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Let $(X_t)_{t \geq 0}$ be a non-singular diffusion on $\mathbb{R}$ which vanishes at zero and is not necessarily recurrent. Let ν be a probability measure on $\mathbb{R}$ having strictly positive density. Necessary and sufficient conditions on ν are given such that there exists a stopping time τ_* of (X_t) solving the Skorokhod embedding problem, i.e. X_{τ_*} has the law ν. Furthermore an explicit construction of τ_* is carried out which is an extension of the Azéma-Yor solution when the process is a recurrent diffusion. In addition, τ_* is characterized uniquely to be the pointwise smallest possible embedding that stochastically maximizes the maximum process of (X_t) up to the time of stopping or stochastically minimizes the minimum process of (X_t) up to the time of stopping, depending on the sign of the mean value of ν.

1. Introduction

Consider a probability measure ν on $\mathbb{R}$ and a non-singular time-homogeneous diffusion $(X_t)_{t \geq 0}$ vanishing at zero. In this paper we consider the problem of embedding the given law ν in the process (X_t) by constructing of a stopping time τ_* of (X_t), i.e. by finding a stopping time τ_* of (X_t) satisfying $X_{\tau_*} \sim \nu$ and determining conditions on ν which make this possible. The problem is known as *Skorokhod embedding problem.*

The proof (see below) leads naturally to explicit construction of an extremal embedding of ν in the following sense. The embedding is an extension of the Azéma-Yor construction [1] that is pointwise the smallest possible embedding that stochastically maximizes $\max_{0 \leq t \leq \tau_*} X_t$ or stochastically minimizes $\min_{0 \leq t \leq \tau_*} X_t$ over all embeddings τ_*, depending on the sign of the mean value of ν.

The Skorokhod embedding problem has been investigated by many authors and was initiated in Skorokhod [16] when (X_t) is Brownian motion. In this case Azéma and Yor [1] (see Rogers [13] for an excursion argument) and Perkins [9] yield two different explicit extremal solutions of the Skorokhod embedding problem in the natural filtration. An extension of the Azéma-Yor embedding when the Brownian motion has an initial law was given in Hobson [6]. The existence of an embedding in a general Markov process was characterized by Rost [15], but no explicit construction of the stopping time was given. Bertoin and Le Jan [3] constructed a new class of embeddings when the process (X_t) is a Hunt process starting at a regular recurrent point. Furthermore Azéma and Yor [1] give an explicit solution when the process (X_t) is a recurrent diffusion. The case where the process (X_t) is Brownian motion with drift (non-recurrent

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diffusion) was studied in Grandits [5] and Peskir [11] and in the latter paper a necessary and sufficient condition on ν is given such that an explicit embedding (an extension of the Azéma-Yor embedding) is possible. More general embedding problems for martingales are considered in Rogers [14] and Brown et al [4].

Applications of Skorokhod embedding problems have gained some interest to option pricing theory. How to design an option given the law of the risk is studied in [10], and bounds on the prices of Lookback options obtained by robust hedging are studied in [7].

This paper was motivated by the works of Grandits [5] and Peskir [11] where they show that an extension of the Azéma-Yor construction is an embedding for the non-recurrent diffusion of Brownian motion with drift. In this paper we extend this solution to general non-recurrent non-singular diffusions. The approach of finding a solution to the Skorokhod problem is the following. First, the initial problem is transformed by composing (X_t) with its scale function into an analogous embedding problem for a continuous local martingale. Secondly, by the time-change given in the construction of the Dambis, Dubins-Schwarz Brownian motion (see [12]) the martingale embedding is shown to be equivalent to embedding in Brownian motion. When (X_t) is Brownian motion we have the embedding given in [1]. This method is well-known (see [1]) and we believe that the results of this paper are known to the specialists in the field, although we could not find it in the literature on Skorokhod embedding problems. The embedding problem for a continuous local martingale has some novelty since the martingale is convergent when the initial diffusion is non-recurrent. Also some properties of the constructed embedding mentioned above are given so to characterize the embedding uniquely. The main emphasis of the paper is on the explicit construction of the embeddings and simplicity of proofs.

2. Formulation of the problem

Let $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ be two Borel functions such that $1/\sigma^2(\cdot)$ and $|\mu(\cdot)|/\sigma^2(\cdot)$ are locally integrable at every point in $\mathbb{R}$. Let $(X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ be the unique weak solution up to an explosion time e of the one-dimensional time-homogeneous stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \quad , \quad X_0 = 0 \tag{2.1}$$

where (B_t) is a standard Brownian motion and $e = \inf\{\, t > 0 : X_t \notin \mathbb{R}\,\}$. In Section 5 the definition, existence and uniqueness of solutions to the stochastic differential equation (2.1) are recalled together with some basic facts on the solutions. For simplicity, the state space of (X_t) is taken to be $I = \mathbb{R}$, but it will be clear that the considerations are generally valid for any state space which is an interval $I = (l, r)$ (see also Section 5).

The scale function of (X_t) is given by

$$S(x) = \int_0^x \exp\left(-2\int_0^u \frac{\mu(r)}{\sigma^2(r)}\,dr \right) du$$

for $x \in \mathbb{R}$. The scale function $S(\cdot)$ has a strictly positive continuous derivative and the second derivative exists almost everywhere. Thus $S(\cdot)$ is strictly increasing with $S(0) = 0$. Define the open interval $J = \big(S(-\infty), S(\infty)\big)$. If $J = \mathbb{R}$ then (X_t) is recurrent and if J is bounded from below or above then (X_t) is non-recurrent (see Proposition 5.3).

Let ν be in the class of probability measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |S(u)| \, \nu(du) < \infty$$

and having a strictly positive density F' where F is the distribution function associated with ν . The assumption that ν has a strictly positive density is made for simplicity. The main problem under consideration in this paper is the following. Given the probability measure ν find a stopping time τ_* of (X_t) satisfying

$$X_{\tau_*} \sim \nu \tag{2.2}$$

and determine necessary and sufficient conditions on ν which make such a construction possible.

1. The first step in finding a solution to the problem (2.2) is to introduce the continuous local martingale $(M_t)_{t\geq 0}$ which shall be used in transforming the original problem into an analogous Skorokhod problem. Let (M_t) be the continuous local martingale given by composing (X_t) with the scale function $S(\cdot)$, i.e.

$$M_t = S(X_t) \; . \tag{2.3}$$

Since $x \mapsto S(x)$ is strictly increasing then Proposition 5.3 ensures that $S(-\infty) < M_t < S(\infty)$ for $t < e$ and if J is bounded from below or above M_t converges to the boundary of J for $t \to e$ and $M_t = M_e$ on $\{e < \infty\}$ for $t \geq e$. By Itô-Tanaka formula it follows that (M_t) is a solution to the stochastic differential equation

$$dM_t = \tilde{\sigma}(M_t) \, dB_t$$

where

$$\tilde{\sigma}(x) = \begin{cases} S'\big(S^{-1}(x)\big) \, \sigma\big(S^{-1}(x)\big) & \text{for} \quad x \in J \\ 0 & \text{else} \; . \end{cases}$$

The quadratic variation process is therefore given by

$$\langle M, M \rangle_t = \int_0^t \tilde{\sigma}^2(M_u) \, du = \int_0^{t \wedge e} \Big(S'(X_u) \, \sigma(X_u) \Big)^2 \, du$$

and it is immediately seen that $t \mapsto \langle M, M \rangle_t$ is strictly increasing for $t < e$. If J is bounded from below or above then $\langle M, M \rangle_e < \infty$, and if $J = \mathbb{R}$ the local martingale (M_t) is recurrent, or equivalently $\langle M, M \rangle_e = \infty$ and $e = \infty$. The process (M_t) does not explode, but the explosion time e for (X_t) can be expressed as $e = \inf\{\, t > 0 \, : \, M_t \notin J \,\}$.

Let G be the distribution function given by

$$G(x) = F\big(S^{-1}(x)\big) \tag{2.4}$$

for $x \in J$ with $G\big(S(-\infty)\big) = 0$ and $G\big(S(\infty)\big) = 1$. Then $x \mapsto G(x)$ is continuous, differentiable and strictly increasing on J . (If Y is a random variable with distribution function F then $S(Y)$ has distribution function G). For a stopping time τ_* of (X_t) it is not difficult to see that $X_{\tau_*} \sim F$ if and only if $M_{\tau_*} \sim G$. Therefore the initial problem (2.2) is analogous to the problem of finding a stopping time τ_* of (M_t) satisfying

$$M_{\tau_*} \sim G \; . \tag{2.5}$$

Moreover if τ_* is an embedding for (M_t) by the above observations it follows that $S(-\infty) < M_{\tau_*} < S(\infty)$ and hence $\tau_* < e$.

2. The second step is to apply time-change and verify that the embedding problem of continuous local martingale (2.5) is equivalent to the embedding problem of Brownian motion. Let (T_t) be the time-change given by

$$T_t = \inf \{ s > 0 \, : \, \langle M, M \rangle_s > t \} = \langle M, M \rangle_t^{-1} \tag{2.6}$$

for $t < \langle M, M \rangle_e$. Define the process $(W_t)_{t \geq 0}$ by

$$W_t = \begin{cases} M_{T_t} & \text{if} \quad t < \langle M, M \rangle_e \\ M_e & \text{if} \quad t \geq \langle M, M \rangle_e \, . \end{cases} \tag{2.7}$$

Since $t \mapsto T_t$ is strictly increasing for $t < \langle M, M \rangle_e$ we have that $\left(\mathcal{F}^M_{T_t} \right) = \left(\mathcal{F}^W_t \right)$. This implies that, if $\tau < \langle M, M \rangle_e$ is a stopping time for (W_t) then T_τ is a stopping time for (M_t) , and vice versa if $\tau < e$ is a stopping time for (M_t) then $\langle M, M \rangle_\tau$ is a stopping time for (W_t) . The process (W_t) is a Brownian motion stopped at $\langle M, M \rangle_e$ according to Dambis, Dubins-Schwarz theorem (see [12]). By the definition of (W_t) it is clear that $\langle M, M \rangle_e = \inf \{ t > 0 \, : \, W_t \notin J \}$ and hence the two processes $(W_t)_{t \geq 0}$ and $(B_{\tau_{S(\infty),S(\infty)} \wedge t})_{t \geq 0}$ have the same law where $\tau_{S(-\infty),S(\infty)} = \inf \{ t > 0 \, : \, B_t \notin J \}$.

From the above observation we deduce that the embedding problem for the continuous local martingale is equivalent to embedding in the stopped Brownian motion, i.e the martingale case (2.5) is equivalent to find a stopping time τ_* of (W_t) satisfying

$$W_{\tau_*} \sim G \, .$$

The method just described will be applied below in Section 4 to find a solution to the initial problem (2.2).

3. Skorokhod embedding in Brownian motion

The above observations indicate that a construction of an embedding in the initial problem (2.2) can be obtained from an embedding in Brownian motion. Therefore an outline of the Azéma-Yor [1] construction of embedding in Brownian motion will be recalled in this section together with some facts of the embedding. These results and facts will be applied in the next section where the construction of the embedding in the initial problem will be carried out.

Let G be the distribution function given in (2.4) i.e. $x \mapsto G(x)$ is continuous, differentiable and strictly increasing on the open interval $J = (\alpha, \beta)$ with $G(\alpha) = 0$ and $G(\beta) = 1$ where $\alpha = S(-\infty)$ and $\beta = S(\infty)$. Furthermore G has finite mean and denote it by

$$m = \int_{\mathbb{R}} u \, dG(u) \, . \tag{3.1}$$

Thus we want to find a stopping time τ_* of (B_t) satisfying

$$B_{\tau_*} \sim G \, . \tag{3.2}$$

Define the two functions

$$c(x) = \int_{\mathbb{R}} \left(u - x \right)^+ dG(u) \quad \text{and} \quad p(x) = \int_{\mathbb{R}} \left(x - u \right)^+ dG(u) \tag{3.3}$$

for $x \in \mathbb{R}$.

It is now possible to present the construction of the Azéma-Yor embedding which is a solution to problem (3.2). If $m \geq 0$, define the increasing function $s \mapsto b_+(s)$ as follows. For $m < s < \beta$ set $b_+(s)$ as the value $z < s$ which minimizes

$$\frac{c(z)}{s-z}$$

and set $b_+(s) = -\infty$ for $s \leq m$ and $b_+(s) = s$ for $s \geq \beta$ (see [4] that $b_+(\cdot)$ is well-defined). Note that $\lim_{s \downarrow m} b_+(s) = \alpha$. The left inverse of $b_+(\cdot)$ is given by

$$b_+^{-1}(x) = \frac{1}{1 - G(x)} \int_x^\infty u \, dG(u) \tag{3.4}$$

for $x < \beta$ and $b_+^{-1}(x) = x$ for $x \geq \beta$. The function $x \mapsto b_+^{-1}(x)$ is the barycentre function of G . Define the stopping time τ_{b_+} by (see Figure 1)

$$\tau_{b_+} = \inf \left\{ t > 0 \, : \, B_t \leq b_+\big(\max{}_{0 \leq r \leq t} B_r\big) \right\} . \tag{3.5}$$

Observe that the stopping time τ_{b_+} can be described by $\tau_{b_+} = \tau_m + \tau_{b_+} \circ \theta_{\tau_m}$ where $\tau_m = \inf\{ t > 0 \, : \, B_t = m \}$ since $b_+(\cdot)$ for $s \leq m$ is defined to be $-\infty$. Similarly, if $m \leq 0$, define the increasing function $s \mapsto b_-(s)$ as follows. For $\alpha < s < m$ set $b_-(s)$ as the value of $z > s$ which minimizes

$$\frac{p(z)}{z-s}$$

and set $b_-(s) = \infty$ for $s \geq m$ and $b_-(s) = s$ for $s \leq \alpha$. Note that $\lim_{s \uparrow m} b_-(s) = \beta$. The left inverse of $b_-(\cdot)$ is given by

$$b_-^{-1}(x) = \frac{1}{G(x)} \int_{-\infty}^x u \, dG(u) \tag{3.6}$$

for $x > \alpha$ and $b_-^{-1}(x) = x$ for $x \leq \alpha$. Define the stopping time τ_{b_-} by

$$\tau_{b_-} = \inf \left\{ t > 0 \, : \, B_t \geq b_-\big(\min{}_{0 \leq r \leq t} B_r\big) \right\} . \tag{3.7}$$

The stopping time τ_{b_-} can be described by $\tau_{b_-} = \tau_m + \tau_{b_-} \circ \theta_{\tau_m}$.

One more observation is needed before stating the result. If τ_* is an embedding of the centered distribution function $x \mapsto G(m+x)$ then the strong Markov property ensures that the stopping time $\tau_m + \tau_* \circ \theta_{\tau_m}$ is an embedding of G where $\tau_m = \inf\{ t > 0 \, : \, B_t = m \}$. The proposition below follows from [1], the above observation and the fact that $(-B_t)$ is Brownian motion.

Proposition 3.1. *Let the distribution function G be given as above. For $m \geq 0$ set $\tau_* = \tau_{b_+}$ given in (3.5) and for $m \leq 0$ set $\tau_* = \tau_{b_-}$ given in (3.7). Then we have that*

$$B_{\tau_*} \sim G \, .$$

The embedding given in Proposition 3.1 has the extremal properties given in the proposition below. Loosely speaking, the proposition says that for $m \geq 0$ the embedding τ_* is pointwise the smallest embedding that stochastically maximizes the maximum process $\max_{0 \leq t \leq \tau_*} B_t$. These properties were observed in [10] for $m = 0$ and in [11] (with drift tending to zero) for non-centered distribution functions. This characterizes τ_* uniquely (called the minimax property in [10] and [11]). For $m \leq 0$ a vice versa result holds for the embedding.

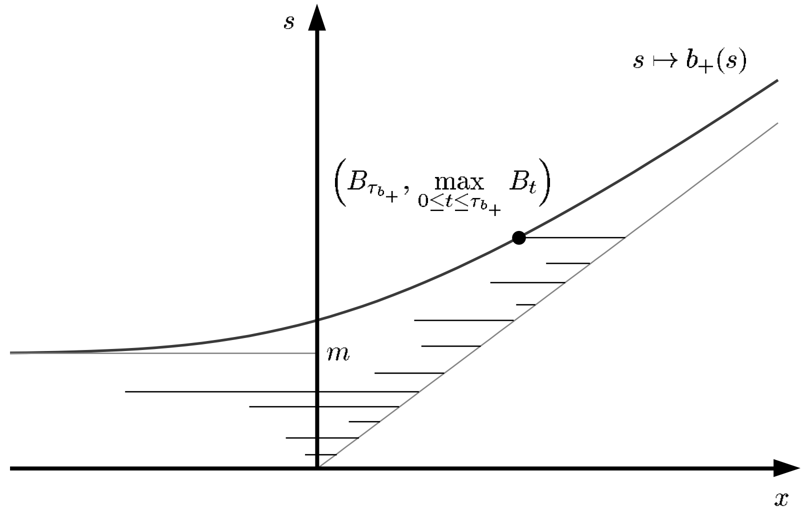

FIGURE 1. A computer drawing of the map $s \mapsto b_+(s)$ where the inverse is given in (3.4) when G is the distribution function of a $N(1,1)$-variable. The above process is $\left(B_t, \max_{0\le r\le t} B_r\right)$. The process can increase in the second component only after hitting the diagonal $x = s$. The stopping time τ_{b_+} given in (3.5) is obtained by stopping the process as soon it hits the boundary $s \mapsto b_+(s)$.

Proposition 3.2. *Under the assumptions of Proposition 3.1, let* τ *be any stopping time of* (B_t) *satisfying*

$$B_\tau \sim G$$

(I). *If* $m \ge 0$ *and* $\mathbf{E}\big(\max_{0\le t\le\tau} B_t\big) < \infty$ *then the following inequality holds*

$$\mathbf{P}\Big(\max_{0\le t\le\tau} B_t \ge s\Big) \le \mathbf{P}\Big(\max_{0\le t\le\tau_*} B_t \ge s\Big) \tag{3.8}$$

for all $s > 0$. *If furthermore* G *satisfies*

$$\int_0^\infty u\log(u)\,dG(u) < \infty \tag{3.9}$$

and the stopping time τ *satisfies* $\max_{0\le t\le\tau} B_t \sim \max_{0\le t\le\tau_*} B_t$ (*i.e. there is equality in* (3.8) *for all* $s > 0$) *then we have*

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

(II). *If* $m \le 0$ *and* $\mathbf{E}\big(\min_{0\le t\le\tau} B_t\big) > -\infty$ *then the following inequality holds*

$$\mathbf{P}\Big(\min_{0\le t\le\tau} B_t \le s\Big) \le \mathbf{P}\Big(\min_{0\le t\le\tau_*} B_t \le s\Big) \tag{3.10}$$

for all $s < 0$. *If furthermore* G *satisfies*

$$\int_{-\infty}^0 u\log(-u)\,dG(u) > -\infty \tag{3.11}$$

and the stopping time τ *satisfies* $\min_{0\le t\le\tau} B_t \sim \min_{0\le t\le\tau_*} B_t$ (*i.e. there is equality in* (3.10) *for all* $s < 0$) *then we have*

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

Remark 3.3. The conditions (3.9) and (3.11) are respectively equivalent to

$$\mathbf{E}\Big(\max_{0\le t\le\tau_*} B_t\Big) < \infty \quad \text{and} \quad \mathbf{E}\Big(\min_{0\le t\le\tau_*} B_t\Big) > -\infty\ .$$

Remark 3.4. For $m \geq 0$ we have that

$$\mathbf{P}\big(\max_{0\leq t\leq \tau_*} B_t \geq s\big) = \inf_{z<s} \frac{c(z)}{s-z} = \exp\Big(-\int_0^s \frac{dr}{r-b_+(r)}\Big)$$

for $s > 0$ and for $m \leq 0$ we have that

$$\mathbf{P}\big(\min_{0\leq t\leq \tau_*} B_t \leq s\big) = \inf_{z>s} \frac{p(z)}{z-s} = \exp\Big(-\int_s^0 \frac{dr}{b_-(r)-r}\Big)$$

for $s < 0$.

Remark 3.5. For $m = 0$, Perkins construction [9] of an embedding σ_* is another extremal embedding which stochastically minimizes $\max_{0\leq t\leq \sigma_*} B_t$ over all embeddings . The construction of the embedding is the following. Define the decreasing function $s \mapsto a_+(s)$ as follows. For $0 < s < \beta$ set $a_+(s)$ as the value $z < s$ which maximizes

$$\frac{c(s)-p(z)}{s-z}$$

and set $a_+(s) = -s$ for $s \geq \beta$. Still for $0 < s < \beta$ the function $a_+(s)$ is the unique root to the equation

$$\frac{c(s)-p(z)}{s-z} = G(z)$$

satisfying $a_+(s) < s$. Define the decreasing function $s \mapsto a_-(s)$ as follows. For $\alpha < s < 0$ set $a_-(s)$ as the value of $z > s$ which maximizes

$$\frac{p(s)-c(z)}{z-s}$$

and set $a_-(s) = -s$ for $s \leq \alpha$. For $\alpha < s < 0$ the function $a_-(s)$ is the unique root to the equation

$$\frac{p(s)-c(z)}{z-s} = 1 - G(z)$$

satisfying $a_-(s) > s$. Define the two stopping times

$$\sigma_{a_+} = \inf\big\{\, t > 0 \,:\, B_t \leq a_+\big(\max_{0\leq r\leq t} B_r\big)\,\big\}$$
$$\sigma_{a_-} = \inf\big\{\, t > 0 \,:\, B_t \geq a_-\big(\min_{0\leq r\leq t} B_r\big)\,\big\}\ .$$

For the stopping time σ_* for (B_t) given by

$$\sigma_* = \sigma_{a_+} \wedge \sigma_{a_-}$$

we have that

$$B_{\sigma_*} \sim G\ .$$

The embedding σ_* can be characterized uniquely in the following way. Let τ be given as in Proposition 3.2 then

$$\mathbf{P}\big(\max_{0\leq t\leq \tau} B_t \geq s\big) \geq \mathbf{P}\big(\max_{0\leq t\leq \sigma_*} B_t \geq s\big) \tag{3.12}$$

for $s > 0$. If there is equality in (3.12) for all $s > 0$ then

$$\tau = \sigma_*\ .$$

Finally we have that

$$\mathbf{P}\big(\max_{0\le t\le \sigma_*} B_t \ge s\big) = 1 - G(s) + \sup_{z<s} \frac{c(s)-p(z)}{s-z}$$
$$= \exp\left(-\int_0^s \frac{dr}{r-a_+(r)}\right) - \int_0^s \exp\left(-\int_u^s \frac{dr}{r-a_+(r)}\right) dG(u)$$

for $s>0$. Thus for any embedding τ given in Proposition 3.2 there is a lower and upper bound of the distribution function of the maximum process $\max_{0\le t\le \tau} B_t$ and the bounds can be attained. There are similar results for the minimum process.

4. Skorokhod embedding in non-singular diffusions

In this section we shall translate the results of the two previous sections into the case of a non-singular diffusion. The main result of this paper is contained in the theorem below.

Let ν be the probability measure on $\mathbb{R}$ with strictly positive density function F' where F is the distribution function associated with ν introduced in Section 2. We shall use the same notation as in Section 2 and 3. Let G be the distribution function given in (2.4). Then m from (3.1) can be rewritten as

$$m = \int_{\mathbb{R}} S(u)\, dF(u)$$

and the two functions $c(\cdot)$ and $p(\cdot)$ from(3.3) can be rewritten as

$$c(x) = \int_{\mathbb{R}} \big(S(u)-x\big)^+ \, dF(u) \quad \text{and} \quad p(x) = \int_{\mathbb{R}} \big(x-S(u)\big)^+ \, dF(u)$$

for $x \in \mathbb{R}$.

With the above notation the construction of the embedding which is a solution to the initial problem (2.2) is the following. If $m\ge 0$, define the increasing function $s\mapsto h_+(s)$ as follows. For $s > S^{-1}(m)$ set $h_+(s)$ as the value of $z<s$ which minimizes

$$\frac{c(S(z))}{S(s)-S(z)}$$

and set $h_+(s) = -\infty$ for $s \le S^{-1}(m)$. Note that $\lim_{s\downarrow S^{-1}(m)} h_+(s) = -\infty$. The left inverse of $h_+(\cdot)$ is given by

$$h_+^{-1}(x) = S^{-1}\left(\frac{1}{1-F(x)} \int_x^\infty S(u)\, dF(u)\right)$$

for all $x\in\mathbb{R}$. Then it is not difficult to see the following connection between $h_+^{-1}(\cdot)$ and $b_+^{-1}(\cdot)$ from (3.4) is valid

$$h_+^{-1}(\cdot) = (S^{-1}\circ b_+^{-1}\circ S)(\cdot)\ . \tag{4.1}$$

Define the stopping time τ_{h_+} by

$$\tau_{h_+} = \inf\big\{\, t>0 \,:\, X_t \le h_+\big(\max_{0\le r\le t} X_r\big)\,\big\}\ . \tag{4.2}$$

Thus by (4.1) and the definition of (M_t) (see (2.3)) it is clear that

$$\tau_{h_+} = \inf\big\{\, t>0 \,:\, M_t \le b_+\big(\max_{0\le r\le t} M_r\big)\,\big\}\ . \tag{4.3}$$

If $m \le 0$, define the increasing function $s \mapsto h_-(s)$ as follows. For $s < S^{-1}(m)$ set $h_-(s)$ as the value of $z > s$ which minimizes

$$\frac{p(S(z))}{S(z) - S(s)}$$

and set $h_-(s) = \infty$ for $s \ge S^{-1}(m)$. Note that $\lim_{s \uparrow S^{-1}(m)} h_-(s) = \infty$. The left inverse of $h_-(\cdot)$ is given by

$$h_-^{-1}(x) = S^{-1}\bigg(\frac{1}{F(x)} \int_{-\infty}^{x} S(u)\, dF(u) \bigg)$$

for all $x \in \mathbb{R}$. Note that the connection between $h_-^{-1}(\cdot)$ and $b_-^{-1}(\cdot)$ from (3.6) is the same as in (4.1). Define the stopping time τ_{h_-} by

$$\tau_{h_-} = \inf \big\{ \, t > 0 \, : \, X_t \ge h_-\big(\min_{0 \le r \le t} X_r \big) \, \big\} \, . \tag{4.4}$$

Again it is clear that

$$\tau_{h_-} = \inf \big\{ \, t > 0 \, : \, M_t \ge b_-\big(\min_{0 \le r \le t} M_r \big) \, \big\} \, .$$

The following theorem is an extension of Proposition 3.1 and states that the above stopping times are solutions to the Skorokhod embedding problem (2.2).

Theorem 4.1. *Let* (X_t) *be a non-singular diffusion vanishes at zero. Let* ν *be a probability measure on* $\mathbb{R}$ *having a strictly positive density* F' *such that*

$$\int_{\mathbb{R}} |S(u)|\, \nu(du) < \infty$$

and set

$$m = \int_{\mathbb{R}} S(u)\, \nu(du) \, .$$

Then there exists a stopping time τ_* *for* (X_t) *such that*

$$X_{\tau_*} \sim \nu$$

if and only if one of the following four cases holds

(i) $S(-\infty) = -\infty$ *and* $S(\infty) = \infty$
(ii) $S(-\infty) = -\infty$, $S(\infty) < \infty$ *and* $m \ge 0$
(iii) $S(-\infty) > -\infty$, $S(\infty) = \infty$ *and* $m \le 0$
(iv) $S(-\infty) > -\infty$, $S(\infty) < \infty$ *and* $m = 0$.

Moreover, if $m \ge 0$ *then* τ_* *is given by* (4.2)*, and if* $m \le 0$ *then* τ_* *is given by* (4.4).

Proof. First to verify that the conditions in cases (i)-(iv) are sufficient, let G be the distribution function given in (2.4). Assume that $m \ge 0$ and let the inverse of $s \mapsto b_+(s)$ be given as in (3.4). Let (W_t) be the process given in (2.7) and define the stopping time $\tilde{\tau}_*$ for (W_t) by

$$\tilde{\tau}_* = \inf \big\{ \, t > 0 \, : \, W_t \le b_+\big(\max_{0 \le r \le t} W_r \big) \, \big\} \, . \tag{4.5}$$

As observed in Section 2 that $\langle M, M \rangle_e = \inf \{ \, t > 0 \, : \, W_t \notin (S(-\infty), S(\infty)) \, \}$ and by the definition of $b_+(\cdot)$ we see that $\tilde{\tau}_* < \langle M, M \rangle_e$ if either $S(-\infty) = -\infty$, or $m = 0$ with $S(-\infty) > -\infty$ and $S(\infty) < \infty$. Therefore in the cases (i), (ii) and (iv) we have that $\tilde{\tau}_* < \langle M, M \rangle_e$. Note that $\tilde{\tau}_* < \langle M, M \rangle_e$ fails in the other cases. The process (W_t) is a

Brownian motion stopped at $\langle M,M\rangle_e$ and hence from Proposition 3.1 we have $W_{\tilde{\tau}_*} \sim G$. Again by an observation in Section 2 the stopping time τ_* for (M_t) given by

$$\tau_* = T_{\tilde{\tau}_*} = \inf\Big\{\, t>0 \,:\, M_t \le b_+\big(\max_{0\le r\le t} M_r\big)\,\Big\}$$

satisfies $M_{\tau_*} = W_{\tilde{\tau}_*} \sim G$ where (T_t) is the time change given in (2.6). From (4.3) we see that τ_* is given in (4.2) and it clearly fulfills $X_{\tau_*} \sim F$. The same arguments hold for $m \le 0$.

The conditions in the cases (i)-(iv) are necessary as well. Indeed, case (i) is trivial because there is no restriction on the class of probability measures we are considering. In case (ii) let τ_* be a stopping time for (X_t) satisfying $X_{\tau_*} \sim F$ or equivalently $M_{\tau_*} \sim G$. Then the process $(M_{\tau_*\wedge t})$ is a continuous local martingale which is bounded from above by $S(\infty) < \infty$. Let $\{\gamma_n\}_{n\ge1}$ be a localization for the local martingale. Applying Fatou's lemma and the optional sampling theorem we have that

$$m = \mathbf{E}\big(M_{\tau_*}\big) \ge \liminf_n \mathbf{E}\big(M_{\tau_*\wedge\gamma_n}\big) = 0\ .$$

The cases (iii) and (iv) are proved in exactly in the same way. Note that (M_t) is a bounded martingale in case (iv). □

Next we explore the properties stated in Proposition 3.2 in the context of a diffusion.

Proposition 4.2. *Under the assumptions of Theorem 4.1, let τ be any stopping time of (X_t) satisfying*

$$X_\tau \sim \nu\ .$$

(I). *If $m \ge 0$ and $\mathbf{E}\big(\max_{0\le t\le\tau} S(X_t)\big) < \infty$ then the following inequality holds*

$$\mathbf{P}\big(\max_{0\le t\le\tau} X_t \ge s\big) \le \mathbf{P}\big(\max_{0\le t\le\tau_*} X_t \ge s\big) \tag{4.6}$$

for all $s \ge 0$. If furthermore ν satisfies

$$\int_0^\infty S(u)\log(S(u))\,\nu(du) < \infty \tag{4.7}$$

and the stopping time τ satisfies $\max_{0\le t\le\tau} X_t \sim \max_{0\le t\le\tau_*} X_t$ (i.e. there is equality in (4.6) for all $s > 0$) then we have

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

(II). *If $m \le 0$ and $\mathbf{E}\big(\min_{0\le t\le\tau} S(X_t)\big) > -\infty$ then the following inequality holds*

$$\mathbf{P}\big(\min_{0\le t\le\tau} X_t \le s\big) \le \mathbf{P}\big(\min_{0\le t\le\tau_*} X_t \le s\big) \tag{4.8}$$

for all $s \le 0$. If furthermore ν satisfies

$$\int_{-\infty}^0 S(u)\log(-S(u))\,\nu(du) > -\infty \tag{4.9}$$

and the stopping time τ satisfies $\min_{0\le t\le\tau} X_t \sim \min_{0\le t\le\tau_*} X_t$ (i.e. there is equality in (4.8) for all $s < 0$) then we have

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

Proof. The two cases (I) and (II) are proved with precisely the same arguments. Therefore we will concentrate on case (I). Let τ be the stopping time given in the proposition. Then we have that $M_\tau \sim G$ and $\mathbf{E}\big(\max_{0\le t\le\tau} M_t\big) < \infty$. Since τ and τ_* are two embeddings we have from Section 2 that the two stopping times $\tilde{\tau}$ and $\tilde{\tau}_*$ for (W_t) given by

$$\tilde{\tau} = \langle M,M\rangle_\tau \quad \text{and} \quad \tilde{\tau}_* = \langle M,M\rangle_{\tau_*}$$

satisfy

$$W_{\tilde{\tau}} \sim W_{\tilde{\tau}_*} \sim G \; .$$

Note that $\tilde{\tau}_*$ is given in (4.5) and that

$$\mathbf{E}\big(\max_{0 \le t \le \tilde{\tau}} W_t \big) = \mathbf{E}\big(\max_{0 \le t \le \tau} M_t \big) < \infty \; .$$

Then Proposition 3.2 gives that

$$\mathbf{P}\big(\max_{0 \le t \le \tilde{\tau}} W_t \ge s \big) \le \mathbf{P}\big(\max_{0 \le t \le \tilde{\tau}_*} W_t \ge s \big)$$

for $s > 0$ and going back we obtain (4.6). The second part is verified with the same arguments. $\square$

Remark 4.3. For $m \ge 0$ we have that

$$\mathbf{P}\big(\max_{0 \le t \le \tau_*} X_t \ge s \big) = \inf_{z<s} \frac{c(S(z))}{S(s) - S(z)} = \exp\bigg(- \int_0^s \frac{dS(u)}{S(u) - S(h_+(u))} \bigg)$$

for $s > 0$ and the condition (4.7) is trivial when $S(\cdot)$ is bounded from above. For $m \le 0$ we have that

$$\mathbf{P}\big(\min_{0 \le t \le \tau_*} X_t \le s \big) = \inf_{z>s} \frac{p(S(z))}{S(z) - S(s)} = \exp\bigg(- \int_s^0 \frac{dS(u)}{S(h_-(u)) - S(u)} \bigg)$$

for $s < 0$ and the condition (4.9) is trivial when $S(\cdot)$ is bounded from below.

Remark 4.4. If $m = 0$ we have another extremal embedding σ_* of Perkins [9] which stochastically minimizes $\max_{0 \le t \le \tau_*} X_t$. The construction of the embedding is the following. Define the decreasing function $s \mapsto g_+(s)$ as follows. For $s > 0$ set $g_+(s)$ as the value of $z < s$ which maximizes

$$\frac{c(S(s)) - p(S(z))}{S(s) - S(z)}$$

For $s > 0$ the function $g_+(s)$ is the unique root to the equation

$$\frac{c(S(s)) - p(S(z))}{S(s) - S(z)} = F(z)$$

satisfying $g_+(s) < s$. Define the decreasing function $s \mapsto g_-(s)$ as follows. For $s < 0$ set $s \mapsto g_-(s)$ as the value of $z > s$ which maximizes

$$\frac{p(S(s)) - c(S(z))}{S(z) - S(s)}$$

For $s < 0$ the function $g_-(s)$ is the unique root to the equation

$$\frac{p(S(s)) - c(S(z))}{S(z) - S(s)} = 1 - F(z)$$

satisfying $g_-(s) > s$. Define the two stopping times

$$\sigma_{g_+} = \inf \big\{ \, t > 0 \, : \, X_t \le a_+\big(\max_{0 \le r \le t} X_r \big) \, \big\}$$
$$\sigma_{g_-} = \inf \big\{ \, t > 0 \, : \, X_t \ge a_-\big(\min_{0 \le r \le t} X_r \big) \, \big\} \; .$$

For the stopping time σ_* for (X_t) given by

$$\sigma_* = \sigma_{g_+} \wedge \sigma_{g_-}$$

we have that

$$X_{\sigma_*} \sim F \; .$$

The embedding σ_* can be characterized uniquely in the following way. If τ is given as in Proposition 4.2 then

$$\mathbf{P}\big(\max_{0\le t\le \tau} X_t \ge s\big) \ge \mathbf{P}\big(\max_{0\le t\le \sigma_*} X_t \ge s\big) \tag{4.10}$$

for $s>0$. If there is equality in (4.10) for all $s>0$ then

$$\tau = \sigma_* \ .$$

Finally we have that

$$\begin{aligned}\mathbf{P}\big(\max_{0\le t\le \sigma_*} X_t \ge s\big) &= 1 - G(s) + \sup_{z<s} \frac{c(S(s)) - p(S(z))}{S(s)-S(z)}\\ &= \exp\Big(-\int_0^s \frac{dS(r)}{S(r)-S(g_+(r))}\Big) - \int_0^s \exp\Big(-\int_u^s \frac{dS(r)}{S(r)-S(g_+(r))}\Big)\, dF(u)\end{aligned}$$

for $s>0$. There are similar results for the minimum process.

5. Appendix: Stochastic differential equation

This Section presents results on existence and uniqueness and various aspects of solutions of the one-dimensional time-homogeneous stochastic differential equation (2.1). For a survey and proofs of these results see Karatzas and Shreve [8].

Let $I=(l,r)$ with $-\infty \le l < r \le \infty$. Consider the non-singular stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t \tag{5.1}$$

where $\mu : I \to \mathbb{R}$ and $\sigma : I \to (0,\infty)$ are Borel functions.

Definition 5.1. A weak solution in the interval I up to an explosion time e of the one-dimensional time-homogeneous stochastic differential equation (5.1) is a triple $((X_t),(B_t))_{t\ge 0}$, $(\Omega,\mathcal{F},\mathbf{P})$ and $(\mathcal{F}_t)$ satisfying the following three conditions.

(i) $(\Omega,\mathcal{F},\mathbf{P})$ is a probability space and $(\mathcal{F}_t)$ is a filtration of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions.

(ii) $(X_t)_{t\ge 0}$ is a continuous $(\mathcal{F}_t)$-adapted, $[l,r]$-valued process with $X_0 \in [l,r]$ $\mathbf{P}$-a.s. and $(B_t)_{t\ge 0}$ is a standard $(\mathcal{F}_t)$-Brownian motion.

(iii) For all $n\ge 1$ we have

$$\int_0^{e_n\wedge t} \Big(\big|\mu(X_u)\big| + \sigma^2(X_u)\Big)\, du < \infty \quad \mathbf{P}\text{-a.s.}$$

for all $0\le t<\infty$ and

$$\big(X_{e_n\wedge t}\big)_{t\ge 0} = \Big(X_0 + \int_0^t \mu(X_u)\mathbf{1}_{\{u\le e_n\}}\, du + \int_0^t \sigma(X_u)\mathbf{1}_{\{u\le e_n\}}\, dB_u\Big)_{t\ge 0} \quad \mathbf{P}\text{-a.s.}$$

where $e_n = \inf\{\, t>0 \,:\, X_t \notin (l_n,r_n)\,\}$ and $l<l_n<r_n<r$ with $l_n \downarrow l$ and $r_n \uparrow r$ for $n\to\infty$.

The explosion time e for the process (X_t) is defined as

$$e := \inf\{\, t>0 \,:\, X_t \notin (l,r)\,\} = \lim_{n\to\infty} e_n \ .$$

Since $\sigma(\cdot)$ is strictly positive (non-singular) we have the following sharp sufficient conditions for existence and uniqueness of solutions of (5.1).

Theorem 5.2. *If for all $x \in I$ there exists $\epsilon > 0$ such that*

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1+|\mu(u)|}{\sigma^2(u)}\, du < \infty \tag{5.2}$$

then for every initial distribution of X_0, the stochastic equation (5.1) *has a weak solution in I up to an explosion time e, and this solution is unique in the sense of probability law.*

Assume that $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the condition (5.2) and let $\mathbf{P}_x$ denote the probability measure when $X_0 = x$. The scale function of (X_t) is given by

$$S(x) = \int_{x_0}^{x} \exp\Big(-2 \int_{x_0}^{u} \frac{\mu(r)}{\sigma^2(r)}\, dr \Big)\, du$$

for $x \in I$ and some $x_0 \in I$. The next proposition states necessary and sufficient condition for the process (X_t) to be recurrent.

Proposition 5.3. *Let (X_t) be a weak solution in the interval I of the stochastic differential equation* (5.1). *We distinguish four cases.*

(i) *If $S(l+) = -\infty$ and $S(r-) = \infty$, then*

$$\mathbf{P}_x\big(e = \infty\big) = \mathbf{P}_x\big(\limsup_{t\uparrow\infty} X_t = r\big) = \mathbf{P}_x\big(\liminf_{t\uparrow\infty} X_t = l\big) = 1$$

for all $x \in I$. In particular, the process (X_t) is recurrent, i.e. $\mathbf{P}_x(\tau_y < \infty) = 1$ for all $x, y \in I$ where $\tau_y = \inf\{\, t > 0 \,:\, X_t = y \,\}$.

(ii) *If $S(l+) > -\infty$ and $S(r-) = \infty$, then $\lim_{t\uparrow e} X_t$ exists $\mathbf{P}_x$-a.s. and*

$$\mathbf{P}_x\big(\lim_{t\uparrow e} X_t = l\big) = \mathbf{P}_x\big(\sup_{0\le t<e} X_t < r\big) = 1$$

for all $x \in I$.

(iii) *If $S(l+) = -\infty$ and $S(r-) < \infty$, then $\lim_{t\uparrow e} X_t$ exists $\mathbf{P}_x$-a.s. and*

$$\mathbf{P}_x\big(\lim_{t\uparrow e} X_t = r\big) = \mathbf{P}_x\big(\inf_{0\le t<e} X_t > l\big) = 1$$

for all $x \in I$.

(iv) *If $S(l+) > -\infty$ and $S(r-) < \infty$, then $\lim_{t\uparrow e} X_t$ exists $\mathbf{P}_x$-a.s. and*

$$\mathbf{P}_x\big(\lim_{t\uparrow e} X_t = l\big) = 1 - \mathbf{P}_x\big(\lim_{t\uparrow e} X_t = r\big) = \frac{S(r-) - S(x)}{S(r-) - S(l+)}$$

for all $x \in I$.

The process (X_t) is non-recurrent in cases (ii), (iii) *and* (iv)

For completeness, to give necessary and sufficient conditions for non-explosion we need to introduce the following function

$$\kappa(x) = 2 \int_{x_0}^{x} \frac{S(x) - S(u)}{S'(u)\sigma^2(u)}\, du$$

for $x \in I$.

Proposition 5.4. (*Feller's Test for Explosions.*) *Let* (X_t) *be a weak solution in the interval* I *of the stochastic differential equation* (5.1). *We distinguish three cases.*

(i) *If* $\kappa(l+) = \kappa(r-) = \infty$ *then* $\mathbf{P}_x\big(e = \infty\big) = 1$ *for all* $x \in I$.

(ii) *If* $\kappa(l+) < \infty$ *or* $\kappa(r-) < \infty$ *then* $\mathbf{P}_x\big(e = \infty\big) < 1$ *for all* $x \in I$.

(iii) *We have* $\mathbf{P}_x\big(e < \infty\big) = 1$ *if and only if one of the following conditions holds.*

  (a) $\kappa(r-) < \infty$ *and* $\kappa(l+) < \infty$ (*in this case* $\mathbf{E}_x(e) < \infty$) .

  (b) $\kappa(r-) < \infty$ *and* $S(l+) = -\infty$.

  (c) $\kappa(l+) < \infty$ *and* $S(r-) = \infty$.

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