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Abstract

The paper discusses a new class of models for spatio-temporal Cox point processes. In these models, the driving field is defined by means of an integral of a weight function with respect to a Lévy basis. The relations to other Cox process models studied previously are discussed and formulas for the 1st and 2nd order characteristics are derived.

Keywords: Lévy bases; Cox processes; Spatio-temporal point processes.

1 Introduction

Cox point processes constitute one of the most important and versatile classes of point process models for clustered point patterns. In this paper, we concentrate on Lévy based Cox processes for which the driving field (i.e. the random intensity function) can be expressed in terms of an integral of a weight function with respect to a Lévy basis. (The term Lévy basis is used for a special type of independently scattered random measure.) The suggested model class includes and generalizes several type of models recently studied in the literature ([3], [6], [7]).

Further, we will employ the concept of ambit processes to define the spatio-temporal version of Lévy based Cox processes. The idea of using ambit processes for describing spatio-temporal phenomena arose out of recent studies of turbulence velocity fields in 3-D and was also successfully used in the modelling of tumour growth ([1], [2], [5]). We will here use the ambit processes for defining the driving field of a spatio-temporal Lévy based Cox process. The resulting point process will thus inherit the non-trivial causal-type correlation structure of the ambit process.

In Section 2 we introduce the basic ingredients of our model, define the model and explain its relations to other classes of Cox point process models. In Section 3 we study the first order structure and the correlation structure of the model. For proofs and further details we refer to the forthcoming publication [4].

2 Definition and basic properties

2.1 Lévy bases

Here we recall only the basic facts about Lévy bases and integration with respect to them which we will need in the sequel. For a more deatiled account, see [2] and references therein.

Let $(\Upsilon, \mathcal{A})$ be a measurable space and let $L = \{L(A), A \in \mathcal{A}\}$ be a Lévy basis defined on this space, i.e. an independently scattered random measure such that the distribution of any $L(A)$ is infinitely divisible.

For a random variable X, let us denote the cumulant function $\log \mathbb{E}\,(\mathrm{e}^{ivX})$ by $C(v \ddagger X)$. When L is a Lévy basis, the cumulant function of $L(A)$ can by the Lévy-Khintchine representation be written as

$$C(v \ddagger L(A)) = iva(A) - \frac{1}{2}v^2 b(A) + \int_{\mathbb{R}} (\mathrm{e}^{ivr} - 1 - ivr\mathbf{1}_{[-1,1]}(r))\, U(\mathrm{d}r, A), \quad (1)$$

where a is a signed measure on $\mathcal{A}$, b is a measure on $\mathcal{A}$, $U(\mathrm{d}r, A)$ is a Lévy measure on $\mathbb{R}$ for each fixed $A \in \mathcal{A}$ and a measure on $\mathcal{A}$ for fixed $\mathrm{d}r$. The measure U is referred to as the generalized Lévy measure and L is said to have the characteristic (a, b, U). If $b = 0$ then L is a Lévy jump basis, while L is a Gaussian Lévy basis if $U = 0$. Any Lévy basis L can always be written as a sum of a Gaussian Lévy basis and an independent Lévy jump basis.

For managing the spatial structure of the Lévy basis it is important that we can assume without loss of generality (for details see [8]) that there exists a measure μ on Υ such that the generalized Lévy measure U factorizes as

$$U(\mathrm{d}r, \mathrm{d}\eta) = V(\mathrm{d}r, \eta)\mu(\mathrm{d}\eta),$$

where $V(\mathrm{d}r, \eta)$ is a Lévy measure for fixed η. Moreover a and b are absolutely continuous with respect to μ, i.e.

$$a(\mathrm{d}\eta) = \tilde{a}(\eta)\mu(\mathrm{d}\eta), \qquad b(\mathrm{d}\eta) = \tilde{b}(\eta)\mu(\mathrm{d}\eta).$$

Considering now the random variable with the cumulant function

$$C(v \ddagger L'(\eta)) = iv\tilde{a}(\eta) - \frac{1}{2}v^2 \tilde{b}(\eta) + \int_{\mathbb{R}} (\mathrm{e}^{ivr} - 1 - ivr\mathbf{1}_{[-1,1]}(r))\, V(\mathrm{d}r, \eta), \quad (2)$$

we get a disintegrated representation of the Lévy basis

$$C(v \ddagger L(\mathrm{d}\eta)) = C(v \ddagger L'(\eta))\, \mu(\mathrm{d}\eta). \quad (3)$$

In case $V(\cdot, \eta)$, $\tilde{a}(\eta)$ and $\tilde{b}(\eta)$ do not depend on η neither does $L'(\eta)$ and the Lévy basis L is called *factorizable*. If moreover $\Upsilon \subset \mathbb{R}^n$ and the measure μ is proportional to the Lebesgue measure, then L is called *homogeneous* and all the finite dimensional distributions of L are translation invariant.

If f is a nonrandom measurable function on Υ which is integrable with respect to the Lévy basis L (see [8] for details), the cumulant function of the integral $\int_\Upsilon f\,\mathrm{d}L$ can be expressed by means of the cumulant function of L. The fundamental relation is

$$C\left(v \ddagger \int_\Upsilon f\,\mathrm{d}L\right) = \int_\Upsilon C(vf(\eta) \ddagger L'(\eta))\,\mu(\mathrm{d}\eta). \tag{4}$$

Example 1 – Gaussian Lévy basis

If L is a Gaussian Lévy basis with characteristic $(a, b, 0)$, then for each set $A \in \mathcal{A}$ the random variable $L(A)$ is $N(a(A), b(A))$ distributed. For the disintegrated representation we obtain $L'(\eta) \sim N(\tilde{a}(\eta), \tilde{b}(\eta))$ and thus

$$C(v \ddagger \int_\Upsilon f\,\mathrm{d}L) = iv \int_\Upsilon f(\eta)\,a(\mathrm{d}\eta) - \frac{1}{2}v^2 \int_\Upsilon f(\eta)^2\,b(\mathrm{d}\eta).$$

It follows that

$$\int_\Upsilon f\,\mathrm{d}L \sim N\Big(\int_\Upsilon f(\eta)\,a(\mathrm{d}\eta), \int_\Upsilon f(\eta)^2\,b(\mathrm{d}\eta)\Big).$$

The basis is factorizable when $\tilde{a}$ and $\tilde{b}$ are constant.

Example 2 – Lévy jump bases

The simplest Lévy jump basis is the Poisson basis for which $L(A) \sim Po(\mu(A))$. This basis has characteristic $(\mu, 0, \delta_1(\mathrm{d}r)\mu(\mathrm{d}\eta))$, where δ_c denotes the Dirac measure concentrated in c. Note that $\tilde{a}(\eta) = 1$. The random variable $L'(\eta)$ has a $Po(1)$ distribution.

A broad class of Lévy jump bases are the so-called G–Lévy bases with

$$V(\mathrm{d}r, \eta) = \mathbf{1}_{\mathbb{R}_+}(r)\frac{r^{-\alpha-1}}{\Gamma(1-\alpha)}\,\mathrm{e}^{-\theta(\eta)r}\,\mathrm{d}r \qquad \text{and} \qquad \tilde{a}(\eta) = \int_0^1 \frac{r^{-\alpha}}{\Gamma(1-\alpha)}\mathrm{e}^{-\theta(\eta)r}\,\mathrm{d}r,$$

where α is an index in $(0, 1)$ or $(-\infty, 0]$ and θ a function on Υ with values in $[0, \infty)$ or $(0, \infty)$, depending on the range of α. Γ denotes the Gamma function. Particularly for $\alpha = 0$ we obtain a Gamma basis with $L'(\eta) \sim \Gamma(1, \theta(\eta))$, and for $\alpha = \frac{1}{2}$ we obtain an Inverse Gaussian basis with $L'(\eta) \sim IG(1, \theta(\eta))$.

2.2 Ambit processes

Let $\mathcal{S} \subseteq \mathbb{R}^d$ be a d-dimensional Borel set and L be a Lévy basis on

$$(\Upsilon, \mathcal{A}) = (\mathcal{S} \times \mathbb{R}, \mathcal{B}(\mathcal{S} \times \mathbb{R})),$$

where $\mathcal{B}$ denotes the Borel σ-algebra. Each point $(x, t) \in \Upsilon$ is associated with an ambit set $A_t(x) \subset \mathcal{S} \times (-\infty, t]$ which defines the causal correlation cone.

The spatio-temporal process $\{\rho(x, t) : (x, t) \in \Upsilon\}$ (the ambit process) is then for each point (x, t) defined as an integral of a deterministic non-negative jointly measurable weight function $f((x, t), (y, s)) = f_{(x,t)}(y, s)$ over an attached ambit set $A_t(x)$ with respect to the Lévy basis, i.e.

$$\rho(x, t) = \int_{A_t(x)} f_{(x,t)}(y, s)L(\mathrm{d}(y, s)). \tag{5}$$

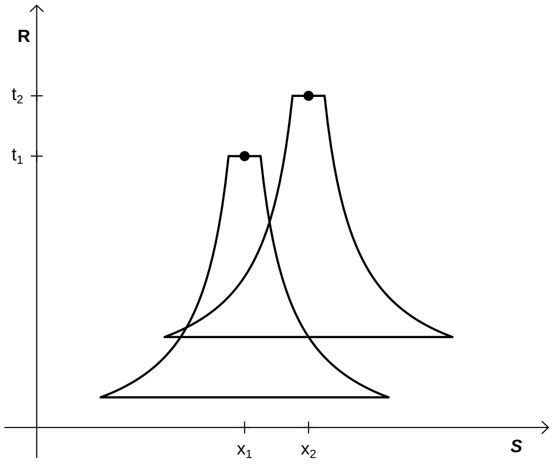

Figure 1: Illustration of the idea of an ambit set. Two ambit sets $A_{t_1}(x_1)$ and $A_{t_2}(x_2)$ are shown.

Thus the ambit set $A_t(x)$ determines the part of L that influences the behaviour of ρ in the point (x,t). In the special case when $A_t(x) = \{(y,s) : (y-x, s-t) \in A_0(0)\}$ for all (x,t) we will call the family of ambit sets *homogeneous*.

By changing the three ingredients of the model, the Lévy basis, the ambit sets and the weight functions, it is possible to obtain various correlation structures for the process ρ. This will be studied in more detail in [4]. In what follows we will suppose that all the ambit sets as well as all weight functions are uniformly bounded.

2.3 Lévy driven and log Lévy driven spatio-temporal Cox processes

Suppose now we have a well defined spatio-temporal random field $\{\rho(x,t) : (x,t) \in \Upsilon\}$ defined by equation (5). We can use it in two ways for defining a spatio-temporal Cox process X on Υ. If the field is non-negative and locally integrable we can define the driving intensity $\lambda(x,t)$ of the Lévy driven Cox process X directly by

$$\lambda(x,t) = \rho(x,t). \tag{6}$$

Another possibility is to define a log Lévy driven Cox process by

$$\lambda(x,t) = \exp(\rho(x,t)), \tag{7}$$

again under the assumption, that $\lambda(x,t)$ is locally integrable on Υ.

The specific choices of the ambit sets and the weight functions enable us to model a wide range of different non-trivial spatio-temporal dependence structures of the ambit process ρ and thus also of the generated Cox process. The stationary case can be characterized as follows.

Lemma 1 *The random field ρ and thus also the Cox process defined by* (6) *or* (7) *is stationary on $\Upsilon = \mathbb{R}^d \times \mathbb{R}$ if the Lévy basis L is homogeneous, the family of ambit sets is homogeneous and the weight function satisfies* $f_{(x,t)}(y,s) = f_{(0,0)}(y-x, s-t)$.

For specific choices of the Lévy basis in the model we can obtain special classes of Cox processes which were studied earlier in the literature. Thus for the Lévy driven Cox processes the non-negativity of $\lambda = \rho$ implies zero Gaussian part and a particular structure of the jump part of the Lévy basis. Thus the following result can be shown:

Lemma 2 *The Lévy driven Cox processes are identical to the class of shot noise Cox processes defined in [6].*

The class of log Lévy driven Cox processes (7) with Gaussian Lévy basis is a subclass of the class of log Gaussian Cox processes introduced in [7]. The most promising new class of processes appear to be the log Lévy driven Cox processes with both jump and Gaussian part.

3 First and second order properties

3.1 Lévy driven spatio-temporal Cox processes

The formulas for the intensity function and the pair correlation function of a Lévy driven Cox process X on a set $\Upsilon = \mathcal{S} \times \mathbb{R}$ can be derived by using the properties of the Cox process and differentiation of the fundamental relation (4).

Theorem 3 *Let X be a Lévy driven Cox process with driving intensity* (6) *and suppose that*

$$\int_{A_t(x)} \int_{\mathbb{R}_+} f_{(x,t)}(y,s) r \, V(\mathrm{d}r, (y,s)) \, \mu(\mathrm{d}(y,s)) < \infty.$$

Then, its intensity function Λ is given by

$$\Lambda(x,t) = \mathbb{E}\,\rho(x,t) = \int_{A_t(x)} f_{(x,t)}(y,s) \, \mathbb{E}\,(L'((y,s))) \, \mu(\mathrm{d}(y,s)). \tag{8}$$

If moreover

$$\int_{A_t(x)} \int_{\mathbb{R}_+} (f_{(x,t)}(y,s) r)^2 \, V(\mathrm{d}r, (y,s)) \, \mu(\mathrm{d}(y,s)) < \infty,$$

for each $(x,t) \in \Upsilon$, then the pair correlation function $g((x_1,t_1),(x_2,t_2))$ of X is given by

$$g((x_1,t_1),(x_2,t_2)) = \\ 1 + \frac{\int_{A_{t_1}(x_1) \cap A_{t_2}(x_2)} f_{(x_1,t_1)}(y,s) f_{(x_2,t_2)}((y,s)) \; Var(L'((y,s))) \, \mu(\mathrm{d}(y,s))}{\Lambda(x_1,t_1) \, \Lambda(x_2,t_2)}. \tag{9}$$

Note that $g((x_1,t_1),(x_2,t_2)) = 1$ if $A_{t_1}(x_1) \cap A_{t_2}(x_2) = \emptyset$. Accordingly the range of the correlations in the model is determined by the ambit sets.

Corollary 4 *If X satisfies the assumptions of Lemma 1 and $\mathbb{E}\,L'$ and $Var\,L'$ exist then we can write $\mu(\mathrm{d}(y,s)) = K\,\mathrm{d}(y,s)$ for some $K > 0$ and it holds*

$$\Lambda(x,t) = \Lambda = K\,I\,\mathbb{E}\,L', \tag{10}$$

$$g((x_1,t_1),(x_2,t_2)) = 1 + \frac{Var\,L'}{(\mathbb{E}\,L')^2}\,\frac{1}{K}\,\frac{I(x_2-x_1,t_2-t_1)}{I^2}, \tag{11}$$

where

$$I = \int_{A_0(0)} f_{(0,0)}(y,s)\,\mathrm{d}(y,s),$$

and

$$I(x_2-x_1,t_2-t_1) = \int_{A_0(0)\cap A_{t_2-t_1}(x_2-x_1)} f_{(0,0)}(y,s) f_{(0,0)}(y-(x_2-x_1), s-(t_2-t_1))\,\mathrm{d}(y,s).$$

Thus for such stationary Lévy driven Cox process we can see nicely how the effects due to the disintegrated parts of the Lévy basis and the integrals of the weight function on the ambit sets combine in the pair correlation function.

Let us end this section by considering the cumulative point process

$$X^C(B) = X(B\times[T_0,T_1]) \qquad B\subset\mathcal{S},\ T_0 < T_1 \in \mathbb{R}.$$

Because of the linear structure of the Lévy driven Cox process model the cumulative process is also a Lévy driven Cox process and its driving intensity has the form

$$\lambda^C(x) = \int_{T_0}^{T_1}\int_{A_t(x)} f_{(x,t)}(y,s)L(\mathrm{d}(y,s))\,\mathrm{d}t = \int_{\Upsilon} f_x^C(y,s)L(\mathrm{d}(y,s)),$$

with a new weight function

$$f_x^C(y,s) = \int_{T_0}^{T_1} \mathbf{1}((y,s)\in A_t(x))\, f((x,t),(y,s))\,\mathrm{d}t$$

defined for each $x\in\mathcal{S}$. Consequently we can obtain formulas for the intensity and pair correlation function of the cumulative process X^C in the same way as we did for the original process X.

3.2 Log Lévy driven spatio-temporal Cox processes

Let X be a log Lévy driven Cox process on a set $\Upsilon = \mathcal{S}\times\mathbb{R}$ with driving intensity given by (7) and denote its n-th order product densities by $m^{(n)}$. Then if they exist they can be computed using the fundamental relation (4) for the kumulant function $K(v \ddagger X) = C(-iv \ddagger X)$.

Theorem 5 *If the n-th order product density of a log Lévy driven Cox process X on Υ exists, then*

$$m^{(n)}((x_1,t_1),\ldots,(x_n,t_n)) = \mathbb{E}\prod_{i=1}^{n}\lambda(x_i,t_i)$$
$$= \exp\left(\int_\Upsilon K\Big(\sum_{i=1}^{n}\mathbf{1}_{A_{t_i}(x_i)}(y,s)f_{(x_i,t_i)}(y,s)\ddagger L'((y,s))\Big)\mu(\mathrm{d}(y,s))\right).$$

Thus in particular we obtain the equations below for the intensity and pair-correlation function of X.

Theorem 6 *Let X be a log Lévy driven Cox process and suppose that its first and second order product densities exist. Then*

$$\Lambda(x,t) = \exp\left(\int_\Upsilon K(\mathbf{1}_{A_t(x)}(y,s)\,f_{(x,t)}(y,s)\ddagger L'((y,s)))\,\mu(\mathrm{d}(y,s))\right), \qquad (12)$$

and

$$g((x_1,t_1),(x_2,t_2)) \qquad (13)$$
$$= \exp\left(\int_{A_{t_1}(x_1)\cap A_{t_2}(x_2)}\Big[K\big(f_{(x_1,t_1)}(y,s)+f_{(x_2,t_2)}(y,s)\ddagger L'((y,s))\big)\right.$$
$$\left. - K\big(f_{(x_1,t_1)}(y,s)\ddagger L'((y,s))\big) - K\big(f_{(x_2,t_2)}(y,s)\ddagger L'((y,s))\big)\Big]\,\mu(\mathrm{d}(y,s))\right).$$

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