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Abstract

We determine the variance-optimal hedge when the logarithm of the underlying price follows a process with stationary independent increments in discrete or continuous time. Although the general solution to this problem is known as backward recursion or backward stochastic differential equation, we show that for this class of processes the optimal endowment and strategy can be expressed more explicitly. The corresponding formulas involve the moment resp. cumulant generating function of the underlying process and a Laplace- or Fourier-type representation of the contingent claim. An example illustrates that our formulas are fast and easy to evaluate numerically.

1 Introduction

A basic problem in mathematical finance is how an option writer can hedge her risk by trading only in the underlying. This question is well understood in frictionless complete markets. It suffices to buy the replicating portfolio in order to completely offset the risk. This elegant approach works well in the standard Black-Scholes or Cox-Ross-Rubinstein setup, but not much beyond.

On the other hand, it has often been reported that real market data exhibits heavy tails and volatility clustering. Two common ways to account for such phenomena are some sort of stochastic volatility or jump processes or a combination of both. In this paper, we adopt the second approach and assume that the logarithmic stock price follows a general process with stationary, independent increments, either in discrete or continuous time. Processes

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of this type play by now an important role in the modelling of financial data (cf. Madan & Seneta (1990), Eberlein & Keller (1995), Eberlein et al. (1998), Barndorff-Nielsen (1998)).

Since replicating portfolios typically do not exist in such incomplete markets, one has to choose alternative criteria for reasonable hedging strategies. If you want to be as safe as in the complete case, you should invest in a *superhedging* strategy (cf. e.g. El Karoui & Quenez (1995)). In this case you may "suffer" profits but no losses at maturity of the derivative, which is very agreeable. On the other hand, even for simple European call options only trivial superhedging strategies exist in a number of reasonable market models ("buy the stock", cf. Eberlein & Jacod (1997), Frey & Sin (1999), Cvitanić et al. (1999)).

Alternatively, you may maximize some expected utility among all portfolios that differ only in the underlying and have a fixed position in the contingent claim. Variations of this approach have been investigated by Föllmer & Leukert (2000), Kallsen (1998, 1999), Cvitanić et al. (2001), Delbaen et al. (2002).

In this paper, we follow a third popular suggestion, namely to minimize some form of quadratic risk (cf. Föllmer & Sondermann (1986), Duffie & Richardson (1991), Schweizer (1994), and Schweizer (2001) for an overview). This can be interpreted as a special case of the second approach if we allow for quadratic utility functions.

Quadratic hedging comes about in two different flavours: *local risk-minimization* as in Föllmer & Schweizer (1989), Schweizer (1991) and *global risk-minimization* (i.e. *variance-optimal hedging*, *mean-variance hedging*) as in Duffie & Richardson (1991), Schweizer (1994). Roughly speaking, one may say that locally optimal strategies are relatively easily to compute but hard to interpret economically whereas the opposite is true for the globally optimal hedge. This paper focuses on the second problem but as a by-product, we also obtain the locally optimal Föllmer-Schweizer hedge. In discounted terms, the global problem can be stated as follows: If H denotes the payoff of the option and S the underlying's price process, try to minimize the squared L^2-distance

$$E\left((c + G_T(\vartheta) - H)^2\right) \tag{1.1}$$

over all initial endowments $c \in \mathbb{R}$ and all in some sense admissible trading strategies ϑ. Here, $G_T(\vartheta) = \int_0^T \vartheta_t dS_t$ (resp. $G_T(\vartheta) = \sum_{n=1}^T \vartheta_n \Delta S_n$ in discrete time) denotes the cumulative gains from trade up to time T. The idea is obviously to approximate the claim as closely as possible in an L^2 sense. Even though one may argue that one should not punish gains, the clarity and simplicity of this criterion is certainly appealing. Since it is harder to explain, we do not discuss local risk-minimization here, but refer instead to Schweizer (2001).

By way of duality, quadratic optimization problems are related to (generally signed) martingale measures, namely the *Föllmer-Schweizer* or *minimal martingale measure* for local and the *variance-optimal martingale measure* for global optimization. A similar duality has been established and exploited in many recent papers on related problems of utility maximization or portfolio optimization (cf. Foldes (1990, 1992), He and Pearson (1991a,b), Karatzas et al. (1991), Cvitanić & Karatzas (1992), Pliska (1997), Kramkov & Schachermayer (1999), Cvitanić et al. (2001), Schachermayer (2001), Kallsen (2000), Goll and Kallsen (2000, 2003)). Roughly speaking, the minimal martingale measure is the martingale measure whose density can be written as $1 + \int_0^T \vartheta_t dM_t$ for some ϑ, where M denotes the martingale part in the Doob-Meyer decomposition of S. The integrand ϑ can be determined relatively easily in terms of the local behaviour of S, which may be given by a stochastic differential equation or by one-step transition probabilities in discrete time. By contrast, the variance-optimal martingale measure is characterized by a density of the form $c + \int_0^T \vartheta_t dS_t$ for some $c \in \mathbb{R}$ and some (generally different) integrand ϑ. Here, it is usually much harder to determine ϑ. This holds with one notable exception, namely if the so-called *mean-variance tradeoff (MVT)* process is deterministic, in which case both measures coincide. More specifically, the integrands ϑ above tally because the difference $\int_0^T \vartheta_t dS_t - \int_0^T \vartheta_t dM_t$ is a constant and can be moved to c. In this case of deterministic MVT, globally risk-minimizing hedging strategies can be computed from locally risk-minimizing ones. The setup in this paper is among the few models of practical importance where the condition of deterministic MVT naturally holds.

The process formed by conditional expectation of the option's payoff under the minimal resp. variance-optimal martingale measure can be interpreted as a derivative price process. In jump-type models one has to be careful at this point because these measures are generally signed and may lead to arbitrage. Therefore, we do not pursue this topic further although this "price process" is implicitly calculated in the paper.

Even in the case of deterministic MVT, the actual computation of variance-optimal hedging strategies involves the joint predictable covariation of the option's "price process" and the underlying stock. For general claims, it may not seem evident how to obtain this covariation. It can be computed quite easily if the payoff is of exponential type e^{zX_T}, where $X := \log(\frac{S}{S_0})$ denotes the process with stationary, independent increments driving the stock price S. The reason is that the "price process" for such exponential payoffs under the variance-optimal martingale measure is again the exponential of a process with stationary independent increments, which leads to handy formulas for the corresponding hedge. Since the optimality criterion in (1.1) is based on an L^2-distance, the resulting hedging strategy is linear in the option. This suggests to write an arbitrary claim as a linear combination of exponential payoffs. Put differently, we work with the inverse Laplace (or Fourier) transform of the option. This will be done in Section 2 for discrete-time and in Section 3 for continuous-time processes, respectively. One could go even one step further and generalize the results to arbitrary processes with independent increments for they still share the important property of deterministic MVT. However, we chose not to do so in order not to drown the arguments in technicalities and because this more general class plays a minor role in applications. Since the first version of this paper circulated, the idea to use Fourier or Laplace transforms with Lévy processes has been applied independently in the framework of option pricing by Carr & Madan (1999) as well as Raible (2000) and, very recently, in the context of quadratic hedging by Černý (2004).

Section 4 illustrates the application of the results. We compare the variance-optimal hedge of a European call in a pure-jump Lévy process model to the Black-Scholes hedge as a benchmark. Since the results in the subsequent sections rely heavily on bilateral Laplace transforms, the appendix contains a summary of important results in this context.

To keep the presentation and notation simple, we confine ourselves to one single underlying. Extensions to the multivariate case and to path-dependent claims will be provided elsewhere. For unexplained notation we refer the reader to standard textbooks on stochastic calculus as e.g. Protter (1992) or Jacod & Shiryaev (2003).

2 Discrete time

Let $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\in\{0,1,\dots,N\}}, P)$ denote a filtered probability space and $X = (X_n)_{n=0,1,\dots,N}$ a real-valued process with stationary, independent increments in the sense that

1. X is adapted to the filtration $(\mathscr{F}_n)_{n\in\{0,1,\dots,N\}}$,
2. $X_0 = 0$,
3. $\Delta X_n := X_n - X_{n-1}$ has the same distribution for $n = 1, \dots, N$,
4. ΔX_n is independent of $\mathscr{F}_{n-1}$ for $n = 1, \dots, N$.

We consider a non-dividend paying stock whose discounted price process S is of the form

$$S_n = S_0 \exp(X_n)$$

with some constant $S_0 > 0$. We assume that $E(S_1^2) = S_0^2 E(e^{2X_1}) < \infty$, which implies that the moment generating function $m : z \mapsto E(e^{zX_1})$ is defined at least for $z \in \mathbb{C}$ with $0 \leq \mathrm{Re}(z) \leq 2$. Moreover, we exclude the degenerate case that S is deterministic. Put differently, $\mathrm{Var}(e^{X_1}) = m(2) - m(1)^2$ does not vanish.

Our goal is to determine the variance-optimal hedge for a European-style contingent claim on the stock expiring at N with discounted payoff H. Mathematically, H denotes a square-integrable, $\mathscr{F}_N$-measurable random variable of the form $H = f(S_N)$ for some function $f : (0, \infty) \to \mathbb{R}$. More specifically, we assume that f is of the form

$$f(s) = \int s^z \Pi(dz) \tag{2.1}$$

for some finite complex measure Π on a strip $\{z \in \mathbb{C} : R' \leq \mathrm{Re}(z) \leq R\}$ where $R', R \in \mathbb{R}$ are chosen such that $E(e^{2R'X_1}) < \infty$ and $E(e^{2RX_1}) < \infty$. Typically we choose $R' = R$, i.e. Π is concentrated on the straight line $R + i\mathbb{R}$.

Remark. Loosely speaking, the option's payoff at maturity is written as a linear combination of powers of the underlying or exponentials of X. Put differently, its payoff function is a kind of inverse Mellin or Laplace transform of the measure Π. To be more specific, let us consider the case $R' = R$. Denote by ℓ the inverse Laplace transform of Π in the sense that $\ell(x) = \int_{R-i\infty}^{R+i\infty} e^{zx}\Pi(dz)$ for $x \in \mathbb{R}$. Then

$$H = f(S_N) = f(\exp(X_N + \log(S_0))) = \ell(X_N + \log(S_0)).$$

Up to a factor e^{Rx}, the function ℓ is just the characteristic function of a measure on the real line (namely the measure ν with $\Pi(B) = \nu(R+iB)$ for Borel sets $B \subset \mathbb{R}$). The reason to consider $R \neq 0$ is simply that ℓ cannot be written as the characteristic function of a finite measure for important claims as e.g. European calls.

The variance-optimal hedge minimizes the L^2-distance between the option's payoff and the terminal value of the hedging portfolio. To be more specific, define the set Θ of *admissible* strategies as the set of all predictable processes ϑ such that the *cumulative gains* $G_n(\vartheta) := \sum_{k=1}^n \vartheta_k \Delta S_k$ are square-integrable for $n = 1, \dots, N$. We call $\varphi \in \Theta$ *variance-optimal hedging strategy* and $V_0 \in \mathbb{R}$ *variance-optimal initial capital* if $c = V_0$ and $\vartheta = \varphi$ minimize the expected squared hedging error

$$E\left((c + G_N(\vartheta) - H)^2\right) \tag{2.2}$$

over all initial endowments $c \in \mathbb{R}$ and all admissible strategies $\vartheta \in \Theta$.

In our framework the variance-optimal hedge and its corresponding hedging error can be determined quite explicitly:

Theorem 2.1 *The variance-optimal initial capital V_0 and the variance-optimal hedging strategy φ are given by*

$$V_0 = H_0 \tag{2.3}$$

and the recursive expression

$$\varphi_n = \xi_n + \frac{\lambda}{S_{n-1}} \left(H_{n-1} - V_0 - G_{n-1}(\varphi)\right), \tag{2.4}$$

where the processes $(H_n), (\xi_n)$ and the constant λ are defined by

$$\begin{aligned} g(z) &:= \frac{m(z+1) - m(1)m(z)}{m(2) - m(1)^2}, \\ h(z) &:= m(z) - (m(1) - 1)g(z), \\ \lambda &:= \frac{m(1) - 1}{m(2) - 2m(1) + 1}, \\ H_n &:= \int S_n^z h(z)^{N-n} \Pi(dz), \\ \xi_n &:= \int S_{n-1}^{z-1} g(z) h(z)^{N-n} \Pi(dz). \end{aligned} \tag{2.5}$$

The optimal hedge V_0, φ is unique up to a null set.

Remark. One may also consider a similar problem where the initial endowment $c = V_0$ is fixed and the mean squared difference in (2.2) is minimized only over the strategies $\vartheta \in \Theta$. This *risk-minimizing hedging strategy* for given V_0 is determined as in Theorem 2.1 but now V_0 in (2.4) denotes the given initial capital instead of the solution to (2.3).

Theorem 2.2 *The variance of the hedging error $E((V_0 + G_N(\varphi) - H)^2)$ in Theorem 2.1 equals*

$$J_0 := \int\int J_0(y,z)\Pi(dy)\Pi(dz),$$

where

$$a(y,z) := h(y)h(z)\frac{m(2) - m(1)^2}{m(2) - 2m(1) + 1},$$

$$b(y,z) := m(y+z) - \big(m(2)m(y)m(z) - m(1)m(y+1)m(z) - m(1)m(y)m(z+1) + m(y+1)m(z+1)\big)\left(m(2) - m(1)^2\right)^{-1},$$

$$J_0(y,z) := \begin{cases} S_0^{y+z} b(y,z)\dfrac{a(y,z)^N - m(y+z)^N}{a(y,z) - m(y+z)} & \text{if } a(y,z) \neq m(y+z) \\ S_0^{y+z} b(y,z) N m(y+z)^{N-1} & \text{if } a(y,z) = m(y+z). \end{cases}$$

The proofs of Theorems 2.1 and 2.2 are to be found at the end of this subsection.

The basic example is of course the European call option $H = (S_N - K)^+$. Its integral representation (2.1) is provided by the following

Lemma 2.3 *Let $K > 0$. For arbitrary $R > 1, s > 0$, we have*

$$(s-K)^+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)}dz.$$

PROOF. For $\mathrm{Re}(z) > 1$ we have

$$\int_{-\infty}^{\infty} (e^x - K)^+ e^{-zx}dx = \frac{K^{1-z}}{z(z-1)}.$$

The assertion follows now from Theorem A.3. □

The representation of some other payoffs can be found in the appendix.

Remark.

1. Using $\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$ and substituting $z-1$ for z we can write the variance-optimal initial capital for the European call option as

$$V_0 = S_0\Psi^{(1)}\left(\log\left(\frac{S_0}{K}\right)\right) - K\Psi^{(0)}\left(\log\left(\frac{S_0}{K}\right)\right)$$

with

$$\Psi^{(j)}(x) := \frac{1}{2\pi i}\int_{R-j-i\infty}^{R-j+i\infty} h(z+j)^N \frac{e^{zx}}{z}dz.$$

This resembles the pricing formulas for European calls in the Black-Scholes and the Cox-Ross-Rubinstein model. But note that $\Psi^{(j)}(x)$ may not be a distribution function in general.

2. For the application of Lemma 2.3 we need slightly more than second moments of X_1 and hence S_N. This seems counter-intuitive because the payoff grows only linearly in S_N. It is in fact possible to derive the optimal hedge in the case where only second moments exist. The idea is to consider the difference of the call and the stock (cf. (A.3)). Since the stock itself corresponds to the unit mass $\Pi = \varepsilon_1$, one immediately obtains an integral representation (2.1) of the call in the strip $0 \leq \mathrm{Re}(z) \leq 1$.

The remainder of this subsection is devoted to the proofs of Theorems 2.1 and 2.2. As it has been noted by Schweizer (1995), the variance-optimal hedge can be obtained from the option's Föllmer-Schweizer decomposition if the so-called mean-variance tradeoff process of the option is deterministic. The latter is defined as

$$K_n := \sum_{k=1}^{n} \frac{(E(\Delta S_k | \mathscr{F}_{k-1}))^2}{\mathrm{Var}(\Delta S_k | \mathscr{F}_{k-1})} = \frac{(m(1)-1)^2}{m(2) - m(1)^2} n.$$

The Föllmer-Schweizer decomposition plays a key role in the determination of locally risk-minimizing strategies in the sense of Föllmer & Schweizer (1989), Schweizer (1991) and it is defined as follows.

Definition 2.4 Denote by $S = S_0 + M + A$ the Doob decomposition of S into a martingale M and a predictable process A. The sum $H = H_0 + \sum_{n=1}^{N} \xi_n \Delta S_n + L_N$ is called *Föllmer-Schweizer decomposition* of $H \in L^2(P)$ if H_0 is $\mathscr{F}_0$-measurable, $\xi \in \Theta$, and L is a square-integrable martingale with $L_0 = 0$ that is orthogonal to M (in the sense that LM is a martingale). We will use this notion as well if H, H_0, ξ, L are complex-valued.

In discrete time any square-integrable random variable admits such a decomposition, which can be found by a backward recursion (cf. Schweizer (1995), Proposition 2.6). However, since this method does not yield a closed-form solution in our framework, we do not use these results. Instead we proceed in two steps. Firstly, we determine the Föllmer-Schweizer decomposition for options whose payoff is of power type. Secondly, we consider claims which are linear combinations of such options in the sense of (2.1). Here, we rely on the linearity of the Föllmer-Schweizer decomposition in the claim.

Lemma 2.5 *Let $z \in \mathbb{C}$ with $S_1^z \in L^2(P)$. Then $H(z) = S_N^z$ admits a Föllmer-Schweizer decomposition $H(z) = H(z)_0 + \sum_{n=1}^{N} \xi(z)_n \Delta S_n + L(z)_N$, where*

$$H(z)_n = h(z)^{N-n} S_n^z,$$

$$\xi(z)_n = g(z) h(z)^{N-n} S_{n-1}^{z-1},$$

$$L(z)_n = H(z)_n - H(z)_0 - \sum_{k=1}^{n} \xi(z)_k \Delta S_k, \tag{2.6}$$

and $g(z), h(z)$ are defined in Theorem 2.1.

Proof. The statement could be derived from Proposition 2.6 and Lemma 2.7 of Schweizer (1995) but it is easier to prove it directly.

Since S_1^z is square-integrable, all the involved expressions are well defined. From (2.6) it follows that

$$\Delta L(z)_n = S_{n-1}^z h(z)^{N-n}\left(e^{z\Delta X_n} - h(z) - g(z)(e^{\Delta X_n}-1)\right). \tag{2.7}$$

Since

$$E\left(e^{z\Delta X_n} - h(z) - g(z)(e^{\Delta X_n}-1)\right) = m(z) - h(z) - g(z)(m(1)-1) = 0, \tag{2.8}$$

this implies that $E(\Delta L(z)_n|\mathscr{F}_{n-1}) = 0$ and hence $L(z)$ is a martingale.

The Doob decomposition $S = S_0 + M + A$ of S satisfies

$$\Delta A_n = E(\Delta S_n|\mathscr{F}_{n-1}) = S_{n-1}(m(1)-1) \tag{2.9}$$

and hence $\Delta M_n = S_{n-1}\left(e^{\Delta X_n} - m(1)\right)$. In view of (2.7) we obtain

$$\Delta M_n \Delta L(z)_n = S_{n-1}^{z+1} h(z)^{N-n}\left(e^{\Delta X_n} - m(1)\right)\left(e^{z\Delta X_n} - h(z) - g(z)(e^{\Delta X_n}-1)\right).$$

From

$$\begin{aligned} &E\left(e^{\Delta X_n}(e^{z\Delta X_n} - h(z) - g(z)(e^{\Delta X_n}-1))\right)\\ &\quad = m(z+1) - h(z)m(1) - g(z)m(2) + g(z)m(1)\\ &\quad = 0 \end{aligned}$$

and (2.8) it follows that $E(\Delta M_n \Delta L(z)_n|\mathscr{F}_{n-1}) = 0$ and hence $ML(z)$ is a martingale as well. □

Proposition 2.6 *Any contingent claim $H = f(S_N)$ as in the beginning of this subsection admits a Föllmer-Schweizer decomposition $H = H_0 + \sum_{n=1}^N \xi_n \Delta S_n + L_N$. Using the notation of the previous lemma, it is given by*

$$\begin{aligned} H_n &= \int H(z)_n \Pi(dz),\\ \xi_n &= \int \xi(z)_n \Pi(dz),\\ L_n &= \int L(z)_n \Pi(dz) = H_n - H_0 - \sum_{k=1}^n \xi_k \Delta S_k. \end{aligned}$$

Moreover, the processes $(H_n), (\xi_n), (L_n)$ are real-valued.

Proof. Firstly, note that $\int E(|\Delta L(z)_n|^2)|\Pi|(dz) < \infty$, where $|\Pi|$ denotes the total variation measure of Π in the sense of Rudin (1987), Section 6.1. From Fubini's theorem we conclude that

$$\begin{aligned} E(\Delta L_n 1_B) &= E\left(\int \Delta L(z)_n \Pi(dz) 1_B\right)\\ &= \int E(\Delta L(z)_n 1_B)\Pi(dz) = 0 \end{aligned}$$

for any $B \in \mathscr{F}_{n-1}$. Hence L is a martingale. Similarly, it is shown that ML is a martingale as well. The assertion concerning the decomposition follows from Lemma 2.5.

Since H and S_n are real-valued, we have

$$0 = (H_0 - \overline{H}_0) + \sum_{n=1}^{N} (\xi_n - \overline{\xi}_n)\Delta S_n + (L_N - \overline{L}_N),$$

which implies that $0 = \mathrm{Im}(H_0) + \sum_{n=1}^{N} \mathrm{Im}(\xi_n)\Delta S_n + \mathrm{Im}(L_N)$. Since the Föllmer-Schweizer decomposition of 0 is unique (cf. Monat & Stricker (1995), Theorem 3.4), we have that H_0, ξ_n, L_n are real-valued for $n = 1, \dots, N$. □

Finally, we apply the preceding results to determine the variance-optimal hedge.

Proof of Theorem 2.1. As it is observed by Schäl (1994), Proposition 5.5, the process S has deterministic mean-variance tradeoff. From Proposition 2.6 and Schweizer (1995), Theorem 4.4 it follows that the variance-optimal hedging strategy φ satisfies

$$\varphi_n = \xi_n + \lambda_n (H_{n-1} - H_0 - G_{n-1}(\varphi)),$$

with

$$\lambda_n := \frac{\Delta A_n}{E(\Delta S_n^2 | \mathscr{F}_{n-1})} = \frac{\lambda}{S_{n-1}}$$

(cf. (2.9)). Moreover, the variance-optimal initial capital equals V_0.

For the uniqueness statement suppose that $\widetilde{V}_0 \in \mathbb{R}$, $\widetilde{\varphi} \in \Theta$ lead to a variance-optimal hedge as well. Define $\widehat{V}_0 := \frac{1}{2}(V_0 + \widetilde{V}_0)$ and $\widehat{\varphi} := \frac{1}{2}(\varphi + \widetilde{\varphi}) \in \Theta$. It is easy to verify that

$$E\big((\widehat{V}_0 + G_N(\widehat{\varphi}) - H)^2\big) < \tfrac{1}{2} E\big((V_0 + G_N(\varphi) - H)^2\big) + \tfrac{1}{2} E\big((\widetilde{V}_0 + G_N(\widetilde{\varphi}) - H)^2\big)$$

if $V_0 + G_N(\varphi)$ and $\widetilde{V}_0 + G_N(\widetilde{\varphi})$ do not coincide almost surely. Hence

$$V_0 - \widetilde{V}_0 + G_N(\varphi - \widetilde{\varphi}) = 0.$$

In particular, $G_N(\varphi - \widetilde{\varphi})$ is $\mathscr{F}_{N-1}$-measurable. We obtain

$$\begin{aligned} 0 &= \mathrm{Var}(G_N(\varphi - \widetilde{\varphi}) | \mathscr{F}_{N-1}) \\ &= \mathrm{Var}((\varphi - \widetilde{\varphi})_N \Delta S_N | \mathscr{F}_{N-1}) \\ &= ((\varphi - \widetilde{\varphi})_N S_{N-1})^2 (m(2) - m(1)^2), \end{aligned}$$

which implies that $(\varphi - \widetilde{\varphi})_N = 0$ almost surely. By induction, we conclude that $(\varphi - \widetilde{\varphi})_n = 0$ for $n = N-1, \dots, 1$ and hence also $V_0 = \widetilde{V}_0$.

The remark following Theorem 2.1 follows from Schweizer (1995), Proposition 4.3. □

Proof of Theorem 2.2. According to Schweizer (1995), Theorem 4.4, the variance of the hedging error equals

$$\sum_{n=1}^{N} E\left((\Delta L_n)^2\right) \prod_{k=n+1}^{N} (1-\lambda_k \Delta A_k) \tag{2.10}$$

with $\lambda_k = \frac{\lambda}{S_{k-1}}$ and ΔA_k as in (2.9). Since $\Delta L_n = \int \Delta L(z)_n \Pi(dz)$, we have that

$$(\Delta L_n)^2 = \int\int \Delta L(y)_n \Delta L(z)_n \Pi(dy)\Pi(dz)$$

and hence

$$E\left((\Delta L_n)^2\right) = \int\int E\left(\Delta L(y)_n \Delta L(z)_n\right) \Pi(dy)\Pi(dz) \tag{2.11}$$

by Fubini's theorem. Equation (2.7) implies

$$\begin{aligned}\Delta L(y)_n \Delta L(z)_n = {} & S_{n-1}^{y+z} h(y)^{N-n} h(z)^{N-n} \left(e^{y\Delta X_n} - h(y) - g(y)(e^{\Delta X_n}-1)\right) \\ & \times \left(e^{z\Delta X_n} - h(z) - g(z)(e^{\Delta X_n}-1)\right).\end{aligned}$$

Since $E(S_{n-1}^{y+z}) = S_0^{y+z} m(y+z)^{n-1}$ etc., we have

$$E\left(\Delta L(y)_n \Delta L(z)_n\right) = S_0^{y+z} (h(y)h(z))^{N-n} m(y+z)^{n-1} b(y,z)$$

with

$$\begin{aligned}b(y,z) = {} & m(y+z) - m(y)(h(z)-g(z)) - m(y+1)g(z) \\ & - (h(y)-g(y))(m(z)-h(z)+g(z)-g(z)m(1)) \\ & - g(y)\big(m(z+1) - m(1)(h(z)-g(z)) - g(z)m(2)\big).\end{aligned}$$

This expression coincides actually with $b(y,z)$ in the statement of the theorem. Consequently, we have shown

$$\begin{aligned}& \sum_{n=1}^{N} E\left(\Delta L(y)_n \Delta L(z)_n\right) \prod_{k=n+1}^{N} (1-\lambda_k \Delta A_k) \\ & \quad = S_0^{y+z} b(y,z) a(y,z)^{N-1} \sum_{n=1}^{N} \left(\frac{m(y+z)}{a(y,z)}\right)^{n-1} \\ & \quad = S_0^{y+z} b(y,z) \frac{a(y,z)^N - m(y+z)^N}{a(y,z) - m(y+z)}\end{aligned}$$

unless the denominator vanishes in the last equation. In view of (2.10) and (2.11), we are done. □

Let us briefly discuss the structure of the variance-optimal hedge. The process ξ in the Föllmer-Schweizer decomposition coincides with the locally risk-minimizing strategy. The process $H_n = H_0 + \sum_{k=1}^n \xi_k \Delta S_k + L_n$ appearing on the right-hand side of the Föllmer-Schweizer decomposition may be interpreted as a "price process" of the option. However, since this process may generate arbitrage, one should be careful with this interpretation. But note that the difference between the locally and globally optimal hedging strategy in (2.4) is proportionate to the difference between this "option price" H_{n-1} and the investor's current wealth.

3 Continuous time

We turn now to the continuous-time counterpart of the previous section. Similarly as before, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ denotes a filtered probability space and $X = (X_t)_{t\in[0,T]}$ a real-valued process with stationary, independent increments (*PIIS, Lévy process*) in the sense that

1. X is adapted to the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ and has càdlàg paths,
2. $X_0 = 0$,
3. the distribution of $X_t - X_u$ depends only on $t - u$ for $0 \leq u \leq t \leq T$,
4. $X_t - X_u$ is independent of $\mathscr{F}_u$ for $0 \leq u \leq t \leq T$.

As in the discrete-time case, the distribution of the whole process X is determined by the law of X_1. The latter is an infinitely divisible distribution which can be expressed in terms of its Lévy-Khinchine representation. Alternatively, one may characterize it by its *cumulant generating function*, i.e. by the continuous mapping $\kappa : D \to \mathbb{C}$ with $E(e^{zX_t}) = e^{t\kappa(z)}$ for $z \in D := \{z \in \mathbb{C} : E(e^{\mathrm{Re}(z)X_1}) < \infty\}$ and $t \in \mathbb{R}_+$. For details on Lévy processes and unexplained notation we refer the reader to Protter (1992), Sato (1999), and Jacod & Shiryaev (2003).

The discounted price process S of the non-dividend paying stock under consideration is supposed to be of the form

$$S_t = S_0 \exp(X_t)$$

with some constant $S_0 > 0$. Again, we assume that $E(S_1^2) = S_0^2 E(e^{2X_1}) < \infty$, which means that $z \in D$ for any complex number z with $0 \leq \mathrm{Re}(z) \leq 2$. Moreover, we exclude the degenerate case that S is deterministic, i.e. we have $\kappa(2) - 2\kappa(1) \neq 0$.

As in Section 2 we consider an option with discounted payoff $H = f(S_T)$ where f is given in terms of a finite complex measure Π (cf. (2.1)). The choice of the set of *admissible trading strategies* is a delicate point in continuous time. Following Schweizer (1994), Section 1, we choose the set

$$\Theta := \left\{ \vartheta \in L(S) : \int_0^\cdot \vartheta_t dS_t \in \mathscr{H}^2 \right\},$$

which is well suited for quadratic optimization problems. Here, the space $\mathscr{H}^2$ of semimartingales is defined as follows:

Definition 3.1 For any real- or complex-valued special semimartingale Y with canonical decomposition $Y = Y_0 + N + B$, we define

$$\|Y\|_{\mathscr{H}^2} := \|Y_0\|_2 + \left\| \sqrt{[N, \overline{N}]_T} \right\|_2 + \|\mathrm{var}(B)_T\|_2,$$

where $\mathrm{var}(B)$ denotes the variation process of B and $\|\cdot\|_2$ the L^2-norm. By $\mathscr{H}^2$ we denote the set of all real- or complex-valued special semimartingales Y with $\|Y\|_{\mathscr{H}^2} < \infty$.

In our setup, this set can be expressed more easily as follows:

Lemma 3.2

$$\Theta = \left\{ \vartheta \textit{ predictable process: } E\left(\int_0^T |\vartheta_t|^2 S_{t-}^2 dt \right) < \infty \right\}$$

PROOF. From Lemma 3.6 below we conclude that $A_t = \kappa(1) \int_0^t S_{u-} du$ and

$$\langle M, M \rangle_t = (\kappa(2) - 2\kappa(1)) \int_0^t S_{u-}^2 du \tag{3.1}$$

for the canonical decomposition $S = S_0 + M + A$ of the special semimartingale S. Hence we have

$$A_t = \int_0^t \lambda_u d\langle M, M \rangle_u \tag{3.2}$$

with $\lambda_u := \frac{\lambda}{S_{u-}}$ and $\lambda := \frac{\kappa(1)}{\kappa(2) - 2\kappa(1)}$. Therefore, the mean-variance tradeoff process

$$K_t = \int_0^t \lambda_u^2 d\langle M, M \rangle_u = \frac{\kappa(1)^2}{\kappa(2) - 2\kappa(1)} t$$

in the sense of Schweizer (1994), Section 1 is deterministic and bounded. According to Schweizer (1994), Lemma 2, we have that $\vartheta \in \Theta$ holds if and only if ϑ is predictable and $E(\int_0^T |\vartheta_t|^2 d\langle M, M \rangle_t) < \infty$. Since

$$\int_0^T |\vartheta_t|^2 d\langle M, M \rangle_t = (\kappa(2) - 2\kappa(1)) \int_0^T |\vartheta_t|^2 S_{t-}^2 dt,$$

the assertion follows. □

If we denote by $G_t(\vartheta) := \int_0^t \vartheta_u dS_u$ the *cumulative gains process* of $\vartheta \in \Theta$, then the *variance-optimal initial capital* and *variance-optimal hedging strategy* can be defined as in the previous section (with T instead of N).

The following characterizations of the variance-optimal hedge and its expected squared error correspond to Theorems 2.1 and 2.2. They are proved at the end of this subsection.

Theorem 3.3 *The variance-optimal initial capital* V_0 *and the variance-optimal hedging strategy* φ *are given by*

$$V_0 = H_0$$

and the expression

$$\varphi_t = \xi_t + \frac{\lambda}{S_{t-}}(H_{t-} - V_0 - G_{t-}(\varphi)), \tag{3.3}$$

where the processes $(H_t), (\xi_t)$ *and the constant* λ *are defined by*

$$\begin{aligned}
\gamma(z) &:= \frac{\kappa(z+1) - \kappa(z) - \kappa(1)}{\kappa(2) - 2\kappa(1)}, \\
\eta(z) &:= \kappa(z) - \kappa(1)\gamma(z), \\
\lambda &:= \frac{\kappa(1)}{\kappa(2) - 2\kappa(1)}, \\
H_t &:= \int S_t^z e^{\eta(z)(T-t)} \Pi(dz), \\
\xi_t &:= \int S_{t-}^{z-1} \gamma(z) e^{\eta(z)(T-t)} \Pi(dz).
\end{aligned} \tag{3.4}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ *is unique up to some* $(P(d\omega) \otimes dt)$*-null set.*

The remark following Theorem 2.1 on *risk-minimizing hedging* for fixed initial endowment V_0 applies in continuous time as well.

Theorem 3.4 *The variance of the hedging error* $E((V_0 + G_T(\varphi) - H)^2)$ *in Theorem 3.3 equals*

$$J_0 := \int\int J_0(y,z)\Pi(dy)\Pi(dz),$$

where

$$\alpha(y,z) := \eta(y) + \eta(z) - \frac{\kappa(1)^2}{\kappa(2) - 2\kappa(1)},$$

$$\begin{aligned}
\beta(y,z) := &\ \kappa(y+z) - \kappa(y) - \kappa(z) \\
&- \frac{(\kappa(y+1) - \kappa(y) - \kappa(1))(\kappa(z+1) - \kappa(z) - \kappa(1))}{\kappa(2) - 2\kappa(1)},
\end{aligned}$$

$$J_0(y,z) := \begin{cases} S_0^{y+z}\beta(y,z)\dfrac{e^{\alpha(y,z)T} - e^{\kappa(y+z)T}}{\alpha(y,z) - \kappa(y+z)} & \text{if } \alpha(y,z) \neq \kappa(y+z), \\[2ex] S_0^{y+z}\beta(y,z)Te^{\kappa(y+z)T} & \text{if } \alpha(y,z) = \kappa(y+z). \end{cases}$$

Remark. If (μ, σ^2, ν) denotes the Lévy-Khinchine triplet of X (relative to the truncation function $x \mapsto x1_{\{|x|\leq 1\}}$), then we have

$$\kappa(z) = \mu z + \frac{\sigma^2}{2} z^2 + \int \left(e^{zx} - 1 - zx1_{\{|x|\leq 1\}}\right) \nu(dx)$$

for $z \in D$ (cf. Sato (1999), Theorem 25.17). In particular, we have $\kappa(z) = \mu z + \frac{\sigma^2}{2} z^2$ for Brownian motion. Note that

$$\Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \frac{e^{(x-\mu)z + \frac{\sigma^2}{2} z^2}}{z} dz$$

for any $R > 0$, where Φ denotes the cumulative distribution function of $N(0,1)$. Using the same decomposition and substitution as in the remark following Lemma 2.3, one easily shows that V_0 and φ in Theorem 3.3 coincide with the Black-Scholes price and the replicating strategy in the case of a European call H and Brownian motion X. This does not come at a surprise because perfect hedging is clearly variance-optimal.

The hedging strategy φ in Theorem 3.3 is given in *feedback* form, i.e. it is only known in terms of its own gains from trade up to time t. From a practical point of view, these gains are obviously known to the trader. However, they cannot be computed recursively as in the discrete-time case. Therefore, one may prefer an explicit expression for $G_t(\varphi)$ from a mathematical point of view. It is provided by the following

Theorem 3.5 *Suppose that $P(\Delta X_t = \log(1 + 1/\lambda)$ for some $t \in (0, T]) = 0$. Then the gains process of the variance-optimal hedging strategy φ in Theorem 3.3 is of the form*

$$G_t(\varphi) = \mathscr{E}(-\lambda \widetilde{X})_t \left(\int_0^t \frac{\xi_u S_{u-} - \lambda(H_{u-} - V_0)}{\mathscr{E}(-\lambda \widetilde{X})_{u-}} \, dY_u \right),$$

where the processes $\widetilde{X}, Y$ are defined as

$$\begin{aligned} \widetilde{X}_t &:= \mathscr{L}(S)_t := \int_0^t \frac{1}{S_{u-}} \, dS_u, \\ Y_t &:= \widetilde{X}_t + \int_0^t \frac{\lambda}{1 - \lambda \Delta \widetilde{X}_u} \, d[\widetilde{X}, \widetilde{X}]_u. \end{aligned} \tag{3.5}$$

Remark. The condition on X is equivalent to assuming that the Lévy measure of X puts no mass on $\log(1 + 1/\lambda)$. This holds for any model of practical importance. Moreover, observe that $\widetilde{X}, Y$ are both Lévy processes (cf. Kallsen & Shiryaev (2002), Lemma 2.7 and straightforward calculations). Recall that the stochastic exponential $\mathscr{E}(U)$ of a real-valued Lévy process or any other semimartingale U can be written explicitly as

$$\mathscr{E}(U) = \exp\Big(U_t - \frac{1}{2}[U, U]_t\Big) \prod_{u \leq t} (1 + \Delta U_u) \exp\Big(-\Delta U_u + \frac{1}{2}(\Delta U_u)^2\Big)$$

(cf. Protter (1992), Theorem II.36).

The remainder of this section is devoted to the proof of Theorems 3.3–3.5. The approach parallels the one in the previous section. As before, we determine the Föllmer-Schweizer decomposition of the claim and apply results that relate this decomposition to the variance-optimal hedge.

Lemma 3.6 *Let $z \in \mathbb{C}$ with $S^z_T \in L^2(P)$. Then S^z is a special semimartingale whose canonical decomposition $S^z_t = S^z_0 + M(z)_t + A(z)_t$ satisfies*

$$A(z)_t = \kappa(z) \int_0^t S^z_{u-} du \tag{3.6}$$

and

$$\langle M(z), M \rangle_t = (\kappa(z+1) - \kappa(z) - \kappa(1)) \int_0^t S^{z+1}_{u-} du, \tag{3.7}$$

where $M = M(1)$ corresponds to $z = 1$ as in the proof of Lemma 3.2.

PROOF. Note that almost by definition of the cumulant generating function, $N(z)_t := e^{-\kappa(z)t} S^z_t$ is a martingale. Integration by parts yields $S^z_t = e^{\kappa(z)t} N(z)_t = S^z_0 + M(z)_t + A(z)_t$ with $M(z)_t = \int_0^t e^{\kappa(z)s} dN(z)_u$ and $A(z)$ as claimed. Moreover, we have

$$\begin{aligned}
[M(z), M]_t &= [S^z, S]_t \\
&= S^{z+1}_t - S^{z+1}_0 - \int_0^t S^z_{u-} dS_u - \int_0^t S_{u-} dS^z_u \\
&= M(z+1)_t - \int_0^t S^z_{u-} dM_u - \int_0^t S_{u-} dM(z)_u + (\kappa(z+1) - \kappa(z) - \kappa(1)) \int_0^t S^{z+1}_{u-} du.
\end{aligned}$$

Note that the first three terms on the right-hand side are local martingales. Since $\langle M(z), M \rangle$ is the predictable part of finite variation of the special semimartingale $[M(z), M]$, Equation (3.7) follows. □

Definition 3.7 Denote by $S = S_0 + M + A$ the canonical special semimartingale decomposition of S into a local martingale M and a predictable process of finite variation A. The sum $H = H_0 + \int_0^T \xi_t dS_t + L_T$ is called *Föllmer-Schweizer decomposition* of $H \in L^2(P)$ if H_0 is $\mathscr{F}_0$-measurable, $\xi \in \Theta$, and L is a square-integrable martingale with $L_0 = 0$ that is orthogonal to M (in the sense that LM is a local martingale). We will use this notion as well if H, H_0, ξ, L are complex-valued.

The existence of a Föllmer-Schweizer decomposition was established in Schweizer (1994), Theorem 15 in our case of bounded mean-variance tradeoff. It can be expressed in terms of a backward stochastic differential equation. Since the latter may be hard to solve, we do not use this result. Instead, we prove directly that the continuous-time limit of the expressions in Section 2 leads to a Föllmer-Schweizer decomposition.

Lemma 3.8 *Let $z \in \mathbb{C}$ with $S_T^z \in L^2(P)$. Then $H(z) = S_T^z$ admits a Föllmer-Schweizer decomposition $H(z) = H(z)_0 + \int_0^T \xi(z)_t dS_t + L(z)_T$, where*

$$\begin{aligned} H(z)_t &:= e^{\eta(z)(T-t)} S_t^z, \\ \xi(z)_t &:= \gamma(z) e^{\eta(z)(T-t)} S_{t-}^{z-1}, \\ L(z)_t &:= H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u, \end{aligned} \tag{3.8}$$

and $\gamma(z), \eta(z)$ are defined in Theorem 3.3. Moreover, M is a square-integrable martingale and hence $L(z)M$ is a martingale.

Proof.
Partial integration and (3.6) yield

$$H(z)_t = H(z)_0 + \int_0^t e^{\eta(z)(T-s)} dM(z)_u + (\kappa(z) - \eta(z)) \int_0^t e^{\eta(z)(T-s)} S_{u-}^z du$$

and

$$\int_0^t \xi(z)_u dS_u = \int_0^t \xi(z)_u dM_u + \kappa(1)\gamma(z) \int_0^t e^{\eta(z)(T-s)} S_{u-}^z du.$$

Since $\kappa(z) - \eta(z) - \kappa(1)\gamma(z) = 0$, the predictable part of finite variation in the special semimartingale decomposition of $L(z)$ vanishes and we have

$$L(z)_t = \int_0^t e^{\eta(z)(T-s)} dM(z)_u - \int_0^t \xi(z)_u dM_u, \tag{3.9}$$

which implies that $L(z)$ is a local martingale.
From (3.7) for z and 1 instead of z it follows that

$$\begin{aligned} \langle L(z), M\rangle_t &= \int_0^t e^{\eta(z)(T-s)} d\langle M(z), M\rangle_u - \int_0^t \xi(z)_u d\langle M, M\rangle_u \\ &= \Big(\kappa(z+1) - \kappa(z) - \kappa(1) - \gamma(z)(\kappa(2) - 2\kappa(1))\Big) \int_0^t e^{\eta(z)(T-s)} S_{u-}^{z+1} du \\ &= 0. \end{aligned}$$

Consequently, $L(z)M$ is a local martingale as well.
Similar calculations yield

$$\begin{aligned} \langle L(z), \overline{L(z)}\rangle_t &= \langle L(z), L(\overline{z})\rangle_t \\ &= \left(\kappa(2\mathrm{Re}(z)) - 2\mathrm{Re}(\kappa(z)) - \frac{|\kappa(z+1) - \kappa(z) - \kappa(1)|^2}{\kappa(2) - 2\kappa(1)}\right) \\ &\quad \times \int_0^t e^{2\mathrm{Re}(\eta(z))(T-s)} S_{u-}^{2\mathrm{Re}(z)} du \end{aligned} \tag{3.10}$$

and

$$\int_0^T |\xi(z)_t|^2 S_{t-}^2 dt = \left|\frac{\kappa(z+1)-\kappa(z)-\kappa(1)}{\kappa(2)-2\kappa(1)}\right|^2 \int_0^T e^{2\mathrm{Re}(\eta(z))(T-t)} S_{t-}^{2\mathrm{Re}(z)} dt. \tag{3.11}$$

Since

$$E(S_{t-}^{2\mathrm{Re}(z)}) = E(S_t^{2\mathrm{Re}(z)}) = S_0^{2\mathrm{Re}(z)} e^{t\kappa(2\mathrm{Re}(z))} \leq S_0^{2\mathrm{Re}(z)} \left(1 \vee e^{T\kappa(2\mathrm{Re}(z))}\right) < \infty, \tag{3.12}$$

it follows that $E(\langle L(z), \overline{L(z)}\rangle_T) < \infty$. Therefore L is a square-integrable martingale.

Similarly, (3.11) and (3.12) yield that $\xi \in \Theta$. Equations (3.7) and (3.12) for 1 instead of z imply that M is a square-integrable martingale. □

Lemma 3.9 *There exist constants $c_1, \ldots, c_5 \geq 0$ such that*

$$\mathrm{Re}(\eta(z)) \leq c_1 \tag{3.13}$$

$$0 \leq \kappa(2\mathrm{Re}(z)) - 2\mathrm{Re}(\kappa(z)) - \frac{|\kappa(z+1)-\kappa(z)-\kappa(1)|^2}{\kappa(2)-2\kappa(1)} \leq -c_2\mathrm{Re}(\eta(z)) + c_3 \tag{3.14}$$

$$|\gamma(z)|^2 \leq -c_4\mathrm{Re}(\eta(z)) + c_5$$

for any $z \in \mathbb{C}$ with $R' \leq \mathrm{Re}(z) \leq R$, where γ, η are defined as in Theorem 3.3.

Proof. Since κ is continuous, there is a constant $c_6 \geq 0$ such that

$$|\kappa(2\mathrm{Re}(z))| \leq 2c_6 \tag{3.15}$$

for any z with $R' \leq \mathrm{Re}(z) \leq R$. Since $\langle L(z), \overline{L(z)}\rangle$ is increasing, (3.10) yields

$$\kappa(2\mathrm{Re}(z)) - 2\mathrm{Re}(\kappa(z)) - \frac{|\kappa(z+1)-\kappa(z)-\kappa(1)|^2}{\kappa(2)-2\kappa(1)} \geq 0.$$

In particular

$$\mathrm{Re}(\kappa(z)) \leq \tfrac{1}{2}\kappa(2\mathrm{Re}(z)) \leq c_6$$

and

$$\frac{|\kappa(z+1)-\kappa(z)-\kappa(1)|^2}{\kappa(2)-2\kappa(1)} \leq 2c_6 - 2\mathrm{Re}(\kappa(z)), \tag{3.16}$$

which implies

$$|\kappa(1)\gamma(z)|^2 \leq c_7 - c_8\mathrm{Re}(\kappa(z)) \leq c_9^2 + \tfrac{1}{4}(\mathrm{Re}(\kappa(z)))^2 \leq \left(\left|\tfrac{1}{2}\mathrm{Re}(\kappa(z))\right| + c_9\right)^2$$

for some $c_7, c_8 \geq 0$ and $c_9 := \sqrt{c_7 + 4c_8^2}$. This yields

$$\begin{aligned}\mathrm{Re}(\eta(z)) &= \mathrm{Re}(\kappa(z)) - \mathrm{Re}(\kappa(1)\gamma(z)) \\ &\leq \mathrm{Re}(\kappa(z)) + |\kappa(1)\gamma(z)| \\ &\leq c_{10} + \tfrac{1}{2}\mathrm{Re}(\kappa(z)) \\ &\leq c_9 + 2c_6 =: c_1\end{aligned} \tag{3.17}$$

with $c_{10} := c_9 + \frac{3}{2}c_6$. Inequality (3.16) also yields

$$|\gamma(z)|^2 \leq c_{11} - \frac{c_4}{2}\mathrm{Re}(\kappa(z))$$

for some $c_{11}, c_4 \geq 0$, which, together with (3.17), leads to

$$|\gamma(z)|^2 \leq c_{11} - c_4(\mathrm{Re}(\eta(z)) - c_{10}) = c_5 - c_4\mathrm{Re}(\eta(z))$$

with $c_5 := c_{11} + c_4 c_{10}$. Finally, the second inequality in (3.14) follows from (3.15), (3.17), and $\kappa(2) - 2\kappa(1) \geq 0$. □

Proposition 3.10 *Any contingent claim $H = f(S_T)$ as in the beginning of this subsection admits a Föllmer-Schweizer decomposition $H = H_0 + \int_0^T \xi_t dS_t + L_T$. Using the notation of Lemma 3.8, it is given by*

$$H_t = \int H(z)_t \Pi(dz), \tag{3.18}$$

$$\xi_t = \int \xi(z)_t \Pi(dz),$$

$$L_t = \int L(z)_t \Pi(dz) = H_t - H_0 - \int_0^t \xi_u dS_u.$$

Moreover, the processes $(H_t), (\xi_t), (L_t)$ are real-valued.

Proof. Let $z \in \mathbb{C}$ with $R' \leq \mathrm{Re}(z) \leq R$. Since $|H(z)_t|^2 = e^{2\mathrm{Re}(\eta(z))(T-t)} S_t^{2\mathrm{Re}(z)}$, we have that $E(|H(z)_t|^2)$ is bounded by some constant which depends only on R, R' (cf. (3.12) and (3.13)). It follows that H_t is a well-defined square-integrable random variable. Similarly, (3.10), (3.12), and Lemma 3.9 yield after straightforward calculations that

$$E\left(|L(z)_t|^2\right) = E\left(\langle L(z), \overline{L(z)}\rangle_t\right) \leq E\left(\langle L(z), \overline{L(z)}\rangle_T\right)$$

is bounded as well by such a constant. Therefore, L_t is a well-defined square-integrable random variable as well. Finally, (3.11) and Lemma 3.9 yield that $E(|\xi(z)_t S_{t-}|^2)$ and also $E(\int_0^T |\xi(z)_u|^2 S_{u-}^2 du)$ are bounded by some constant which depends only on t, R, R'. Therefore ξ is well defined and $\xi \in \Theta$ by Lemma 3.2. The same Fubini-type argument as in discrete time shows that $E((L_t - L_u)1_B) = 0$ and $E((M_t L_t - M_u L_u)1_B) = 0$ for $u \leq t$, $B \in \mathscr{F}_u$ (cf. Proposition 2.6). Hence L is a square-integrable martingale which is orthogonal to M. To be precise, we interpret L as the up to indistinguishability unique modification whose paths are almost surely càdlàg (cf. Protter (1992), Corollary I.1). By Fubini's theorem for stochastic integrals (cf. Protter (1992), Theorem IV.46), we have

$$\int \int_0^t \xi(z)_u dS_u \Pi(dz) = \int_0^t \int \xi(z)_u \Pi(dz) dS_u = \int_0^t \xi_u dS_u.$$

Together with (3.18) and (3.8) it follows that H_0, ξ, L do indeed provide a Föllmer-Schweizer decomposition of H. As in the proof of Proposition 2.6, the uniqueness of the real-valued Föllmer-Schweizer decomposition yields that the processes (H_t), (ξ_t), (L_t) are real-valued. □

PROOF OF THEOREM 3.3. According to the proof of Lemma 3.2, the mean-variance tradeoff process of S in the sense of Schweizer (1995), Section 1 equals

$$K_t = \frac{\kappa(1)^2}{\kappa(2) - 2\kappa(1)} t = \int_0^t \frac{\lambda}{S_{u-}} dA_u.$$

In view of Proposition 3.10, the optimality follows from Theorem 3 and Corollary 10 of Schweizer (1994).

As in the proof of Theorem 2.1 it follows that $V_0 = \widetilde{V}_0$ and $G_T(\varphi) = G_T(\widetilde{\varphi})$ if $\widetilde{V}_0, \widetilde{\varphi}$ denote another variance-optimal hedge. Observe that the local martingale $N_t := -\int_0^t \lambda_u dM_u$ satisfies $\langle N, N\rangle_T = \int_0^T \lambda_u^2 d\langle M, M\rangle_u = K_T$ where λ_u is defined as in the proof of Lemma 3.2. From Choulli et al. (1998), Propositions 3.7, 3.9(ii) and the remark after Definition 5.4, it follows that $G(\varphi - \tilde{\varphi})$ is a $\mathscr{E}(N)$-martingale in the sense of that paper. By Proposition 3.12(i) in the same paper, it is determined by its terminal value $G_T(\varphi - \widetilde{\varphi}) = 0$, i.e. $G_t(\varphi - \widetilde{\varphi}) = 0$ for any $t \in [0, T]$. Hence

$$\begin{aligned} 0 &= E\left([G(\varphi - \widetilde{\varphi}), G(\varphi - \widetilde{\varphi})]_T\right) \\ &= E\left(\int_0^T (\varphi - \widetilde{\varphi})_t^2 d[S, S]_t\right) \\ &= E\left(\int_0^T (\varphi - \widetilde{\varphi})_t^2 d[M, M]_t\right) \\ &= E\left(\int_0^T (\varphi - \widetilde{\varphi})_t^2 d\langle M, M\rangle_t\right) \\ &= (\kappa(2) - 2\kappa(1)) E\left(\int_0^T \frac{(\varphi - \widetilde{\varphi})_t^2}{S_{t-}^2} dt\right). \end{aligned}$$

This implies that $\varphi_t(\omega) = \widetilde{\varphi}_t(\omega)$ outside some $(P(d\omega) \otimes dt)$-null set. □

PROOF OF THEOREM 3.4. Similarly as in Lemma 3.6, it is shown that

$$\langle M(y), M(z)\rangle_t = (\kappa(y + z) - \kappa(y) - \kappa(z)) \int_0^t S_{u-}^{y+z} du.$$

From (3.9), $\langle L(y), M\rangle = 0$, and (3.7) it follows that

$$\begin{aligned}\langle L(y), L(z)\rangle_t &= \int_0^t e^{(\eta(y)+\eta(z))(T-s)} d\langle M(y), M(z)\rangle_u \\ &\quad - \int_0^t \gamma(z) e^{(\eta(y)+\eta(z))(T-s)} S_{u-}^{z-1} d\langle M(y), M\rangle_u \\ &= \beta(y,z) \int_0^t e^{(\eta(y)+\eta(z))(T-s)} S_{u-}^{y+z} du. \end{aligned} \tag{3.19}$$

Consequently,

$$\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t = \beta(y,z) \int_0^T S_{t-}^{y+z} e^{\alpha(y,z)(T-t)} dt, \tag{3.20}$$

where K denotes the mean-variance tradeoff process as in the proof of Lemma 3.2. Since $E(S_{t-}^{y+z}) = S_0^{y+z} e^{\kappa(y+z)t}$, an application of Fubini's theorem yields

$$E\left(\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t\right) = S_0^{y+z} \beta(y,z) \int_0^T e^{\kappa(y+z)t+\alpha(y,z)(T-t)} dt,$$

which equals $J_0(y,z)$.

Observe that

$$\operatorname{Re}\langle L(y), L(z)\rangle = \frac{1}{2}\Big(\big\langle L(y)+L(\overline{z}), \overline{L(y)+L(\overline{z})}\,\big\rangle - \big\langle L(y), \overline{L(y)}\,\big\rangle - \big\langle L(z), \overline{L(z)}\,\big\rangle\Big)$$

and

$$\begin{aligned}&\big\langle L(y)+L(\overline{z}), \overline{L(y)+L(\overline{z})}\,\big\rangle \\ &\quad \le \big\langle L(y)+L(\overline{z}), \overline{L(y)+L(\overline{z})}\,\big\rangle + \big\langle L(y)-L(\overline{z}), \overline{L(y)-L(\overline{z})}\,\big\rangle \\ &\quad = 2\big\langle L(y), \overline{L(y)}\,\big\rangle + 2\big\langle L(z), \overline{L(z)}\,\big\rangle.\end{aligned}$$

In the proof of Proposition 3.10 we noted that $E(\langle L(z), \overline{L(z)}\rangle_T)$ and hence also the expected total variation of $\operatorname{Re}(\langle L(y), L(z)\rangle_t)$ is bounded by some constant which depends only on R, R'. By replacing $L(\overline{z})$ with $iL(\overline{z})$, it follows analogously that the total variation of $\operatorname{Im}(\langle L(y), L(z)\rangle_t)$ is bounded by a similar constant. Therefore

$$\int\int \langle L(y), L(z)\rangle_t \Pi(dy)\Pi(dz)$$

is a well-defined continuous, predictable, complex-valued process of finite variation. Since

$$L_t^2 = \int\int L(y)_t L(z)_t \Pi(dy)\Pi(dz),$$

an application of Fubini's theorem yields that

$$L_t^2 - \int\int \langle L(y), L(z)\rangle_t \Pi(dy)\Pi(dz)$$

is a martingale. This implies

$$\langle L, L\rangle_t = \int\int \langle L(y), L(z)\rangle_t \Pi(dy)\Pi(dz)$$

by definition of the predictable quadratic variation. Another application of Fubini's theorem yields

$$\int_0^T e^{-(K_T-K_t)} d\langle L, L\rangle_t = \int\int\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t \Pi(dy)\Pi(dz)$$

and hence

$$\begin{aligned} E\left(\int_0^T e^{-(K_T-K_t)} d\langle L, L\rangle_t\right) &= \int\int E\bigg(\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t\bigg)\Pi(dy)\Pi(dz) \\ &= \int\int J_0(y,z)\Pi(dy)\Pi(dz). \end{aligned}$$

By Schweizer (1994), Corollary 9, the left-hand side of the previous equation equals the variance of the hedging error. □

Finally, we prove the explicit representation of the gains process.

PROOF OF THEOREM 3.5. By (3.3), $G(\varphi)$ solves the stochastic differential equation

$$\begin{aligned} G_t(\varphi) &= \int_0^t \left(\xi_u + \frac{\lambda(H_{u-} - V_0)}{S_{u-}}\right) dS_u - \int_0^t \frac{\lambda}{S_{u-}} G_{u-}(\varphi) dS_u \\ &= \int_0^t (\xi_u S_{u-} + \lambda(H_{u-} - V_0)) d\widetilde{X}_u + \int_0^t G_{s-}(\varphi) d(-\lambda\widetilde{X})_u. \end{aligned}$$

By Jacod (1979), (6.8) this equation has a unique solution, which is given by

$$\begin{aligned} G_t(\varphi) &= \mathscr{E}(-\lambda\widetilde{X})_t \\ &\quad \times \left(\int_0^t \frac{\xi_u S_{u-} - \lambda(H_{u-} - V_0)}{\mathscr{E}(-\lambda\widetilde{X})_{u-}} d\widetilde{X}_u + \int_0^t \frac{\xi_u S_{u-} - \lambda(H_{u-} - V_0)}{\mathscr{E}(-\lambda\widetilde{X})_u} d[\widetilde{X}, \lambda\widetilde{X}]_u\right). \end{aligned}$$

Since $\mathscr{E}(-\lambda\widetilde{X})_u = (1 - \lambda\Delta\widetilde{X}_u)\mathscr{E}(-\lambda\widetilde{X})_{u-}$, the assertion follows. □

4 Examples with numerical illustrations

In this section we illustrate how the approach is applied to concrete models that are considered in the literature. As an example we provide numerical results for the normal inverse Gaussian model. The other setups lead to similar figures.

4.1 Discrete-time hedging in the Black-Scholes model

Suppose the underlying follows geometric Brownian motion with annual drift parameter μ and volatility σ. The continuously compounded riskless interest rate is denoted by r. If there are N trading days per year (e.g. $N = 252$), then the discounted daily log returns are normally distributed with mean $(\mu - r - \sigma^2/2)/N$ and variance σ^2/N.

Let us consider an option expiring in T trading days from now. If trading is restricted to times kT/n for $k = 0, 1, \ldots, N$, the market becomes incomplete. Theorem 2.1 applies with the moment generating function

$$m(z) = \exp\left(\left(\left(\mu - r - \frac{\sigma^2}{2}\right)z + \frac{\sigma^2 z^2}{2}\right)\frac{T}{Nn}\right). \tag{4.1}$$

If continuous trading is permitted, the Black-Scholes market is complete. Hence the hedging error is exactly zero. The variance-optimal capital and hedging strategy are given by the Black-Scholes price and delta hedging, respectively. It can be verified easily that this agrees in fact with the formulas in Theorems 3.3 and 3.4, where the relevant cumulant function is

$$\kappa(z) = \frac{1}{N}\left(\left(\mu - r - \frac{\sigma^2}{2}\right)z + \frac{\sigma^2 z^2}{2}\right).$$

4.2 Merton's jump-diffusion with normal jumps

In the jump-diffusion model considered by Merton (1976), the logarithmic stock price is modelled as a Brownian motion with drift μ and volatility σ plus occasional jumps from an independend compound Poisson process with intensity λ. A particularly simple and popular case is obtained when the jumps are normally distributed, say with mean β and variance δ.

$$m(z) = \exp\left(\left((\mu - r)z + \frac{\sigma^2 z^2}{2} + \lambda(e^{\beta z + \delta^2 z^2/2} - 1)\right)\frac{T}{Nn}\right)$$

and

$$\kappa(z) = \frac{1}{N}\left((\mu - r)z + \frac{\sigma^2 z^2}{2} + \lambda\left(e^{\beta z + \delta^2 z^2/2} - 1\right)\right),$$

respectively. Note that Merton uses a slightly different parameterization.

4.3 Hyperbolic, NIG, and VG models

The hyperbolic, normal inverse Gaussian (NIG), and the variance gamma (VG) Lévy processes are subfamilies or limiting cases of the class of generalized hyperbolic models, which all fit in the general framework of this paper. We refer to Eberlein & Raible (2001) for further details. For the choice of parameters $\alpha, \beta, \delta, \mu, \sigma, \nu, \vartheta$ below time is measured in days rather than years.

4.3.1 Hyperbolic model

The moment generating function in the hyperbolic case is of the form

$$m(z) = \left(\frac{\sqrt{\alpha^2 - \beta^2}}{\sqrt{\alpha^2 - (\beta + z)^2}} \frac{K_1\left(\delta\sqrt{\alpha^2 - (\beta + z)^2}\right)}{K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} e^{(\mu - \frac{r}{N})z} \right)^{\frac{T}{n}}, \tag{4.2}$$

where K_1 denotes the modified Bessel function of the third kind with index 1. Some care has to be taken if T/n is not an integer. The T/n-th power in (4.2) is in fact the distinguished T/n-th power (cf. Sato (1999), Section 7). The cumulant function equals

$$\kappa(z) = \mathrm{Ln}\left(\frac{\sqrt{\alpha^2 - \beta^2}}{\sqrt{\alpha^2 - (\beta + z)^2}} \frac{K_1\left(\delta\sqrt{\alpha^2 - (\beta + z)^2}\right)}{K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} e^{(\mu - \frac{r}{N})z} \right).$$

Here Ln denotes the distinguished logarithm, see Sato (1999), Section 7.

4.3.2 Normal inverse Gaussian model

The moment generating function of the normal inverse Gaussian model is given by

$$m(z) = \exp\left(\left(\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2} \right) + \left(\mu - \frac{r}{N} \right) z \right) \frac{T}{n} \right).$$

Consequently, the cumulant function equals

$$\kappa(z) = \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2} \right) + \left(\mu - \frac{r}{N} \right) z.$$

4.3.3 Variance gamma model

As final example let us consider the variance gamma model as described in Madan et al. (1998), based on the VG Lévy process with parameters σ, ν, ϑ plus a linear drift with rate μ. The discounted returns for intervals of length T/n have the moment generating function

$$m(z) = \left(e^{(\mu - \frac{r}{N})z} \left(\frac{1 - \nu\vartheta - \frac{1}{2}\nu\sigma^2}{1 - \nu\vartheta z - \frac{1}{2}\nu\sigma^2 z^2} \right)^{\frac{1}{\nu}} \right)^{\frac{T}{n}}.$$

The cumulant function needed for continuous-time hedging is given by

$$\kappa(z) = \left(\mu - \frac{r}{N} \right) z + \frac{1}{\nu} \ln\left(\frac{1 - \nu\vartheta - \frac{1}{2}\nu\sigma^2}{1 - \nu\vartheta z - \frac{1}{2}\nu\sigma^2 z^2} \right).$$

4.4 Numerical illustration

Figures 1–3 illustrate the results for a European call in the normal inverse Gaussian model, compared to Black-Scholes as a benchmark. The daily parameters of the normal inverse distribution, namely $\alpha = 75.49$, $\beta = -4.089$, $\delta = 0.012$, $\mu = 0$, were estimated by Rydberg (1997) for Deutsche Bank. The parameters for the benchmark Gaussian model are chosen such that both models lead to returns of the same mean and variance. The annual continuously compounded interest rate is set to 4%. We consider a European call option with strike price 100 maturing in three months from now. Figure 1 shows the variance-optimal initial capital as a function of the stock price in the NIG model for both continuous and weekly rebalancing of the hedging portfolio. The Black-Scholes price is plotted as well for comparison. One may observe that the three curves cannot be distinguished by eye, i.e. they do not differ much in absolute terms. A similar picture is obtained for the hedge ratio at time 0 as a function of the initial stock price (cf. Figure 2).

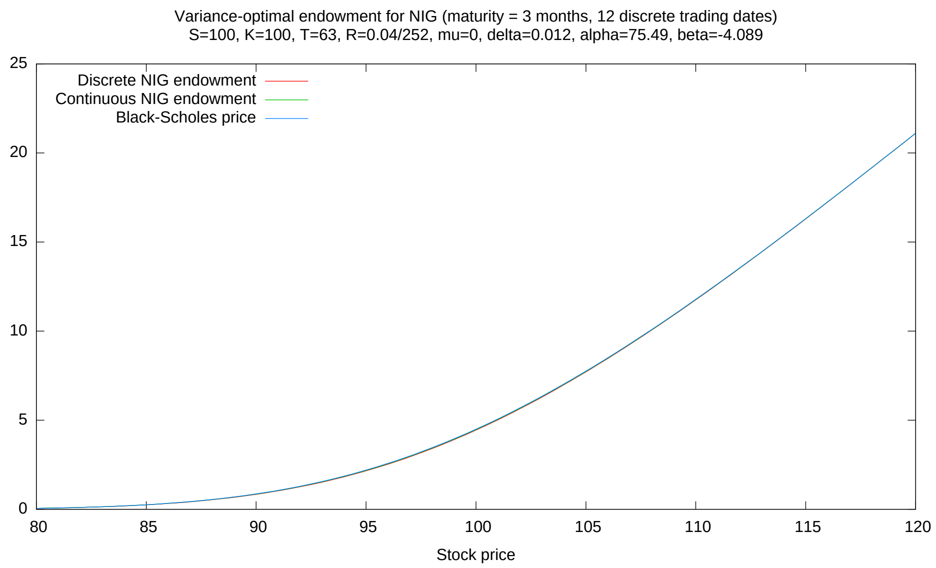

Figure 1: Variance-optimal initial capital for normal inverse Gaussian returns

The Black-Scholes delta provides a good proxy for the optimal hedge in the NIG model for both continuous and weekly rebalancing. As a result one may say that the Black-Scholes approach produces a reasonable hedge for the European call even if real data follows this rather different jump-type model. The similarity ceases to hold when it comes to the hedging error, which vanishes in a true Black-Scholes world. Figure 3 shows the variance of the hedging error as a function of the number of trades. E.g., weakly rebalancing of the hedging portfolio corresponds to 12 trades. The horizontal line in Figure 3 indicates the hedging error for continuous rebalancing in the NIG model. The two decreasing curves refer to the discrete hedging error in the NIG and the Gaussian case, respectively. In the latter case it converges to 0, which is the error in the limiting Black-Scholes model. As

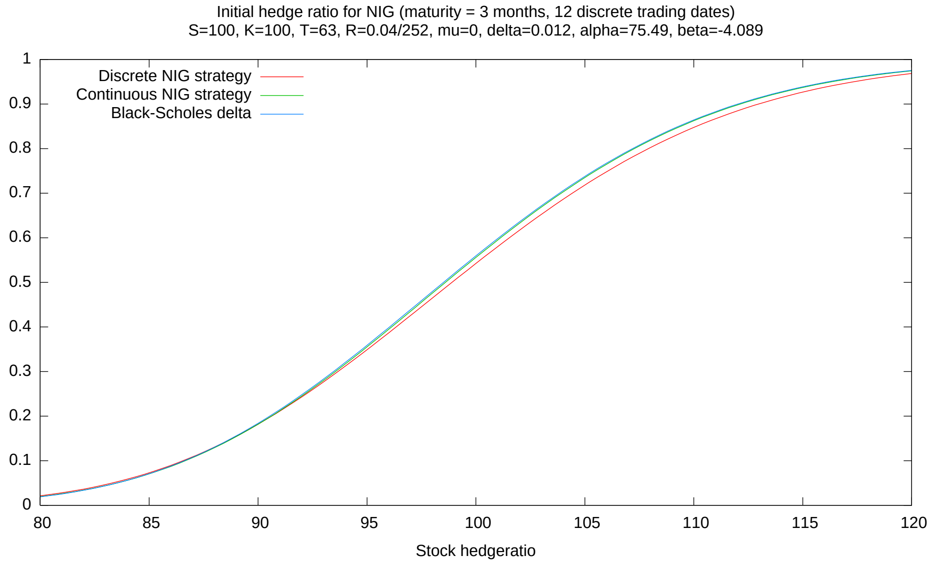

Figure 2: Variance-optimal initial hedge for normal inverse Gaussian returns

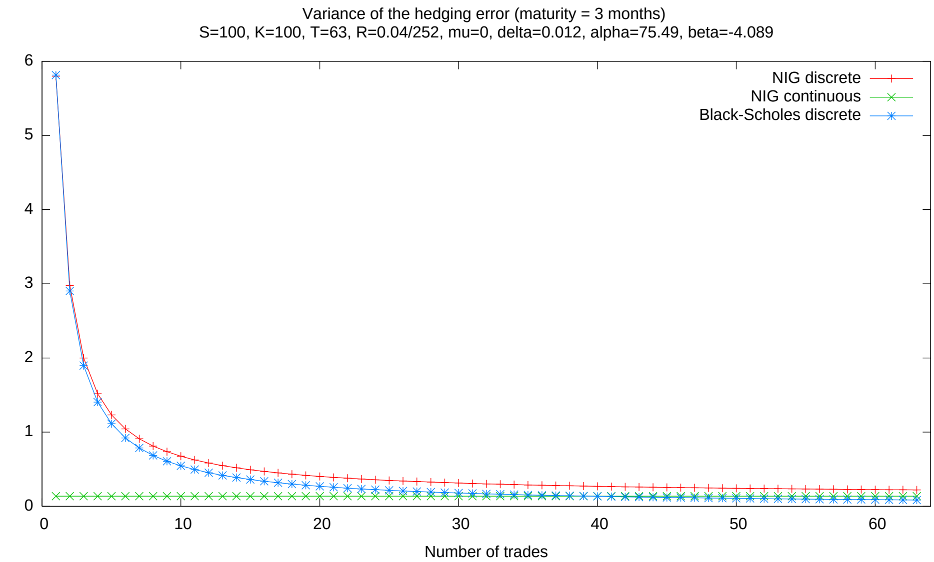

Figure 3: Variance of the hedging error for normal inverse Gaussian returns

far as the size is concerned, the variance of the error in the weekly rebalanced NIG setup ($0.584 = 0.76^2$) equals approximately the sum of the error in the corresponding Gaussian model (0.453, due to discrete rather than perfect hedging) and the inherent error in the continuous-time NIG model (0.137, due to incompleteness from jumps). The standard deviation 0.76 of the hedging error in the discrete NIG case may be compared to the Black-Scholes price 4.50 of the option.

A Bilateral Laplace transforms

Definition A.1 Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function. The *(bilateral) Laplace transform* $\widetilde{f}$ is given by

$$\widetilde{f}(z) = \int_{-\infty}^{+\infty} f(x)e^{-zx}dx \tag{A.1}$$

for any $z \in \mathbb{C}$ such that the integral exists.

The Laplace transform $\widetilde{f}$ is also denoted by $\mathscr{L}[f(x); z]$ or by $\mathscr{L}_{II}[f(x); z]$ when it is necessary to distinguish the bilateral from the usual (unilateral) Laplace transform. The latter is defined by the same integral, but starting from 0 instead of $-\infty$.

We say that the Laplace transform $\widetilde{f}(z)$ exists if the Laplace transform integral (A.1) converges absolutely, or, in other words, if it exists as a proper Lebesgue integral as opposed to an improper integral. The following lemma shows that the domain of a Laplace transform is always a vertical strip in the complex plain. It may be empty, degenerate to a vertical line, a closed or open left or right half-plane, or all of $\mathbb{C}$.

Lemma A.2 *Suppose that $\widetilde{f}(a)$ and $\widetilde{f}(b)$ exist for real numbers $a \leq b$. Then $\widetilde{f}(z)$ exists for any $z \in \mathbb{C}$ with $a \leq \mathrm{Re}(z) \leq b$.*

Proof. This is obvious because $|f(x)e^{-zx}| = |f(x)|e^{-\mathrm{Re}(z)x} \leq |f(x)e^{-ax}| + |f(x)e^{-bx}|$. □

From

$$\widetilde{f}(u + iv) = \int_{-\infty}^{+\infty} f(x)e^{-(u+iv)x}dx = \int_{-\infty}^{+\infty} e^{ux}f(-x)e^{ixv}dx \tag{A.2}$$

we see that $\mathscr{L}[f(x); u+iv] = \mathscr{F}[e^{ux}f(-x); v]$, where the last expression denotes the Fourier transform of the function $x \mapsto e^{ux}f(-x)$. Hence all properties of the bilateral Laplace transform can be reformulated in terms of the Fourier transform and vice versa.

There are many inversion formulas for the Laplace transform known in the literature. We will use the so-called Bromwich inversion integral, which can be justified by the following theorem.

Theorem A.3 *Suppose that the Laplace transform $\widetilde{f}(R)$ exists for $R \in \mathbb{R}$.*

1. *If $v \mapsto \widetilde{f}(R+iv)$ is integrable, then $x \mapsto f(x)$ is continuous and*

$$f(x) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \widetilde{f}(z) e^{zx} dz, \qquad \text{for } x \in \mathbb{R}.$$

2. *If f is of finite variation on any compact interval, then*

$$\lim_{\varepsilon \to 0} \frac{1}{2}(f(x+\varepsilon) + f(x-\varepsilon)) = \lim_{c \to \infty} \frac{1}{2\pi i} \int_{R-ic}^{R+ic} \widetilde{f}(z) e^{zx} dz, \qquad \text{for } x \in \mathbb{R}.$$

PROOF. The first statement follows from Rudin (1987), Theorem 9.11 and (A.2). For the second assertion cf. Doetsch (1971), Satz 4.4.1. □

Let us consider the Laplace transform representations of a number of simple payoff functions. They are mostly taken from Raible (2000) and they can be derived by straightforward calculations from Theorem A.3. Interestingly, the put option payoff is expressed by the same integral as the call, but with the vertical line of integration to the left of zero, i.e.

$$(K-s)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \qquad (R < 0).$$

A related example is the payoff

$$(s-K)^+ - s = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \qquad (0 < R < 1). \tag{A.3}$$

While this does not correspond to an option arising in practice, it can be used to compute the variance-optimal hedge for calls and puts in a situation when the moment or cumulant function of the underlying exists in $0 \leq \mathrm{Re}(z) \leq 2$, but in no larger strip. This is actually the natural minimal integrability requirement in the present setup.

The *power call* (cf. Reed (1995)) can be represented by

$$((s-K)^+)^2 = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{2K^{2-z}}{z(z-1)(z-2)} dz \qquad (R > 2),$$

which generalizes to higher integer powers as

$$((s-K)^+)^n = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{n! K^{n-z}}{z(z-1)\cdots(z-n)} dz \qquad (R > n),$$

and even to arbitrary powers $\alpha > 1$ by

$$((s-K)^+)^\alpha = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z K^{\alpha-z} B(\alpha+1, z-\alpha)dz \qquad (R > \alpha),$$

where B denotes the Euler beta function, which can be expressed by the more familiar Euler gamma function,

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The *self-quanto call* can be written as

$$(s-K)^+ s = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{(z-1)(z-2)}dz \qquad (R > 2).$$

The *digital option* with payoff function $f(s) = 1_{[K,\infty)}(s)$ coincides almost surely with the payoff function

$$f(s) = \tfrac{1}{2}1_{\{K\}}(s) + 1_{(K,\infty)}(s) \tag{A.4}$$

if the law of S_N resp. S_T has no atoms. Using Statement 2 in Theorem A.3, the latter can be expressed as

$$f(s) = \lim_{c\to\infty} \frac{1}{2\pi i}\int_{R-ic}^{R+ic} s^z \frac{K^{-z}}{z}dz \qquad (R > 0). \tag{A.5}$$

This suggests to apply the results of the previous sections to the measure

$$\Pi(dz) = \frac{1}{2\pi i}\frac{K^{-z}}{z}dz \tag{A.6}$$

in the case of the digital option. However, this measure is not of finite variation. Nevertheless, the main statements remain valid if we interpret the integrals as Cauchy principal value integrals.

Lemma A.4 *Theorems 2.1, 2.2, and 3.3–3.5 hold for the digital option (A.4) and the measure (A.6) if the integrals are interpreted in the principal value sense, i.e.*

$$H_n := P\text{-}\lim_{c\to\infty}\int_{R-ic}^{R+ic} S_n^z h(z)^{N-n}\Pi(dz), \tag{A.7}$$

$$\xi_n := P\text{-}\lim_{c\to\infty}\int_{R-ic}^{R+ic} S_{n-1}^{z-1} g(z) h(z)^{N-n}\Pi(dz), \tag{A.8}$$

$$J_0 := \lim_{c\to\infty}\int_{R-ic}^{R+ic}\int_{R-ic}^{R+ic} \mathrm{Re}(J_0(y,z))\Pi(dy)\Pi(dz) \tag{A.9}$$

etc., where P-lim refers to convergence in probability. In continuous time, the corresponding limit for $\xi_t(\omega)$ is to be interpreted in $(P(d\omega)\otimes dt)$-measure.

Proof. We will show the assertion in the continuous-time setting. The discrete-time case follows similarly.

Step 1: For $c \in \mathbb{R}_+$ define $H^{(c)} := f^c(S_T)$ with

$$f^c(s) := \int_{R-ic}^{R+ic} s^z \Pi(dz).$$

Since $\overline{\frac{K^{-z}}{2\pi i z}} = -\frac{K^{-\bar{z}}}{2\pi i \bar{z}}$, it follows that $H^{(c)}$ is real-valued. For $s \in \mathbb{R}_+$ we have

$$\begin{aligned} f(s) - f^c(s) &= \lim_{c' \to \infty} \frac{1}{2\pi i} \left(\int_{R+ic}^{R+ic'} \left(\frac{s}{K}\right)^z \frac{1}{z} dz + \int_{R-ic'}^{R-ic} \left(\frac{s}{K}\right)^z \frac{1}{z} dz \right) \\ &= \lim_{c' \to \infty} \frac{1}{\pi} \int_c^{c'} \operatorname{Re}\left(\frac{(s/K)^{R+ix}}{R+ix} \right) dx. \end{aligned}$$

The integrand equals

$$\left(\frac{s}{K}\right)^R \left(\frac{R \cos(x \log(\frac{s}{K}))}{R^2 + x^2} + \frac{R^2 \sin(x \log(\frac{s}{K}))}{(R^2 + x^2)x} - \frac{\sin(x \log(\frac{s}{K}))}{x} \right)$$

Since $\sup_{c \in \mathbb{R}_+} |\int_c^\infty \frac{\sin(x)}{x} dx| < \infty$ (cf. Abramowitz & Stegun (1968), Section 5.2), it follows that

$$\sup_{c \in \mathbb{R}_+} |f(s) - f^c(s)| \leq u s^R$$

for some $u \in \mathbb{R}_+$. Consequently, $(H^{(c)} - H)^2 \leq u^2 S_T^{2R} \in L^2$ for any $c \in \mathbb{R}_+$, which implies that $H^{(c)} \stackrel{c \to \infty}{\to} H$ in L^2 by dominated convergence.

Step 2: Denote by $H = \widetilde{H}_0 + \int_0^T \widetilde{\xi}_t dS_t + \widetilde{L}_T$ the Föllmer-Schweizer decomposition of H, which exists e.g. by Monat & Stricker (1995), Theorem 3.4. Moreover, let $H_t^{(c)}, \xi_t^{(c)}, L_t^{(c)}$ be defined as in Proposition 3.10 for the claim $H^{(c)}$. By Theorem 3.8 in Monat & Stricker (1995), we have $H_0^{(c)} \to \widetilde{H}_0$,

$$E\left(\int_0^T (\xi_t^{(c)} - \widetilde{\xi}_t)^2 d\langle M, M \rangle_t \right) \to 0, \tag{A.10}$$

and $E((L_T^{(c)} - \widetilde{L}_T)^2) \to 0$ for $c \to \infty$. Since $L^{(c)}, L$ are martingales, this implies $L_t^{(c)} \to \widetilde{L}_t$ in L^2 and hence in probability for any $t \in [0, T]$. Together with (3.2), we obtain

$$\int_0^t (\xi_u^{(c)} - \widetilde{\xi}_u) dM_u \to 0,$$
$$\int_0^t (\xi_u^{(c)} - \widetilde{\xi}_u) dA_u \to 0,$$

and hence

$$\int_0^t \xi_u^{(c)} dS_u \to \int_0^t \widetilde{\xi}_u dS_u$$

in probability for any $t \in [0,T]$. Moreover, we have $\xi^{(c)} \to \widetilde{\xi}$ in measure relative to $P(d\omega) \otimes dt$ (cf. (A.10) and (3.1)). Together, we obtain that $\widetilde{H}_0, \widetilde{\xi}_t, \widetilde{L}_t$ coincide with the expressions in Proposition 3.10 for H if the integrals are interpreted in the principal value sense. Theorems 3.3 and 3.5 now follow precisely as in Section 3.

Step 3: Denote by $J_0^{(c)}, \widetilde{J}_0$ the variance of the hedging error for $H^{(c)}$ and H, respectively. In a Hilbert space the mapping $x \mapsto \|x - P(x)\|^2$ is continuous if P denotes the projection on some closed subspace. Hence $J_0^{(c)} \to \widetilde{J}_0$ for $c \to \infty$. Since Theorem 3.4 is applicable to $H^{(c)}$, it follows that $\widetilde{J}_0$ concides with J_0 in (A.9). □

The *log contract* of Neuberger (1994) does not seem to fit into this framework as the logarithm has no Laplace transform. Nevertheless we can express it as a difference of two payoffs, namely its positive and negative part. The former has a Laplace transform for $\mathrm{Re}(z) > 0$, the latter for $\mathrm{Re}(z) < 0$ and we have

$$\log(s) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{1}{z^2} dz - \frac{1}{2\pi i}\int_{R'-i\infty}^{R'+i\infty} s^z \frac{1}{z^2} dz$$

with $R' < 0$ and $R > 0$.

Finally, let us emphasize again that the whole approach is linear in the claim. Hence, we immediately obtain the variance-optimal hedge for any linear combination of the payoffs above, as e.g. bull and bear spreads, collars, etc.

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