently, a_N is given as

$$\int_{i\mathbb{R}_+}\Big(\int_{O(1)\times O(m+1)}\Big|\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(\exp(-Y)k,\tau) - \sum_{l=0}^{N}\frac{1}{l!}\mathrm{d}\pi_\tau^{O(1,m+1)}(Y)^l\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(k,\tau)\Big|^2\,\mathrm{d}k\Big)|\chi(\tau)|^2\,\mathrm{d}m_{\sigma,\mu}(\tau)$$

up to a constant factor, where $\mathrm{d}k$ is the Haar measure on $O(1)\times O(m+1)$. Since $\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}$ is analytic on $O(1,m+1)\times i\mathbb{R}_+$, the sequence

$$\sum_{l=0}^{N}\frac{1}{l!}\mathrm{d}\pi_\tau^{O(1,m+1)}(Y)^l\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(k,\tau)$$

converges uniformly to $\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(\exp(-Y)k,\tau)$ on the compact set $(k,\tau) \in (O(1)\times O(m+1))\times \operatorname{supp}\chi$, which proves $a_N \to 0$. ☐

To verify the intertwining condition (iv) we first prove the intertwining property for each single space $L^2(\mathbb{R}^m, |x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)$ for fixed τ by embedding it into the $\mathbb{C}$-antilinear algebraic dual of the Harish-Chandra module $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$ of K-finite vectors. For $\tau \in T(\sigma,\mu)$ and $X \in \mathcal{U}(\mathfrak{h})$ let

$$f_{\tau,X}(x) := (\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x), \qquad x \in \mathbb{R}^m\setminus\{0\}.$$

Proposition 4.4. *Let $X \in \mathcal{U}(\mathfrak{h})$, $\phi \in \mathcal{H}^k(\mathbb{R}^{n-m})$ and $g \in L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$.*

(i) For every $\tau \in T(\sigma,\mu)$ the integral

$$\int_{\mathbb{R}^n} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau) f_{\tau,X}(x)\phi(y)\overline{g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y$$

converges absolutely and defines a continuous function in τ.

(ii) For every $\tau \in T(\sigma,\mu)$ and $j = 1,\ldots,m$ we have

$$\int_{\mathbb{R}^n} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau} f_{\tau,X})(x)\phi(y)\overline{g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^n} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau) f_{\tau,X}(x)\phi(y)\overline{(\mathcal{B}_j^{n,\sigma}g)(x,y)}|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y. \quad (4.2)$$

Proof. We first note that by (1.13) the function $f_{\tau,X}(x)$ is a linear combination of functions of the form

$$f(x)=\widetilde{K}_{-\frac{\tau}{2}+a}(|x|)|x|^{2a}p(x)$$

for $a\in\mathbb{N}_0$ and $p\in\mathbb{C}[x]$ with coefficients depending smoothly on τ. Therefore we may replace $f_{\tau,X}(x)$ by one of these functions $f(x)$. For the same reason we may assume that

$$g(x,y)=\widetilde{K}_{-\frac{\sigma}{2}+b}(|(x,y)|)|(x,y)|^{2b}q(x,y)$$

for some $b\in\mathbb{N}_0$ and $q\in\mathbb{C}[x,y]$.

(i) By (B.1) and (B.2) there exists a continuous function $C_1(\tau)>0$ on $T(\sigma,\mu)$ and $N_1>0$ such that

$$|\widetilde{K}_{-\frac{\tau}{2}+a}(t)t^{2a}|\leq C_1(\tau)(1+t)^{N_1}e^{-t},\qquad t>0.$$

For the hypergeometric function we have by (B.5) and (B.8) (checking the cases $\tau\in i\mathbb{R}_+$ and $\tau\in(\operatorname{Re}\sigma-\mu-4\mathbb{N}_0)\cap\mathbb{R}_+$ separately)

$$|F(t,\tau)|\leq C_2(\tau)(1+t)^{\frac{\operatorname{Re}\sigma-\operatorname{Re}\tau-\mu}{4}},\qquad t>0,$$

for some continuous function $C_2(\tau)>0$ on $T(\sigma,\mu)$. We estimate

$$\begin{aligned}|p(x)|&\leq C_3(1+|x|)^{N_2},\\ |\phi(y)|&\leq C_4|y|^k\leq C_4|(x,y)|^k,\\ |q(x,y)|&\leq C_5(1+|(x,y)|)^{N_3}.\end{aligned}$$

Further, for the K-Bessel function of parameter $-\frac{\sigma}{2}+b$ we find by (B.1) and (B.2) that

$$|\widetilde{K}_{-\frac{\sigma}{2}+b}(t)t^{2b}|\leq C_6t^{-\delta}(1+t)^{N_4}e^{-t},\qquad t>0,$$

for some arbitrarily small $\delta>0$ (covering the possible log-term for $\sigma=b=0$) and $N_4>0$. Hence we obtain

$$\begin{aligned}&\left||x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y)\overline{g(x,y)}\right|\\ &\quad\leq C_1(\tau)C_2(\tau)C_3C_4C_5C_6|x|^{\frac{\operatorname{Re}\sigma-\operatorname{Re}\tau-\mu}{2}}\left(1+\frac{|y|^2}{|x|^2}\right)^{\frac{\operatorname{Re}\sigma-\operatorname{Re}\tau-\mu}{4}}\\ &\qquad\times(1+|x|)^{N_1+N_2}e^{-|x|}|(x,y)|^{k-\delta}(1+|(x,y)|)^{N_3+N_4}e^{-|(x,y)|}\\ &\quad\leq C(\tau)|(x,y)|^{\frac{\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}-\delta}(1+|(x,y)|)^Ne^{-|(x,y)|}\end{aligned}$$

with $C(\tau)=C_1(\tau)C_2(\tau)C_3C_4C_5C_6$ and $N=N_1+N_2+N_3+N_4$. This is integrable on $\mathbb{R}^n$ with respect to the measure $|(x,y)|^{-\operatorname{Re}\sigma}$ if and only if

$$\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}-\delta>-n.$$

Since δ can be chosen arbitrarily small this is equivalent to

$$n-\operatorname{Re}\sigma+m-\operatorname{Re}\tau>0.$$

But since $\operatorname{Re}\sigma<n$ and $\operatorname{Re}\tau<m$ this inequality holds true for all $\tau\in T(\sigma,\mu)$ and therefore the integral converges absolutely. Moreover, we even have $n-\operatorname{Re}\sigma+m-\operatorname{Re}\tau>n-\operatorname{Re}\sigma>0$ for all τ and hence the convergence is uniformly in τ varying in a compact subset of $T(\sigma,\mu)$, which finishes the proof of (i).

(ii) First recall from Proposition 2.1 that

$$|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau}f)(x)\phi(y)=\mathcal{B}_j^{n,\sigma}\left[|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y)\right].$$

Therefore we have to show that

$$\int_{\mathbb{R}^n}\mathcal{B}_j^{n,\sigma}\Phi(x,y)\cdot\overline{g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y \stackrel{!}{=}\int_{\mathbb{R}^n}\Phi(x,y)\cdot\overline{\mathcal{B}_j^{n,\sigma}g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y, \tag{4.3}$$

where we abbreviate

$$\Phi(x,y)=|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y).$$

The operator $\mathcal{B}_j^{n,\sigma}$ is formally self-adjoint with respect to $|(x,y)|^{-\operatorname{Re}\sigma}$ since $\mathrm{d}\rho_\sigma^G(N_j)=-i\mathcal{B}_j^{n,\sigma}$ is, as part of the Lie algebra action, formally skew-adjoint on $C_c^\infty(\mathbb{R}^n\setminus\{0\})\subseteq L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)^\infty$. Therefore it remains to show that we can integrate by parts without leaving any boundary terms. Fix $j\in\{1,\ldots,m\}$ and consider the domain

$$\Omega_{j,\varepsilon}:=\{(x,y)\in\mathbb{R}^n:|x_j|>\varepsilon\}\subseteq\mathbb{R}^n$$

for $\varepsilon>0$. Clearly $\mathbb{R}^n\setminus\bigcup_{\varepsilon>0}\Omega_{j,\varepsilon}$ is of measure zero and hence (4.3) is equivalent to

$$\lim_{\varepsilon\to0}\int_{\Omega_{j,\varepsilon}}\mathcal{B}_j^{n,\sigma}\Phi(x,y)\cdot\overline{g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y \stackrel{!}{=}\lim_{\varepsilon\to0}\int_{\Omega_{j,\varepsilon}}\Phi(x,y)\cdot\overline{\mathcal{B}_j^{n,\sigma}g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.4}$$

On $\Omega_{j,\varepsilon}$ both $|x|$ and $|(x,y)|$ are bounded from below by ε. Hence, by (B.3) and (B.6), all factors in the integrand

$$|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y)\overline{g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma}$$

can be arbitrarily often differentiated in x and y and the result is a smooth function on $\overline{\Omega_{j,\varepsilon}}$. Since further the hypergeometric function grows at most polynomially and the K-Bessel functions decay exponentially near ∞, all such differentiated terms decay exponentially as $|(x,y)|\to\infty$ and are hence integrable on $\overline{\Omega_{j,\varepsilon}}$. Therefore we can arbitrarily integrate by parts and all intermediate integrals exist. It remains to show that for $\varepsilon\to0$ all boundary terms that occur while integrating by parts vanish. By the asymptotic behaviour of the K-Bessel functions at ∞ the boundary terms at ∞ always vanish. Hence, by the choice of $\Omega_{j,\varepsilon}$, the only boundary terms that occur are for derivatives in x_j at $x_j=\pm\varepsilon$. Therefore we only need to consider the parts

$$x_j\frac{\partial^2}{\partial x_j^2},\ \frac{\partial}{\partial x_j}\quad\text{and}\quad E\frac{\partial}{\partial x_j}$$

of $\mathcal{B}_j^{n,\sigma}$. We treat these three parts separately. Here we start with the right hand side of (4.4) and then integrate by parts once or twice.

(a) $\frac{\partial}{\partial x_j}$. The boundary terms that occur when integrating by parts are (up to multiplication with a constant) of the form

$$\int_{\mathbb{R}^{n-1}} \Big(\Phi(x',\varepsilon,y)\overline{g(x',\varepsilon,y)}|(x',\varepsilon,y)|^{-\operatorname{Re}\sigma} - \Phi(x',-\varepsilon,y)\overline{g(x',-\varepsilon,y)}|(x',-\varepsilon,y)|^{-\operatorname{Re}\sigma}\Big)\,\mathrm{d}x'\,\mathrm{d}y$$

where we write $x=(x',x_j)$ with $x'=(x_1,\ldots,\widehat{x_j},\ldots,x_m)\in\mathbb{R}^{m-1}$. The integrand obviously converges pointwise almost everywhere to 0 as $\varepsilon\to 0$ and it suffices to find an integrable function independent of ε dominating the integrand to apply the Dominated Convergence Theorem. For this note that in both $\Phi(x,y)$ and $g(x,y)$ the only terms dependent on the sign of x_j are the polynomials $p(x)$ and $q(x,y)$, respectively. Using the same estimates as in the proof of part (i) we find that

$$\begin{aligned}&\left|\Phi(x',\varepsilon,y)\overline{g(x',\varepsilon,y)}|(x',\varepsilon,y)|^{-\operatorname{Re}\sigma} - \Phi(x',-\varepsilon,y)\overline{g(x',-\varepsilon,y)}|(x',-\varepsilon,y)|^{-\operatorname{Re}\sigma}\right|\\ &\quad\le C|(x',\varepsilon,y)|^{\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}-\delta}(1+|(x',\varepsilon,y)|)^N e^{-|(x',\varepsilon,y)|}\\ &\qquad\times\left|p(x',\varepsilon)\overline{q(x',\varepsilon,y)}-p(x',-\varepsilon)\overline{q(x',-\varepsilon,y)}\right|\end{aligned}$$

for some $N>0$ and an arbitrarily small $\delta>0$. Now note that $p(x',\varepsilon)\overline{q(x',\varepsilon,y)}-p(x',-\varepsilon)\overline{q(x',-\varepsilon,y)}$ is an odd polynomial in ε and hence of the form $\varepsilon\cdot r(x',\varepsilon,y)$. For the extra ε from this observation we use the estimate $|\varepsilon|\le|(x',\varepsilon,y)|$. We further estimate $|r(x',\varepsilon,y)|\le C'(1+|(x',\varepsilon,y)|)^{N'}$ for some $C',N'>0$ and find (assuming $\varepsilon\le 1$)

$$\le CC'|(x',\varepsilon,y)|^{\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1-\delta}(1+|(x',1,y)|)^{N+N'}e^{-|(x',y)|}.$$

Now suppose the exponent $\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1$ is ≤ 0. Then we can estimate

$$\le CC'|(x',y)|^{\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1-\delta}(1+|(x',1,y)|)^{N+N'}e^{-|(x',y)|},$$

which is independent of $\varepsilon\in(0,1)$ and integrable on $\mathbb{R}^{n-1}$ for small $\delta>0$ since $\operatorname{Re}\sigma<n$ and $\operatorname{Re}\tau<m$. If the exponent $\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1-\delta$ is positive the estimate $\varepsilon\le 1$ also yields a dominant integrable function independent of ε. Therefore, in both cases we can apply the Dominated Convergence Theorem and obtain that as $\varepsilon\to 0$ the boundary terms vanish.

(b) $x_j\frac{\partial^2}{\partial x_j^2}$. Integrating by part once gives (up to multiplication by a constant) the boundary terms

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\Bigg(&\Phi(x',\varepsilon,y)\left(x_j\overline{\frac{\partial g}{\partial x_j}(x,y)}\right)_{x_j=\varepsilon}|(x',\varepsilon,y)|^{-\operatorname{Re}\sigma}\\ &-\Phi(x',-\varepsilon,y)\left(x_j\overline{\frac{\partial g}{\partial x_j}(x,y)}\right)_{x_j=-\varepsilon}|(x',-\varepsilon,y)|^{-\operatorname{Re}\sigma}\Bigg)\,\mathrm{d}x'\,\mathrm{d}y.\end{aligned}\tag{4.5}$$

We have

$$g(x,y)=\widetilde{K}_{-\frac{\sigma}{2}+b}(|(x,y)|)|(x,y)|^{2b}q(x,y)$$

and use the product rule to find $x_j \frac{\partial g}{\partial x_j}(x,y)$. The first term is by (B.3)

$$\begin{aligned} -\frac{x_j^2}{2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(|(x,y)|)|(x,y)|^{2b}q(x,y) \\ = -\frac{x_j^2}{2|(x,y)|^2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(|(x,y)|)|(x,y)|^{2(b+1)}q(x,y) \end{aligned}$$

and putting $x_j = \pm\varepsilon$ gives

$$= -\frac{\varepsilon^2}{2|(x',\varepsilon,y)|^2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(|(x',\varepsilon,y)|)|(x',\varepsilon,y)|^{2(b+1)}q(x',\pm\varepsilon,y).$$

Again ε^2 can be estimated by $|(x',\varepsilon,y)|^2$ and we find that

$$\frac{\varepsilon^2}{2|(x',\varepsilon,y)|^2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(|(x',\varepsilon,y)|)|(x',\varepsilon,y)|^{2(b+1)} \quad \text{and} \quad \widetilde{K}_{-\frac{\sigma}{2}+b}(|(x,y)|)|(x,y)|^{2b}$$

satisfy the same estimates (see part (i)). The same argument applies to the other two terms in the product rule. Therefore the same argument as in (a) yields the vanishing of the boundary terms (4.5). Similar arguments yield the vanishing of the boundary terms that occur when integrating by parts for the second time. For this note that the formal adjoint of $\frac{\partial}{\partial x_j}$ on $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$ is $-\frac{\partial}{\partial x_j} + (\operatorname{Re}\sigma)\frac{x_j}{|(x,y)|^2}$. Both summands are treated separately as above.

(c) $E\frac{\partial}{\partial x_j}$. We have

$$E\frac{\partial}{\partial x_j} = x_j\frac{\partial^2}{\partial x_j^2} + \sum_{k\neq j} x_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j} + \sum_k y_k \frac{\partial}{\partial y_k}\frac{\partial}{\partial x_j}.$$

The first term was already treated in part (b). For the other two terms note that we can first integrate by parts the derivatives with respect to x_k ($k \neq j$) and y_k without any boundary terms occurring. Secondly, integration by parts of the derivative with respect to x_j is dealt with as in part (b). This finishes the proof. □

Remark 4.5. It is necessary in the proof of Proposition 4.4 (ii) to restrict integration to the domain $\Omega_{j,\varepsilon}$. This is because the operator $\mathcal{B}_j^{n,\sigma}$ is of second order and we have to integrate by parts twice. The intermediate result, i.e. after integrating by parts once, may not be integrable on $\mathbb{R}^n$ and hence we need to restrict to a subdomain on which these intermediate results are integrable. The same problem occurs when one considers the two summands $x_j\Delta$ and $-(2E-\sigma+n)\frac{\partial}{\partial x_j}$ separately. Here the integral over $\mathbb{R}^n$ for each of the two summands may not converge while the integral for the sum $\mathcal{B}_j^{n,\sigma}$ does by Proposition 4.4 (i) converge.

Remark 4.6. Part (i) of Proposition 4.4 constructs an embedding of

$$L^2(\mathbb{R}^m, |x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)_{K\cap O(1,m+1)} \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m})$$

into the $\mathbb{C}$-antilinear algebraic dual of $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$ for every $\tau \in T(\sigma,\mu)$. By part (ii) this embedding is $\mathfrak{h}$-equivariant.

Let us now continue the proof of Theorem 4.1 by showing property (iv) in Lemma 4.2. Let $v_1 \in V_1$ and $v_2 \in V_2$. Suppose that

$$v_1(x,\tau,y) = f_{\tau,X}(x)\phi(y)\chi(\tau) \quad \text{and} \quad v_2(x,y) = g(x,y)$$

with $X \in \mathcal{U}(\mathfrak{h})$, $\chi \in C_c(T(\sigma,\mu))$, $\phi \in \mathcal{H}^k(\mathbb{R}^{n-m})$, and $g \in L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$. We have

$$\begin{aligned}
&(\varphi(\mathrm{d}\rho_1(N_j)v_1) \mid v_2)_{\mathcal{H}_2} \\
&\quad = -i\int_{\mathbb{R}^n}\int_{T(\sigma,\mu)} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau} f_{\tau,X})(x)\phi(y)\overline{g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma} \\
&\qquad\qquad \chi(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = -i\int_{T(\sigma,\mu)}\int_{\mathbb{R}^n} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau} f_{\tau,X})(x)\phi(y)\overline{g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma} \\
&\qquad\qquad \chi(\tau)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}m_{\sigma,\mu}(\tau),
\end{aligned}$$

where we were able to change the order of integration, because by Proposition 4.4 (i) the inner integral in the last line converges absolutely and is continuous in τ and the integration is only over the compact subset $\operatorname{supp}\chi \subseteq T(\sigma,\mu)$. Now, by Proposition 4.4 (ii) we find

$$\begin{aligned}
&\quad = -i\int_{T(\sigma,\mu)}\int_{\mathbb{R}^n} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau) f_{\tau,X}(x)\phi(y)\overline{\mathcal{B}_j^{n,\sigma} g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma} \\
&\qquad\qquad \chi(\tau)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}m_{\sigma,\mu}(\tau) \\
&\quad = -i\int_{\mathbb{R}^n}\int_{T(\sigma,\mu)} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau) f_{\tau,X}(x)\phi(y)\overline{\mathcal{B}_j^{n,\sigma} g(x,y)}|(x,y)|^{-\operatorname{Re}\sigma} \\
&\qquad\qquad \chi(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = (\varphi(v_1) \mid \mathrm{d}\rho_2(N_j)v_2)_{\mathcal{H}_2},
\end{aligned}$$

again using Proposition 4.4 (i) to change the order of integration. This finally shows property (iv) of Lemma 4.2 and we obtain that $\varphi = \Psi(\sigma,k)$ intertwines the group action of N_H and hence of H. Thus the proof of Theorem 4.1 is complete. □

We obtain the whole spectral decomposition of $\rho^G_{\sigma,\varepsilon}|_H$ from (4.1) and Theorem 4.1.

Theorem 4.7. *For $\sigma \in i\mathbb{R} \cup (-n,n)$ the representation $\rho^G_{\sigma,\varepsilon}$ decomposes under the restriction to $H = O(1,m+1) \times O(n-m)$ as*

$$\begin{aligned}
\rho^G_{\sigma,\varepsilon}\big|_H \cong \sum_{k=0}^{\infty}{}^{\oplus}\Big(&\int_{i\mathbb{R}_+}^{\oplus} \rho^{O(1,m+1)}_{\tau,\varepsilon+k}\,\mathrm{d}\tau \\
&\oplus \bigoplus_{j\in\mathbb{Z}\cap\left[0,\frac{|\operatorname{Re}\sigma|-n+m-2k}{4}\right)} \rho^{O(1,m+1)}_{|\operatorname{Re}\sigma|-n+m-2k-4j,\varepsilon+k}\Big) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}).
\end{aligned}$$

5 Intertwining operators in the non-compact picture

In Proposition 2.1 we explicitly found an intertwining operator $C^\infty(\mathbb{R}^m \setminus \{0\}) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) \to C^\infty(\mathbb{R}^n \setminus \{x = 0\})$. In the Fourier transformed picture this operator is given by

$$A(\sigma,\tau)(f\otimes\phi)(x,y) = |x|^{\frac{\sigma-\tau-\mu}{2}}\,{}_2F_1\left(\frac{\mu-\sigma+\tau}{4},\frac{\mu-\sigma-\tau}{4};\frac{\mu}{2};-\frac{|y|^2}{|x|^2}\right) f(x)\phi(y),$$

where again $\mu = 2k+n-m$. In Proposition 4.4 we even showed that for fixed $\sigma \in i\mathbb{R}\cup(-n,n)$, $k \in \mathbb{N}_0$ and $\tau \in T(\sigma, 2k-n+m)$ the operator $A(\sigma,\tau)$ is intertwining between the Harish-Chandra module of $\rho^{O(1,m+1)}_{\tau,\varepsilon+k} \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m})$ and the $\mathbb{C}$-antilinear algebraic dual of the Harish-Chandra module of $\rho^{O(1,n+1)}_{\sigma,\varepsilon}$. We now find a formal expression for this intertwiner in the non-compact picture.

Consider the following diagram

$$\begin{array}{ccc} C_c^\infty(\mathbb{R}^m\setminus\{0\})\otimes\mathcal{H}^k(\mathbb{R}^{n-m}) & \xrightarrow{A(\sigma,\tau)} & \mathcal{S}'(\mathbb{R}^n) \\ \downarrow{\scriptstyle \mathcal{F}_{\mathbb{R}^m}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathcal{F}_{\mathbb{R}^n}} \\ \mathcal{F}_{\mathbb{R}^m}C_c^\infty(\mathbb{R}^m\setminus\{0\})\otimes\mathcal{H}^k(\mathbb{R}^{n-m}) & \xrightarrow[I(\sigma,\tau)]{} & \mathcal{S}'(\mathbb{R}^n). \end{array}$$

We extend the operator $A(\sigma,\tau)$ for all $\sigma,\tau\in\mathbb{C}$ and determine the operator $I(\sigma,\tau)$ for $\operatorname{Re}\sigma \ll \operatorname{Re}\tau \ll 0$. We have

$$\begin{aligned}
&\mathcal{F}_{\mathbb{R}^n}A(\sigma,\tau)(f\otimes\phi)(\xi,\eta)\\
&\quad= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{n-m}} e^{-ix\cdot\xi-iy\cdot\eta}|x|^{\frac{\sigma-\tau-\mu}{2}}\\
&\qquad\qquad\times {}_2F_1\left(\frac{\mu-\sigma+\tau}{4},\frac{\mu-\sigma-\tau}{4};\frac{\mu}{2};-\frac{|y|^2}{|x|^2}\right) f(x)\phi(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

We first calculate the integral over $y\in\mathbb{R}^{n-m}$. Using Appendix B.4 and the integral formula (B.9) we find

$$\begin{aligned}
&(2\pi)^{-\frac{n-m}{2}}\int_{\mathbb{R}^{n-m}} e^{-iy\cdot\eta}{}_2F_1\left(\frac{\mu-\sigma+\tau}{4},\frac{\mu-\sigma-\tau}{4};\frac{\mu}{2};-\frac{|y|^2}{|x|^2}\right)\phi(y)\,\mathrm{d}y\\
&\quad= i^{-k}\phi(\eta)|\eta|^{-\frac{\mu-2}{2}}\int_0^\infty J_{\frac{\mu-2}{2}}(|\eta|s){}_2F_1\left(\frac{\mu-\sigma+\tau}{4},\frac{\mu-\sigma-\tau}{4};\frac{\mu}{2};-\frac{s^2}{|x|^2}\right)s^{\frac{\mu}{2}}\,\mathrm{d}s\\
&\quad= i^{-k}\phi(\eta)|\eta|^{-\frac{\mu-2}{2}}\frac{2^{\frac{\sigma+2}{2}}\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})}|x|^{\frac{\mu-\sigma}{2}}|\eta|^{-\frac{\sigma+2}{2}}K_{\frac{\tau}{2}}(|x|\cdot|\eta|).
\end{aligned}$$

If we let

$$\psi(x,\eta) := \frac{2^{\frac{\sigma+2}{2}}i^{-k}\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})}|\eta|^{-\frac{\sigma+\mu}{2}}|x|^{-\frac{\tau}{2}}K_{\frac{\tau}{2}}(|x|\cdot|\eta|)$$

then we find that

$$\begin{aligned}\mathcal{F}_{\mathbb{R}^n} A(\sigma,\tau)(f\otimes\phi)(\xi,\eta) &= \mathcal{F}_{\mathbb{R}^m}(f\cdot\psi(-,\eta))(\xi)\cdot\phi(\eta)\\ &= (\mathcal{F}_{\mathbb{R}^m}\psi(-,\eta) * \mathcal{F}_{\mathbb{R}^m} f)(\xi)\cdot\phi(\eta).\end{aligned}$$

Therefore we compute, using again Appendix B.4 and the integral formula (B.10) (noticing that $K_\nu(x) = K_{-\nu}(x)$)

$$\begin{aligned}&(\mathcal{F}_{\mathbb{R}^m}\psi(-,\eta))(\xi)\\ &= \frac{2^{\frac{\sigma+2}{2}} i^{-k}\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})}|\eta|^{-\frac{\sigma+\mu}{2}}|\xi|^{-\frac{m-2}{2}}\int_0^\infty J_{\frac{m-2}{2}}(|\xi|s)s^{-\frac{\tau}{2}}K_{\frac{\tau}{2}}(|\eta|s)s^{\frac{m}{2}}\,\mathrm{d}s\\ &= \frac{2^{\frac{\sigma-\tau+m}{2}} i^{-k}\Gamma(\frac{\mu}{2})\Gamma(\frac{m-\tau}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})}|\eta|^{-\frac{\sigma+\tau+\mu}{2}}(|\xi|^2+|\eta|^2)^{\frac{\tau-m}{2}}.\end{aligned}$$

Altogether we see that $I(\sigma,\mu)$ is a partial convolution operator combined with a multiplication operator

$$I(\sigma,\tau)(f\otimes\phi)(\xi,\eta) = \mathrm{const}\cdot|\eta|^{-\frac{\sigma+\tau+\mu}{2}}\phi(\eta)\int_{\mathbb{R}^m}(|\xi-\xi'|^2+|\eta|^2)^{\frac{\tau-m}{2}}f(\xi')\,\mathrm{d}\xi'.$$

For $m=n-1$ this operator appears in [10] as a special case. This expression for $I(\sigma,\tau)$ is valid for $\operatorname{Re}\sigma \ll \operatorname{Re}\tau \ll 0$. It has a holomorphic extension to all $\sigma,\tau\in\mathbb{C}$ for $f\in\mathcal{F}_{\mathbb{R}^m}C_c^\infty(\mathbb{R}^m\setminus\{0\})$.

A Decomposition of principal series

We give a short alternative proof for the decomposition of the principal series $\pi^G_{\sigma,\varepsilon}$, $\sigma\in i\mathbb{R}$, $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$, into irreducible H-representations. This decomposition turns out to be essentially equivalent to the Plancherel formula for $L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)$, where $\mathcal{L}'_\delta$ are the line bundles over the Riemannian symmetric space $O(1,m+1)/(O(1)\times O(m+1))$ induced by the characters $(a,g)\mapsto a^\delta$ of $O(1)\times O(m+1)$, $\delta\in\mathbb{Z}/2\mathbb{Z}$.

Consider the flag variety $X=G/P$. Since $G/P\cong K/M$ we can identify X with the unit sphere $S^n\subseteq\mathbb{R}^{n+1}$. For this we define a G-action on S^n by the formula

$$g\circ x := \frac{\mathrm{pr}_x(g(1,x))}{\mathrm{pr}_0(g(1,x))}, \qquad x\in S^n,$$

where $\mathrm{pr}_0:\mathbb{R}^{n+2}\to\mathbb{R}$ and $\mathrm{pr}_x:\mathbb{R}^{n+2}\to\mathbb{R}^{n+1}$ denote the projections onto the first coordinate and the last $n+1$ coordinates, respectively, and $g(1,x)$ is the usual action of g on $(1,x)\in\mathbb{R}\times\mathbb{R}^{n+1}\cong\mathbb{R}^{n+2}$. Then it is easy to prove the following:

Lemma A.1. *The operation $\circ$ defines a transitive group action of G on S^n. The stabilizer of the point $e_{n+1}=(0,\ldots,0,1)\in S^n$ is equal to the parabolic subgroup P. The maximal compact subgroup K also acts transitively on S^n and the stabilizer subgroup of the point x_0 is equal to M.*

Let us consider a slightly different embedding of $O(1,m+1)\times O(n-m)$ into $G=O(1,n+1)$. Let

$$H' := \{\mathrm{diag}(g,h) : g\in O(1,m+1), h\in O(n-m)\}.$$

Then clearly H and H' are conjugate and hence the branching to H is equivalent to the branching to H'. We shall therefore only deal with H' in this section.

Lemma A.2. *Under the action $\circ$ of the group H' the sphere S^n decomposes into the two orbits*

$$\begin{aligned}\mathcal{O}_0 &:= H'\circ e_1 = \{(x',0) : x'\in S^m\},\\ \mathcal{O}_1 &:= H'\circ e_{n+1} = \{(x',x'')\in S^n : x'\in\mathbb{R}^{m+1}, x''\in\mathbb{R}^{n-m}, x''\neq 0\}.\end{aligned}$$

The orbit $\mathcal{O}_1$ is open and dense in S^n. The isotropy group of e_{n+1} in H' is

$$S=\{(a,g,h,a) : a\in O(1), g\in O(m+1), h\in O(n-m-1)\}.$$

Now consider the realization of $\pi_{\sigma,\varepsilon}$ in the compact picture, i.e. on $L^2(G/P,\mathcal{L}_{\sigma,\varepsilon})$, where $\mathcal{L}_{\sigma,\varepsilon}$ denotes the line bundle over G/P associated to the character $man\mapsto \xi_\varepsilon(m)a^{\sigma+\rho}$ of P. Since the orbit $\mathcal{O}_1\subseteq G/P$ is open and dense we have

$$L^2(G/P,\mathcal{L}_{\sigma,\varepsilon})\cong L^2(\mathcal{O}_1,\mathcal{L}_{\sigma,\varepsilon}|_{\mathcal{O}_1}).$$

Now the stabilizer S of $eP\in G/P$ in H is contained in P and hence the restriction of the line bundle $\mathcal{L}_{\sigma,\varepsilon}$ to $\mathcal{O}_1\cong H'/S$ is induced by the restriction of the corresponding character of P to S which is simply $\xi_\varepsilon|_S$. Therefore we find

$$L^2(G/P,\mathcal{L}_{\sigma,\varepsilon})\cong L^2(\mathcal{O}_1,\mathcal{L}_\varepsilon),$$

where $\mathcal{L}_\varepsilon$ is the line bundle over $\mathcal{O}_1\cong H'/S$ induced by the character $\xi_\varepsilon|_S$. Using the decomposition of $L^2(S^{n-m-1})$ into spherical harmonics we find

$$L^2(\mathcal{O}_1,\mathcal{L}_\varepsilon)\cong \sum_{k=0}^{\infty}{}^{\oplus} L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_{\varepsilon+k})\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m})$$

as H'-representations, where for $\delta\in(\mathbb{Z}/2\mathbb{Z})$ we denote by $\mathcal{L}'_\delta$ the line bundle over the symmetric space $O(1,m+1)/(O(1)\times O(m+1))$ induced by the character $(a,g)\mapsto a^\delta$ of $O(1)\times O(m+1)$. Together we obtain

$$\pi^G_{\sigma,\varepsilon}\big|_H\cong \sum_{k=0}^{\infty}{}^{\oplus} L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_{\varepsilon+k})\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m})$$

and hence the decomposition of $\pi^G_{\sigma,\varepsilon}|_H$ into irreducible H-representations is equivalent to the decomposition of $L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)$ into irreducible $O(1,m+1)$-representations, $\delta\in\mathbb{Z}/2\mathbb{Z}$. Since $O(1,m+1)/(O(1)\times O(m+1))$ is a Riemannian symmetric space of rank one the decomposition of $L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)$ is well-known and given by

$$L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)\cong\int_{i\mathbb{R}_+}^{\oplus}\pi^{O(1,m+1)}_{\tau,\delta}\,\mathrm{d}\tau,$$

the unitary isomorphism established by the spherical Fourier transform. This proves Theorem 4.7 for the special case $\sigma\in i\mathbb{R}$.

B Special functions

For the sake of completeness we collect here the necessary formulas for certain special functions needed in this paper.

B.1 The *K*-Bessel function

We renormalize the classical K-Bessel function $K_\alpha(z)$ by

$$\widetilde{K}_\alpha(z) := \left(\frac{z}{2}\right)^{-\alpha} K_\alpha(z).$$

Then $\widetilde{K}_\alpha(z)$ solves the differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \frac{2\alpha+1}{z}\frac{\mathrm{d}u}{\mathrm{d}z} - u = 0.$$

It has the following asymptotic behaviour as $x \to 0$ (see [18, Chapters III & VII]):

$$\widetilde{K}_\alpha(x) = \begin{cases} \frac{\Gamma(\alpha)}{2}\left(\frac{x}{2}\right)^{-2\alpha} + o(x^{-2\alpha}), & \text{for } \operatorname{Re}\alpha > 0, \\ -\log\left(\frac{x}{2}\right) + o\left(\log\left(\frac{x}{2}\right)\right), & \text{for } \operatorname{Re}\alpha = 0, \\ \frac{\Gamma(-\alpha)}{2} + o(1), & \text{for } \operatorname{Re}\alpha < 0. \end{cases} \tag{B.1}$$

Further, as $x \to \infty$ we have

$$\widetilde{K}_\alpha(x) = \frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{-\alpha-\frac{1}{2}} e^{-x}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right). \tag{B.2}$$

For the derivative of $\widetilde{K}_\alpha(z)$ the following identity holds (see [18, equation III.71 (6)]):

$$\frac{\mathrm{d}}{\mathrm{d}z}\widetilde{K}_\alpha(z) = -\frac{z}{2}\widetilde{K}_{\alpha+1}(z). \tag{B.3}$$

B.2 The Gauß hypergeometric function

Consider the classical Gauß hypergeometric function

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. The function ${}_2F_1(a,b;c;z)$ is holomorphic in z for $z \notin [1,\infty)$ and meromorphic in the parameters $a,b,c \in \mathbb{C}$. It solves the differential equation

$$(1-z)z\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + (c-(a+b+1)z)\frac{\mathrm{d}u}{\mathrm{d}z} - abu = 0. \tag{B.4}$$

The following formula allows to study the asymptotic behaviour of the Gauß hypergeometric function near $x = \infty$ (see [5, equation 9.132 (2)]):

$$\begin{aligned} {}_2F_1(a,b;c;z) = &\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,a-c+1;a-b+1;\tfrac{1}{z}) \\ &+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,b-c+1;b-a+1;\tfrac{1}{z}). \end{aligned} \tag{B.5}$$

Both summands on the right hand side of (B.5) are generically linear independent solutions of (B.4). Their Wronskian is given by

$$\begin{aligned} W(z^{-a}{}_2F_1(a,a-c+1;a-b+1;-\tfrac{1}{z}),z^{-b}{}_2F_1(b,b-c+1;b-a+1;-\tfrac{1}{z})) \\ = (a-b)(1+z)^{c-a-b-1}z^{-c}. \end{aligned}$$

The following simple formula for the derivative of the hypergeometric function holds:

$$\frac{\mathrm{d}}{\mathrm{d}z}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z). \tag{B.6}$$

We recall Kummer's transformation formula (see [5, equation 9.131 (1)]):

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z). \tag{B.7}$$

For $a \in -\mathbb{N}_0$ the hypergeometric function ${}_2F_1(a,b;c;z)$ degenerates to a polynomial which can be expressed in terms of the Jacobi polynomials $P_n^{(a,b)}(z)$ (see [5, equation 8.962 (1)]):

$${}_2F_1(-n,b;c;z) = \frac{n!}{(c)_n}P_n^{(c-1,b-c-n)}(1-2z), \qquad n \in \mathbb{N}_0, \tag{B.8}$$

where

$$P_n^{(a,b)}(z) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+n+1)_k(a+k+1)_{n-k}}{k!}\left(\frac{1-z}{2}\right)^k.$$

B.3 Integral formulas

We consider the J-Bessel function $J_\nu(z)$ and the K-Bessel function $K_\nu(z)$. For the J-Bessel function and the hypergeometric function the following integral formula holds for $y > 0$, $\operatorname{Re}\lambda > 0$ and $-1 < \operatorname{Re}\nu < 2\max(\operatorname{Re}\alpha, \operatorname{Re}\beta) - \frac{3}{2}$ (see [5, equation 7.542 (10)])

$$\begin{aligned} &\int_0^\infty {}_2F_1(\alpha,\beta;\nu+1;-\lambda^2x^2)J_\nu(xy)x^{\nu+1}\,\mathrm{d}x \\ &\qquad = \frac{2^{\nu-\alpha-\beta+2}\Gamma(\nu+1)}{\lambda^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}y^{\alpha+\beta-\nu-2}K_{\alpha-\beta}\left(\frac{y}{\lambda}\right). \end{aligned} \tag{B.9}$$

For the J-Bessel function and the K-Bessel function we have the following integral formula for $\operatorname{Re}\mu > |\operatorname{Re}\nu| - 1$ and $\operatorname{Re} b > |\operatorname{Im} a|$ (see [5, equation 6.576 (7)])

$$\int_0^\infty x^{\mu+\nu+1}J_\mu(ax)K_\nu(bx)\,\mathrm{d}x = 2^{\mu+\nu}a^\mu b^\nu\frac{\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}. \tag{B.10}$$

B.4 Fourier and Hankel transform

Let $\mathcal{F}_{\mathbb{R}^n}$ denote the Euclidean Fourier transform on $\mathbb{R}^n$ as defined in (1.3). Let $k \in \mathbb{N}_0$ and $\phi \in \mathcal{H}^k(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}_+, r^{n+2k-1}\,\mathrm{d}r)$ denote by $f \otimes \phi \in L^2(\mathbb{R}^n)$ the function

$$(f \otimes \phi)(x) := f(|x|)\phi(x), \qquad x \in \mathbb{R}^n.$$

Then by [15, Chapter IV, Theorem 3.10]

$$\mathcal{F}_{\mathbb{R}^n}(f \otimes \phi) = i^{-k}(\mathcal{H}_{\frac{n+2k-2}{2}} f) \otimes \phi,$$

where $\mathcal{H}_\nu$ is the modified Hankel transform of parameter $\nu \geq -\frac{1}{2}$

$$\mathcal{H}_\nu f(r) = r^{-\nu} \int_0^\infty J_\nu(rs) f(s) s^{\nu+1}\,\mathrm{d}s,$$

which is a unitary isomorphism (up to a scalar multiple) on $L^2(\mathbb{R}_+, r^{2\nu+1}\,\mathrm{d}r)$.

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