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Abstract

In this paper we offer a systematic survey and comparison of the Esscher martingale transform for linear processes, the Esscher martingale transform for exponential processes, and the minimal entropy martingale measure for exponential Lévy models and present some new results in order to give a complete characterization of those classes of measures. We illustrate the results with several concrete examples in detail.

Key words: Esscher transform, minimal entropy, martingale measures, Lévy processes.

1 Introduction

Lévy processes combine great flexibility with analytical tractability for financial modelling. Essential features of asset returns like heavy tails, aggregational Gaussianity, and discontinuous price movements are captured by simple exponential Lévy models, that are a natural generalization of the famous geometric Brownian motion. More realistic dependence structures, volatility clustering etc. are easily described by models based on Lévy processes.

Typically such models create incomplete markets; that means that there exist infinitely many martingale measures and equivalent to the physical measure describing the underlying price evolution. Each of them corresponds to a set of derivatives prices compatible with the no arbitrage requirement. Thus derivatives prices are not determined by no arbitrage, but depend on investors preferences. Consequently one approach to find the "correct" equivalent martingale measure, consists in trying to identify a utility function describing the investors preferences. It has been shown in many interesting cases, maximizing utility admits a dual formulation: to find an equivalent martingale measure minimizing some kind of distance to the physical probability measure given, [BF02].

For exponential utility the dual problem is the minimization of relative entropy [Fri00]. Therefore the minimal entropy martingale measures has attracted considerable interest both, in a general, abstract setting, but also for the concrete exponential Lévy models.

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Another popular choice for an equivalent martingale measure in the framework of exponential Lévy processes is based on the Esscher transform, see [GS94].

The Esscher transform approach has been used to study the minimal entropy martingale measure by [Cha99], [FM03], and [ES05]. It turned out, that this Esscher martingale measure is different from the Esscher martingale measure of [GS94], and there was some confusion in the literature.

In the paper [KS02], the authors introduce the *Esscher martingale measure for exponential processes* and the *Esscher martingale transform for linear processes* to distinguish the two kinds of Esscher transforms and clarify the issue.

In [ES05] the authors provide the main results on the minimal entropy martingale measure for exponential Lévy processes in rigorous way, the relation to the Esscher martingale transform for the linear processes, an explanation of the structure preservation property of the minimal entropy martingale measure, a generalization to the multivariate case, and an application to a particular stochastic volatility model.

In the present note we present in a detailed and systematic way both the Esscher martingale transform for the exponential and the linear processes in the simple and concrete setting of exponential Lévy models.

Then we provide the converse of some of the statements contained in [ES05], that allows a complete characterization of the minimal entropy martingale measure by the Esscher martingale transform for linear processes. We discuss in particular the case when the minimal entropy martingale measure does not exist, and illustrate that in this case the entropy has an infimum that is not attained. We think this could be relevant for counterexamples related to the dual problem of exponential utility maximization.

We also present applications of the theory developed to some specific parametric models, namely the normal inverse Gaussian Lévy process, the variance gamma Lévy process, and for illustrative purposes, a simple Poisson difference model, where all calculations can be performed in elementary and explicit way.

In Section 2 we will discuss the Esscher transform for Lévy processes, the exponential and logarithmic transforms, and both kinds of Esscher martingale measures.

In Section 3 the results about the minimal entropy martingale measure and the relation with the Esscher martingale transform for linear processes will be recalled, and some new results will be provided.

In Section 4 the examples are discussed in detail.

For the clarity of exposition and the continuity of the treatment we will postpone longer proofs to the appendix.

2 The Esscher transform

2.1 The Esscher transform for random variables

The Esscher transform is originally a transformation of distribution functions: Given a distribution function $F(x)$ and a parameter θ the Esscher transform $F^\theta(x)$ is defined by

$$dF^\theta(x) = \frac{e^{\theta x} dF(x)}{\displaystyle\int e^{\theta y} dF(y)}, \tag{2.1}$$

provided the integral exists. If $F(x)$ admits a density $f(x)$ then $F^\theta(x)$ has the density

$$f^\theta(x) = \frac{e^{\theta x} f(x)}{\displaystyle\int e^{\theta y} f(y) dy}. \tag{2.2}$$

The transformation is named in honor of the Swedish actuary Fredrik Esscher, who introduced it for a special case in [Ess32]. See [BE65, Section 13] for the early history and further references. In the statistical literature the transformation is known as *exponential tilting*.

The Esscher transform of probability measures is defined analogously: Given a probability space $(\Omega, \mathcal{F}, P)$, a random variable X, and a parameter θ the Esscher transform P^θ, sometimes also called *Esscher measure*, is defined by

$$dP^\theta = \frac{e^{\theta X} dP}{E[e^{\theta X}]}, \tag{2.3}$$

provided the expectation exists. This transformation depends on the parameter θ and the random variable X. It should be specified clearly which θ and X are used, when talking about *the* Esscher transform of P or *the* Esscher measure.

2.2 The Esscher transform for a Lévy process

The Esscher transform generalizes naturally to probability spaces carrying Lévy processes. In the following let '$\stackrel{\text{d}}{=}$' denote equality in distribution. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, $(\mathcal{F}_t)_{t\geq 0}$ a filtration, satisfying the usual conditions, and $(X_t)_{t\geq 0}$ is a Lévy process, in the sense that

1. X has independent increments, i.e., $X_{t_2} - X_{t_1}$ is independent of $\mathcal{F}_{t_1}$ for all $0 \leq t_1 \leq t_2$.
2. X has stationary increments, i.e, we have $X_{t_2} - X_{t_1} \stackrel{\text{d}}{=} X_{t_2 - t_1}$ for all $0 \leq t_1 \leq t_2$.
3. $X_0 = 0$ a.s.
4. $(X_t)_{t\geq 0}$ is stochastically continuous.
5. $(X_t)_{t\geq 0}$ has càdlàg paths.

We shall also speak of a Lévy process $(X_t)_{0\leq t\leq T}$, where $T > 0$ is a finite horizon, and the meaning of this terminology is apparent, cf. [KS91, Definition 1.1, p.47].

To fix notation, let us recall a few concepts and facts related to Lévy processes. There is a cumulant function $\kappa(z)$, that is defined at least for $z \in \mathbb{C}$ with $\Re z = 0$, such that

$$E[e^{zX_t}] = e^{\kappa(z)t}. \tag{2.4}$$

Let us fix a truncation function $h(x)$. This can be any function with compact support that satisfies $h(x) = x$ in a neighborhood of $x = 0$, for example $h(x) = xI_{|x|\leq 1}$, but sometimes other choices are possible and simpler. The Lévy-Kintchine formula asserts

$$\kappa(z) = bz + c\frac{z^2}{2} + \int (e^{zx} - 1 - h(x)z) U(dx), \tag{2.5}$$

where $b \in \mathbb{R}$, $c \geq 0$, and U a positive measure on $\mathbb{R} \setminus \{0\}$, called the Lévy measure. It satisfies

$$\int (1 \wedge x^2) U(dx) < \infty. \tag{2.6}$$

We call (b, c, U) the Lévy triplet of X. We note, that b depends on h, but not c and U. When $E[X_1^2] < \infty$ we may take $h(x) = x$. If the process X is of finite variation, which is equivalent to

$$\int_{|x| \leq 1} |x| U(dx) < \infty, \tag{2.7}$$

we can also use $h(x) = 0$.

Theorem 1. *Suppose $T > 0$ and $\theta \in \mathbb{R}$ such that*

$$E[e^{\theta X_1}] < \infty. \tag{2.8}$$

Then

$$\frac{dP^\theta}{dP} = e^{\theta X_T - \kappa(\theta) T} \tag{2.9}$$

defines a probability measure P^θ such that $P^\theta \sim P$ and $(X)_{0 \leq t \leq T}$ is a Lévy process under P^θ with triplet $(b^\theta, c^\theta, U^\theta)$ given by

$$b^\theta = b + \theta c + \int (e^{\theta x} - 1) h(x) U(dx), \tag{2.10}$$

$$c^\theta = c, \tag{2.11}$$

$$U^\theta(dx) = e^{\theta x} U(dx). \tag{2.12}$$

Proof: [Shi99, Theorem 2, Section VII.3c, p.685] □

Let us denote expectation with respect to P^θ by E^θ. We have $E^\theta[e^{zX_t}] = e^{\kappa^\theta(z)t}$ for $0 \leq t \leq T$, where

$$\kappa^\theta(z) = \kappa(z + \theta) - \kappa(\theta). \tag{2.13}$$

Remark 1. *Let us write $Q \overset{loc}{\sim} P$ if Q is a probability measure such that $Q|_{\mathcal{F}_T} \sim P|_{\mathcal{F}_T}$ for all $T \geq 0$. If we do not consider $T > 0$ as a fixed number in the previous theorem, but set*

$$\frac{dP_T^\theta}{dP} = e^{\theta X_T - \kappa(\theta) T} \tag{2.14}$$

for all $T \geq 0$, then $(P_T^\theta)_{T \geq 0}$ defines a consistent family of measures. With the usual, additional technical assumptions to apply the Kolmogoroff consistency theorem we can define a measure $P^\theta \overset{loc}{\sim} P$, such that $(X_t)_{t \geq 0}$ becomes a Lévy process with triplet $(b^\theta, c^\theta, U^\theta)$ as above.

The measure P^θ, if it exists, is called Esscher transform of P, or Esscher measure. Let us stress, that it depends on the Lévy process X and on the parameter θ. Again, it should be specified clearly which θ and X are used, when talking about *the* Esscher transform of P. In a more explicit notation we could write

$$P^\theta = P^{\theta \cdot X}. \tag{2.15}$$

The Esscher transform for Lévy processes and its application to option pricing was pioneered by [GS94]. The Esscher transforms for a Lévy process are also studied in statistics as an *exponential family of processes*, see [KL89].

2.3 The Esscher martingale transforms in option pricing

In the context of option pricing only one particular choice of the parameter θ is of interest: The one, such that the discounted asset price becomes a martingale under P^θ. To emphasize this aspect, that particular Esscher transform is called the Esscher *martingale* transform.

In the option pricing literature two variants have been used, corresponding to two different choices of the Lévy process X. Their close relation was clarified in [KS02]. In that paper the authors introduced the names Esscher martingale transform for *linear* processes and Esscher martingale transform for *exponential* processes to distinguish the two variants. Moreover they generalized both concepts to arbitrary semimartingales, the most general class of processes for (mainstream) continuous-time finance. We will discuss both transforms for Lévy processes in detail in the next two subsections. In Section 4 two concrete examples are worked out. At present we think of X as a Lévy process, but the following definitions and properties of the exponential and logarithmic transform apply to resp. hold true for arbitrary semimartingales starting at zero.

Suppose $S_0 > 0$ is a constant, and the process $(S_t)_{t\geq 0}$ defined by

$$S_t = S_0 e^{X_t} \tag{2.16}$$

is modelling the discounted price of a traded asset. By Itô's formula we obtain the stochastic differential equation

$$dS_t = S_{t-} d\tilde{X}_t, \tag{2.17}$$

where $(\tilde{X}_t)_{t\geq 0}$ is given by

$$\tilde{X}_t = \int_0^t S_{u-}^{-1} dS_u. \tag{2.18}$$

The process $\tilde{X}$ is called the *exponential transform* of X. Thus we can also write

$$S_t = S_0 \mathcal{E}(\tilde{X})_t. \tag{2.19}$$

We observe

$$\Delta\tilde{X}_t = e^{\Delta X_t} - 1 \tag{2.20}$$

and thus $\Delta\tilde{X} > -1$. Conversely, if $(\tilde{X}_t)_{t\geq 0}$ satisfies $\Delta\tilde{X} > -1$ then the process $(X_t)_{t\geq 0}$ defined by

$$X_t = \ln \mathcal{E}(\tilde{X})_t \tag{2.21}$$

is called the *logarithmic transform* of $\tilde{X}$. Clearly the exponential and logarithmic transform are inverse operations.

Theorem 2. *Suppose X is a Lévy process, then its exponential transform $\tilde{X}$ is a Lévy process with $\Delta\tilde{X} > -1$. Suppose conversely $\tilde{X}$ is a Lévy process with $\Delta\tilde{X} > -1$ then its logarithmic transform X is a Lévy process. The characteristic triplets (b, c, U) and $(\tilde{b}, \tilde{c}, \tilde{U})$ with respect to the truncation function h are related by*

$$\tilde{b} = b + \frac{1}{2}c + \int (h(e^x - 1) - h(x))U(dx) \tag{2.22}$$

$$\tilde{c} = c \tag{2.23}$$

$$\tilde{U}(dx) = (U \circ g^{-1})(dx), \tag{2.24}$$

resp.

$$b = \tilde{b} - \frac{1}{2}\tilde{c} - \int (h(x) - h(\ln(1+x)))\tilde{U}(dx) \tag{2.25}$$

$$c = \tilde{c} \tag{2.26}$$

$$U(dx) = (\tilde{U} \circ \tilde{g}^{-1})(dx), \tag{2.27}$$

where

$$g(x) = e^x - 1, \qquad \tilde{g}(x) = \ln(1+x). \tag{2.28}$$

Proof: [KS02, Lemma 2.7.2, p.400] □

Note that actually $\tilde{g} = g^{-1}$. Let us recall a few auxiliary results and some properties for later usage. The Lévy measure U admits a density iff $\tilde{U}$ does, and

$$\tilde{u}(x) = \frac{1}{1+x} u(\ln(1+x)) \tag{2.29}$$

for $x > -1$, resp.

$$u(x) = e^x \tilde{u}(e^x - 1) \tag{2.30}$$

for $x \in \mathbb{R}$. We have the following properties:

- X is a compound Poisson process iff $\tilde{X}$ is,
- X is increasing resp. decreasing iff $\tilde{X}$ is so,
- X has finite variation iff $\tilde{X}$ has,
- X has infinite variation iff $\tilde{X}$ has.

For all $z \leq 0$ we have

$$E[e^{z\tilde{X}_1}] < \infty. \tag{2.31}$$

Let us conclude this subsection with some (heuristic) intuition: The right tail of $\tilde{X}_t$ is much heavier than the right tail of X_t. The left tail of $\tilde{X}_t$ is *very* light. Unless the right tail of X_t is *extraordinarily* light we have $E[e^{z\tilde{X}_t}] = \infty$ for all $z > 0$.

2.3.1 The Esscher martingale transform for exponential Lévy processes

Theorem 3. *Suppose $T > 0$ and there exists $\theta^\sharp \in \mathbb{R}$ such that*

$$E[e^{\theta^\sharp X_T}] < \infty, \qquad E[e^{(\theta^\sharp + 1)X_T}] < \infty, \tag{2.32}$$

and the equation

$$\kappa(\theta^\sharp + 1) - \kappa(\theta^\sharp) = 0 \tag{2.33}$$

holds. Then

$$\frac{dP^\sharp}{dP} = e^{\theta^\sharp X_T - \kappa(\theta^\sharp)T}, \tag{2.34}$$

defines an equivalent martingale measure for $(S_t)_{0\le t\le T}$. *The process* $(X_t)_{0\le t\le T}$ *is a Lévy process under* $P^\sharp$ *with characteristic triplet* $(b^\sharp, c^\sharp, U^\sharp)$, *where*

$$b^\sharp = b + c\theta^\sharp + \int (e^{\theta^\sharp x} - 1)h(x)U(dx), \tag{2.35}$$

$$c^\sharp = c, \tag{2.36}$$

$$U^\sharp(dx) = e^{\theta^\sharp x}U(dx). \tag{2.37}$$

Proof: This follows from [KS02, Theorem 4.1, p.421], combined with Theorem 1 above. □

Let us denote expectation with respect to $P^\sharp$ by $E^\sharp$. We have $E^\sharp[e^{zX_t}] = e^{\kappa^\sharp(z)t}$ for $0 \le t \le T$, where

$$\kappa^\sharp(z) = \kappa(z + \theta^\sharp) - \kappa(\theta^\sharp). \tag{2.38}$$

The measure $P^\sharp$ is called the Esscher martingale transform for the exponential Lévy process e^X. If no $\theta^\sharp \in \mathbb{R}$ satisfying (2.32) and (2.33) exists, we say that the Esscher martingale transform for the exponential Lévy process e^X does not exist. In the notation above $P^\sharp = P^{\theta^\sharp}$ where X is used in the Esscher transform, or more explicitly, $P^\sharp = P^{\theta^\sharp \cdot X}$.

Remark 2. *The first condition in (2.32) is required to assure that* $P^\sharp$ *exists, the second to assure that the asset price process* S *is integrable under* $P^\sharp$. *The conditions are equivalent to*

$$\int_{x<-1} e^{\theta^\sharp x}U(dx) < \infty, \qquad \int_{x>1} e^{(\theta^\sharp+1)x}U(dx) < \infty. \tag{2.39}$$

Condition (2.33) *assures that* X *is a martingale under* $P^\sharp$.

2.3.2 The Esscher martingale transform for linear Lévy processes

In view of equation (2.17) finding an equivalent (local) martingale measure for S is equivalent to finding an equivalent (local) martingale measure for $\tilde{X}$.

Remark 3. *Actually the term* local *is redundant in the context of Lévy processes. It can be shown, that any Lévy process and any ordinary or stochastic exponential of a Lévy process, that is a local martingale (or even a sigmamartingale), is automatically a martingale. This observation is related to the property, that the first jump time of a Poisson process is a totally inaccessible stopping time and one cannot control the size of the last jump for a Lévy process stopped at a stopping time.*

Theorem 4. *Suppose* $T > 0$ *and there exists* $\theta^* \in \mathbb{R}$ *such that*

$$E[|\tilde{X}_T|e^{\theta^*\tilde{X}_T}] < \infty, \tag{2.40}$$

and the equation

$$\tilde{\kappa}'(\theta^*) = 0 \tag{2.41}$$

holds. Then

$$\frac{dP^*}{dP} = e^{\theta^*\tilde{X}_T - \tilde{\kappa}(\theta^*)T}, \tag{2.42}$$

defines an equivalent martingale measure for $(S_t)_{0\le t\le T}$. *The process* $(X_t)_{0\le t\le T}$ *is a Lévy process under* P^* *with characteristic triplet* (b^*, c^*, U^*), *where*

$$b^* = b + \theta^* c - \int (h(x)e^{\theta^*(e^x-1)} - h(e^x-1))U(dx), \tag{2.43}$$

$$c^* = c, \tag{2.44}$$

$$U^*(dx) = e^{\theta^*(e^x-1)}U(dx). \tag{2.45}$$

Proof: This follows from [KS02, Theorem 4.4, p.423], combined with Theorem 1 and Theorem 2 above. □

Let us denote expectation with respect to P^* by E^*. We have $E^*[e^{zX_t}] = e^{\kappa^*(z)t}$ for $0 \le t \le T$, but in this case we do not have a simpler expression for the cumulant function $\kappa^*(z)$ than the Lévy-Kintchine formula

$$\kappa^*(z) = b^* z + c^* \frac{z^2}{2} + \int (e^{xz} - 1 - h(x)z)U^*(dx). \tag{2.46}$$

The measure P^* is called the Esscher martingale transform for the linear Lévy process $\tilde{X}$. If no $\theta^* \in \mathbb{R}$ satisfying (2.40) and (2.41) exists, we say that the Esscher martingale transform for the linear Lévy process $\tilde{X}$ does not exist. In the notation above $P^* = P^{\theta^*}$ where $\tilde{X}$ is used in the Esscher transform, or more explicitly, $P^* = P^{\theta^* \cdot \tilde{X}}$.

Remark 4. *The condition* (2.40) *assures that* P^* *exists and that the asset price process* S *is integrable under* P^*. *The condition is equivalent to*

$$\int_{x>1} e^{\theta^* e^x} U(dx) < \infty. \tag{2.47}$$

Condition (2.41) *assures that* $\tilde{X}$, *and thus also* S, *is a martingale under* P^*.

2.4 Relations between the Esscher and other structure preserving martingale measures for exponential Lévy models

The Lévy-Itô decomposition tells us, that any Lévy process X with triplet (b, c, U) can be written as

$$X_t = bt + X_t^c + \int_0^t \int h(x)(\mu - \nu)(dx, ds) + \int_0^t \int (x - h(x))\mu(dx, ds), \tag{2.48}$$

with $X_t^c = \sqrt{c}W_t$, where W is a standard Brownian motion, with $\mu(dx, dt)$ the jump measure of X, and $\nu(dx, dt) = U(dx)dt$ its compensator.

Remark 5. *The first double integral on the right hand side of (2.48) is the* stochastic integral with respect to a compensated random measure, *see [JS87, Definition II.1.27, p.72] or [HWY92, p.301] for a precise description. Alternatively, one can avoid this slightly technical concept from stochastic calculus for general semimartingales and rewrite the expression as an explicit limit in terms of compound Poisson approximations to* X, *see [Sat99, Section 6.33, p.217] and [CS02]. If* X *is of finite variation, then we have*

$$\int_0^t \int h(x)(\mu - \nu)(ds, dx) = \sum_{s\le t} h(\Delta X_s) - t \int h(x)U(dx). \tag{2.49}$$

Suppose $(X_t)_{0\le t\le T}$ is a Lévy process under P and also under another measure $P^\dagger$. Then we call the change of measure, or just the measure $P^\dagger$, *structure preserving*, if $(X_t)_{0\le t\le T}$ is a Lévy process under $P^\dagger$.

Theorem 5. *Suppose $T > 0$, $\psi \in \mathbb{R}$, and $y : \mathbb{R} \to (0,\infty)$ is a function satisfying*

$$\int (\sqrt{y(x)} - 1)^2 U(dx) < \infty. \tag{2.50}$$

Then

$$\tilde{N}_t = \psi X_t^c + \int_0^t \int (y(x) - 1)(\mu - \nu)(ds, dx) \tag{2.51}$$

is well-defined and

$$\frac{dP^\dagger}{dP} = \mathcal{E}(\tilde{N})_T \tag{2.52}$$

defines a measure $P^\dagger$ such that $P^\dagger \sim P$ and $(X_t)_{0\le t\le T}$ is a Lévy process under $P^\dagger$ with characteristic triplet $(b^\dagger, c^\dagger, U^\dagger)$, where

$$b^\dagger = b + c\psi + \int h(x)(y(x) - 1)U(dx) \tag{2.53}$$

$$c^\dagger = c \tag{2.54}$$

$$U^\dagger(dx) = y(x)U(dx). \tag{2.55}$$

A Proof is sketched in the appendix. □

It can be shown, that for $\mathcal{F}$ being the natural filtration of X, all structure preserving measures are as in the theorem above.

Let us denote expectation with respect to $P^\dagger$ by $E^\dagger$. We have $E^\dagger[e^{zX_t}] = e^{\kappa^\dagger(z)t}$ for $0 \le t \le T$, where

$$k^\dagger(z) = \left(b + c\psi + \int h(x)(y(x) - 1)U(dx)\right) z + c\frac{z^2}{2} + \int (e^{zx} - 1 - h(x)z)y(x)U(dx). \tag{2.56}$$

The process $(S_t)_{0\le t\le T}$ is a martingale under $P^\dagger$ if

$$\int_{x>1} e^x y(x)U(dx) < \infty \tag{2.57}$$

and

$$b + c(\psi + \tfrac{1}{2}) + \int ((e^x - 1)y(x) - h(x))U(dx) = 0. \tag{2.58}$$

Thus we see, that the Esscher transform for exponential Lévy processes uses the function $y(x) = e^{\theta^\sharp x}$. The Esscher transform for linear Lévy processes uses $y(x) = e^{\theta^*(e^x-1)}$. Structure preserving measure changes have

$$y(x) = \frac{dU^\dagger}{dU}(x). \tag{2.59}$$

3 The minimal entropy martingale measure for exponential Lévy models

3.1 Definition of the minimal entropy martingale measure

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and Q is another probability measure on $(\Omega, \mathcal{F})$. The relative entropy $I(Q, P)$ of Q with respect to P is defined by

$$I(Q,P) = \begin{cases} E_P\left[\frac{dQ}{dP} \ln\left(\frac{dQ}{dP}\right)\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Note, that even if $Q \ll P$ it might be the case that $I(Q, P) = +\infty$.

Suppose S is a stochastic process on $(\Omega, \mathcal{F}, P)$ modelling discounted asset prices. Let

$$\mathcal{Q}_a(S) = \left\{Q \ll P \;\middle|\; S \text{ is a local } Q\text{-martingale}\right\}. \tag{3.2}$$

A probability measure $\hat{P} \in \mathcal{Q}_a(S)$ is called *minimal entropy martingale measure* for S, if it satisfies

$$I(\hat{P}, P) = \min_{Q \in \mathcal{Q}_a(S)} I(Q, P) \tag{3.3}$$

The minimum entropy martingale measure and related issues in a general semimartingale setting have been introduced and thoroughly investigated in [Fri00], [BF02], [GR02], and [CM03]. Note that many general results require locally bounded asset price processes, and this is not the case for most Lévy processes of interest in our context.

Suppose $\mathcal{G}$ is a sub-sigmaalgebra of $\mathcal{F}$. Then we set

$$I_{\mathcal{G}}(Q, P) = I(Q|_{\mathcal{G}}, P|_{\mathcal{G}}). \tag{3.4}$$

When working with a filtration $(\mathcal{F}_t)$ sometimes the notation

$$I_t(Q, P) = I_{\mathcal{F}_t}(Q, P), \tag{3.5}$$

is used and (I_t) is called the *entropy process.*

3.2 Main results on the minimum entropy martingale measure for exponential Lévy processes

The minimal entropy martingale measure for exponential Lévy processes has been studied by [Cha99] under the assumption of the existence of exponential moments. More general results are provided in [FM03]. In this section we summarize their results, and add a small contribution, namely the converse statement of the main result by [ES05], that allows a complete characterization of the minimal entropy martingale measure as the Esscher transform for the linear Lévy process $\tilde{X}$ in the univariate case.

We also discuss the case when the minimal entropy martingale measure does not exist, and we compute the infimum of the entropies, that is not attained in this case. This discussion could provide a basis for counterexamples in the context of dual problems related to maximization of exponential utility.

Let us first summarize a few explicit computations for the entropy.

Theorem 6. *Suppose $P^\sharp$ is the Esscher martingale transform for the exponential Lévy process e^X, then*

$$I(P^\sharp, P) = (\theta^\sharp \kappa'(\theta^\sharp) - \kappa(\theta^\sharp))T. \tag{3.6}$$

Suppose P^* is the Esscher martingale transform for the linear Lévy process $\tilde{X}$, then

$$I(P^*, P) = -\tilde{\kappa}(\theta^*)T. \tag{3.7}$$

Suppose $P^\dagger$ is an equivalent martingale measure for e^X, that corresponds to the deterministic and time-independent Girsanov parameters (ψ, y) with respect to X, then

$$I(P^\dagger, P) = \left[\frac{1}{2}c\psi^2 + \int (y(x)\ln(y(x)) - y(x) + 1)\, U(dx)\right] T. \tag{3.8}$$

Proof: This is a reformulation of [CT04, Proposition 9.10, p.312]. □

A key assumption in the general theory is, that there is at least one equivalent martingale measure with finite entropy. The next theorem shows, that, except for trivial cases, when no equivalent martingale measure exists, this assumption is satisfied for exponential Lévy models

Theorem 7. *Suppose the Lévy process X is increasing or decreasing, but not constant, then e^X admits arbitrage. Otherwise e^X admits no free lunch with vanishing risk, and there is an equivalent martingale measure for e^X with finite entropy, such that X remains a Lévy process.*

Proof: This theorem is proved in [Jak02] and [CS02], except for the assertion on finite entropy. This is done in the appendix. See also [EJ97]. □

Now we are ready to state the main result, the characterization of the minimum entropy martingale measure for the exponential Lévy process e^X as the Esscher transform for the linear Lévy process $\tilde{X}$.

Theorem 8. *The minimum entropy martingale measure for the exponential Lévy process e^X exists iff the Esscher martingale measure for the linear Lévy process $\tilde{X}$ exists. If both measures exist, they coincide.*

Proof: In view of the previous theorem we can reformulate Theorem A of [ES05] for a real-valued Lévy process X as follows: Suppose the minimal entropy martingale measure $\hat{P}$ for the exponential Lévy process e^X exists. Then X is a Lévy process under $\hat{P}$. Theorem B of [ES05] says: If the Esscher martingale measure for the linear Lévy process $\tilde{X}$ exists, then it is the minimum entropy martingale measure for the exponential Lévy process e^X. In the appendix we show the converse: If the minimum entropy martingale measure for the exponential Lévy process e^X exists, then it is the Esscher martingale measure for the linear Lévy process $\tilde{X}$. □

Let us now discuss existence. There is $\bar{\theta} \in [0, +\infty]$ such that

$$E[|\tilde{X}_1|e^{\theta\tilde{X}_1}] < \infty \quad \forall\theta < \bar{\theta} \tag{3.9}$$

and

$$E[|\tilde{X}_1|e^{\theta\tilde{X}_1}] = \infty \quad \forall\theta > \bar{\theta} \tag{3.10}$$

The expectation $E[|\tilde{X}_1|e^{\bar{\theta}\tilde{X}_1}]$ can be finite or infinite. Let us use the convention $\tilde{\kappa}'(\bar{\theta}) = +\infty$ if $E[|\tilde{X}_1|e^{\bar{\theta}\tilde{X}_1}] = +\infty$. If $E[|\tilde{X}_1|e^{\bar{\theta}\tilde{X}_1}] < +\infty$ then trivially $E[e^{\bar{\theta}\tilde{X}_1}] < +\infty$ and $\tilde{\kappa}(\bar{\theta})$ is a well-defined finite number.

Corollary 1. *If*

$$\inf_{\theta<0} \tilde{\kappa}'(\theta) \leq 0 \tag{3.11}$$

and

$$\tilde{\kappa}'(\bar{\theta}) \geq 0 \tag{3.12}$$

then the minimum entropy martingale measure for e^X *exists and coincides with the Esscher martingale measure for the linear Lévy process* $\tilde{X}$.

Let us now discuss non-existence: If X is decreasing or increasing, but not constant we have arbitrage, so let us exclude those trivial cases.

Theorem 9. *Suppose the Lévy process* X *is neither increasing nor decreasing and*

$$\tilde{\kappa}'(\bar{\theta}) < 0. \tag{3.13}$$

Then the minimum entropy martingale measure does not exist,

$$\inf_{Q\in\mathcal{Q}_a(S)} I(Q,P) = -\tilde{\kappa}(\bar{\theta})T \tag{3.14}$$

and there is a sequence of structure preserving equivalent martingale measures P^n, *such that*

$$\lim_{n\to\infty} I(P^n,P) = \inf_{Q\in\mathcal{Q}_a(S)} I(Q,P). \tag{3.15}$$

Proof: The proof is given in the appendix □

Interpretation: in the above situation the process e^X is a supermartingale and we must shift mass to the right. However this has to be done by reweighing the jumps with $y(x) = e^{\theta(e^x-1)}$. Taking $\theta = \bar{\theta}$ is not enough, but taking any $\theta > \bar{\theta}$ is too much, as integrability is lost. A more decent choice of $y(x)$ is required.

Remark 6. *So far we studied the minimum entropy martingale measure on a fixed horizon* T, *that was implicit in the notation. To discuss the dependence on the horizon let us briefly introduce the following more explicit notation: Let*

$$\mathcal{Q}^a_T(S) = \left\{ Q \ll P \;\middle|\; (S_t)_{0\leq t\leq T} \text{ is a local } Q\text{-martingale} \right\}. \tag{3.16}$$

A probability measure $\hat{P}_T \in \mathcal{Q}^a_T(S)$ *is called* minimal entropy martingale measure *for the process* S *and horizon* T, *if it satisfies*

$$I_T(\hat{P}_T,P) = \min_{Q\in\mathcal{Q}^a_T(S)} I_T(Q,P). \tag{3.17}$$

If we consider the problem for $0 < t \leq T$, *then it follows that*

$$P^*_t = P^*_T|_{\mathcal{F}_t}. \tag{3.18}$$

4 Exponential Lévy Examples

4.1 The normal inverse Gaussian Lévy process

The normal inverse Gaussian distribution NIG$(\mu, \delta, \alpha, \beta)$ with parameter range

$$\mu \in \mathbb{R}, \quad \delta > 0, \quad \alpha > 0, \quad -\alpha \leq \beta \leq \alpha. \tag{4.1}$$

is defined by the probability density

$$p(x) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}. \tag{4.2}$$

Here K_1 is the modified Bessel function of second kind and order 1, also known as Macdonald function. The cumulant function is

$$\kappa(z) = \mu z + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+z)^2}\right), \tag{4.3}$$

and it exists for

$$-\alpha - \beta \leq \Re(z) \leq \alpha - \beta. \tag{4.4}$$

The Lévy density is

$$u(x) = \frac{\delta\alpha}{\pi} e^{\beta x}|x|^{-1}K_1(\alpha|x|). \tag{4.5}$$

If $(X_t)_{t\geq 0}$ denotes a Lévy process, such that $X_1 \sim$ NIG$(\mu, \delta, \alpha, \beta)$, then $X_t \sim$ NIG$(\mu t, \delta t, \alpha, \beta)$ for all $t > 0$. Using the asymptotics [AS65, 9.7.2, p.378]

$$K_1(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\left(1 + O\left(\frac{1}{z}\right)\right) \qquad (z \to \infty). \tag{4.6}$$

we see that

$$p(x) \sim \begin{cases} A_1 x^{-3/2} e^{-(\alpha-\beta)x} & x \to +\infty \\ A_2 (-x)^{-3/2} e^{(\alpha+\beta)x} & x \to -\infty \end{cases} \tag{4.7}$$

where

$$A_1 = \sqrt{\frac{\alpha}{2\pi}} e^{\delta\sqrt{\alpha^2-\beta^2}+(\alpha-\beta)\mu}, \qquad A_2 = \sqrt{\frac{\alpha}{2\pi}} e^{\delta\sqrt{\alpha^2-\beta^2}-(\alpha+\beta)\mu}. \tag{4.8}$$

This shows that $p(x)$ has semi-heavy tails, except for the following two extremal cases: If $\beta = \alpha$ then the right tail is heavy, if $\beta = -\alpha$ then the left tail is heavy. If $|\beta| < \alpha$ then

$$\mathbb{E}[X_1] = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}, \qquad \mathbb{V}[X_1] = \frac{\delta\alpha^2}{\sqrt{\alpha^2-\beta^2}^3}. \tag{4.9}$$

If $|\beta| = \alpha$ those moments do not exist. Using the asymptotics [AS65, 9.6.11 and 9.6.7, p.375]

$$K_1(z) = z^{-1}\left(1 + O\left(z^2 \ln z\right)\right) \qquad (z \to 0). \tag{4.10}$$

we obtain

$$u(x) \sim \frac{\delta}{\pi} x^{-2} \qquad x \to 0, \tag{4.11}$$

and this shows that the NIG Lévy process has infinite variation. We will also use

$$u(x) \sim \begin{cases} \delta\sqrt{\frac{\alpha}{2\pi}} x^{-3/2} e^{-(\alpha-\beta)x} & x \to +\infty \\ \delta\sqrt{\frac{\alpha}{2\pi}} (-x)^{-3/2} e^{(\alpha+\beta)x} & x \to -\infty. \end{cases} \tag{4.12}$$

4.1.1 The Esscher transform for the exponential NIG process

Proposition 1. *If*

$$0 < \alpha < \frac{1}{2} \quad or \quad \alpha \geq \frac{1}{2},\ |\mu| > \delta\sqrt{2\alpha - 1} \tag{4.13}$$

then the Esscher martingale measure $P^\sharp$ for the exponential process e^X does not exist. If

$$\alpha \geq \frac{1}{2},\ |\mu| \leq \delta\sqrt{2\alpha - 1} \tag{4.14}$$

then the Esscher martingale measure $P^\sharp$ for the exponential process e^X does exist. The Esscher parameter is then

$$\theta^\sharp = -\beta - \frac{1}{2} - \frac{\mu}{2\delta}\sqrt{\frac{4\alpha^2\delta^2}{\mu^2 + \delta^2} - 1}. \tag{4.15}$$

and X is under $P^\sharp$ a $\mathrm{NIG}(\mu, \delta, \alpha, \beta^\sharp)$ *process, where*

$$\beta^\sharp = -\frac{1}{2} - \frac{\mu}{2\delta}\sqrt{\frac{4\alpha^2\delta^2}{\mu^2 + \delta^2} - 1}. \tag{4.16}$$

Proof: The Esscher transform P^θ for the exponential NIG process exists always for

$$-\alpha - \beta \leq \theta \leq \alpha - \beta. \tag{4.17}$$

The process X is a $\mathrm{NIG}(\mu, \delta, \alpha, \beta + \theta)$ process under P^θ. If $0 < \alpha < \frac{1}{2}$, then no P^θ produces integrability for e^X, and thus $P^\sharp$ does not exist. If $\alpha \geq \frac{1}{2}$, the Esscher transform P^θ exists and e^X is integrable under P^θ for

$$-\alpha - \beta \leq \theta \leq \alpha - \beta - 1. \tag{4.18}$$

The function

$$f(\theta) = \kappa(\theta + 1) - \kappa(\theta) \tag{4.19}$$

is increasing on $[-\alpha - \beta, \alpha - \beta - 1]$ with

$$f(-\alpha - \beta) = \mu - \delta\sqrt{2\alpha - 1}, \qquad f(\alpha - \beta - 1) = \mu - \delta\sqrt{2\alpha - 1}. \tag{4.20}$$

Thus if $|\mu| > \delta\sqrt{2\alpha - 1}$ then $P^\sharp$ does not exist. If $\mu \leq \delta\sqrt{2\alpha - 1}$ then there is a solution, that can be computed explicitly as (4.16). Looking at the new cumulant function gives the law of X under $P^\sharp$. □

4.1.2 The Esscher transform for the linear process

Proposition 2. *If*

$$0 < \alpha < \frac{1}{2}, \quad or \quad \alpha \geq \frac{1}{2},\ \alpha - 1 < \beta \leq \alpha, \quad or \quad \alpha \geq \frac{1}{2},\ -\alpha \leq \beta \leq \alpha - 1, \tag{4.21}$$

$$\mu \geq \delta(\sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2}) \tag{4.22}$$

then the Esscher martingale measure P^* for the linear process $\tilde{X}$, and thus the minimal entropy martingale measure for e^X does exist. If

$$\alpha \geq \frac{1}{2},\ -\alpha \leq \beta \leq \alpha - 1,\ \mu < \delta(\sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2}) \tag{4.23}$$

then the Esscher martingale measure P^* for the linear process $\tilde{X}$, and thus the minimal entropy martingale measure for e^X does not exist.

Proof: The Esscher transform for the linear process $\tilde{X}$ exists for $\vartheta \leq 0$. We cannot simplify the integral representation for the cumulant function and its derivative, and we have to solve the martingale equation for ϑ numerically. For $0 < \alpha < \frac{1}{2}$, or if $\alpha \geq \frac{1}{2}$ and $\alpha - 1 < \beta \leq \alpha$, we obtain from the results above, that

$$\lim_{\vartheta \to -\infty} \tilde{\kappa}'(\vartheta) = -\infty, \qquad \lim_{\vartheta \to -0} \tilde{\kappa}'(\vartheta) = +\infty, \tag{4.24}$$

thus, there is always a solution, and P^* exists. If $\alpha \geq \frac{1}{2}$ and $-\alpha \leq \beta \leq \alpha - 1$, we obtain from the results above, that

$$\lim_{\vartheta \to -\infty} \tilde{\kappa}'(\vartheta) = -\infty, \qquad \lim_{\vartheta \to -0} \tilde{\kappa}'(\vartheta) = \mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}). \tag{4.25}$$

Thus, if $\mu < \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2})$ then P^* does not exist, while for $\mu \geq \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2})$ it exists. □

4.1.3 Structure preserving measure changes

Any function $y(x)$ with

$$\int (\sqrt{y(x)} - 1)^2 e^{\beta x} |x|^{-1} K_1(\alpha|x|) dx < \infty \tag{4.26}$$

gives a structure preserving change of measure. If $(X_t)_{0 \leq t \leq T} \sim \text{NIG}(\mu, \delta, \alpha, \beta)$ under P, and $(X_t)_{0 \leq t \leq T} \sim \text{NIG}(\mu', \delta', \alpha', \beta')$ under P', and $P' \sim P$, then this implies $\mu' = \mu$ and $\delta' = \delta$. This change of measure is characterized by the function

$$y(x) = e^{(\beta' - \beta)x} \frac{\alpha' K_1(\alpha'|x|)}{\alpha K_1(\alpha|x|)}. \tag{4.27}$$

The martingale condition is

$$\mu + \delta\big[\sqrt{\alpha'^2 - \beta'^2} - \sqrt{\alpha'^2 - (\beta'+1)^2}\big] = 0. \tag{4.28}$$

Conversely, all structure preserving equivalent measure changes are of this type. This illustrates, that there are structure preserving changes of measure, that are not Esscher transforms.

4.2 The variance gamma Lévy process

The variance gamma distribution $VG(\mu, \lambda, \gamma, \beta)$ with parameters

$$\mu \in \mathbb{R}, \quad \lambda > 0, \quad \gamma > 0, \quad \beta \in \mathbb{R} \tag{4.29}$$

is defined by the probability density

$$p(x) = \sqrt{\frac{2}{\pi}} \frac{\gamma^\lambda}{\Gamma(\lambda) \sqrt{\beta^2 + 2\gamma}^{\lambda - 1/2}} e^{\beta(x-\mu)} |x - \mu|^{\lambda - 1/2} K_{\lambda - 1/2}\left(|x - \mu| \sqrt{\beta^2 + 2\gamma}\right). \tag{4.30}$$

The cumulant function is

$$\kappa(z) = \mu z + \lambda \ln\left(\frac{\gamma}{\gamma - \beta z - z^2/2}\right) \tag{4.31}$$

and it exists for

$$-\beta - \sqrt{\beta^2 + 2\gamma} < \Re(z) < -\beta + \sqrt{\beta^2 + 2\gamma}. \tag{4.32}$$

The Lévy density is

$$u(x) = \lambda |x|^{-1}(e^{-c_1 x} I_{x>0} + e^{c_2 x} I_{x<0} \tag{4.33}$$

where

$$c_1 = -\beta + \sqrt{\beta^2 + 2\gamma}, \qquad c_2 = \beta + \sqrt{\beta^2 + 2\gamma}. \tag{4.34}$$

If (X_t) denotes a Lévy process, such that $X_1 \sim VG(\mu, \lambda, \gamma, \beta)$, then $X_t \sim VG(\mu t, \lambda t, \gamma, \beta)$ for all $t > 0$. We have

$$\mathbb{E}[X_1] = \frac{\lambda\beta}{\gamma}, \qquad \mathbb{V}[X_1] = \frac{\lambda}{\gamma}\left(1 + \frac{\beta^2}{\gamma}\right). \tag{4.35}$$

Using again the asymptotics [AS65, 9.7.2, p.378] we see that

$$p(x) \sim \begin{cases} A_1 x^{\lambda - 1/2}(e^{-c_1 x} & x \to +\infty \\ \\ A_2 x^{\lambda - 1/2} e^{c_2 x} & x \to -\infty \end{cases} \tag{4.36}$$

with some constants A_1 and A_2.

The Lévy density has the asymptotics

$$u(x) = \lambda |x|^{-1}(1 + \mathcal{O}(|x|)) \qquad (x \to 0) \tag{4.37}$$

so the process is of infinite activity and of finite variation.

4.2.1 The Esscher transform for the exponential process

Proposition 3. *If*

$$\beta^2 + 2\gamma \le \frac{1}{4} \tag{4.38}$$

then the Esscher martingale measure $P^\sharp$ for the exponential process e^X does not exist. If $\beta^2 + 2\gamma > \frac{1}{4}$ then the Esscher martingale measure $P^\sharp$ for the exponential process e^X does exist. The Esscher parameter is then

$$\theta^\sharp = -\beta - \frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1 + \beta^2\varepsilon^2 - \epsilon + 2\gamma\epsilon^2} \tag{4.39}$$

where

$$\varepsilon = 1 - e^{\mu/\lambda} \tag{4.40}$$

and X is under $P^\sharp$ a $VG(\mu, \lambda, \gamma^\sharp, \beta^\sharp)$ process, where

$$\gamma^\sharp = \gamma - \beta\theta^\sharp - \theta^{\sharp 2}/2 \tag{4.41}$$

and

$$\beta^\sharp = \beta + \theta^\sharp. \tag{4.42}$$

Proof: The Esscher transform P^θ for the exponential VG process exists always for

$$-\beta - \sqrt{\beta^2 + 2\gamma} < \theta < -\beta + \sqrt{\beta^2 + 2\gamma}. \tag{4.43}$$

The process X is a $VG(\mu, \delta, \alpha, \beta + \theta)$ process under P^θ. If $\beta^2 + 2\gamma \le \frac{1}{4}$, then no such P^θ grants integrability for e^X, and thus $P^\sharp$ does not exist. If $\beta^2 + 2\gamma > \frac{1}{4}$, the Esscher transform P^θ exists and e^X is integrable under P^θ for

$$-\beta - \sqrt{\beta^2 + 2\gamma} < \theta < -\beta - 1 + \sqrt{\beta^2 + 2\gamma}. \tag{4.44}$$

The function

$$f(\theta) = \kappa(\theta + 1) - \kappa(\theta) \tag{4.45}$$

is increasing on $(-\beta - \sqrt{\beta^2 + 2\gamma}, -\beta - \sqrt{\beta^2 + 2\gamma})$ with $f(\theta)$ tending to $-\infty$ resp. $+\infty$ for θ tending to the left resp. right endpoint of this interval. Thus there is a solution, that can be computed explicitly as (4.39). By looking at the new cumulant function we can identify the law of X under $P^\sharp$. □

4.2.2 The Esscher transform for the linear process

The Esscher martingale transform for the linear process and thus the minimal entropy martingale measure has been discussed in [FM03, Example 3.3, p.524].

4.2.3 Structure preserving measure changes

Any function $y(x)$ with

$$\int (\sqrt{y(x)} - 1)^2 u(x) dx < \infty \tag{4.46}$$

gives a structure preserving change of measure. If $(X_t)_{0 \le t \le T} \sim VG(\mu, \lambda, \gamma, \beta)$ under P, and $(X_t)_{0 \le t \le T} \sim VG(\mu^\dagger, \lambda^\dagger, \gamma^\dagger, \beta^\dagger)$ under $P^\dagger$, and $P^\dagger \sim P$, then this implies $\mu^\dagger = \mu$ and $\lambda^\dagger = \lambda$. This change of measure is characterized by the function

$$y(x) = e^{-(c_1^\dagger - c_1)x} I_{\{x<0\}} + e^{(c_2^\dagger - c_2)x} I_{\{x<0\}}. \tag{4.47}$$

where

$$c_1^\dagger = -\beta^\dagger + \sqrt{\beta^{\dagger 2} + 2\gamma^\dagger}, \qquad c_2^\dagger = \beta^\dagger + \sqrt{\beta^{\dagger 2} + 2\gamma^\dagger}. \tag{4.48}$$

The martingale condition is

$$\mu + \lambda \ln \left(\frac{\gamma^\dagger}{\gamma^\dagger - \beta^\dagger - 1/2} \right) = 0. \tag{4.49}$$

4.3 The Poisson difference model

This model is not commonly used, but we think it is not completely unrealistic, at least in comparison to other models, and allows the most explicit calculations.

Suppose returns are given by

$$X_t = \mu t + \alpha_1 N_t^1 - \alpha_2 N_t^2, \tag{4.50}$$

where N^1 and N^2 are two independent standard Poisson processes with intensity $\lambda_1 > 0$ resp. $\lambda_2 > 0$, and $\mu \in \mathbb{R}$ and $\alpha_1 > 0$ and $\alpha_2 > 0$ are parameters. Let us call this the Poisson difference model $DP(\mu, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$. We have

$$E[X_t] = (\mu + \alpha_1 \lambda_1 - \alpha_2 \lambda_2)\, t \tag{4.51}$$

and

$$V[X_t] = (\alpha_1^2\lambda_1^2 + \alpha_2^2\lambda_2^2)t. \tag{4.52}$$

The cumulant function is

$$\kappa(z) = \mu z + \lambda_1(e^{\alpha_1 z} - 1) + \lambda_2(e^{-\alpha_2 z} - 1). \tag{4.53}$$

Alternatively, this model can be described as compound Poisson processes

$$X_t = \mu t + \sum_{k=1}^{N_t} Y_k. \tag{4.54}$$

Here N is a standard Poisson process with intensity

$$\lambda = \lambda_1 + \lambda_2 \tag{4.55}$$

and $(Y_k)_{k\geq 1}$ is an independent iid sequence with

$$P[Y_k = \alpha_1] = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \quad P[Y_k = -\alpha_2] = \frac{\lambda_2}{\lambda_1 + \lambda_2}. \tag{4.56}$$

For numerical illustration we take annual parameters

$$\mu = 0, \quad \alpha_1 = 0.001, \quad \alpha_2 = 0.001, \quad \lambda_1 = 20050, \quad \lambda_2 = 19950, \tag{4.57}$$

and we assume 250 trading days. This yields daily returns with mean 0.0004 and standard deviation 0.01265. In Figure 1 the histogram for daily returns is shown, in Figure 2 on page 19 an intra-day path simulation is displayed.

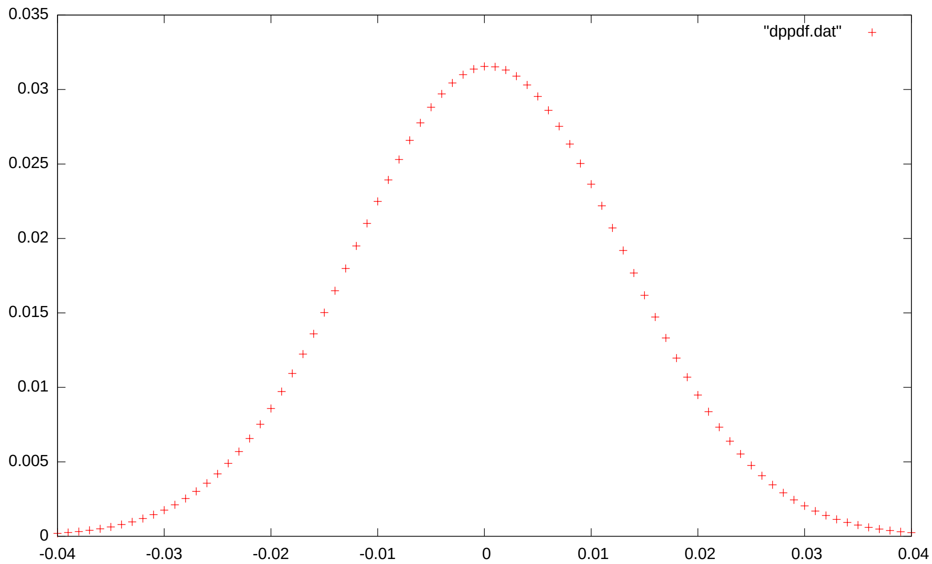

Figure 1: Probability function for the distribution of daily returns in the Poisson difference model with $\mu = 0$, $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\lambda_1 = 20050$, $\lambda_2 = 19950$.

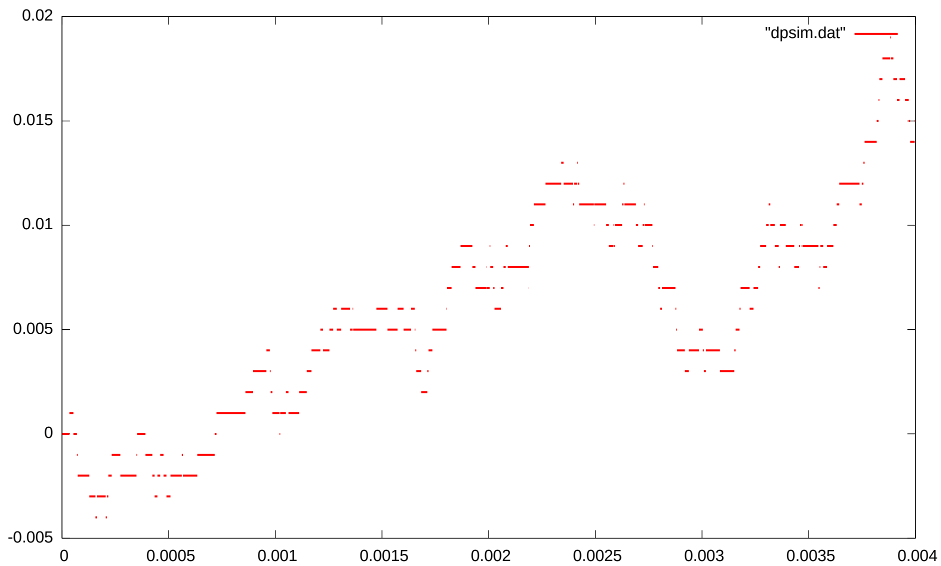

Figure 2: A path simulation for one day in the Poisson difference model with $\mu = 0$, $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\lambda_1 = 20050$, $\lambda_2 = 19950$.

4.3.1 The Esscher transform for exponential processes

The Esscher transform for exponential processes exists always, and the parameter satisfies

$$\mu(\theta+1) + \lambda_1(e^{\alpha_1(\theta+1)} - 1) + \lambda_2(e^{-\alpha_2(\theta+1)} - 1) = \mu\theta + \lambda_1(e^{\alpha_1\theta} - 1) + \lambda_2(e^{-\alpha_2\theta} - 1). \quad (4.58)$$

If $\mu = 0$, which we will assume from now on, this equation can be solved elementarily and we obtain

$$\theta^\sharp = \frac{1}{\alpha_1 + \alpha_2} \ln\left[\frac{\lambda_2(1 - e^{-\alpha_2})}{\lambda_1(e^{\alpha_1} - 1)}\right]. \quad (4.59)$$

Under $P^\sharp$ we have $X \sim DP(\lambda_1^\sharp, \lambda_2^\sharp, \alpha_1, \alpha_2)$ where

$$\lambda_1^\sharp = \lambda_1 \left[\frac{\lambda_1(1 - e^{-\alpha_2})}{\lambda_2(e^{\alpha_1} - 1)}\right]^{\frac{\alpha_1}{\alpha_1+\alpha_2}}, \qquad \lambda_2^\sharp = \lambda_2 \left[\frac{\lambda_2(1 - e^{-\alpha_2})}{\lambda_1(e^{\alpha_1} - 1)}\right]^{-\frac{\alpha_2}{\alpha_1+\alpha_2}}. \quad (4.60)$$

The entropy is

$$I_T(P^\sharp, P) = (\theta^\sharp \kappa'(\theta^\sharp) - \kappa(\theta^\sharp))T. \quad (4.61)$$

4.3.2 The Esscher transform for linear processes

The exponential transform of X is

$$\tilde{X}_t = \tilde{\alpha}_1 N_t^1 - \tilde{\alpha}_2 N_t^2, \quad (4.62)$$

where

$$\tilde{\alpha}_1 = e^{\alpha_1} - 1, \qquad \tilde{\alpha}_2 = 1 - e^{-\alpha_2}. \quad (4.63)$$

Thus $\tilde{X} \sim DP(\lambda_1, \lambda_2, \tilde{\alpha}_1, \tilde{\alpha}_2)$, and the cumulant function is

$$\tilde{\kappa}(z) = \lambda_1(e^{\tilde{\alpha}_1 z} - 1) + \lambda_2(e^{-\tilde{\alpha}_2 z} - 1). \quad (4.64)$$

The solution to $\tilde{\kappa}'(\theta) = 0$ is

$$\theta^* = \frac{1}{e^{\alpha_1} - e^{-\alpha_2}} \ln \left[\frac{\lambda_2(1 - e^{-\alpha_2})}{\lambda_1(e^{\alpha_1} - 1)} \right]. \tag{4.65}$$

Under $\tilde{P}^*$ we have $X \sim DP(\lambda_1^*, \lambda_2^*, \alpha_1, \alpha_2)$ where

$$\lambda_1^* = \lambda_1 \left[\frac{\lambda_2(1 - e^{-\alpha_2})}{\lambda_1(e^{\alpha_1} - 1)} \right]^{\frac{e^{\alpha_1}-1}{e^{\alpha_1}-e^{-\alpha_2}}}, \qquad \lambda_2^* = \lambda_2 \left[\frac{\lambda_2(1 - e^{-\alpha_2})}{\lambda_1(e^{\alpha_1} - 1)} \right]^{-\frac{1-e^{-\alpha_2}}{e^{\alpha_1}-e^{-\alpha_2}}}. \tag{4.66}$$

The entropy is

$$I_T(P^*, P) = -\tilde{\kappa}(\theta^*)T. \tag{4.67}$$

A Proofs

A.1 Proof of Theorem 5

With y satisfying (2.50) we can define $\tilde{N}$ according to (2.51). The process $\tilde{N}$ is a Lévy process and a martingale with $\tilde{N}_0 = 0$. In this case it is known that $\mathcal{E}(\tilde{N})$ is a proper martingale, and $E[\mathcal{E}(\tilde{N})_T] = 1$, so (2.52) indeed defines a probability measure $P^\dagger$.

To see that $(X_t)_{0 \le t \le T}$ is a Lévy process with triplet $(b^\dagger, c^\dagger, U^\dagger)$, let us define N as the logarithmic transform of $\tilde{N}$. We know, that N is also a Lévy process. This can be used in an easy calculation to show that the characteristic functions of the finite dimensional distributions have the required structure. □

Remark 7. *Similar theorems on the change of measure for Lévy processes have been proved and are available in many textbooks and articles, for example [Sat99, Theorem 33.1, p.218], [EJ97], [ES05]. They differ slightly with respect to our statement. For example some start with $P^\dagger \ll P$ given, while we want to construct $P^\dagger$ from given Girsanov parameters (ψ, y). Some other use the canonical setting to achieve a measure $P^\dagger \overset{loc}{\sim} P$, such that $(X_t)_{t \ge 0}$ is a Lévy process, etc. Therefore we provided a sketch of the proof for our formulation.*

A.2 Proof of Theorem 7

In this proof we use the truncation functions

$$h_a(x) = x I_{\{|x| \le a\}} \tag{A.1}$$

for $a > 0$ and denote the first characteristic with respect to h_a by b_a. So we have for the cumulant function

$$\kappa(z) = b_a z + c\frac{z^2}{2} + \int (e^{zx} - 1 - h_a(x)z) U(dx). \tag{A.2}$$

Using a structure preserving change of measure $P \mapsto P'$ with deterministic Girsanov parameters (ψ, y) the new triplet (b'_a, c', U') with respect to h_a is given by

$$b'_a = b_a + c\psi + \int h_a(x)(y(x) - 1) U(dx), \tag{A.3}$$

$$c' = c, \tag{A.4}$$

$$U'(dx) = y(x) U(dx). \tag{A.5}$$

The new cumulant function is

$$\kappa'(z) = b'_a z + c' \frac{z^2}{2} + \int (e^{zx} - 1 - h_a(x) z) U'(dx). \tag{A.6}$$

The martingale condition is $\kappa'(1) = 0$, which means

$$b_a + c\left(\psi + \frac{1}{2}\right) + \int \left((e^x - 1) y(x) - h_a(x)\right) U(dx) = 0. \tag{A.7}$$

The entropy is

$$I(P', P) = \frac{1}{2} c \psi^2 + \int \left(y(x) \ln(y(x)) - y(x) + 1\right) U(dx). \tag{A.8}$$

Remark 8. *We have for $y \geq 0$ the inequality*

$$(\sqrt{(}y) - 1)^2 \leq y \ln y - y + 1, \tag{A.9}$$

and thus, if a function $y(x)$ satisfies

$$\int (y(x) \ln y(x) - y(x) + 1) U(dx) < \infty, \tag{A.10}$$

then this implies the integrability condition (2.50) for in the corresponding structure preserving change of measure.

Now we follow [CS02, p.18f] and consider six cases.

Case I. Suppose there exists $a > 0$ such that $U((-\infty, a)) > 0$ and $U((a, +\infty)) > 0$, i.e., there are positive and negative jumps. Then we choose

$$\psi = 0, \qquad y(x) = \begin{cases} \alpha & x < -a \\ 1 & |x| \leq a \\ \beta e^{-2x} & x > a, \end{cases} \tag{A.11}$$

where α and β are finite, positive constants, determined as follows: If

$$b_a + c/2 + \int_{\{|x| \leq a\}} (e^x - 1 - x) U(dx) \leq 0 \tag{A.12}$$

then

$$\alpha = 1, \qquad \beta = -\frac{b_a + c/2 + \int_{\{|x| \leq a\}} (e^x - 1 - x) U(dx) + \int_{\{x < -a\}} (e^x - 1) U(dx)}{\int_{\{x > a\}} (e^x - 1) e^{-2x} U(dx)}, \tag{A.13}$$

otherwise

$$\alpha = -\frac{b_a + c/2 + \int_{\{|x| \leq a\}} (e^x - 1 - x) U(dx) + \int_{\{x > a\}} (e^x - 1) e^{-2x} U(dx)}{\int_{\{x < -a\}} (e^x - 1) U(dx)}, \qquad \beta = 1. \tag{A.14}$$

The entropy is

$$I(P', P) = \int_{\{x < -a\}} (\alpha \ln \alpha - \alpha + 1) U(dx) + \int_{\{x > a\}} \left(\beta e^{-2x} (\ln \beta - 2x) - \beta e^{-2x} + 1\right) U(dx), \tag{A.15}$$

which is, in view of the integrability properties of $U(dx)$, clearly finite.

Case II. Suppose $\nu((-\infty,0)) = 0$ and $\int_{0<x\le 1} xU(dx) = \infty$. Then we can find $a > 0$ such that $\nu((a,+\infty)) > 0$ and

$$b_a + \frac{c}{2} + \int_{\{0<x\le a\}} (e^x - 1 - x)U(dx) < 0. \tag{A.16}$$

We use

$$\psi = 0, \qquad y(x) = \begin{cases} 1 & x \le a \\ \beta e^{-2x} & x > a, \end{cases} \tag{A.17}$$

where β is a finite, positive constants, determined as

$$\beta = -\frac{b_a + \frac{c}{2} + \int_{\{0<x\le a\}} (e^x - 1 - x)U(dx)}{\int_{\{x>a\}} (e^x - 1)e^{-2x}U(dx)}. \tag{A.18}$$

Obviously the entropy is finite.

Case III. Suppose $U((-\infty,0)) = 0$, $\int_{0<x\le 1} xU(dx) < \infty$, and $c > 0$. We take

$$\psi = -\frac{1}{2} - \frac{1}{c}\left[b_1 + \int\limits_{\{0<x\le 1\}} (e^x - 1 - x)U(dx) + \int\limits_{\{x>1\}} (e^x - 1)e^{-2x}U(dx) \right] \tag{A.19}$$

and

$$y(x) = \begin{cases} 1 & x \le 1 \\ e^{-2x} & x > 1, \end{cases} \tag{A.20}$$

The entropy is finite.

Case IV. Suppose $U((-\infty,0)) = 0$, $U((0,+\infty)) > 0$, $\int_{0<x\le 1} xU(dx) < \infty$, $c = 0$, $b_0 < 0$. We can find $a > 0$, such that $U((a,+\infty)) > 0$ and

$$b_a + \int\limits_{\{0<x\le 1\}} (e^x - 1 - x)U(dx) < 0. \tag{A.21}$$

We proceed as in case II.

Case V. This case corresponds to a subordinator and is of no concern to us.

Case VI. This case covers Brownian motion, and the entropy is clearly finite.

Let us now consider the cases, where the Lévy measure is concentrated on the negative real line.

Case II'. Suppose $\nu((0,+\infty)) = 0$ and $\int_{-1\le x<0} xU(dx) = -\infty$. As

$$b_a = b_1 - \int\limits_{\{-1\le x<-a\}} xU(dx) \tag{A.22}$$

we can find $a > 0$ such that

$$b_a + \frac{c}{2} + \int_{\{-a\le x<0\}} (e^x - 1 - x)U(dx) > 0. \tag{A.23}$$

We use

$$\psi = 0, \qquad y(x) = \begin{cases} \alpha & x \le -a \\ 1 & x > a, \end{cases} \tag{A.24}$$

where α is a finite, positive constants, determined as

$$\alpha = -\frac{b_a + \frac{c}{2} + \int_{\{-a \le x < 0\}} (e^x - 1 - x)U(dx)}{\int_{\{x < -aa\}} (e^x - 1)U(dx)}. \tag{A.25}$$

Obviously the entropy is finite.

Case III'. Suppose $U((0,+\infty)) = 0$, $\int_{-1 \le x < 0} xU(dx) > -\infty$, and $c > 0$. We take

$$\psi = -\frac{1}{2} - \frac{1}{c}\left[b_1 + \int\limits_{\{-1 \le x < 0\}} (e^x - 1 - x)U(dx) + \int\limits_{\{x < -1\}} (e^x - 1)U(dx)\right] \tag{A.26}$$

and

$$y(x) = 1. \tag{A.27}$$

The entropy is finite.

Case IV'. Suppose $U((0,+\infty)) = 0$, $U((-\infty,0)) > 0$, $\int_{-1 \le x < 0} xU(dx) > -\infty$, $c = 0$, $b_0 > 0$. We can find $a > 0$, such that $U((-\infty,a)) > 0$ and

$$b_a + \int\limits_{\{-1 \le x < 0\}} (e^x - 1 - x)U(dx) > 0. \tag{A.28}$$

We proceed as in case II'.

Case V'. This case corresponds to the negative of a subordinator and is of no concern to us. □

A.3 Proof of Theorem 8

To prove Theorem 8 we first show two lemmas.

Lemma 1. *Suppose X is neither decreasing nor increasing. Then*

$$\inf_{\theta < 0} \tilde{\kappa}'(\theta) < 0. \tag{A.29}$$

Proof: Suppose that X has no negative jumps, no Brownian component, and finite variation. Then

$$\tilde{\kappa}'(\vartheta) = \tilde{b} + \int_0^{+\infty} (e^{\vartheta(e^x - 1)}(e^x - 1) - h(e^x - 1))U(dx). \tag{A.30}$$

We can apply the Monotone Convergence Theorem. If X has negative jumps then there is a number $\epsilon > 0$, such that

$$\int_{-\infty}^{-\epsilon} (e^x - 1)U(dx) < 0. \tag{A.31}$$

If X has a Brownian component, then $\tilde{X}$ has the same, and its second characteristic satisfies $\tilde{c} > 0$. Suppose $\vartheta \le -\epsilon$ and let us use from now on in this proof the truncation function $h(x) = xI|x| \leq 1$. We have

$$\tilde{\kappa}'(\vartheta) = \tilde{b} + \tilde{c}\vartheta + \int_{-\infty}^{-\epsilon} (e^{\vartheta(e^x-1)} - 1)(e^x - 1)U(dx)$$
$$+ \int_{-\epsilon}^{\ln 2} (e^{\vartheta(e^x-1)} - 1)(e^x - 1)U(dx) + \int_{\ln 2}^{+\infty} e^{\vartheta(e^x-1)}(e^x - 1)U(dx) \tag{A.32}$$
$$\leq \tilde{b} + \tilde{c}\vartheta + (e^{\vartheta(e^{-\epsilon}-1)} - 1) \int_{-\infty}^{-\epsilon} (e^x - 1)U(dx) + \int_{\ln 2}^{+\infty} e^{-\epsilon(e^x-1)}(e^x - 1)U(dx). \tag{A.33}$$

This follows from elementary inequalities for the first and third integrand in (A.32), and the observation that the second integrand is negative. Recalling $\tilde{c} \geq 0$ we obtain the desired limit. Suppose now that X has no negative jumps, no Brownian component, but infinite variation. Then

$$\int_0^{\ln 2} (e^x - 1)U(dx) = \infty. \tag{A.34}$$

We have

$$\tilde{\kappa}'(\vartheta) = \tilde{b} + \int_0^{\ln 2} (e^{\vartheta(e^x-1)} - 1)(e^x - 1)U(dx) + \int_{\ln 2}^{+\infty} e^{\vartheta(e^x-1)}(e^x - 1)U(dx). \tag{A.35}$$

Applying Fatou's Lemma to the first integral, and the Monotone Convergence Theorem to the second we obtain the desired conclusion for $\theta \to -\infty$. □

Lemma 2. *Suppose X is neither increasing nor decreasing and the jumps of X are bounded from above. Then $\bar{\theta} = +\infty$ and*

$$\sup_{\theta > 0} \tilde{\kappa}'(\theta) \geq 0. \tag{A.36}$$

Proof: Suppose the Lévy process X is neither increasing nor decreasing, and its jumps are bounded from above. Then the jumps of $\tilde{X}$ are bounded from above and below and $\tilde{X}$ has moments of all orders. Thus we can work with the truncation function $h(x) = x$. Under the given assumptions $E[|\tilde{X}_1|e^{\theta \tilde{X}_1}] < \infty$ for all $\theta \in \mathbb{R}$. Suppose the jumps of $\tilde{X}$ are bounded by $r > 1$. We have

$$\tilde{\kappa}(\theta) = \tilde{b}\theta + \tilde{c}\frac{\theta^2}{2} + \int_{(-1,0)} (e^{\theta x} - 1 - x\theta)\tilde{U}(dx) + \int_{(0,r]} (e^{\theta x} - 1 - x\theta)\tilde{U}(dx) \tag{A.37}$$

and

$$\tilde{\kappa}'(\theta) = \tilde{b} + \tilde{c}\theta + \int_{(-1,0)} (e^{\theta x} - 1)x\tilde{U}(dx) + \int_{(0,r]} (e^{\theta x} - 1)x\tilde{U}(dx) \tag{A.38}$$

Case (i): Suppose there is a diffusion component or there are positive jumps. Then

$$\lim_{\theta \to +\infty} \tilde{c}\theta + \int_{(0,r]} (e^{\theta x} - 1)x\tilde{U}(dx) = +\infty \tag{A.39}$$

while $\int_{(0,r]} (e^{\theta x} - 1)x\tilde{U}(dx)$ remains bounded as $\theta \to +\infty$. So $\tilde{\kappa}'(\theta) \to +\infty$ as $\theta \to +\infty$.

Case (ii): Suppose there is no diffusion component and there are no positive jumps. Then

$$\tilde{\kappa}'(\theta) = \tilde{b} + \int_{(-1,0)} (e^{\theta x} - 1)x\tilde{U}(dx) \tag{A.40}$$

and

$$\lim_{\theta\to+\infty} \tilde{\kappa}'(\theta) = \tilde{b} - \int_{(-1,0)} x\tilde{U}(dx). \tag{A.41}$$

As we are working with $h(x) = x$ the expression on the right hand side is the linear drift of $\tilde{X}$. Since we assumed that X, thus $\tilde{X}$ is not decreasing, this quantity has to be positive. □

To complete the proof of Theorem 8 we need the following proposition.

Proposition 4. *Suppose the minimum entropy martingale measure for the exponential Lévy process e^X exists. Then it is the Esscher martingale transform for the linear Lévy process $\tilde{X}$.*

Proof: Suppose X is neither increasing nor decreasing and its jumps are bounded from above From Lemma 2 we see, that the Esscher martingale measure P^* for the linear process $\tilde{X}$ exists. By [ES05, Theorem B] we conclude the minimal entropy measure exists and coincides with P^*.

It remains to treat the case, when the jumps of X are not bounded from above. For ease of notation and without loss of generality we assume $T = 1$. Suppose the minimum entropy measure exits. By [ES05, Theorem B] it is obtained via a structure preserving change of measure with deterministic and time-independent Girsanov parameters (ψ_0, y_0) with respect to X. They satisfy the martingale constraint

$$b + c\Big(\psi_0 + \frac{1}{2}\Big) + \int ((e^x - 1)y_0(x) - h(x))U(dx) = 0 \tag{A.42}$$

and the minimal entropy is

$$I(0) = \frac{1}{2}c\psi_0^2 + \int (y_0(x)\ln y_0(x) - y_0(x) + 1)\, U(dx). \tag{A.43}$$

Suppose A is an arbitrary compact subset of $\mathbb{R}\setminus\{0\}$. Since $y_0(x) > 0$ U-a.e. and $y\ln y - y + 1 \leq y$ for $y \geq e^2$ we can find $r_1 > (\max A)_+$ and $r_2 > r_1$ such that

$$0 < \int_B (e^x - 1)y_0(x)U(dx) < \infty, \tag{A.44}$$

where $B = [r_1, r_2]$. Let

$$\alpha = \int_A (e^x - 1)y_0(x)U(dx), \qquad \beta = \int_B (e^x - 1)y_0(x)U(dx), \tag{A.45}$$

and set

$$y_\delta(x) = \left(1 + \delta I_A(x) - \frac{\delta\alpha}{\beta} I_B(x)\right) y_0(x). \tag{A.46}$$

The pair (ψ_0, y_δ) is for

$$|\delta| < \frac{\beta}{1 + |\alpha|} \tag{A.47}$$

the Girsanov pair corresponding to changing to an equivalent martingale measure. The entropy

$$I(\delta) = \int (y_\delta(x)\ln y_\delta(x) - y_\delta(x) + 1)\, U(dx) \tag{A.48}$$

must have a minimum at $\delta = 0$. By splitting the integral into contributions from A, B, and $\mathbb{R} \setminus (A \cup B)$ we can justify by elementary arguments differentiation under the integral sign. We have $I'(0) = 0$, with

$$I'(0) = \int \ln y_0(x) \left(I_A(x) - \frac{\alpha}{\beta} I_B(x) \right) y_0(x) U(dx). \tag{A.49}$$

In a similar way we can check $I''(0) > 0$. We can rewrite (A.49) as

$$\int_A y_0(x) \ln y_0(x) U(dx) = \frac{\alpha}{\beta} \int_B y_0(x) \ln y_0(x) U(dx). \tag{A.50}$$

Let

$$\theta = \frac{1}{\beta} \int_B y_0(x) \ln y_0(x) U(dx). \tag{A.51}$$

Then (A.50) can we rewritten as

$$\int_A (\ln y_0(x) - \theta(e^x - 1)) y_0(x) U(dx) = 0. \tag{A.52}$$

Now θ depends on B, and thus to some extent on A. But we can use the same B, thus the same θ for any compact subset $A' \subseteq A$. This implies

$$\ln y_0(x) = \theta(e^x - 1) \tag{A.53}$$

U-a.e. on A. By considering an increasing sequence of compact sets approaching $\mathbb{R} \setminus \{0\}$ we see that (A.53) holds U-a.e. That shows, that y_0 corresponds to the Esscher martingale transform for the linear Lévy process $\tilde{X}$. □

A.4 Proof of Theorem 9

We know from the assumptions, that $\tilde{\kappa}'(\bar{\theta}) < 0$. This implies $E[|\tilde{X}_1| e^{\bar{\theta}\tilde{X}_1}] < \infty$. Thus $E[|\tilde{X}_1|] < \infty$, or equivalently, $\int_{x>1} x \tilde{U}(dx) < \infty$. We also have $E[e^{\bar{\theta}\tilde{X}_1}] < \infty$. Let us use $h(x) = x I_{|x| \leq 1}$ as truncation function. We consider changes of measure with Girsanov parameters with respect to $\tilde{X}$ given by

$$\psi = \bar{\theta}, \qquad y(x) = \begin{cases} e^{\bar{\theta}x} & x \leq 1 \\ e^{\theta_n x} & 1 < x \leq n \\ 1 & x > n, \end{cases} \tag{A.54}$$

with θ_n to be defined by the martingale condition as follows: The function

$$\tilde{f}(\theta) = \tilde{b} + \tilde{c}(\bar{\theta} + 1/2) + \int_{x<1} (x e^{\bar{\theta}x} - h(x)) \tilde{U}(dx) + \int_{1<x\leq n} x e^{\theta x} \tilde{U}(dx) + \int_{x>n} x \tilde{U}(dx) \tag{A.55}$$

is increasing in θ. We have $\tilde{f}(\bar{\theta}) < 0$ and $\tilde{f}(\theta) \to +\infty$ as $\theta \to +\infty$, at least for sufficiently large n. If we define θ_n to be a solution to $\tilde{f}(\theta_n) = 0$, then θ_n is decreasing to $\bar{\theta}$ as $n \to \infty$. Let P^n denote the corresponding measure.

We have seen above $\int_{x>1} x \tilde{U}(dx) < \infty$, thus $\int_{x>n} x \tilde{U}(dx)$ vanishes as $n \to \infty$. From $f(\theta_n) = 0$ we conclude

$$\lim_{n\to\infty} \int_{1<x\leq n} x e^{\theta x} \tilde{U}(dx) = - \left[\tilde{b} + \tilde{c}(\bar{\theta} + 1/2) + \int_{x<1} (x e^{\bar{\theta}x} - h(x)) \tilde{U}(dx) \right]. \tag{A.56}$$

Since $\theta_n > \bar{\theta}$ the integrand in the following integral is nonnegative, and by the Fatou Lemma

$$\lim_{n\to\infty} \int (e^{\theta_n x} - e^{\bar{\theta}x}) I_{\{1<x\leq n\}}(x)\tilde{U}(dx) = 0, \tag{A.57}$$

and thus

$$\lim_{n\to\infty} \int_{1<x\leq n} e^{\theta_n x}\tilde{U}(dx) = \int e^{\bar{\theta}x}\tilde{U}(dx). \tag{A.58}$$

Without loss of generality let us assume $T = 1$. Then the entropy is

$$I(P^n, P) = \frac{1}{2}c\bar{\theta}^2 + \int_{x\leq 1} (e^{\bar{\theta}x}(\bar{\theta}x - 1) + 1)\tilde{U}(dx) + \int_{1<x\leq n} (e^{\theta_n x}(\theta_n x - 1) + 1)\tilde{U}(dx). \tag{A.59}$$

$$\begin{aligned} &\int_{1<x\leq n} (e^{\theta_n x}(\theta_n x - 1) + 1)\tilde{U}(dx) \qquad\qquad (A.60) \\ &\quad = \int\limits_{1<x\leq n} \tilde{U}(dx) + \int_{1<x\leq n} e^{\theta_n x}\tilde{U}(dx) + \theta_n \int\limits_{1<x\leq n} x\tilde{U}(dx) - \int\limits_{1<x\leq n} e^{\theta_n x}\tilde{U}(dx). \end{aligned}$$

Letting $n \to \infty$ we obtain from the previous arguments

$$\lim_{n\to\infty} I(P^n, P) = -\tilde{\kappa}(\bar{\theta}). \tag{A.61}$$

So we have proved that the value $\tilde{\kappa}(\bar{\theta})T$ is approached by the entropy of a sequence of equivalent martingale measures. Let us now show that this value is actually a lower bound for the relative entropy. We follow the proof of Theorem 3.1 in[FM03, p.520]: Suppose $Q \ll P$ is a probability measure, such that $(\tilde{X})_{0\leq t\leq T}$ is a local martingale under Q. Let τ_n be a localizing sequence of stopping times, taking values in $[0, T]$ and tending Q-a.s. to T. For $m \geq 1$ let

$$\check{X}_t^m = \sum_{s\leq t} \Delta\tilde{X}_s I_{\{1<\Delta\tilde{X}_s\leq m\}} \tag{A.62}$$

and

$$\bar{X}_t^m = \tilde{X}_t - \check{X}_t^m. \tag{A.63}$$

Then $\check{X}^m$ and $\bar{X}^m$ are two independent Lévy processes with cumulant functions

$$\check{\kappa}_m(z) = \int_{(1,m]} (e^{zx} - 1)\tilde{U}(dx) \tag{A.64}$$

and

$$\bar{\kappa}_m(z) = \tilde{\kappa}(z) - \check{\kappa}_m(z). \tag{A.65}$$

We consider an arbitrary sequence θ_m increasing to $\bar{\theta}$, such that $\theta_m < \bar{\theta}$ for all $m \geq 1$. We can find $m_0 \geq 1$ such that $\tilde{U}((1, m]) > 0$ for all $m \geq m_0$. Let us define now for $m \geq m_0$ the measures R^m by

$$\frac{dR^m}{dP} = e^{N_T^m}, \tag{A.66}$$

where

$$N_t^m = \theta_m\tilde{X}_t + \epsilon_m\check{X}_t^m - \bar{\kappa}_m(\theta_m)t - \check{\kappa}_m(\theta_m + \epsilon_m)t \tag{A.67}$$

and $\epsilon_m > 0$ is chosen to satisfy

$$E^{R^m}[\tilde{X}_T] = 0. \tag{A.68}$$

Clearly ϵ_m decreases as $m \to \infty$. We observe

$$\ln \left. \frac{dR_m}{dP} \right|_{\mathcal{F}_{\tau_n}} = N^m_{\tau_n}. \tag{A.69}$$

We have

$$I_{\mathcal{F}_T}(Q|P) \geq I_{\mathcal{F}_{\tau_n}}(Q|P) \geq E^Q \left[\ln \left(\left. \frac{dR_m}{dP} \right|_{\mathcal{F}_{\tau_n}} \right) \right] = E^Q[N^m_{\tau_n}]. \tag{A.70}$$

The first and the second inequalities follow from well-known properties of the entropy, see [FM03, Lemma 2.1 (2–3), p.314f]. Now $\tilde{X}$ stopped at τ_n is a martingale under Q and thus $E^Q[\tilde{X}_{\tau_n}] = 0$. The process $\check{X}^m$ is nonnegative, and so

$$E^Q[N^m_{\tau_n}] \geq -(\bar{\kappa}_m(\theta_m) + \check{\kappa}_m(\theta_m + \epsilon_m))E^Q[\tau_n]. \tag{A.71}$$

We have $E^Q[\tau_n] \to T$ by dominated convergence. Finally,

$$\bar{\kappa}_m(\theta_m) + \check{\kappa}_m(\theta_m + \epsilon_m) = \tilde{\kappa}(\theta_m) - \check{\kappa}_m(\theta_m) + \check{\kappa}_m(\theta_m + \epsilon_m) \tag{A.72}$$

and

$$\check{\kappa}_m(\theta_m + \epsilon_m) - \check{\kappa}_m(\theta_m) = \int\limits_{1<x\leq m} (e^{\epsilon_m x} - 1)e^{\theta_m x}\tilde{U}(dx). \tag{A.73}$$

The integrand is nonnegative, and another application of Fatou's Lemma shows that this integral vanishes as $m \to \infty$. As $\tilde{\kappa}(\theta_m) \to \tilde{\kappa}(\bar{\theta})$ for $m \to \infty$ we conclude

$$I(Q,P) \geq -\tilde{\kappa}(\bar{\theta})T \tag{A.74}$$

and we are done. □

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