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Sums of Dependent Lognormal Random Variables: Asymptotics and Simulation

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Abstract

Let $(Y_1, \ldots, Y_n)$ have a joint n-dimensional Gaussian distribution with a general mean vector and a general covariance matrix, and let $X_i = \mathrm{e}^{Y_i}$, $S_n = X_1 + \cdots + X_n$. The asymptotics of $\mathbb{P}(S_n > x)$ as $n \to \infty$ is shown to be the same as for the independent case with the same lognormal marginals. Further, a number of simulation algorithms based on conditional Monte Carlo ideas are suggested and their efficiency properties studied.

Key words: Bounded relative error, complexity, conditional Monte Carlo, Gaussian copula, logarithmic efficiency, rare event, simulation, subexponential distribution, tail asymptotics, value at risk

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1 Introduction

Tail probabilities $\mathbb{P}(S_n > x)$ of a sum $S_n = X_1 + \cdots + X_n$ of heavy-tailed risks $X_1, \ldots, X_n$ is of major importance in applied probability and its applications in risk management, such as the determination of risk measures like the value-at-risk (VaR) for given portfolios of risks, evaluation of credit risk, aggregate claims distributions in insurance, operational risk ([13], [18] etc. Under the assumption of independence among the risks, the situation is well understood. In particular, from the very definition of subexponential distributions, given identical marginal distributions, the maximum among the involved risks determines the distribution of the sum and, on the other hand, for non-identical marginals the distribution of the sum is determined by the component with the heaviest tail (see e.g. Asmussen [4, Ch.IX]).

Over the last few years, several results in this direction have been developed. A survey and some new results are given in Albrecher & Asmussen [2]. For regularly varying marginals and $n = 2$, [2] gives bounds in terms of the tail dependence coefficient

$$\lambda = \lim_{u \to 1} \mathbb{P}\big(F_2(X_2) > u \,\big|\, F_1(X_1) > u\big)$$

and it is noted that the asymptotics of $\mathbb{P}(S_n > x)$ is the same as in the independent case when $\lambda = 0$. For general discussion of bounds, see further Denuit *et al.* [14], Cossette *et al.* [12] and Embrechts & Puccetti [16]. Finally Wüthrich [23] and Alink *et al.* [3] gave sharp asymptotics of the tail of S_n in the case of an Archimedean copula.

The overall picture is that, except for some special cases, the situation seems best understood with regularly varying marginals. This paper deals with the basic case of lognormal marginals with a multivariate Gaussian copula, which appears particularly important for applications to insurance and finance. That is, we can write $X_k = \mathrm{e}^{Y_k}$ where the random vector $(Y_1, \ldots, Y_n)$ has a multivariate Gaussian distribution with $\mathbb{E}\, Y_k = \mu_k$, $\mathbb{V}\mathrm{ar}\ Y_k = \sigma_k^2$ and $\mathbb{C}\mathrm{ov}(Y_k, Y_\ell) = \sigma_{k\ell}$ (here $\sigma_{kk} = \sigma_k^2$). The marginal density of X_k is

$$\frac{1}{x\sqrt{2\pi}\sigma_k} \exp\Big\{ - \frac{(\log x - \mu_k)^2}{2\sigma_k^2} \Big\}$$

Our investigations go in two directions, asymptotics of $\mathbb{P}(S_n > x)$ as $x \to \infty$, and how to develop simulation algorithms which are efficient for large x. We next state and discuss our results in these two directions. Numerical illustrations are then given in Section 4, together with a discussion of the main findings, and the proofs of the theoretical results can be found in Sections 5–6.

2 Asymptotic Results

The asymptotics of $\mathbb{P}(S_n > x)$ is a problem with a well-known solution when the X_k are independent. More precisely, let

$$\sigma^2 = \max_{k=1,\ldots,n} \sigma_k^2\,, \quad \mu = \max_{k:\,\sigma_k^2 = \sigma^2} \mu_k\,, \quad m_n = \#\left\{k : \sigma_k^2 = \sigma^2,\ \mu_k = \mu\right\}. \tag{1}$$

Since the tail behaviour is well-known to be

$$\mathbb{P}(X_k > x) \;\sim\; \frac{\sigma_k}{\sqrt{2\pi}\log(x-\mu_k)} \exp\Big\{-\frac{(\log x-\mu_k)^2}{2\sigma_k^2}\Big\}$$

it follows that the lognormal distribution $\overline{F}_{\mu,\sigma}(x)$ with parameters μ, σ given by (1) is the heaviest among the marginal distributions of the X_k. Therefore by standard subexponential theory

$$\mathbb{P}(S_n > x) \;\sim\; m\overline{F}_{\mu,\sigma}(x) \;\sim\; \frac{\sigma m_n}{\sqrt{2\pi}(\log x-\mu)} \exp\Big\{-\frac{(\log x-\mu)^2}{2\sigma^2}\Big\}$$

where $\overline{F}_{\mu,\sigma}(x)$ is the heaviest tail and m_n the number of summands with this tail.

Our main asymptotic result on $\mathbb{P}(S_n > x)$ states that this remains true in the dependent setting under consideration:

Theorem 2.1. *Let $X_k = e^{Y_k}$ where the Y_k's have multivariate normal distribution with $\mu_k = \mathbb{E}[Y_k]$ and $\sigma_k^2 = \mathbb{V}\mathrm{ar}\,[Y_k]$. Let $S_n = X_1 + \cdots + X_n$ and let σ^2, μ, m_n be defined by* (1)*. Then*

$$\mathbb{P}(S_n > x) \sim m_n\overline{F}_{\mu,\sigma}(x) \tag{2}$$

Remark 2.1. As will be seen, our proof that $m_n\overline{F}_{\mu,\sigma}(x)$ is an asymptotic lower bound for $\mathbb{P}(S_n > x)$ essentially just uses the well-known tail independence of Gaussian copulas. It seems therefore reasonable to ask whether Theorem 2.1 remains valid when considering the same marginals but a different copula with tail independence. An example in Albrecher & Asmussen [2] shows that this is not the case. □

Remark 2.2. Our numerical results show that the approximation in Theorem 2.1 is rather accurate but tends to underestimate. This could be explained by some r.v. X_i with tails being only slightly lighter than the heaviest one. This suggests considering the adjusted approximation

$$\mathbb{P}(S_n > x) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > x) \tag{3}$$

which obviously has the same asymptotics. □

3 Simulation Algorithms

We next consider simulation of $\mathbb{P}(S_n > x)$. By an *estimator*, we understand a r.v. $Z(x)$ that can be generated by simulation and is unbiased, i.e. $\mathbb{E}\,Z(x) = \mathbb{P}(S_n > x)$. The standard efficiency concepts in rare event simulation are bounded relative error, meaning

$$\limsup_{x\to\infty} \frac{\mathbb{V}\mathrm{ar}\;Z(x)}{\mathbb{P}(S_n > x)^2} < \infty$$

and the slightly weaker concept of logarithmic efficiency, where one only requires

$$\limsup_{x\to\infty} \frac{\mathbb{V}\mathrm{ar}\;Z(x)}{\mathbb{P}(S_n > x)^{2-\epsilon}} = 0$$

for all $\epsilon > 0$, or, equivalently, that

$$\limsup_{x\to\infty} \frac{|\log \mathbb{V}\text{ar } Z(x)|}{2|\log \mathbb{P}(S_n > x)|} \geq 1$$

For background, see Asmussen & Rubinstein [10], Heidelberger [21] and Asmussen & Glynn [8].

For the i.i.d. case, the first logarithmically efficient algorithm with heavy tails was given by Asmussen & Binswanger [6] in the regularly varying case. It simulates $X_1, \ldots, X_n$, forms the order statistics $X_{(1)} < \cdots < X_{(n)}$, discards the largest and returns the conditional Monte Carlo estimator

$$Z_1(x) = \mathbb{P}\big(S_n > n \,\big|\, X_{(1)}, \ldots, X_{(n-1)}\big) = \frac{\overline{F}((x - S_{(n-1)}) \vee X_{(n-1)})}{\overline{F}(X_{(n-1)})}$$

where $S_{(n-1)} = X_{(1)} + \cdots + X_{(n-1)}$. For the i.i.d. lognormal case, logarithmic efficiency was established in Binswanger [11]. The algorithm generalizes immediately to the dependent setting, with the modification that the conditional expectation comes out in a different way. For each k, let F_k^c denote the conditional distribution of X_k given the X_ℓ with $\ell \neq k$. Well-known formulas for conditioning in the normal distribution show that F_k^c is a lognormal distribution with parameters μ_k^c, σ_k^{2c} where

$$\mu_1^c = \mu_1 + \big(\sigma_{12} \ldots \sigma_{1n}\big)\Sigma_1^{-1}\big(\log X_2 - \mu_2 \ldots \log X_n - \mu_n\big)^t$$

$$\sigma_1^{2c} = \sigma_1^2 - \big(\sigma_{12} \ldots \sigma_{1n}\big)\Sigma_1^{-1}\begin{pmatrix} \sigma_{12} \\ \vdots \\ \sigma_{1n} \end{pmatrix} \quad \text{where} \quad \Sigma_1 = \begin{pmatrix} \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \ddots & \vdots \\ \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}$$

and similar formulas hold for $k > 1$. Let K be the k with $X_k = X_{(n)}$.

Theorem 3.1. *Let $X_1, \ldots, X_n$ be lognormal random variables with Gaussian copula, $X_{(1)}, \ldots, X_{(n)}$ the corresponding order statistics, $S_{(n-1)} = X_{(1)} + \cdots + X_{(n-1)}$, then*

$$Z_2(x) = \frac{\overline{F}^c_K((x - S_{(n-1)}) \vee X_{(n-1)})}{\overline{F}^c_K(X_{(n-1)})} \tag{4}$$

is an unbiased estimator of $\mathbb{P}(S_n > x)$ and

$$\limsup_{x\to\infty} \frac{\mathbb{V}\text{ar } Z(x)}{\mathbb{P}(S_n > x)^{2-\epsilon}} = 0$$

for all $\epsilon > 2\rho/(1+\rho)$ where $\rho = \max\{\mathbb{C}\text{orr}(X_i, X_j) : i \neq j\}$.

An inspection of the proof in Section 6 shows that the lower bound on ϵ cannot be improved. That is, the algorithm provides variance reduction but less and less so as ρ increases. Of course, the case $\rho = 0$ covers logarithmic efficiency in the independent case shown in [11].

A somewhat different conditional Monte Carlo idea was suggested by Asmussen & Kroese [9], who used a (trivial) symmetry argument to note that the following estimator is unbiased in the i.i.d. case. It simulates $X_1, \ldots, X_{n-1}$, takes the maximum

M_{n-1} and returns the estimator $n\overline{F}(M_{n-1} \vee (x - S_{n-1}))$. The next modification provides a first approach to the exchangeable case:

$$Z_3(x) = n\overline{F}^c(M_{n-1} \vee (x - S_{n-1})) \tag{5}$$

where $F^c = F_1^c = \cdots = F_n^c$. The algorithm is shown to have bounded relative error in the i.i.d. case with regularly varying marginals in [9]. That the same conclusion is true with lognormal marginals is a special case of the following result and was also shown independently by Kortschak [22] (but note that only independence is considered there). In fact:

Theorem 3.2. *The estimator $Z_3(x)$ has bounded relative error in the exchangeable lognormal case.*

We next consider the extension for the general non-exchangeable dependent case. For each i, simulate independently the set $\{X_{j,i} : j \neq i\}$ with the same distribution as $\{X_j : j \neq i\}$, let $M_{n,i} = \max\{X_{j,i} : j \neq i\}$, $S_{n,i} = \sum\limits_{j \neq i} X_{j,i}$ and return

$$Z_4(x) = \sum_{i=1}^{n} \overline{F}_i^c(M_{n,i} \vee (x - S_{n,i})) \tag{6}$$

A faster and simpler version of this algorithm is obtained when simulating just the set $\{X_1, \ldots, X_n\}$ and considering $M_{n,i} = \max\{X_j : j \neq i\}$ and $S_{n,i} = \sum_{j \neq i} X_j$. In practice, the first version showed smaller variances.

Theorem 3.3. *Both versions of the estimator $Z_4(x)$ are unbiased and have bounded relative error.*

4 Numerical Examples

Example 1. We considered 10 lognormal r.v. with multivariate Gaussian copula and parameters $\mu = i - 10$, $\sigma_i^2 = i$ and $\sigma_{ij} = 0.4\sigma_i\sigma_j$ (The value 0.4 of the correlations is often claimed to be typical for financial data).

The first graph in figure 1 contains the approximations in Proposition 2.1 and Remark 2.2 and the simulated values for the tail probability $\mathbb{P}(S_{10} > x)$ with the associated 95% confidence interval corresponding to $R = 5000$ replications using the first version of the estimator $Z_4(x)$.

The second graph shows the relative difference (difference among the two approximations divided by the largest one) goes to 0. The graph is compatible with the (obvious) fact that the relative diffence goes to 0, but shows that the convergence is slow in the present case, as must be expected from the fact that the second largest variance 9 is quite close to the largest one 10.

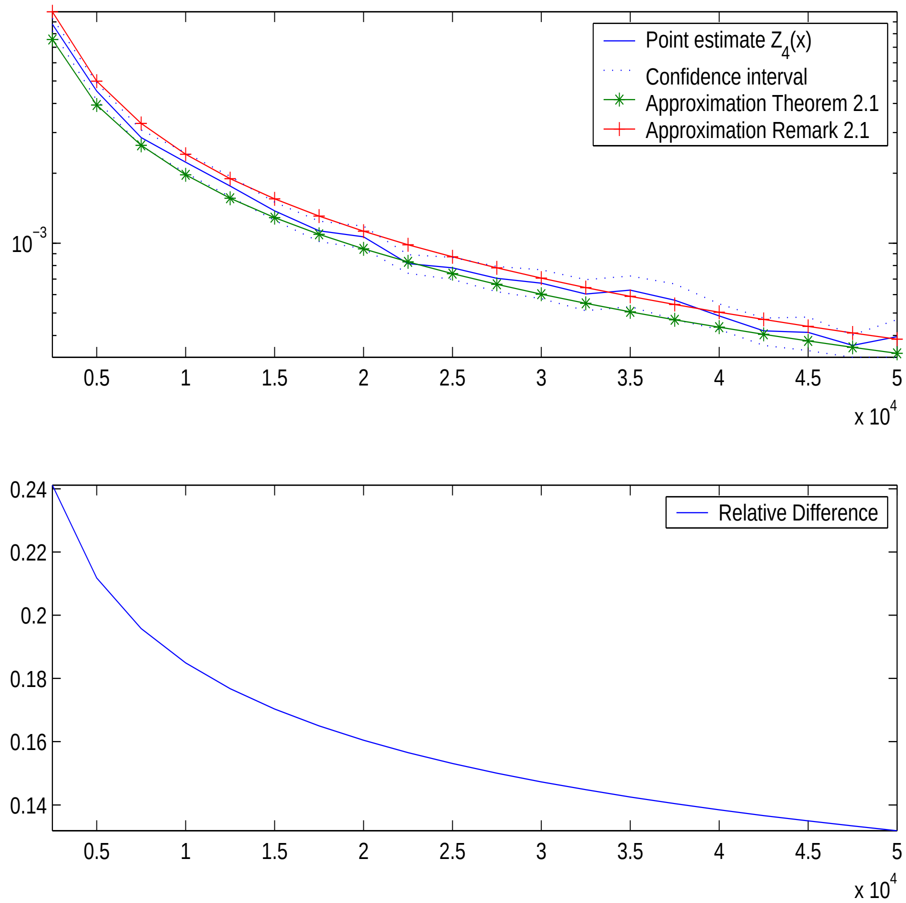

Figure 1: Tail probability $\mathbb{P}(S_{10} > x)$.
Comparison of approximations and simulations.

Example 2. We use the same $n = 10$ and the same parameters as in Example 1.

Figure 2 shows the comparison of the estimator $Z_2(x)$ and the two versions of the estimator $Z_4(x)$ using $R = 5000$ replications (no importance sampling technique used). A general algorithm by Glasserman, Heidelberger & Shahabuddin [20] was used for comparison purposes using $R = 50.000$ replications. The particular interest in this comparison is that [20] appears to be more or less the only attempt to perform efficient rare event simulation in a depending setting incorporating the present problem of evaluating $\mathbb{P}(S_n > x)$. The idea behind the algorithm is a delta-gamma approximation which leads to an importance sampling scheme requiring a rootfinding. We refer to [20] for the details for the general case; the implementation for our dependent sum setting is straightforward and we omit the details.

The four panels of Figure 2 give the point estimates, the estimates of the variance $\mathbb{V}\mathrm{ar}\ Z_i(x)$ and the similar quantity for the GHS estimator, the observed execution times and finally the observed execution times multiplied by the variance estimates. The interpretation of this last quantity is a variance per unit computer time, so that the comparisons of the different algorithms give a measure of how efficiently they use computer time.

We note in particular in the third panel that all of our $Z_i(x)$ algorithms have an execution time which is negligible compared to the GHS algorithm (the difference can by far be explained by the different values of R). Variances and times elapsed are similar for our $Z_i(x)$. However, the implementation of the second version of $Z_4(x)$ requires less effort.

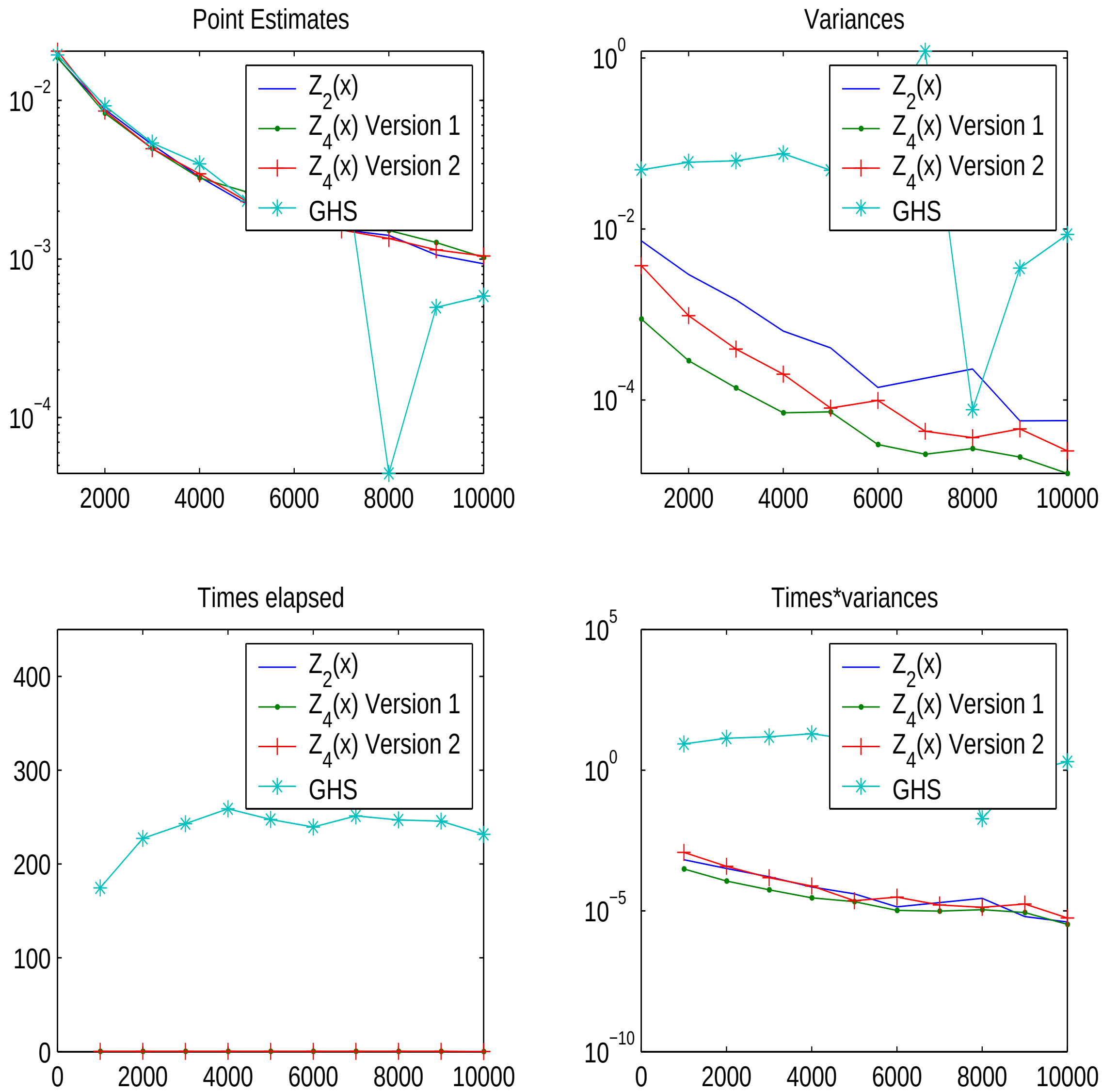

Figure 2: Tail probability $\mathbb{P}(S_{10} > x)$.
Comparison of estimators $Z_2(x)$, $Z_4(x)$ and GHS.

Example 3. In the third example we consider 100 lognormal r.v. with multivariate gaussian copula. Let $F_i \sim N(i - 9, i + 1)$ with $i = 0, \ldots, 9$. In the example, we replicated each F_i 10 times by letting $X_{10i+1}, \ldots, X_{10(i+1)}$ have common distribution F_i. All correlations were again set to 0.4.

Figure 3 provides a comparison of the estimators $Z_2(x)$ and the two versions of $Z_4(x)$ using $R = 10.000$ replications. Although the estimator $Z_2(x)$ has a bigger variance, it provides a considerable faster algorithm than any version of $Z_4(x)$, and for this example it is the most efficient in terms of time-variance comparisons.

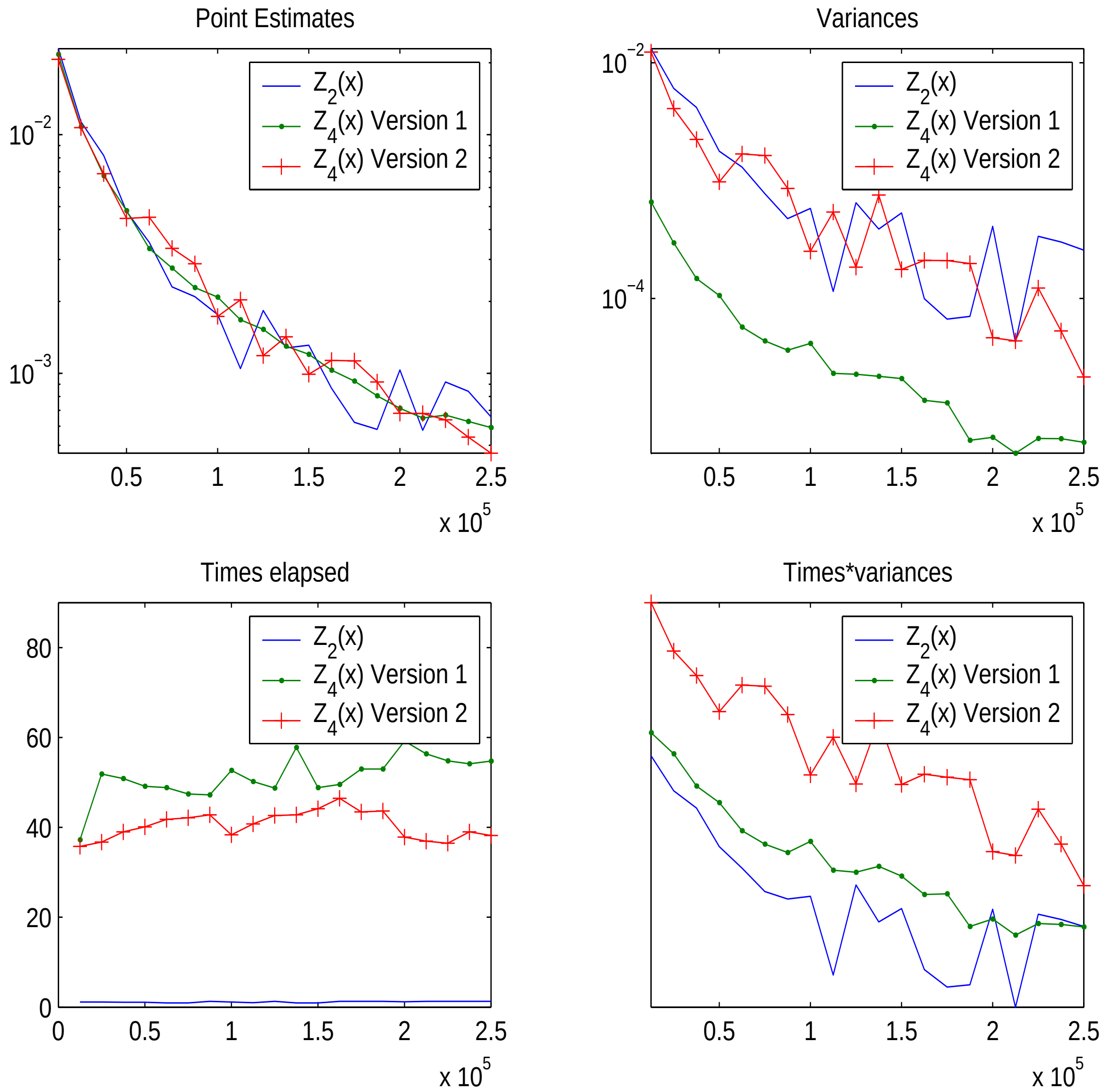

Figure 3: Tail probability $\mathbb{P}(S_{100} > x)$.
Comparison of estimators $Z_2(x)$ and $Z_4(x)$.

Example 4. Consider two lognormal r.v. $X_1 = \mathrm{e}^{Y_1}, X_2 = \mathrm{e}^{Y_2}$, with multivariate Gaussian copula and parameters $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$ and $\sigma_{12} = 0.7$. We approximate the tail probability $\mathbb{P}(S_2 > x)$ using the estimator $Z_3(x)$ plus importance sampling for X_2. The new measure employed for sampling was a lognormal r.v. with parameters $\mu = \sigma_{12}x$ and $\sigma^2 = 1$. This is motivated by the general principle of choosing the importance distribution so close to the conditional distribution given the are event as possible and the fact that Y_2 given $S_2 > x$ is of order $= \sigma_{12}x$, as shown by a heuristical calculation in [2].

Results with $R = 5.000$ replications are shown in Figure 4. For this particular example, the comparison is favorable: lower variance and shorter simulation times are observed in the new algorithm.

The overall picture of the simulation experiments is that the algorithms proposed

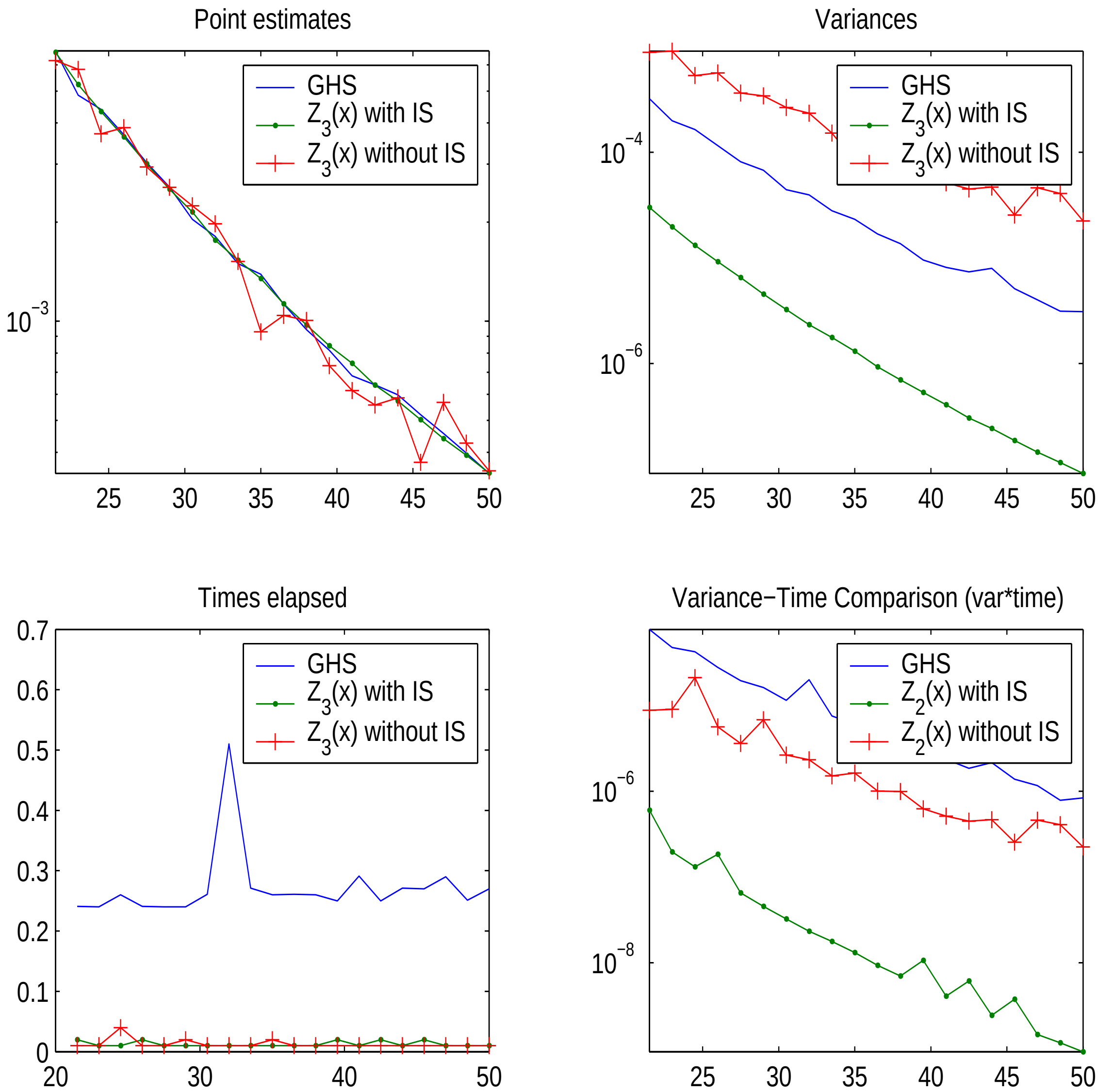

Figure 4: Tail probability $\mathbb{P}(S_2 > x)$

in Section 3 all perform better in terms of the variance per time criterion than the GHS algorithm. To this it should be added that our algorithms also are much simpler to program. Once this is said, one should of course note that the GHS algorithm applies to settings much more general than the present one of dependent sums.

Our estimators $Z_2(x)$, $Z_3(x)$ and $Z_4(x)$ (in both versions) performed in the examples in a rather similar way, even if our theoretical results show that $Z_2(x)$ has to be inferior in the limit $x \to \infty$. As shown for a simple case in Example 4, there is also potential to combine with other variance reduction methods.

5 Proof of Asymptotic Results

Proof of Theorem 2.1. W.l.o.g. consider that $X_1, \dots, X_n$ are ordered such that $X_1 \sim F_{\mu,\sigma}$ and $\mu = 0$ (otherwise replace X_i and x by $X_i e^{-\mu}$ and $xe^{-\mu}$). We use induction. The case $n = 1$ is straightforward. Assume the theorem is true for $n-1$. We need the following lemmas:

Lemma 5.1. *Let* $0 < \beta < 1$. *Then*

$$\mathbb{P}(S_{n-1} > x - x^\beta) \sim \mathbb{P}(S_{n-1} > x)$$

as $x \to \infty$.

Proof. By the hypothesis of the induction, we have

$$\mathbb{P}(S_{n-1} > x - x^\beta) \sim \frac{\sigma_1 m_{n-1}}{\sqrt{2\pi}\log(x - x^\beta)} \exp\{-\frac{\log^2(x - x^\beta)}{2\sigma_1^2}\}$$

Now, just note that

$$\begin{aligned}
\log(x - x^\beta) &= \log x + \log(1 - 1/x^{1-\beta}) \sim \log x \\
\log^2(x - x^\beta) &= \log^2 x + \log^2(1 - 1/x^{1-\beta}) + 2\log x \log(1 - 1/x^{1-\beta}) \\
&= \log^2 x + o(1) - \frac{2\log x}{x^{1-\beta}} = \log^2 x + o(1)\,.
\end{aligned}$$

□

Lemma 5.2. *There exists* $0 < \beta < 1$ *such that*

$$\mathbb{P}(S_{n-1} > x^\beta, X_n > x^\beta)$$

is asymptotically dominated by $\mathbb{P}(X_1 > x)$ *as* $x \to \infty$.

Proof. If $\sigma_1 > \sigma_n$ choose $\frac{\sigma_n}{\sigma_1} < \beta < 1$. Then $\mathbb{P}(S_n > x^\beta, X_n > x^\beta) \le \mathbb{P}(X_n > x^\beta)$, and this has lighter tail than $\mathbb{P}(X_1 > x)$ since $\beta > \frac{\sigma_n}{\sigma_1}$.

If $\sigma_1 = \sigma_n$ choose β such that $\beta^2 > \max\{1/2, \gamma\}$ and $(\beta - \frac{\gamma}{\beta})^2 + \beta^2 > 1$ where $\gamma = \max\{\frac{\sigma_{kn}}{\sigma_n^2}\}$ (observe that this is possible since $\gamma \in (-1, 1)$). Let $\alpha = 1/\beta$ and consider

$$\begin{aligned}
\mathbb{P}(S_{n-1} > x^\beta, X_n > x^\beta) =\ & \mathbb{P}(S_{n-1} > x^\beta, x^\alpha > X_n > x^\beta) \\
& + \mathbb{P}(S_{n-1} > x^\beta, X_n > x^\alpha) \\
\le\ & \mathbb{P}(S_{n-1} > x^\beta, x^\alpha > X_n > x^\beta) + \mathbb{P}(X_n > x^\alpha)
\end{aligned}$$

It follows that $\mathbb{P}(X_n > x^\alpha)$ is asymptotic dominated by $\mathbb{P}(X_1 > x)$ since $\alpha > 1$ and $\sigma_1 = \sigma_n$. Denote $\overline{Y}^c(y) = (Y_1, \dots, Y_{n-1} | Y_n = y)$ and the corresponding definitions for $X^c(y)$ and $S^c_{n-1}(y)$ and consider

$$\mathbb{P}(S_{n-1} > x^\beta, x^\alpha > X_n > x^\beta) = \int_{\beta\log x}^{\alpha\log x} \mathbb{P}(S^c_{n-1}(y) > x^\beta) f_{Y_n}(y)dy$$

Using the standard fact that $Y^c(y) \sim N(\{\mu_i + \frac{\sigma_{in}}{\sigma_n^2}(y-\mu_n)\}_i, \{\sigma_{ij} - b_{ij}\}_{ij})$, where $b_{ij} = \frac{\sigma_{in}\sigma_{jn}}{\sigma_n^2}$, we can bound the last by

$$\int\limits_{\beta \log x}^{\alpha \log x} \mathbb{P}(S_{n-1}^c(0)e^{\gamma y} > x^\beta) f_{Y_n}(y)dy \tag{7}$$

Let K be the index of the $Y_k^c(y)$ with the heaviest tail. If $\gamma < 0$, then last expression is bounded by $\mathbb{P}(S_{n-1}^c(0) > x^\beta)\,\mathbb{P}(X_n > x^\beta)$ which is asymptotically bounded by $m^c\,\mathbb{P}(X_K^c(0) > x^\beta)\,\mathbb{P}(X_n > x^\beta)$ (hypothesis of induction), and this is asymptotically bounded by $m^c\,\mathbb{P}^2(X_1 > x^\beta)$ (because X_K^c and X_n do not have heavier tails than X_1). It follows that

$$m^c\,\mathbb{P}^2(X_1 > x^\beta) \sim \frac{\sigma_1 m^c}{\log^2 x^\beta} \exp\Big\{-\frac{2\beta^2 \log^2 x}{2\sigma_1^2}\Big\}$$

is asymptotically dominated by $\mathbb{P}(X_1 > x)$ since $2\beta^2 > 1$.

Now, consider the case $\gamma > 0$, then the expression (7) is bounded by:

$$\mathbb{P}(S_{n-1}^c(0)e^{\gamma \log x^\alpha} > x^\beta)\,\mathbb{P}(X_n > x^\beta) = \mathbb{P}(S_{n-1}^c(0) > x^{\beta-\gamma\alpha})\,\mathbb{P}(X_n > x^\beta)$$

By the hypothesis of induction last expression is asymptotically equivalent to

$$m^c\,\mathbb{P}(X_K^c(0) > x^{\beta-\gamma\alpha})\,\mathbb{P}(X_n > x^\beta)$$

which is asymptotically bounded above by

$$m^c\,\mathbb{P}(X_1 > x^{\beta-\gamma\alpha})\,\mathbb{P}(X_1 > x^\beta)$$

Observe that $\beta - \gamma\alpha = \beta - \frac{\gamma}{\beta} > 0$ since we choose $\beta^2 > \gamma$. It follows that

$$m^c\,\mathbb{P}(X_n > x^{\beta-\gamma\alpha})\,\mathbb{P}(X_n > x^\beta) \sim \frac{\sigma_1^2 m^c}{2\pi \log x^{\beta-\gamma\alpha} \log x^\beta} \exp\Big\{-\frac{[(\beta-\gamma\alpha)^2 + \beta^2]\log^2 x}{2\sigma_1^2}\Big\}$$

is asymptotically dominated by $\mathbb{P}(X_1 > x)$ since $(\beta - \gamma/\beta)^2 + \beta^2 > 1$. $\square$

For an asymptotic lower bound of $\mathbb{P}(S_n > x)$, consider

$$\begin{aligned}
\mathbb{P}(S_n > x) &\geq \sum_i \mathbb{P}(X_i > x, X_j < x; j \neq i) \\
&= \sum_i \mathbb{P}(X_i > x) - \mathbb{P}(\bigcup_{j\neq i}\{X_i > x, X_j > x\}) \\
&\geq \sum_i \mathbb{P}(X_i > x) - \sum_{j\neq i} \mathbb{P}(X_i > x, X_j > x) \\
&= \sum_i \mathbb{P}(X_i > x) - \sum_{j\neq i} \mathbb{P}(X_j > x | X_i > x)\,\mathbb{P}(X_i > x) \\
&= \sum_i \mathbb{P}(X_i > x) - \sum_{j\neq i} o(1)\,\mathbb{P}(X_i > x) \sim m_n\,\mathbb{P}(X_1 > x)
\end{aligned}$$

where we use the fact that in the multivariate case X_i and X_j are tail independent.

For an asymptotic upper bound, choose β as in the hypotesis of Lemma 5.2 and consider

$$\begin{aligned}\mathbb{P}(S_n > x) &\leq \mathbb{P}(S_n > x, S_{n-1} < x^\beta) + \mathbb{P}(S_n > x, S_{n-1} > x^\beta, X_n < x^\beta) \\ &\quad + \mathbb{P}(S_n > x, S_{n-1} > x^\beta, X_n > x^\beta) \\ &\leq \mathbb{P}(X_n > x - x^\beta) + \mathbb{P}(S_{n-1} > x - x^\beta) + \mathbb{P}(S_{n-1} > x^\beta, X_n > x^\beta)\end{aligned}$$

We use succesively Lemma 5.1 twice, the hypotesis of induction and Lemma 5.2 to get the asymptotic upper bound

$$\begin{aligned}&\mathbb{P}(X_n > x) + \mathbb{P}(S_{n-1} > x) + \mathbb{P}(S_{n-1} > x^\beta, X_n > x^\beta) \\ &\quad \sim \mathbb{P}(X_n > x) + m_{n-1}\,\mathbb{P}(X_1 > x) + \mathbb{P}(S_{n-1} > x^\beta, X_n > x^\beta) \\ &\quad \sim \mathbb{P}(X_n > x) + m_{n-1}\,\mathbb{P}(X_1 > x) \sim m_n\,\mathbb{P}(X_1 > x)\end{aligned}$$

completing the proof. □

6 Proofs of Efficiency of the Simulation Algorithms

Proof of Theorem 3.1. Let $F_{\mu,\sigma}$ be the distribution of the lognormal random variable with the heaviest tail. W.l.o.g. consider that $\mu = 0$ (otherwise replace X_i and x by $X_i e^{-\mu}$ and $x e^{-\mu}$). We start by proving that estimator $Z_2(x)$ is unbiased. Let $\mathcal{F}_{(n-1)} = \sigma(K, X_i : i \neq K)$, hence

$$\begin{aligned}\mathbb{P}(S_n > x) &= \mathbb{E}[\mathbb{P}(S_n > x | \mathcal{F}_{(n-1)})] \\ &= \mathbb{E}[\mathbb{P}(X_{(n)} + S_{(n-1)} > x | \mathcal{F}_{(n-1)})] \\ &= \mathbb{E}[\mathbb{P}(X_K > x - S_{(n)} | \mathcal{F}_{(n-1)})] \\ &= \mathbb{E}\left[\frac{\overline{F}^c_K((x - S_{(n-1)}) \vee X_{(n-1)})}{\overline{F}^c_K(X_{(n-1)})}\right]\end{aligned}$$

To prove the claimed efficiency properties of $Z_2(x)$, we need the following lemmas:

Lemma 6.1. *Let F_1 and F_2 lognormal distributions such that F_2 has a heavier tail than F_1. Then, there exists $c \in \mathbb{R}$ such that*

$$\frac{\overline{F}_1(x)}{\overline{F}_1(y)} \leq c \frac{\overline{F}_2(x)}{\overline{F}_2(y)}$$

for all $y \leq x$.

Proof. Let $\lambda_1(x)$, $\lambda_2(x)$ the corresponding failure rate functions and consider the next function

$$\begin{aligned}[\lambda_1(t) - \lambda_2(t)]^+ &= \lambda_1(t) - \lambda_2(t) + [\lambda_2(t) - \lambda_1(t)]\ \mathbb{I}_{\{t:\lambda_1(t)<\lambda_2(t)\}}(t) \\ &\leq \lambda_1(t) - \lambda_2(t) + \lambda_2(t)\,\mathbb{I}_{\{t:\lambda_1(t)<\lambda_2(t)\}}(t)\end{aligned}$$

We claim that $\{t : \lambda_1(t) < \lambda_2(t)\} \subseteq [0, y_0]$ for some $y_0 \in \mathbb{R}^+$. Consider the case where $\sigma_1 < \sigma_2$. We use the tail asymptotics of $\lambda(x)$ to obtain

$$\lim_{x\to\infty} \frac{\lambda_1(x)}{\lambda_2(x)} = \lim_{x\to\infty} \frac{\frac{\log x}{x\sigma_1^2}}{\frac{\log x}{x\sigma_2^2}} = \frac{\sigma_2^2}{\sigma_1^2} > 1$$

proving our claim. Let's turn to the case $\sigma_1 = \sigma_2$ and $\mu_1 < \mu_2$. It will be enough to check that $\lambda_2(x) \le \lambda_1(x)$ or equivalently that $\lambda(x, \mu)$ is a decreasing of μ:

$$\begin{aligned}
\lambda'(x,\mu) &= \frac{\frac{\log x-\mu}{\sigma^2} f(x,\mu)\overline{F}(x,\mu) - f(x,\mu) \int_x^\infty \frac{\log t-\mu}{\sigma^2} f(t,\mu)dt}{\overline{F}^2(t,\mu)} \\
&= \frac{\log x f(x,\mu)\overline{F}(x,\mu) - f(x,\mu) \int_x^\infty \log t f(t,\mu)dt}{\sigma^2\overline{F}^2(t,\mu)}
\end{aligned}$$

The last expression is negative since

$$\int_x^\infty \log t f(t,\mu)dt > \log x \int_x^\infty f(t,\mu)dt = \log x \overline{F}(x)$$

Now, $\lambda_2(t)$ is real-valued on the closed interval $[0, y_0]$ and hence bounded by continuity. So, we get the next inequality

$$-\lambda_1(t) \le -\lambda_2(t) + c_1 \mathbb{I}_{[0,y_0]}(t)$$

Then

$$\begin{aligned}
\frac{\overline{F}_1(x)}{\overline{F}_1(y)} = \exp\Big\{ -\int_y^x \lambda_1(t)dt \Big\} &\le \exp\Big\{ -\int_y^x \lambda_2(t)dt + \int_y^x c_1 \mathbb{I}_{[0,y_0]}(t)dt \Big\} \\
&\le \exp\Big\{ -\int_y^x \lambda_2(t)dt + \int_0^{y_0} c_1 dt \Big\} \\
&= \exp\Big\{ \log \frac{\overline{F}_2(x)}{\overline{F}_2(y)} + c_2 \Big\} = c\frac{\overline{F}_2(x)}{\overline{F}_2(y)}
\end{aligned}$$

□

Recall that $\overline{F}_{\mu,\sigma}$ is the distribution of the random variable with the heaviest tail.

Lemma 6.2. *Let $\rho = \max\{\mathbb{C}\mathrm{orr}(X_i, X_j) : i \ne j\}$. Then, $\mathbb{V}\mathrm{ar}\ Z_2(x)$ is asymptotically bounded by*

$$h(x)\overline{F}_{\mu,\sigma}(x/n)\big[c_3|\log \overline{F}_{\mu,\sigma}(x/n)| + c_4\big]$$

where

$$h(x) = \sqrt{\frac{(1+\rho)}{2\pi(1-\rho)}} \frac{\sigma}{x} \exp\Big\{ -\frac{\log^2 x}{2\sigma^2}\Big(\frac{1-\rho}{1+\rho}\Big)\Big\}$$

and c_3, c_4 are positive constants.

Proof. Consider:

$$\mathbb{E}\Big[\frac{\overline{F}^{c\ 2}_K((x-S_{n-1})\vee X_{(n-1)})}{\overline{F}^{c\ 2}_K(X_{(n-1)})}\Big] = \mathbb{E}\Big[\frac{\overline{F}^{c\ 2}_K((x-S_{(n-1)})\vee X_{(n-1)})}{\overline{F}^{c\ 2}_K(X_{(n-1)})}; X_{(n-1)} < \frac{x}{n}\Big]$$
$$+\,\mathbb{E}\Big[\frac{\overline{F}^{c\ 2}_K((x-S_{(n-1)})\vee X_{(n-1)})}{\overline{F}^{c\ 2}_K(X_{(n-1)})}; X_{(n-1)} > \frac{x}{n}\Big]$$

If $X_{(n-1)} < x/n$ then $\overline{F}^c_K(x-S_{(n-1)}) < \overline{F}^c_K(x/n)$, so the last expression can be bounded by

$$\mathbb{E}\Big[\frac{\overline{F}^{c\ 2}_K(x/n)}{\overline{F}^{c\ 2}_K(X_{(n-1)})}; X_{(n-1)} < \frac{x}{n}\Big] + \mathbb{E}\Big[1; X_{(n-1)} > \frac{x}{n}\Big]$$

By Lemma 6.1 this can be bounded by

$$c_1\mathbb{E}\Big[\frac{\overline{F}^2_{\mu,\sigma}(x/n)}{\overline{F}^2_{\mu,\sigma}(X_{(n-1)})}; X_{(n-1)} < \frac{x}{n}\Big] + \mathbb{E}\Big[1; X_{(n-1)} > \frac{x}{n}\Big]$$
$$= c_1\overline{F}^2_{\mu,\sigma}(x/n)\int_0^{x/n}\frac{f_{(n-1)}(y)}{\overline{F}^2_{\mu,\sigma}(y)}dy + \overline{F}_{(n-1)}(x/n)$$

Using partial integration we can write this as

$$c_1\overline{F}^2_{\mu,\sigma}(x/n)\Big[-\frac{\overline{F}_{(n-1)}(y)}{\overline{F}^2_{\mu,\sigma}(y)}\Big|_0^{x/n} + 2\int_0^{x/n}\frac{\overline{F}_{(n-1)}(y)f(y)}{\overline{F}^3_{\mu,\sigma}(y)}dy\Big] + \overline{F}_{(n-1)}(x/n)$$
$$\le c_1\overline{F}^2_{\mu,\sigma}(x/n)\Big[1 + 2\int_0^{x/n}\frac{\overline{F}_{(n-1)}(y)f(y)}{\overline{F}^3_{\mu,\sigma}(y)}dy\Big] + \overline{F}_{(n-1)}(x/n) \qquad (8)$$

Let Y_1, Y_2 be lognormal random variables with common distribution $F_{\mu,\sigma}$ and $\mathbb{C}\mathrm{orr}(Y_1, Y_2) = \rho$ (as defined in the hypothesis of the Lemma). Observe that the following inequalities hold asymptotically

$$\overline{F}_{(n-1)}(x) \le \sum_{i>j}\mathbb{P}(X_i > x, X_j > x) \le \sum_{i>j}\mathbb{P}(Y_1 > x, Y_2 > x)$$
$$= \sum_{i>j}\mathbb{P}(Y_2 > x|Y_1 > x)\,\mathbb{P}(Y_1 > x) = \sum_{i>j}\mathbb{P}(Y_2 > x|Y_1 > x)\overline{F}_{\mu,\sigma}(x)$$
$$\le n(n-1)\,\mathbb{P}(Y_2 > x|Y_1 > x)\overline{F}_{\mu,\sigma}(x) \sim c_2 h(x)\overline{F}_{\mu,\sigma}(x)$$

The last is true since $\mu = 0$ and therefore

$$\mathbb{P}(Y_2 > x|Y_1 > x) \sim 2\,\mathbb{P}(Y_2 > x|Y_1 = x) \sim \sqrt{\frac{2(1+\rho)}{\pi(1-\rho)}}\frac{\sigma}{x}\exp\Big\{-\frac{\log^2 x}{2\sigma^2}\Big(\frac{1-\rho}{1+\rho}\Big)\Big\}$$

Hence, equation (8) is asymptotically bounded by

$$c_1\overline{F}^2_{\mu,\sigma}(x/n)\left[1+2c_2\int_0^{x/n}\frac{h(x)}{\overline{F}^2_{\mu,\sigma}(y)}f_{\mu,\sigma}(y)dy\right]+c_2h(x/n)\overline{F}_{\mu,\sigma}(x/n)$$

Observe that $h(x)/\overline{F}_{\mu,\sigma}(x)\to\infty$ (since $h(x)$ has heavier tail). So, we obtain the following asymptotic upper bounds

$$c_1\overline{F}^2_{\mu,\sigma}(x/n)\left[1+2c_2\frac{h(x/n)}{\overline{F}_{\mu,\sigma}(x/n)}\int_0^{x/n}\frac{f_{\mu,\sigma}(y)}{\overline{F}_{\mu,\sigma}(y)}dy\right]+c_2h(x/n)\overline{F}_{\mu,\sigma}(x/n)$$

which is asymptotically bounded by

$$h(x/n)\overline{F}_{\mu,\sigma}(x/n)\big[c_3|\log\overline{F}_{\mu,\sigma}(x/n)|+c_4\big]$$

□

By Lemma 6.2:

$$\begin{aligned}\liminf_{x\to\infty}\frac{|\log\mathbb{V}\mathrm{ar}\ Z_2(x)|}{|\log\mathbb{P}^{(2-\epsilon)}(S_n>x)|}&\geq\liminf_{x\to\infty}\frac{|\log(h(x/n)\overline{F}_{\mu,\sigma}(x/n)\big[c_2|\log\overline{F}_{\mu,\sigma}(x/n)|+c_3\big])}{\log(\overline{F}^{(2-\epsilon)}_{\mu,\sigma}(x))}\\&=\liminf_{x\to\infty}\frac{\log(h(x/n)\overline{F}_{\mu,\sigma}(x/n))}{(2-\epsilon)\log(\overline{F}_{\mu,\sigma}(x))}\end{aligned}\tag{9}$$

since

$$\liminf_{x\to\infty}\frac{\log(\big[c_2|\log\overline{F}_{\mu,\sigma}(x/n)|+c_3\big])}{(2-\epsilon)\log(\overline{F}_{\mu,\sigma}(x))}=0$$

Using L'Hopital rule on expression (9) we obtain:

$$\liminf_{x\to\infty}\frac{\log(h(x/n))+\log(\overline{F}_{\mu,\sigma}(x/n))}{(2-\epsilon)\log(\overline{F}_{\mu,\sigma}(x))}=\liminf_{x\to\infty}\frac{\lambda_h(x/n)+\lambda_{\mu,\sigma}(x/n)}{(2-\epsilon)n\lambda_{\mu,\sigma}(x)}\tag{10}$$

where $\lambda_h(x)$ is the failure rate of a lognormal random variable and is well known to be asymptotically equivalent to

$$\frac{\log x}{\sigma^2x}\frac{(1-\rho)}{1+\rho}$$

So the limit in (10) is equal to

$$\begin{aligned}\liminf_{x\to\infty}\frac{\frac{\log(x/n)}{\sigma^2x/n}\frac{(1-\rho)}{1+\rho}+\frac{\log(x/n)}{\sigma^2x/n}}{(2-\epsilon)n\frac{\log x}{\sigma^2x}}&=\liminf_{x\to\infty}\frac{2}{(2-\epsilon)(1+\rho)}\frac{\log(x/n)}{\log x}\\&=\frac{2}{(2-\epsilon)(1+\rho)}\geq1\end{aligned}$$

where we used the hypothesis $\epsilon\geq\frac{2\rho}{1+\rho}$. □

Proof of Theorem 3.2.

$$\limsup_{x\to\infty}\frac{\mathbb{V}\text{ar } Z_3(x)}{\mathbb{P}^2(S_n>x)} \le \limsup_{x\to\infty}\frac{\mathbb{E}(Z_3^2(x))}{\mathbb{P}^2(X_1>x)}$$
$$\le \limsup_{x\to\infty}\frac{(n+1)^2\mathbb{E}(\overline{F}^2(M_n\vee(x-S_n)))}{\overline{F}^2(x)} \tag{11}$$

If $M_n<x/2n$, then $M_n<x/2<x-nM_n<x-S_n$, so

$$\frac{\mathbb{E}[\overline{F}^2(M_n\vee(x-S_n))]}{\overline{F}^2(x)}$$
$$=\frac{\mathbb{E}[\overline{F}^2(M_n\vee(x-S_n)); M_n<x/2n]+\mathbb{E}[\overline{F}^2(M_n\vee(x-S_n)); M_n>x/2n]}{\overline{F}^2(x)}$$
$$\le \mathbb{E}\Big[\frac{\overline{F}^2(x-nM_n)}{\overline{F}^2(x)}; M_n<x/2n\Big]+\mathbb{E}\Big[\frac{\overline{F}^2(M_n)}{\overline{F}^2(x)}; M_n>x/2n\Big] \tag{12}$$

We consider these two terms separately.

First Integral: The first term in (12) is equivalent to

$$\int_0^{x/2}\frac{\overline{F}^2(x-y)}{\overline{F}^2(x)}f_{M_n}(y/n)dy$$
$$=-\frac{n\overline{F}^2(x-y)\overline{F}_{M_n}(y/n)}{\overline{F}^2(x)}\Bigg|_0^{x/2}+2n\int_0^{x/2}\frac{\overline{F}(x-y)f(x-y)}{\overline{F}^2(x)}\overline{F}_{M_n}(y/n)dy$$
$$<\frac{n\overline{F}^2(x)-n\overline{F}^2(x/2)\overline{F}_{M_n}(x/2n)}{\overline{F}^2(x)}+2n^2\int_0^{x/2}\frac{\overline{F}(x-y)f(x-y)}{\overline{F}^2(x)}\overline{F}(y/n)dy$$
$$<n+2n^2\int_0^{x/2}\frac{\overline{F}^2(x-y)}{\overline{F}^2(x)}\frac{f(x-y)}{\overline{F}(x-y)}\frac{\overline{F}(y/n)}{f(y/n)}f(y/n)dy$$

where we used $\overline{F}_{M_n}(x)<n\overline{F}(x)$. For large x the last expression is bounded by

$$n+2n^2c\frac{\log(x/2)}{x/2}\frac{x/2n}{\log(x/2n)}\int_0^{x/2}\frac{\overline{F}^2(x-y)}{\overline{F}^2(x)}f(y/n)dy$$
$$=n+2nc\frac{\log(x/2)}{\log(x/2n)}\int_0^{x/2}\frac{\overline{F}^2(x-y)}{\overline{F}^2(x)}f(y/n)dy$$

This expression remains bounded due to the following Lemma.

Lemma 6.3.

$$\lim_{x\to\infty} \int_0^{x/2} \frac{\overline{F}^2(x-y)}{\overline{F}^2(x)} f(y/n)dy < \infty$$

Proof. Consider

$$\overline{F}(x) = \exp\Big\{-\int_0^x \lambda(t)dt\Big\}$$

where $\lambda(t)$ is the failure rate of the lognormal distribution and by standard subexponential theory we know that $\lambda(t)$ is asymptotically equivalent to $\frac{\log(x)}{\sigma^2 x}$. By choosing $c > \frac{1}{\sigma}^2$ we obtain that $c\frac{\log t}{t}$ is an asymptotic upper bound for $\lambda(t)$, then

$$\begin{aligned}\frac{\overline{F}(x-y)}{\overline{F}(x)} &= \exp\Big\{\int_{x-y}^{x} \lambda(t)dt\Big\} < \exp\Big\{c\log x \int_{x-y}^{x} \frac{1}{t}dt\Big\} \\ &= \exp\Big\{c\log x(\log x - \log(x-y))\Big\}\end{aligned}$$

Using a first order Taylor expansion of log around $(x-y)$ and the fact that it is a concave function we have that $\log x < \log(x-y) + \frac{y}{x-y}$, so the last expression is bounded by $\exp\{c\log x\frac{y}{x-y}\}$.

Consider $x > 1$. We claim that $\{y|\log(2y) > \log x\frac{y}{x-y}\} = [y(x), x/2]$ with $1/2 < y(x) < x/2$. This is true since both functions are increasing and equal when $y = x/2$, but $\log(2y)$ is concave and $\log x\frac{y}{y-x}$ is convex. It is also observed that $y_0 = \sup\{y(x)\} < \infty$ since $y(x) \to 1/2$ as $x \to \infty$. So, we have proved that

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} < c_1 \exp\{c\log y\}$$

when $y \in [y_0, x/2]$. Using this result and considering $x > 2y_0$ we get

$$\int_{y_0}^{x/2} \frac{\overline{F}^2(x-y)}{\overline{F}^2(x)} f(y/n)dy < \int_{y_0}^{\infty} \frac{c_3}{\sqrt{2\pi}} \exp\{c_4\log y - \frac{\log^2 y}{2}\}dy < \infty$$

For $y \in (0, y_0)$ we simply use the bound:

$$\int_0^{y_0} \frac{\overline{F}^2(x-y)}{\overline{F}^2(x)} f(y/n)dy < \frac{\overline{F}^2(x-y_0)}{\overline{F}^2(x)} \to 1$$

□

Second Integral: The second term in (12) is equivalent to:

$$\int_{x/2}^{\infty} \frac{\overline{F}^2(y)}{\overline{F}^2(x)} f_{M_n}(y/n)dy = -\frac{n\overline{F}^2(y)\overline{F}_{M_n}(y/n)}{\overline{F}^2(x)}\Big|_{x/2}^{\infty} - 2n\int_{x/2}^{\infty} \frac{\overline{F}(y)f(y)}{\overline{F}^2(x)}\overline{F}_{M_n}(y/n)dy$$
$$< \frac{n\overline{F}^2(x/2)\overline{F}_{M_n}(x/2n)}{\overline{F}^2(x)} < \frac{n^2\overline{F}^2(x/2)\overline{F}(x/2n)}{\overline{F}^2(x)}$$

which goes 0 as $x \to \infty$. □

Proof of Theorem 3.2 (Version 1). First, we prove that the estimator $Z_4(x)$ is unbiased. Let $\mathbb{E}_{-i}$ the expectation taken over the set of r.v.'s $\{X_{j,i} : j \neq i\}$, then

$$\mathbb{P}(S_{n+1} > x) = \sum_i \mathbb{P}(S_n > x, X_i = M_n) = \sum_i \mathbb{P}[X_i > (M_{n,i} \vee (x - S_{n,i}))]$$
$$= \sum_i \mathbb{E}_{-i}[\overline{F}_i^c(M_{n,i} \vee (x - S_{n,i}))] \tag{13}$$

Next, we will prove that it has bounded relative error.

$$\limsup_{x\to\infty} \frac{\mathbb{V}\text{ar } Z_4(x)}{\mathbb{P}^2(S_n > x)}$$
$$= \limsup_{x\to\infty} \frac{\mathbb{V}\text{ar}\,(\sum_{i=1}^n \overline{F}_i^c(M_{n,i} \vee (x - S_{n,i})))}{\mathbb{P}^2(S_n > x)} = \limsup_{x\to\infty} \frac{\sum_{i=1}^n \mathbb{V}\text{ar}\,(\overline{F}_i^c(M_{n,i} \vee (x - S_{n,i})))}{\mathbb{P}^2(S_n > x)}$$
$$< \limsup_{x\to\infty} \frac{\sum_{i=1}^n \mathbb{E}(\overline{F}_i^{c2}(M_{n,i} \vee (x - S_{n,i})))}{\mathbb{P}^2(S_n > x)} < \limsup_{x\to\infty} \frac{c\sum_{i=1}^n \mathbb{E}(\overline{F}_{\mu,\sigma}^2(M_{n,i} \vee (x - S_{n,i})))}{\overline{F}_{\mu,\sigma}^2(x)}$$

where we used that the sets $\{X_j^i : j \neq i\}$ where simulated independently. Last expression has the same form than expression (11). Proof is completed in the same way but checking that $\overline{F}_{M_{n,i}}(x) < n\overline{F}_{\mu,\sigma}(x)$. □

Proof of Theorem 3.3 (Version 2). Consider $\mathbb{E}_{-i}$ the expectation taken over the set of r. v. $\{X_j : j \neq i\}$.

$$\mathbb{P}(S_n > x) = \sum_i \mathbb{P}(S_n > x, X_i = M_n) = \sum_i \mathbb{P}[X_i > (M_{n,i} \vee (x - S_{n,i}))]$$
$$= \sum_i \mathbb{E}_{-i}[\overline{F}_i^c(M_{n,i} \vee (x - S_{n,i}))] = \sum_i \mathbb{E}[\overline{F}_i^c(M_{n,i} \vee (x - S_{n,i}))]$$
$$= \mathbb{E}\Big[\sum_i \overline{F}_i^c(M_{n,i} \vee (x - S_{n,i}))\Big]$$

proving that it is an unbiased estimator of $\mathbb{P}(S_n > x)$. Now, for the bounded relative error:

$$\limsup_{x\to\infty} \frac{\mathbb{V}\mathrm{ar}\ Z_4(x)}{\mathbb{P}^2(S_n > x)}$$

$$= \limsup_{x\to\infty} \frac{\mathbb{V}\mathrm{ar}\,(\sum_{i=1}^{n} \overline{F}_i^c(M_{n,i} \vee (x - S_{n,i})))}{\mathbb{P}^2(S_n > x)}$$

$$\leq \limsup_{x\to\infty} \frac{\mathbb{E}\Big[\Big(\sum_{i=1}^{n} \overline{F}_i^c(M_{n,i} \vee (x - S_{n,i}))\Big)^2\Big]}{\mathbb{P}^2(S_n > x)}$$

$$\leq \limsup_{x\to\infty} \frac{1}{\mathbb{P}^2(S_n > x)} \mathbb{E}\Bigg[\sum_{i=1}^{n} \overline{F}_i^{c2}(M_{n,i} \vee (x - S_{n,i})) + \sum_{j,k} \overline{F}_j^c(M_{n,j} \vee (x - S_{n,j}))\overline{F}_k^c(M_{n,k} \vee (x - S_{n,k}))\Bigg]$$

we use the Cauchy-Schwarz inequality:

$$\leq \limsup_{x\to\infty} \frac{1}{\mathbb{P}(S_n > x)} \Bigg[\sum_{i=1}^{n} \mathbb{E}\overline{F}_i^{c2}(M_{n,i} \vee (x - S_{n,i})) + \sum_{j,k} \left[\mathbb{E}\overline{F}_j^{c2}(M_{n,j} \vee (x - S_{n,j}))\mathbb{E}\overline{F}_k^{c\,2}(M_{n,k} \vee (x - S_{n,k}))\right]^{\frac{1}{2}}\Bigg]$$

$$\leq \limsup_{x\to\infty} \frac{c\sum_{i=1}^{n} \mathbb{E}(\overline{F}_i^{c2}(M_{n,i} \vee (x - S_{n,i})))}{\mathbb{P}(S_n > x)}$$

$$\leq \limsup_{x\to\infty} \frac{c\sum_{i=1}^{n} \mathbb{E}(\overline{F}_{\mu,\sigma}^2(M_{n,i} \vee (x - S_{n,i})))}{\overline{F}_{\mu,\sigma}^2(x)}$$

The last expression has the same form as the expression in (11). The proof follows in the same way but checking that $\overline{F}_{M_{n,i}}(x) < n\overline{F}_{\mu,\sigma}(x)$. □

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