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Abstract

To a risk model an independent perturbation process is added. If the perturbation process is Brownian motion, Lundberg inequalities and Cramér-Lundberg approximations can be proved. Also the asymptotic behaviour of the ruin probability in the case of heavy claims can be obtained. If, the perturbation is a Lévy process, ladder epochs and ladder heights can be defined. In the stationary case, the distribution of the ladder height are obtained.

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1 Introduction

Risk processes were introduced by Lundberg [14]. He introduced the classical risk process

$$C_t = \sum_{i=1}^{N_t} Y_i - ct \tag{1.1}$$

where $\{N_t\}$ is a Poisson process with rate λ, $\{Y_i\}$ is a sequence of iid positive random variables, independent of $\{N_t\}$, and $c > 0$ is the premium rate. We work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by G the distribution function of the claim sizes, by $\mu = \mathbb{E}[Y]$ its expected value and by $M_Y(r) = \mathbb{E}[e^{rY}]$ its moment generating function. This process was also investigated by Cramér [3]. In this context one is mainly interested in the ruin probabilities

$$\psi(u) = \mathbb{P}[\sup_{t\geq 0} C_t > u] = \mathbb{P}[\tau(u) < \infty]$$

where the ruin time $\tau = \tau(u)$ is defined as $\tau(u) = \inf\{t \geq 0 : C_t > u\}$.

From random walk theory it follows that $\psi(u) = 1$ for all u if $c \leq \lambda\mu$. We therefore assume the *net profit condition* $c > \lambda\mu$, which implies that $\psi(u) \leq \psi(0) = \lambda\mu/c$. We first discuss the light-tail case. Assume that there exists a solution $R \neq 0$ to the equation $\lambda(M_Y(R)-1) = cR$. If R exist it is unique and positive and following *Lundberg inequalities* hold:

$$c_- \mathrm{e}^{-Ru} \leq \psi(u) \leq c_+ \mathrm{e}^{-Ru}\,, \tag{1.2}$$

where $0 \leq c_- \leq c^+ \leq 1$, see [15]. If, moreover, $M_Y'(R) < \infty$ then $c_- > 0$ and there exits a constant c^* such that

$$\psi(u) \sim c^* \mathrm{e}^{-Ru}\,, \tag{1.3}$$

where $f(u) \sim g(u)$ means that $\lim_{u\to\infty} f(u)/g(u) = 1$. Equation (1.3) is called the *Cramér-Lundberg approximation*.

For heavy-tailed claim size distributions the situation is quite different. A distribution function G is called *subexponential* if $\overline{G^{*2}}(x) \sim 2\overline{G}(x)$, where $\overline{G}(x) = 1-G(x)$. We write then $G \in \mathcal{S}$. By $G_I(x) = \mu^{-1}\int_0^x \overline{G}(y)\,\mathrm{d}y$ we denote the *integrated tail distribution*. Note that $G_I(x)$ is the distribution function of the ladder-heights. The following result was proved by Embrechts and Veraverbeke [7]. Assume $G_I \in \mathcal{S}$. Then

$$\psi(u) \sim \frac{\lambda\mu}{c - \lambda\mu}\overline{G_I}(u)\,. \tag{1.4}$$

The above results can be generalized to many other risk models, see for instance [12], [15], [18] and [1].

The classical risk model is of course only an idealization of the real world. The linear premium income means for example that the insurance portfolio is always of the same size and that interest and inflation cancel each other. For this reason, Gerber [11] perturbed the process by Brownian motion and considered

$$X_t = C_t + \eta B_t = \sum_{i=1}^{N_t} Y_i - ct + \eta B_t\,. \tag{1.5}$$

The perturbation can be seen as a fluctuation of the premium income, of the return earned from investment, a fluctuation of the environment determining the claim sizes or claim intensities or fluctuation in the administration costs. Here $\{B_t\}$ is a Brownian motion with $\mathbb{E}[B_t] = 0$ and $\mathbb{E}[B_t^2] = 2t$. (1.5) is the process we want to investigate in this review paper. In Section 2 we will consider the light-tail case and find Lundberg estimates as well as Cramér-Lundberg approximations in the perturbed classical risk model as well as more general models. The heavy-tailed case will be treated in Section 3. There are basically two possibilities to get heavy-tails: heavy-tailed claim sizes (Section 3.1) and perturbation by processes with heavy tails (Section 3.2). This paper gives a survey over the topic, and will therefore contain only a few new results. Most of the results will not be proved here. The interested reader will find the proofs in the original papers.

Below we will work with martingales. For this purpose we need a filtration $\{\mathcal{F}_t\}$, which we will assume to be the smallest right-continuous filtration such that the processes $\{C_t\}$ and $\{B_t\}$ are adapted. If we Markovize the process the filtration has to be slightly increased in order that the additional processes, as for instance $\{\lambda_t\}$ also are adapted. Note that we do not complete $\{\mathcal{F}_t\}$. This is necessary in order that we will be able to change the measure.

Remark. The perturbation considered in this paper is an independent process added to the risk process. Another possibility would be to perturb the parameters and the claim size distribution in a risk process. Such perturbations were considered in [13]. ■

2 Cramér-Lundberg theory

In this section we will start by considering the classical risk model perturbed by Brownian motion. We will use martingales to find Lundberg's inequalities and the

Cramér-Lundberg approximation. In the same way one can obtain the results in other perturbed risk models as well. We will construct the martingales as in [10], but follow the approach of [17].

2.1 The classical model

From Markov process theory we know that if $\{f(X_t)\}$ is a martingale, then the function $f(x)$ has to fulfil the equation

$$\eta^2 f''(x) - cf'(x) + \lambda\Big(\int_0^\infty f(x+y)\,\mathrm{d}G(y) - f(x)\Big) = 0\,, \tag{2.1}$$

see also [11]. If there is a function $f(x)$ with $f(x) = 0$ for $x \le 0$ fulfilling (2.1) for $x > 0$ and $\lim_{x\to\infty} f(x) = 1$, then $f(u) = 1 - \psi(u)$. This gives a possibility to find $\psi(u)$ numerically. We try a function of the form $f(x) = \mathrm{e}^{Rx}$ for all x. This yields

$$\eta^2 R^2 - cR + \lambda(m_Y(R) - 1) = 0\,. \tag{2.2}$$

We call a positive solution R to (2.2) the *adjustment coefficient*. By direct verification it follows that the process $\{\mathrm{e}^{RX_t}\}$ is indeed a martingale. Next we define the new measure $\mathbb{Q}_t$ on $\mathcal{F}_t$ by

$$\mathbb{Q}_t[A] = \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{RX_t}; A]\,.$$

This measure is independent of t, i.e. $\mathbb{Q}_s|_{\mathcal{F}_t} = \mathbb{Q}_t$ for $s \ge t$. Moreover, the measure can be extended to a measure on $\mathcal{F}$. We therefore just write $\mathbb{Q}$ instead of $\mathbb{Q}_t$. If T is a stopping time and $A \subset \{T < \infty\}$ then also $\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{RX_T}; A]$ holds. For an introduction to change of measure techniques see for instance [15] or [17].

Under the measure $\mathbb{Q}$ the process $\{X_t\}$ remains a perturbed risk model. The parameters are $\tilde{c} = c - 2\eta^2 R$, $\tilde{\lambda} = \lambda M_Y(R)$ and $\mathrm{d}\tilde{G}(x) = \mathrm{e}^{Rx}\,\mathrm{d}G(x)/M_Y(R)$. From (2.2) it follows that $\tilde{\lambda}\tilde{\mu} = \lambda M_Y'(R) > c - 2\eta^2 R = \tilde{c}$. Hence the net profit condition is not fulfilled and $\mathbb{Q}[\tau < \infty] = 1$. We can now express the ruin probability in terms of the new measure $\mathbb{Q}$

$$\psi(u) = \mathbb{E}_{\mathbb{P}}[e^{RX_\tau} e^{-RX_\tau}; \tau < \infty] = \mathbb{E}_{\mathbb{Q}}[e^{-RX_\tau}; \tau < \infty] = \mathbb{E}_{\mathbb{Q}}[e^{-R(X_\tau - u)}] \, e^{-Ru} . \tag{2.3}$$

Now we can find Lundberg's inequalities. The upper bound was already proved in [4].

Proposition 2.1. *Let $\{X_t\}$ be a classical risk model perturbed by diffusion. Then*

$$c_- e^{-Ru} < \psi(u) < e^{-Ru} ,$$

where

$$c_- = \inf_{x < x_0} \frac{e^{Rx} \overline{G}(x)}{\int_x^\infty e^{Ry} \, dG(y)}$$

and $x_0 = \sup\{x \geq 0 : G(x) < 1\}$.

Proof. From the definition of τ the upper Lundberg bound $\psi(u) < e^{-Ru}$ follows immediately. Consider now the lower bound. Ruin can happen in two ways, by a jump over u or by the diffusion reaching the level u. If ruin occurs by diffusion then $e^{-R(X_\tau - u)} = 1$. If ruin occurs by a jump then $e^{-R(X_\tau - u)} < 1$. Let us therefore assume that ruin occurs by a jump. If we condition on $u - X_{\tau-} = x$ then the claim Y_+ leading to ruin must be larger than x. Thus

$$\mathbb{E}_{\mathbb{Q}}[e^{-R(X_\tau - u)} \mid u - X_{\tau-} = x] = \mathbb{E}_{\mathbb{Q}}[e^{-R(Y - x)} \mid Y > x] .$$

This yields the lower bound

$$\mathbb{E}_{\mathbb{Q}}[e^{-R(X_\tau - u)}] > \inf_{x < x_0} \mathbb{E}_{\mathbb{Q}}[e^{-R(Y - x)} \mid Y > x] .$$

The right-hand side is c_-. □

As in [4] we could consider the events "ruin occurs by diffusion" and "ruin occurs by jump" separately. Define $\psi_c(u) = \mathbb{P}[\tau(u) < \infty, X_\tau = u]$ and $\psi_d(u) = \mathbb{P}[\tau(u) < \infty, X_\tau > u]$. Then we find the following result.

Proposition 2.2. *Let $\{X_t\}$ be a classical risk model perturbed by diffusion. Then*

$$\psi_c(u) = \mathbb{Q}[X_\tau = u]\, \mathrm{e}^{-Ru}\,, \qquad c_-\mathbb{Q}[X_\tau > u]\, \mathrm{e}^{-Ru} \le \psi_d(u) \le c_+\mathbb{Q}[X_\tau > u]\, \mathrm{e}^{-Ru}\,,$$

where c_- is given in Proposition 2.1 and

$$c_+ = \sup_{x<x_0} \frac{\mathrm{e}^{Rx}\overline{G}(x)}{\int_x^\infty \mathrm{e}^{Ry}\,\mathrm{d}G(y)} \le 1\,.$$

Proof. We find

$$\psi_c(u) = \mathbb{E}_\mathbb{Q}[\mathrm{e}^{-R(X_\tau-u)}; X_\tau = u]\, \mathrm{e}^{-Ru} = \mathbb{Q}[X_\tau = u]\, \mathrm{e}^{-Ru}$$

and

$$\psi_d(u) = \mathbb{E}_\mathbb{Q}[\mathrm{e}^{-R(X_\tau-u)}; X_\tau > u]\, \mathrm{e}^{-Ru} = \mathbb{E}_\mathbb{Q}[\mathrm{e}^{-R(X_\tau-u)} \mid X_\tau > u]\, \mathbb{Q}[X_\tau > u]\, \mathrm{e}^{-Ru}\,.$$

The proof follows now as in Proposition 2.1. □

We now turn to the Cramér-Lundberg approximation. In order to prove the Cramér-Lundberg approximation we need to define ladder epochs. Because for $u = 0$ we have $\tau(u) = 0$, we cannot take $\tau(0)$ as the ladder epoch. Following [4] we define the first *ladder time* as

$$\tau_+ = \inf\{T_i : X_{T_i} > \sup\{X_t : 0 \le t < T_i\}\}\,, \tag{2.4}$$

the time at which a jump leads to a new maximum of the process. The variables $\{T_i\}$ denote the jump times of $\{N_t\}$. We define the part of the *ladder height* due to the diffusion $L_c = \sup\{X_t : 0 \le t < \tau_+\}$ and the part due to the jump $L_d = X_{\tau_+} - L_c$, see also Figure 3.1. We now can prove the Cramér-Lundberg approximations.

Proposition 2.3. (Dufresne and Gerber (1991)) *Let $\{X_t\}$ be a classical risk model perturbed by diffusion. Then*

$$\psi_c(u) \sim \frac{\eta^2 R}{\lambda M_Y'(R) - c - 2\eta^2 R}\, \mathrm{e}^{-Ru}\,, \qquad \psi_d(u) \sim \frac{c - \lambda\mu - \eta^2 R}{\lambda M_Y'(R) - c - 2\eta^2 R}\, \mathrm{e}^{-Ru}\,.$$

Proof. Under $\mathbb{Q}$ let $H(x, y)$ be the joint distribution function of (L_c, L_d), H_c be the distribution function of L_c, H_d be the distribution function of L_d and $\tilde{H}$ be the distribution function of $L_c + L_d$. Then the function $f_c(u) = \mathbb{Q}[X_\tau = u]$ fulfils the renewal equation

$$f_c(u) = 1 - H_c(u) + \int_0^u f_c(u - x)\,\mathrm{d}\tilde{H}(x)\,.$$

The function $f_d(u) = \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-R(X_\tau - u)}; X_\tau > u]$ fulfils

$$f_d(u) = \int_0^u \int_{u-x}^\infty \mathrm{e}^{-R(x+y-u)}\, H(\mathrm{d}x, \mathrm{d}y) + \int_0^u f_d(u - x)\,\mathrm{d}\tilde{H}(x)\,.$$

The result follows now from the key renewal theorem. For the details see [17] and [15]. □

2.2 More general models

We now turn to more general models. We will introduce the different models and give the results. For the details the reader should consult [10] and [17].

2.2.1 Sparre-Andersen model

We now assume that $\{N_t\}$ is a renewal process, whereas $\{Y_i\}$ is still an iid sequence independent of $\{N_t\}$. We denote the inter-arrival distribution by F and let T be a random variable with distribution F. The mean value of T is denoted by λ^{-1}, and we let $\lambda = 0$ if $\mathbb{E}[T] = \infty$. The distribution of T_1 is denoted by F_1, which may be different from F. A special case is the stationary renewal risk model, where $\lambda > 0$ and $F_1(x) = F_I(x) = \lambda \int_0^x \overline{F}(y)\,\mathrm{d}y$. We assume the net profit condition $c > \lambda\mu$. The *adjustment coefficient* R is the strictly positive solution to $M_Y(R)M_T(\eta^2 R^2 - cR) = 1$, if such a solution exists. Here $M_T(r) = \mathbb{E}[\mathrm{e}^{rT}]$. We assume that R exists. Denote by $V_t = T_{N_t+1} - t$ the time remaining to the next claim. Then the process $\{(X_t, V_t)\}$ is a Markov process.

Remark. It would be more natural to consider the process $\{(X_t, W_t)\}$ with $W_t = t - T_{N_t}$ denoting the time elapsed since the last claim. But it turns out, that the corresponding martingale is much more complicated. Even though for our process the filtration $\{\mathcal{F}_t\}$ gives now information about the unobservable event of the next claim, the filtration has no influence on the results, that are much simpler to obtain with the present Markovization. ■

The process

$$\{L_t = (M_{T_1}(\eta^2 R^2 - cR))^{-1} \mathrm{e}^{(\eta^2 R^2 - cR)V_t}\, \mathrm{e}^{RX_t}\} \tag{2.5}$$

is a martingale, see [10]. As before, we use it to change the measure $\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[L_t; A]$ for $A \in \mathcal{F}_t$. The measure can be extended to a measure on $\mathcal{F}$.

Under $\mathbb{Q}$ the process $\{X_t\}$ is again a perturbed renewal risk process. The parameters are $\mathrm{d}\tilde{G}(x) = M_T(\eta^2 R^2 - cR)\mathrm{e}^{Rx}\, \mathrm{d}G(x)$, $\mathrm{d}\tilde{F}(t) = M_Y(R)\mathrm{e}^{(\eta^2 R^2 - cR)t}\, \mathrm{d}F(t)$, $\tilde{c} = c - 2\eta^2 R$ and $\mathrm{d}\tilde{F}_1(t) = (M_{T_1}(\eta^2 R^2 - cR))^{-1}\mathrm{e}^{(\eta^2 R^2 - cR)t}\, \mathrm{d}F_1(t)$. Moreover, under $\mathbb{Q}$ the process has positive drift, i.e. $\mathbb{Q}[\tau < \infty] = 1$.

The ruin probabilities can now be expressed as

$$\psi_c(u) \quad = \quad M_{T_1}(\eta^2 R^2 - cR)\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{(cR - \eta^2 R^2)V_\tau}; X_\tau = 0]\, \mathrm{e}^{-Ru} \tag{2.6}$$

and

$$\psi_d(u) \quad = \quad M_{T_1}(\eta^2 R^2 - cR)M_Y(R)\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-R(X_\tau - u)}; X_\tau < 0]\, \mathrm{e}^{-Ru}\,. \tag{2.7}$$

We see that the expression for $\psi_c(u)$ needs information about V_τ. This can be a problem. If namely T is heavy-tailed, it will be difficult to find an estimate for the expected value. We therefore need a technical condition. For a discussion on this condition see [6].

Assumption 2.1. *Assume that*

$$b = \inf_{x < t_0} \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{(\eta^2 R^2 - cR)(T - x)} \mid T > x] > 0\,,$$

where $t_0 = \sup\{t \geq 0 : F(t) < 1\}$.

Proposition 2.4. (Furrer and Schmidli (1994)) *Let* $\{X_t\}$ *be a perturbed renewal process. Assume* R *is well-defined. Then*

$$M_{T_1}(\eta^2 R^2 - cR) M_Y(R) \mathbb{Q}[X_\tau > u] c_- \, \mathrm{e}^{-Ru} \le \psi_d(u)$$
$$\le \; M_{T_1}(\eta^2 R^2 - cR) M_Y(R) \mathbb{Q}[X_\tau > u] c_+ \, \mathrm{e}^{-Ru}$$

where c_- *and* c_+ *are given in Proposition 2.2. If moreover Assumption 2.1 holds then*

$$M_{T_1}(\eta^2 R^2 - cR) \mathbb{Q}[X_\tau = u] \, \mathrm{e}^{-Ru} \le \psi_c(u) \le M_{T_1}(\eta^2 R^2 - cR) \mathbb{Q}[X_\tau = u] b^{-1} \, \mathrm{e}^{-Ru} \, .$$

□

Proposition 2.5. (Schmidli (1995)) *Let* $\{X_t\}$ *be a perturbed renewal process. Assume* R *is well-defined,* $M_Y'(R) < \infty$ *and Assumption 2.1 holds. Then there exist constants* C_c *and* C_d *independent of the distribution of* T_1 *such that*

$$\psi_c(u) \sim C_c M_{T_1}(\eta^2 R^2 - cR) \, \mathrm{e}^{-Ru} \, , \qquad \psi_d(u) \sim C_d M_{T_1}(\eta^2 R^2 - cR) \, \mathrm{e}^{-Ru} \, .$$

□

The constant C_d can be expressed in terms of the distribution of (L_c, L_d) defined as in Section 2. This is, however, not explicit because the ladder-height distribution is not known.

2.2.2 Björk-Grandell model

In this section we consider a Cox risk model. Let $\{(\Lambda_i, \sigma_i)\}$ be a sequence of independent random vectors. For $i \ge 2$ the distribution of (Λ_i, σ_i) is $F(\ell, t)$, the distribution of (Λ_1, σ_1) is denoted by $F_1(\ell, t)$. We assume $\Lambda_i \ge 0$ and $\sigma_i > 0$. In order to avoid trivialities, $\mathbb{P}[\Lambda_2 > 0] > 0$. The Λ's denote the intensity levels, the

σ's the length of the time a certain intensity holds. Let $\Sigma_0 = 0$ and $\Sigma_i = \Sigma_{i-1} + \sigma_i$. We let $\lambda_t = \Lambda_i$ and $V_t = \Sigma_i - t$ if $\Sigma_{i-1} \le t < \Sigma_i$. The process $\{N_t\}$ is now a *Cox point process* (doubly stochastic point process) with intensity $\{\lambda_t\}$. By (Λ, σ) we denote a generic random vector with distribution $F(\ell, t)$. We assume the net profit condition $c\mathbb{E}[\sigma] > \mathbb{E}[\Lambda\sigma]\mu$.

The *adjustment coefficient* R is the strictly positive solution to $\phi(0, R) = 1$, where

$$\phi(\vartheta, r) = \mathbb{E}[\mathrm{e}^{(\Lambda(M_Y(r)-1)+\eta^2 r^2 - cr - \vartheta)\sigma}] .$$

Let $\phi_1(\vartheta, r) = \mathbb{E}[\mathrm{e}^{(\Lambda_1(M_Y(r)-1)+\eta^2 r^2 - cr - \vartheta)\sigma_1}]$. Then the process

$$\{L_t = (\phi_1(0, R))^{-1} \mathrm{e}^{(\lambda_t(M_Y(R)-1)+\eta^2 R^2 - cR)V_t} \mathrm{e}^{RX_t}\} \tag{2.8}$$

is a martingale provided $\phi_1(0, R) < \infty$. We again define the measure Q on $\mathcal{F}_t$ via the Radon-Nikodym derivative L_t.

Analogously to Section 2.2.1 one obtains the following results, see also [19].

Assumption 2.2. *Assume that*

$$b = \operatorname*{ess\,inf}_{\Lambda \in B} \inf_{t < t_\Lambda}$$

where $B = [0, (cR - \eta^2 R^2)/(M_Y(R) - 1))$ *and* $t_\Lambda = \sup\{t : \mathbb{P}[\sigma > t \mid \Lambda] > 0\}$.

Proposition 2.6. **(Furrer and Schmidli (1994))** *Let* $\{X_t\}$ *be a perturbed Björk-Grandell model and assume that* R *is well-defined. Under Assumption 2.2 there exists a constant* $c(F_1)$ *such that*

$$\psi_c(u) \le c(F_1)\mathbb{Q}[X_\tau = u]\, \mathrm{e}^{-Ru} , \qquad \psi_d(u) \le c(F_1) c_+ \mathbb{Q}[X_\tau > u]\, \mathrm{e}^{-Ru} ,$$

where c_+ *is given in Proposition 2.2. If both*

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\phi(0, r)\right|_{r=R} < \infty \quad \text{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}r}\phi_1(0, r)\right|_{r=R} < \infty \tag{2.9}$$

then there exist a constant $\tilde{c}(F_1)$ such that

$$\psi(u) \geq \tilde{c}(F_1)\,\mathrm{e}^{-Ru}\,.$$

□

Proposition 2.7. (Schmidli (1997)) *Let $\{X_t\}$ be a perturbed Björk-Grandell model and assume that R is well-defined, Assumption 2.2 and (2.9) hold. Then there exist constants C_c and C_d independent of $F_1(\ell, t)$ such that*

$$\psi_c(u) \sim C_c\phi_1(0,R)\,\mathrm{e}^{-Ru}\,, \qquad \psi_d(u) \sim C_d\phi_1(0,R)\,\mathrm{e}^{-Ru}\,.$$

□

2.2.3 Markov modulated risk model

Let $\{Z_t\}$ be a continuous-time Markov chain with state space $E = \{1, \ldots, J\}$ where J is some natural number. Denote the intensity matrix by $\mathbf{\Lambda}$. The stationary initial distribution is denoted by $\boldsymbol{\pi}$. Let λ_i be some non-negative numbers. The process $\{N_t\}$ is a Cox point process with intensity process $\{\lambda_{Z_t}\}$. A claim arriving at time t has distribution $G_{Z_t}(x)$, where $G_i(x)$ are some distribution functions. Given $\{Z_t : t \geq 0\}$, the claim sizes are mutually conditionally independent and conditionally independent of the arrival times. We assume the net profit condition $c > \sum_{i=1}^J \pi_i\lambda_i\mu_i$ where $\mu_i = \int_0^\infty x\,\mathrm{d}F_i(x)$ is the mean value of a claim arriving in state i.

Denote by $\boldsymbol{S}(r)$ the diagonal matrix with entries $S_{ii}(r) = \lambda_i(M_i(r)-1)-cr+\eta^2r^2$, where $M_i(r) = \int_0^\infty \mathrm{e}^{rx}\,\mathrm{d}F_i(x)$ is the moment generating function for claims arriving in state i. Let $\boldsymbol{L}(r) = \mathbf{\Lambda} + \boldsymbol{S}(r)$. Then the matrix $\mathrm{e}^{\boldsymbol{L}(r)}$ has strictly positive entries. Thus the spectral radius $\mathrm{e}^{\theta(r)}$ of $\mathrm{e}^{\boldsymbol{L}(r)}$ is the Perron-Frobenius eigenvalue and the corresponding (right) eigenvector $\boldsymbol{\eta}(r) = (\eta_1(r), \ldots, \eta_J(r))^\top$ has strictly positive entries. In order to simplify notation we norm the eigenvector such that $\sum_{i=1}^J \pi_i \eta_i(r)] = 1$. The function $\theta(r)$ is log-convex. The *adjustment coefficient* is the strictly positive solution to $\theta(R) = 0$, if such a solution exists. Let $\eta_i = \eta_i(R)$. The process

$$L_t = \frac{\eta_{Z_t}}{\mathbb{E}[\eta_{Z_0}]} \mathrm{e}^{RX_t} \tag{2.10}$$

is then a martingale. We define the new measure $\mathbb{Q}$ on $\mathcal{F}_t$ with Radon-Nikodym derivative $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = L_t$. This measure can be extended to the whole σ-algebra $\mathcal{F}$. Under Q the process $\{(Z_t, X_t)\}$ is again a perturbed Markov modulated risk model. The parameters under $\mathbb{Q}$ are $\tilde{\Lambda}_{ij} = (\eta_i)^{-1}\eta_j\Lambda_{ij}$ for $i \neq j$, $\mathbb{Q}[Z_0 = i] = \eta_i \mathbb{P}[Z_0 = i]/\mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]$, $\tilde{\lambda}_i = \lambda_i M_i(R)$, $\mathrm{d}\tilde{G}_i(x) = \mathrm{e}^{Rx}\,\mathrm{d}G_i(x)/M_i(R)$ and $\tilde{c} = c - 2\eta^2 R$. As in the cases discussed before, the process has negative safety loading, and therefore ruin occurs almost surely under $\mathbb{Q}$. We get the formulae

$$\psi_c(u) = \mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathbb{E}_{\mathbb{Q}}[(\eta_{Z_\tau})^{-1}; X_\tau = u]\,\mathrm{e}^{-Ru}, \tag{2.11}$$

$$\psi_d(u) = \mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathbb{E}_{\mathbb{Q}}[(\eta_{Z_\tau})^{-1}\mathrm{e}^{-R(X_\tau - u)}; X_\tau > u]\,\mathrm{e}^{-Ru}\,. \tag{2.12}$$

This yields the Lundberg inequalities.

Proposition 2.8. **(Schmidli (1995))** *Let $\{(Z_t, X_t)\}$ be a perturbed Markov modulated risk model. Then*

$$\frac{\mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathbb{Q}[X_\tau = u]}{\max \eta_i}\,\mathrm{e}^{-Ru} \le \psi_c(u) \le \frac{\mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathbb{Q}[X_\tau = u]}{\min \eta_i}\,\mathrm{e}^{-Ru}$$

and

$$c_- \mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathbb{Q}[X_\tau > u]\,\mathrm{e}^{-Ru} \le \psi_d(u) \le c_+ \mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathbb{Q}[X_\tau > u]\,\mathrm{e}^{-Ru}$$

where

$$c_- = \min_{i \le J} \inf_{x < x_i} \frac{\mathrm{e}^{Rx}\overline{G}_i(x)}{\eta_i \int_x^\infty \mathrm{e}^{Ry}\,\mathrm{d}G_i(y)}, \qquad c_+ = \max_{i \le J} \sup_{x < x_i} \frac{\mathrm{e}^{Rx}\overline{G}_i(x)}{\eta_i \int_x^\infty \mathrm{e}^{Ry}\,\mathrm{d}G_i(y)}$$

and $x_i = \sup\{x \ge 0 : G_i(x) < 1\}$. □

Moreover, from the key renewal theorem the Cramér-Lundberg approximation follows.

Proposition 2.9. (Schmidli (1995)) *Let $\{(Z_t, X_t)\}$ be a perturbed Markov modulated risk model. There are constants C_c and C_d such that $\psi_c \sim C_c \mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathrm{e}^{-Ru}$ and $\psi_d \sim C_d \mathbb{E}_{\mathbb{P}}[\eta_{Z_0}]\mathrm{e}^{-Ru}$. Moreover, $C_d > 0$ and $C_c > 0$ if and only if $M_i'(R) < \infty$ for all i such that $\lambda_i > 0$.* □

This concludes the discussion of the small claim case.

3 The heavy-tailed case

In practice, actuaries are mostly concerned about large claims. They speak about large claims if 20% of the portfolio are responsible for 80% or more of the aggregate claim. As a worst case scenario the Pareto distribution often is used to model the claim sizes, as for instance in property insurance. For motor insurance the log-normal distribution is popular. These two distributions are examples from the class $\mathcal{S}$ of subexponential distributions. This is one possibility to consider heavy tails.

On the other hand, it is also possible that the perturbation is responsible for heavy tails. For instance if the perturbation process is an α-stable Lévy motion, the perturbation process leads to a heavy-tail. It seems unrealistic that the perturbation process is heavier than the underlying (unperturbed) risk process. Indeed, it would be very strange if the investment risk or the risk of the fluctuations in the portfolio would be larger than the insurance risk. However, we will in the discussion below not exclude this case.

3.1 The subexponential case

Let us now assume that the claim sizes are heavy-tailed. This case was first considered in the special case of a classical risk model perturbed by Brownian motion by Veraverbeke [22]. His result was based on Corollary 3.4. Similar results were also proved by Schlegel [16]. We assume here that $\{B_t\}$ is a Brownian motion.

Because below we want to compare the ruin probability of a perturbed model with the ruin probability of the corresponding unperturbed model we will use the notation $\psi_u(u)$ to indicate the unperturbed model. Sometimes we will change the premium density. In order to indicate which premium we use we will write $\psi_u(u, c)$.

In the lemma below we recall some properties of subexponential distributions.

Lemma 3.1.

i) *Let $G \in \mathcal{S}$. Then*

$$\lim_{x\to\infty} \frac{\overline{G}(x-y)}{\overline{G}(x)} = 1$$

for any $y \in \mathbb{R}$.

ii) *Let $G \in \mathcal{S}$. For any $\varepsilon > 0$ one has $\overline{G}(x)e^{\varepsilon x} \to \infty$ as $x \to \infty$.*

iii) *Let $G_1 \in \mathcal{S}$ and G_2 be an arbitrary distribution function. For some $c \in [0, \infty)$ suppose $\overline{G_2}(x)/\overline{G_1}(x) \to c$. Then $\overline{G_1 * G_2}(x)/\overline{G_1}(x) \to 1 + c$. If $c > 0$ then $G_2 \in \mathcal{S}$.*

iv) *Let $G \in \mathcal{S}$. Then $\overline{G^{*n}}(x) \sim n\overline{G}(x)$ for all $n \in \mathbb{N}$.*

v) *Let $G \in \mathcal{S}$. For any $\varepsilon > 0$ there is a constant $D = D(\varepsilon)$ such that for all $n \in N$ and all $x > 0$ we have $\overline{G^{*n}}(x) \le D(1+\varepsilon)^n\overline{G}(x)$.* □

Motivated by the result of [22] we conjecture that the ruin probability is of the form a constant times the integrated tail distribution of the claim sizes. In Veraverbeke's result the constant is continuous in c. This motivates the conditions in Proposition 3.1 below. But we first need the following

Lemma 3.2. *Assume that there exists an $\alpha > 0$ such that*

$$\lim_{u\to\infty} \int_0^u \psi_u(u-x)\alpha \mathrm{e}^{-\alpha x}\,\mathrm{d}x/\psi_u(u) = 1\,.$$

Then

$$\lim_{u\to\infty} \frac{\psi_u(u-x)}{\psi_u(u)} = 1\,.$$

Proof. It is no loss of generality to assume that $x \ge 0$. Then by Fatou's Lemma for any sequence $\{u_n\}$ converging to infinity

$$\begin{aligned} 1 &= \lim_{n\to\infty} \int_0^{u_n} \psi_u(u_n-x)\alpha \mathrm{e}^{-\alpha x}\,\mathrm{d}x/\psi_u(u_n) \\ &\ge \int_0^\infty \varliminf_{n\to\infty} \frac{\psi_u(u_n-x)}{\psi_u(u_n)}\alpha \mathrm{e}^{-\alpha x}\,\mathrm{d}x \\ &\ge \int_0^\infty \alpha \mathrm{e}^{-\alpha x}\,\mathrm{d}x = 1\,. \end{aligned}$$

Assume the assertion were wrong. Then there would be a sequence (u_n) and an $x_0 > 0$ such that

$$\varliminf_{n\to\infty} \frac{\psi_u(u_n-x_0)}{\psi_u(u_n)} \ge a > 1\,.$$

By the monotonicity of ψ_u the above must hold for all $x \ge x_0$. But then

$$\int_0^\infty \varliminf_{n\to\infty} \frac{\psi_u(u_n-x)}{\psi_u(u_n)}\alpha \mathrm{e}^{-\alpha x}\,\mathrm{d}x \ge 1 - \mathrm{e}^{-\alpha x_0} + a\mathrm{e}^{-\alpha x_0} > 1$$

which would be a contradiction. □

We now can state the following

Proposition 3.1. *Consider the perturbed risk model $\{X_t\}$ with premium density c fulfilling the net profit condition and its corresponding unperturbed model. Assume the ultimate ruin probabilities $\psi_u(u, c')$ of the unperturbed model with premium density c' fulfils the following two conditions:*

i) $\lim_{c'\to c}\lim_{u\to\infty} \frac{\psi_u(u,c')}{\psi_u(u,c)} = 1\,,$

ii) $\lim_{u\to\infty} \int_0^u \psi_u(u-x,c')\alpha e^{-\alpha x}\,dx/\psi_u(u,c') = 1$ *for any* $\alpha > 0$ *and all* c' *in a neighbourhood of* c.

Then

$$\lim_{u\to\infty} \frac{\psi(u)}{\psi_u(u)} = 1\,.$$

Proof. From Lemma 3.2 we conclude that

$$\lim_{u\to\infty} \frac{\psi_u(u-x,c')}{\psi_u(u,c')} = 1$$

for all $x \in \mathbb{R}$ and c'. This implies that

$$\lim_{u\to\infty} \frac{e^{-\alpha u}}{\psi_u(u,c')} = 0\,,$$

see [5, p.336]. Let

$$f(c') = \lim_{u\to\infty} \frac{\psi_u(u,c')}{\psi_u(u,c)}\,.$$

Choose an $\varepsilon > 0$ such that the net profit condition for the premium rate $c - \varepsilon$ is fulfilled. Let

$$M_p(c) = \sup_{t\ge 0} \sum_{i=1}^{N_t} Y_i - ct + \eta B_t$$

be the maximum of a perturbed risk model and $M_u(c)$ be the maximum of the corresponding unperturbed process. Let $Z(\varepsilon)$ be an independent exponentially distributed random variable with parameter ε/η^2. Note that $Z(\varepsilon)$ has the same distribution as the maximum of the process $(\eta B_t - \varepsilon t)$. Then

$$M_p(c) \le \sup_{t\ge 0}\Bigl(\sum_{i=1}^{N_t} Y_i - (c-\varepsilon)t\Bigr) + \sup_{t\ge 0}\,(\eta W_t - \varepsilon t) \overset{\mathrm{d}}{=} M_u(c-\varepsilon) + Z(\varepsilon)$$

where $\overset{\mathrm{d}}{=}$ means equal in distribution. Thus

$$\psi(u) \le \psi_u(u,c-\varepsilon) \int_0^u \frac{\psi_u(u-x,c-\varepsilon)}{\psi_u(u,c-\varepsilon)} \frac{\varepsilon}{\eta^2} e^{-\varepsilon x/\eta^2}\,dx + e^{-\varepsilon u/\eta^2}\,.$$

Dividing by $\psi_u(u,c)$ and letting $u \to \infty$ yields

$$\varlimsup_{u\to\infty} \frac{\psi(u)}{\psi_u(u,c)} \le f(c-\varepsilon)$$

and because ε was arbitrary

$$\varlimsup_{u\to\infty} \frac{\psi(u)}{\psi_u(u,c)} \le 1\,.$$

Analogously

$$M_p(c) \ge \sup_{t\ge 0}\Big(\sum_{i=1}^{N_t} Y_i - (c+\varepsilon)t\Big) - \sup_{t\ge 0}\,(-\eta W_t - \varepsilon t) \stackrel{\mathrm{d}}{=} M_u(c+\varepsilon) - Z(\varepsilon)\,.$$

This can be written as

$$\psi(u) \ge \psi_u(u,c+\varepsilon)\int_0^\infty \frac{\psi_u(u+x,c+\varepsilon)}{\psi_u(u,c+\varepsilon)}\,\frac{\varepsilon}{\eta^2}\mathrm{e}^{-\varepsilon x/\eta^2}\,\mathrm{d}x\,.$$

Analogously, using the bounded convergence theorem, it follows that

$$\varliminf_{u\to\infty} \frac{\psi(u)}{\psi_u(u,c)} \ge 1\,.$$

□

We conclude the discussion of subexponential claims with three examples of non-classical risk models that were discussed in [10] and [17]. The examples shall only illustrate how Proposition 3.1 can be applied. It is, of course, possible to treat a much larger class of risk processes. For results in the unperturbed case see for instance [1].

3.1.1 Perturbed Sparre Andersen model

Using Proposition 3.1 Veraverbeke's [22] result becomes easy to prove and can readily be extended to the Sparre Andersen model.

Corollary 3.1. *Let (X_t) be a perturbed Sparre Andersen model. Assume that the distribution G_0 is subexponential. Then under the net profit condition $c > \lambda\mu$*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{G}_0(u)} = \frac{\lambda\mu}{c - \lambda\mu} .$$

Proof. It is proved in [7] that the formula would be correct if $\psi(u)$ is replaced by $\psi_u(u)$. The first condition of Proposition 3.1 is easy to check and the second one follows from Lemma 3.1. Thus the assertion follows from Proposition 3.1. □

3.1.2 Perturbed Björk-Grandell model

Using Proposition 3.1 the main difficulty is to prove the result in the unperturbed case. We consider here only the case of subexponential claim sizes. The cases with subexponential intensity levels and negative safety loading for a subexponential time length follow similarly from the results in the unperturbed case discussed in [1] and [21].

Corollary 3.2. *Let (X_t) be a perturbed Björk-Grandell model. Assume that both G and G_0 are subexponential distributions and that there exists an $\varepsilon > 0$ such that $E[\exp\{\varepsilon\Lambda_i\sigma_i\}] < \infty$. Then under the net profit condition $cE[\sigma_i] > \mu E[\Lambda_i\sigma_i]$*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{G}_0(u)} = \frac{\mu E[\Lambda_i\sigma_i]}{cE[\sigma_i] - \mu E[\Lambda_i\sigma_i]} .$$

Proof. It follows from [1] that the assertion holds if $\psi(u)$ is replaced by $\psi_u(u)$. The assertion follows now as in the proof of Corollary 3.1. □

3.1.3 Perturbed Markov modulated risk model

Consider now the Markov modulated risk model. Let G be distribution function such that

$$\lim_{x\to\infty} \frac{\overline{G}_i(x)}{\overline{G}(x)} = \alpha_i \in [0,\infty)$$

for all $i \le J$.

Corollary 3.3. *Let*

$$\alpha = \sum_{i=1}^{J} \pi_i \lambda_i \alpha_i / (-\Lambda_{ii}) > 0$$

and assume the net profit condition $c > \sum_{i=1}^{J} \pi_i \lambda_i \mu_i$*. If* $G_I(x) \in \mathcal{S}$ *then, independently of the initial distribution of* Z_0*,*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{G}_0(u)} = \frac{\alpha\mu}{(c - \sum_{i=1}^{J} \lambda_i \mu_i) \sum_{i=1}^{J} \pi_i/(-\Lambda_{ii})}$$

where $\mu = \int_0^\infty \overline{G}(x)\,\mathrm{d}x$*.*

Proof. For the unperturbed case the result is proved in [1]. The assertion follows now as in the proof of Corollary 3.1. □

3.2 Perturbation by Lévy processes

Let us now assume that $\{B_t\}$ is a Lévy process. For the approach used in this section we have to assume that $\{B_t\}$ has no downward jumps. Moreover, we assume $\mathbb{E}[|B_t|] < \infty$ and that any drift of the perturbation process is included in the premium rate c, i.e. $\mathbb{E}[B_t] = 0$. Such a perturbation was considered by Furrer [8] and [9]. He let $\{B_t\}$ be an α-stable Lévy motion. A cadlag process $\{B_t\}$ is called a *(standard)* α*-stable Lévy motion* if

i) $B_0 = 0$,

ii) $\{B_t\}$ has independent increments,

iii) For $0 \le s < t$, $B_t - B_s$ has a stable distribution with parameters $\alpha \in (0, 2]$, $\sigma = (t-s)^{1/\alpha}$, $\beta \in [-1, 1]$, i.e. the characteristic function of $B_t - B_s$ is

$$\log \mathbb{E}[\mathrm{e}^{\mathrm{i}r(B_t - B_s)}] = \begin{cases} -\sigma^\alpha |r|^\alpha (1 - \mathrm{i}\beta \operatorname{sign} r \tan(\pi\alpha/2)), & \text{if } \alpha \neq 1, \\ -\sigma |r| (1 + \mathrm{i}\beta 2/\pi \operatorname{sign} r \ln |r|), & \text{if } \alpha = 1. \end{cases}$$

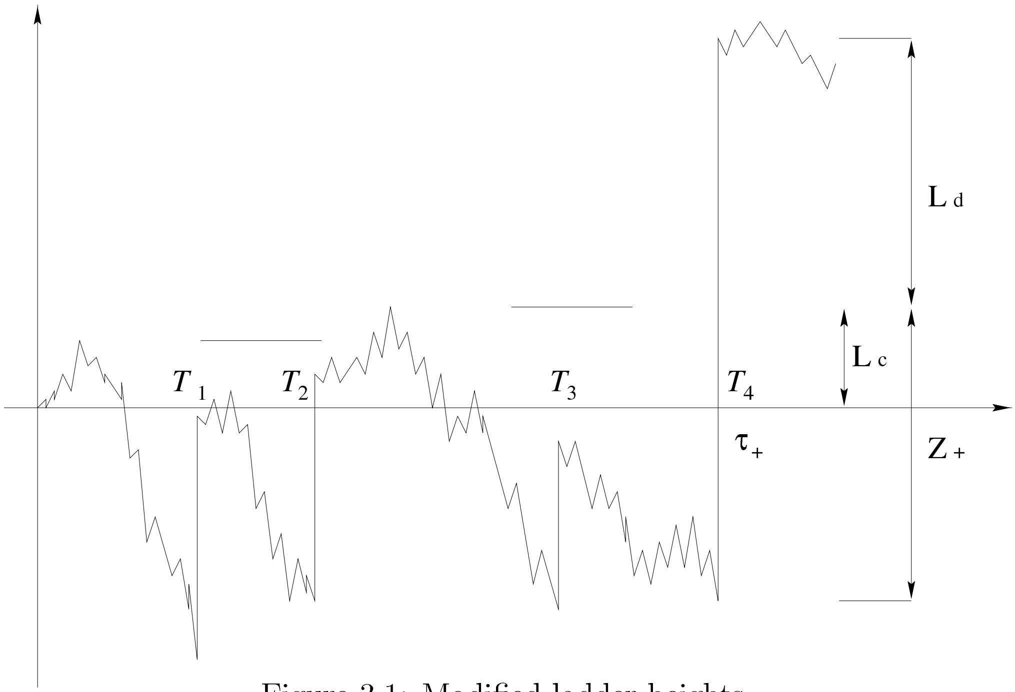

Figure 3.1: Modified ladder heights

If $\alpha = 2$ we get Brownian motion, as considered before. In this case β has no influence. If $\alpha < 2$ then we have to assume $\beta = 1$, otherwise the process would have downward jumps. Moreover, $\mathbb{E}[|B_t|] < \infty$ implies $\alpha > 1$. But note that in this section we allow Lévy processes much more general than α-stable Lévy motions, as for example mixtures of α-stable Lévy motions, compound Poisson processes and limits of compound Poisson processes.

As unperturbed risk model we consider a general stationary and ergodic risk model. This means that we have to start the renewal risk model, the Björk-Grandell model or the Markov modulated risk model in its stationary state. More specifically, let $\mathcal{M} = \{(T_i, Y_i, M_i)\}$ be a stationary and ergodic marked point process where T_i is the i-th claim occurrence time and the corresponding mark (Y_i, M_i) is the claim size and an environmental variable with values in some Polish space $(E, \mathcal{E})$. For example, in a Björk-Grandell model $M_i = (\lambda_{T_i}, V_{T_i})$, the intensity and the time left until the next change of the intensity, in a Markov modulated model, $M_i = Z_{T_i}$. In a Sparre Andersen model no environmental marks are needed, hence $E = \{0\}$.

The process $\{N_t\}$ in (1.5) is defined as $N_t = \sum_{i=1}^{\infty} \mathbb{1}_{0<T_i\le t}$. The claim intensity of the process is $\lambda = \mathbb{E}[N_1]$ and the mean of a typical claim is $\mu = \lambda^{-1}\mathbb{E}[\sum_{i=1}^{N_1} Y_i]$. We assume the net profit condition $\lambda\mu \le c$. Note that, in contrast to the models considered before, we allow here $\lambda\mu = c$, the case of no drift. As in Section 2.1 we define the modified ladder epoch τ_+ by (2.4), the ladder heights as

$$L_c = \sup\{X_t : 0 \le t < \tau_+\}, \qquad L_d = X_{\tau_+} - L_c, \tag{3.1}$$

the variable Z_+ and the claim leading to a new ladder height, and the environmental variable at the ladder time by

$$Z_+ = L_c - S_{(\tau_+)-}, \qquad U_+ = Z_+ + L_d, \qquad M_+ = M_{N_{\tau_+}}. \tag{3.2}$$

These variables are illustrated in Figure 3.1. The variables L_d, Z_+, U_+, M_+ are only well-defined on $\{\tau_+ < \infty\}$, whereas L_c is also defined on $\{\tau_+ = \infty\}$.

In order to formulate Proposition 3.2 below we need the Palm distribution $\mathbb{P}^0$ of $\mathcal{M}$. For an introduction of Palm measures see [15]. Intuitively, the Palm distribution can be seen as conditioning on the event that there is a claim at $t = 0$. In the special case of a Sparre Andersen model, $\mathbb{P}^0$ is the ordinary case, i.e. $F_1(x) = F(x)$. We find the following distributions.

Proposition 3.2. (Schmidli (1998)) *For $\ell_c, \ell_d, z \ge 0$ and $A \in \mathcal{E}$ we have*

$$\mathbb{P}[\tau_+ < \infty, L_c > \ell_c, L_d > \ell_d, Z_+ > z] = c^{-1}\lambda(1-H(\ell_c)) \int_{\ell_d+z}^{\infty} \mathbb{P}^0[U_0 > x, M_0 \in A]\,\mathrm{d}x$$

and

$$\mathbb{P}[\tau_+ = \infty, L_c > \ell_c] = (1 - c^{-1}\lambda\mu)(1 - H(\ell_c))\,,$$

where $H(x)$ is the distribution function of $\sup_{t\ge0} \eta B_t - ct$. $\square$

Remarks.

i) A first result of the above type was proved in [4]. Here $\mathbb{P}[\tau_+ < \infty, L_c > \ell_c, L_d > \ell_d]$ and $\mathbb{P}[\tau_+ = \infty, L_c > \ell_c]$ was found for a classical risk model perturbed by Brownian motion.

ii) The results remain true for the unperturbed case, i.e. if $\eta = 0$. In this case $H(x)$ is degenerate. This result was proved in [2].

iii) Note that the distribution of L_c does not depend on $\{\tau_+ < \infty\}$. Moreover, its distribution is independent on the point process $\mathcal{M}$ and depends only on the perturbation process and the premium density. The probability $\mathbb{P}[\tau_+ < \infty]$ is $\lambda\mu/c$ and depends only on some basic characteristics of the point process and not on the perturbation process. The conditional distribution of L_d, Z_+, U_+ and M_+ given $\tau_+ < \infty$ does not depend on perturbation process. The joint distribution is completely determined by the distribution of a typical mark.

iv) As it follows from the proof of the result in [20], it is important that the process starts in its stationary state, that the perturbation process is a Lévy process independent of the marked point process and that it has no downward jumps. One therefore cannot expect such a result to hold if one of the assumptions is violated. ■

Proposition 3.2 is in particular interesting if one considers a perturbed classical risk model. The following Pollaczek-Khinchin formulae hold.

Corollary 3.4. *Let $\mathcal{M}$ be a compound Poisson process. Then*

$$\begin{aligned}\psi(u) &= (1 - c^{-1}\lambda\mu)\sum_{n=0}^{\infty}\Big(\frac{\lambda\mu}{c}\Big)^n(1 - G_I^{*n} * H^{*(n+1)}(u))\,, \\ \psi_c(u) &= \sum_{n=0}^{\infty}\Big(\frac{\lambda\mu}{c}\Big)^n(G_I^{*n} * H^{*n}(u) - G_I^{*n} * H^{*(n+1)}(u))\,,\end{aligned}$$

$$\psi_d(u) \;=\; \sum_{n=1}^{\infty}\Big(\frac{\lambda\mu}{c}\Big)^n (G_I^{*(n-1)} * H^{*n}(u) - G_I^{*n} * H^{*n}(u))\,.$$

Proof. At the time τ_+ the process is in its stationary state, thus ladder times $\tau_+^{(2)}, \tau_+^{(3)}, \ldots$ can be defined. Thus the number of ladder epochs has a geometric distribution. The formulae follow readily. □

Let us now consider two examples.

3.2.1 Perturbation by Brownian motion

This example is from [4]. If $\{B_t\}$ is a Brownian motion then $\overline{H}(x) = \mathrm{e}^{-c/\eta^2}$. Taking Laplace transforms in Corollary 3.4 gives

$$\int_0^\infty \psi(u)\mathrm{e}^{-su}\,\mathrm{d}u \;=\; \frac{\lambda(\mu - s^{-1}(1-M_Y(-s))) + s\eta^2}{cs + s\eta^2 - \lambda(1-M_Y(-s))}\,, \tag{3.3}$$

$$\int_0^\infty \psi_c(u)\mathrm{e}^{-su}\,\mathrm{d}u \;=\; \frac{s\eta^2}{cs + s\eta^2 - \lambda(1-M_Y(-s))}\,, \tag{3.4}$$

$$\int_0^\infty \psi_d(u)\mathrm{e}^{-su}\,\mathrm{d}u \;=\; \frac{\lambda(\mu - s^{-1}(1-M_Y(-s)))}{cs + s\eta^2 - \lambda(1-M_Y(-s))}\,. \tag{3.5}$$

The above transformations yield explicit formulae if $M_Y(r)$ is a rational function i.e. the claim size distribution is of matrix exponential type, for instance phase type. In this case, the Laplace-transforms (3.3)–(3.5) are rational functions and can therefore be inverted explicitly. An example is discussed in [4].

3.2.2 Perturbation by α-stable Lévy motion

This example is from [8], see also [9]. Suppose now that $\{B(t)\}$ is an α-stable Lévy motion with $\alpha \in (1,2)$. Then $\overline{H}(x) = \sum_{n=0}^{\infty}(\Gamma(1+(\alpha-1)n))^{-1}(-a)^n x^{(\alpha-1)n}$ where $a = c\cos(\pi(1-\alpha/2))\eta^{-\alpha}$. Note that $\alpha = 2$ yields the exponential function considered in Section 3.2.1. We consider now the asymptotic behaviour of $\psi(u)$. First, the asymptotic behaviour of $\overline{H}(x)$ is $\overline{H}(x) \sim (a\Gamma(2-\alpha))^{-1}x^{-(\alpha-1)}$. We consider three cases for the asymptotic behaviour of $G(x)$.

Case $H(x) = o(\overline{G_I}(x))$ and $G_I(x) \in \mathcal{S}$. From Lemma 3.1 it follows that $G_I * H(x) \in \mathcal{S}$ and that $1 - G_I^{*n} * H^{*(n+1)}(u) \leq 1 - G_I^{*(n+1)} * H^{*(n+1)}(u) \leq D(1+\varepsilon)^{n+1}$ for some $0 < \varepsilon < c/(\lambda\mu)$ and some D. From Lemma 3.1, from Corollary 3.4 and from the bounded convergence theorem we find that

$$\psi(u) \sim \psi_d(u) \sim \frac{\lambda\mu}{c - \lambda\mu} \overline{G_I}(u)\,. \tag{3.6}$$

In particular, $\psi_c(u)/\overline{G_I}(u) \to 0$, i.e. for large initial capital it is unlikely that ruin is caused by the perturbation.

Case $\overline{G_I}(x) \sim \kappa x^{-(\alpha-1)}$. Note that the condition is equivalent to $\overline{G}(x) \sim \mu\kappa(\alpha - 1)x^{-\alpha}$. In this case $1 - G_I * H(x) \sim (\kappa + (a\Gamma(2-\alpha))^{-1})x^{-(\alpha-1)}$, implying that $G_I * H(x) \in \mathcal{S}$. As above the bounded convergence theorem can be applied. From Corollary 3.4 and Lemma 3.1 we find

$$\psi(u) \sim \frac{\kappa\lambda\mu a\Gamma(2-\alpha) + c}{(c-\lambda\mu)a\Gamma(2-\alpha)} u^{-(\alpha-1)}\,, \tag{3.7}$$

$$\psi_c(u) \sim \frac{c}{(c-\lambda\mu)a\Gamma(2-\alpha)} u^{-(\alpha-1)}\,, \tag{3.8}$$

$$\psi_d(u) \sim \frac{\kappa\lambda\mu}{c-\lambda\mu} u^{-(\alpha-1)}\,. \tag{3.9}$$

In this case ruin can be caused by the perturbation or by a claim.

Case $\overline{G_I}(x) = o(x^{-(\alpha-1)})$. Similarly as above we get

$$\psi(u) \sim \psi_c(u) \sim \frac{c}{c-\lambda\mu} \overline{H}(u) \sim \frac{c}{(c-\lambda\mu)a\Gamma(2-\alpha)} u^{-(\alpha-1)}\,. \tag{3.10}$$

In particular, $\psi_d(u)u^{\alpha-1} \to 0$, i.e. for large initial capital it is unlikely that ruin is caused by a claim.

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