THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Inhomogeneous Spatial Point Processes, with a view to Spatio-Temporal Modelling

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Research Report
No. 04 I March 2004

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This Thiele Research Report is also Research Report number 442 in the Stochastics Series at Department of Mathematical Sciences, University of Aarhus, Denmark.

Inhomogeneous spatial point processes, with a view to spatio-temporal modelling

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1 Introduction

In recent years, models for inhomogenous spatial point processes have been studied quite intensively, see [1, 2, 3, 4] and references therein. The majority of the inhomogeneous models has been constructed by introducing inhomogeneity into a homogeneous template point process X, defined on a bounded subset $\mathcal{X}$ of $\mathbb{R}^k$.

In most cases it is assumed that the template process is a homogeneous Markov point process with a density f_X with respect to the restriction of the unit rate Poisson point process Π to $\mathcal{X}$. Inhomogeneity may be introduced by using a non-constant first-order term in the density. Quite different approaches are inhomogeneity by independent inhomogeneous thinning and transformation of the template process. Inhomogeneity may also be constructed such that the resulting inhomogeneous process is a locally scaled version of the template process.

The inhomogeneous point processes mentioned above have mainly been studied in the case where the interaction between the points can be characterized as inhibition. The log Gaussian Cox processes constitute a very tractable alternative model class for clustered inhomogeneous point patterns, cf. [5] and [6]. Here, space-time modelling can be developed very elegantly, as demonstrated by an example of modelling a plant population. Spatio-temporal processes are also very important in the modelling of earthquakes. A popular model for clustered patterns in this field was suggested by Hawkes [7]. For a short review on spatio-temporal point processes in environmental statistics, see [8].

In the present paper we will give a short review of recent inhomogeneous purely spatial point processes, with a view to spatio-temporal modelling.

2 Inhomogeneous spatial point processes

In principle, any given homogeneous point process can be turned into an inhomogeneous point process by independent thinning with a survival probability $p(\eta)$

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that depends on the location $\eta \in \mathcal{X}$. As Baddeley et al. show [2], second order functions as Ripley's K-function can be defined for thinned point processes such that they coincide with the corresponding second order functions of the original process. However, thinning changes the interaction structure. Thus, if a very regular point process is subjected to inhomogeneous thinning, regions of low intensity seem to exhibit almost no interaction and look similar to a realization of a Poisson process.

Another method that is applicable on any process is to generate inhomogeneity by a nonlinear transformation of the spatial coordinates. Jensen and Nielsen [3] prove that the process resulting from transformation of a Markov point process is again Markov. Transformation does in general not preserve (local) isotropy of the template process.

Ogata and Tanemura [9] and Stoyan and Stoyan [1] suggest to introduce inhomogeneity into Markov or Gibbs models by location dependent first order interaction. As an example, consider a Strauss template X on $\mathcal{X}$ with parameters $\beta > 0$, $\gamma \in [0, 1]$ and $R > 0$, which is defined by a density

$$f_X(x) \propto \beta^{n(x)} \gamma^{s(x)}, \quad s(x) = \sum_{\eta \neq \xi \in x} \mathbf{1}(\|\eta - \xi\| \leq R), \tag{1}$$

with respect to the unit rate Poisson process on $\mathcal{X}$. The resulting inhomogeneous process has density

$$f_X(x) \propto \prod_{\eta \in x} \beta(\eta) \gamma^{s(x)}, \quad s(x) = \sum_{\eta \neq \xi \in x} \mathbf{1}(\|\eta - \xi\| \leq R) \tag{2}$$

with respect to the unit rate Poisson process. For such an inhomogeneous process, the degree of regularity in the resulting process depends on the intensity as in the case of thinning, described above.

A fourth approach that preserves locally the geometry of the template model, in particular the degree of regularity and also isotropy, was introduced in [4]. It can be applied to models that are specified by a density with respect to the unit rate Poisson process. The idea of the approach is that a location dependent scale factor $c(\eta) > 0$ changes the local specification of the model such that in a neighbourhood of any point $\eta \in \mathcal{X}$, the inhomogeneous process behaves like the template process scaled by the factor $c(\eta)$. This is achieved by defining the locally scaled process X_c by a density $f_{X_c}^{(c)}$ with respect to an inhomogeneous Poisson process of rate $c(\eta)^{-k}$. The density $f_{X_c}^{(c)}$ is obtained (up to a normalizing constant) from the template density f_X by replacing all d-dimensional volume measures ν^d that occur in the definition of f_X by their locally scaled counterparts ν_c^d, where $\nu_c^d(A) := \int_A c(u)^{-d} \nu^d(du)$ for all $A \in \mathcal{B}_k$.

A locally scaled version of the Strauss process has thereby the density

$$f_{X_c}^{(c)}(x) \propto \beta^{n(x)} \gamma^{s_c(x)}, \quad s_c(x) = \sum_{\eta \neq \xi \in x} \mathbf{1}(\nu_c^1([\eta, \xi]) \leq R), \tag{3}$$

where $\nu_c^1([\eta,\xi]) := \int_{[\eta,\xi]} c(u)^{-1}\nu^1(du)$ is the locally scaled length of the segment $[\eta,\xi]$. This modification applies to all Markov point processes where the higher order interaction is a function of pairwise distances. The resulting inhomogeneous point process is again Markov, now with respect to the neighbour relation

$$\eta \sim \xi \quad \Longleftrightarrow \quad \nu_c^1([\eta,\xi]) \leq R.$$

Since evaluation of the integral in the locally scaled length measure may be computationally expensive in the general case, the scaled distance of two points may be approximated by

$$\nu_c^1([\eta,\xi] \approx \frac{\|\eta-\xi\|}{(c(\eta)+c(\xi))/2}. \tag{4}$$

Using (4) in (3), and adjusting the first order term in (3), we get the density f_{X_c} of X_c with respect to the unit rate Poisson process as

$$f_{X_c}(x) \propto \beta^{n(x)}\gamma^{s_c(x)}\prod_{\eta\in x} c(\eta)^{-k}, \quad s_c(x) = \sum_{\eta\neq\xi\in x} \mathbf{1}(\|\eta-\xi\| \leq \tfrac{c(\eta)+c(\xi)}{2}R). \tag{5}$$

As shown in [4], if the scaling function is slowly varying compared to the interaction radius the local intensity in a point η of such a locally scaled process is in good approximation proportional to $c(\eta)^{-k}$. Figure 1 shows a realization of a locally scaled Strauss process.

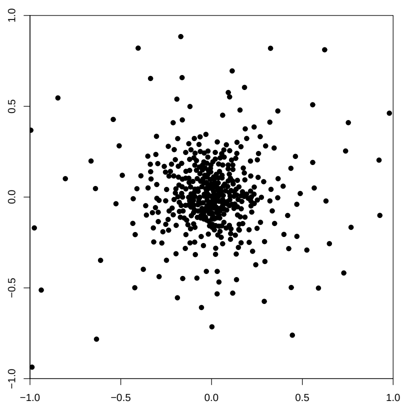

Figure 1: Result of a simulation from a locally scaled Strauss process on $[-1,1]^2$, with parameters $\beta = 100, \gamma = 0.01$, and $R = 0.1$, and scaling function was set to $c(\eta) = 2\|\eta\|^2 + 0.1$.

3 An example of an inhomogeneous space-time point process

Let $Z = \{(\xi, t)\}$ be a space-time point process on a bounded set $\mathcal{X} \times (0, T] \subset \mathbb{R}^k \times \mathbb{R}_+$ and define $Z_{<t} = \{(\eta, s) \in Z, s < t\}$. We let $c_1 : \mathbb{R}^k \to \mathbb{R}$ and $c_2 : \mathbb{R}_+$ be positive and bounded local scaling functions for space and time, respectively. Then a natural extention of the purely spatial locally scaled strauss process is the space-time process Z defined by the following density f_Z of Z with respect to a unit rate Poisson point process on $\mathcal{X} \times (0, T]$

$$f_Z(z) \propto \beta^{n(z)} \gamma^{s_c(z)} \prod_{(\xi,t)\in z} \frac{1}{c_1(\xi)^k c_2(t)},$$

where

$$s_c(z) = \sum_{(\xi,t)\in z} \sum_{(\eta,s)\in z_{<t}} \mathbf{1}[\| \eta - \xi \| \leq c_1(\eta) R].$$

Notice that the distance at which two points η and ξ are defined to be neighbours depends on the location via the scaling function c_1, however now only the scaling function at the "older" point is taken into account.

Figure 2 shows a simulation of the process Z on $[-1, 1]^2 \times (0, 12]$ with $\beta = 100$, $\gamma = 0.01$ and $R = 0.1$. The local scaling functions are defined by $c_1(\xi) = 0.2 + 4 \| \xi \|^2$ and $c_2(t) = 0.2 + 0.05t$. Notice the similarity with Figure 1.

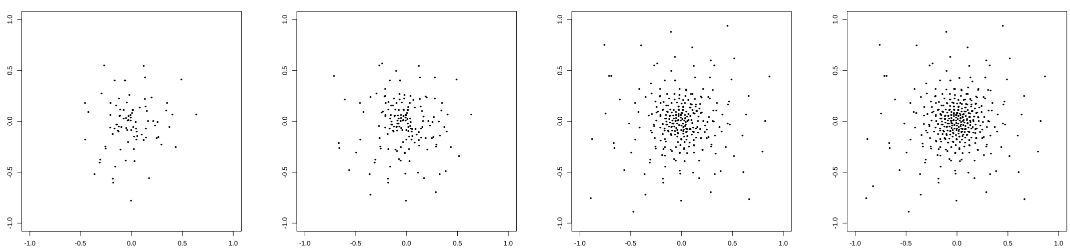

Figure 2: Result of a simulation from a space-time process on $[-1, 1]^2 \times (0, 12]$, with $\beta = 100, \gamma = 0.01$, $R = 0.1$, $c_1(\xi) = 0.2 + 4 \| \xi \|^2$ and $c_2(t) = 0.2 + 0.05t$. The figure shows the point pattern at times $t = 2, 4, 8, 12$.

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