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Probability Measures, Lévy Measures, and Analyticity in Time

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Contents

Abstract

We investigate the relation of the semigroup probability density of an infinite activity Lévy process to the corresponding Lévy density. For subordinators we provide three methods to compute the former from the latter. The first method is based on approximating compound Poisson distributions, the second method uses convolution integrals of the upper tail integral of the Lévy measure, and the third method uses the analytic continuation of the Lévy density to a complex cone and contour integration. As a byproduct we investigate the smoothness of the semigroup density in time. Several concrete examples illustrate the three methods and our results.

1 Introduction

For the infinitely divisible laws there are a number of intriguing and useful relations and points of similarity between the probability measures or probability densities of the laws on the one hand and their associated Lévy measures or Lévy densities on the other.

In particular, if U is the Lévy measure of an infinitely divisible law on $\mathbb{R}^d$ with associated Lévy process $\{X_t\}_{t\geq 0}$ and if $P(\mathrm{d}x;t)$ denotes the law of X_t then (see Sato (1999, Corollary 8.9))

$$\lim_{t\to 0} t^{-1} \int_{\mathbb{R}^d} f(x)P(\mathrm{d}x;t) = \int_{\mathbb{R}^d} f(x)U(\mathrm{d}x) \tag{1}$$

for any function f in the space $C_\#$ of bounded continuous functions on $\mathbb{R}^d$ vanishing in a neighborhood of 0.

The present paper considers the converse problem, of calculating $P(\mathrm{d}x;t)$ from U and, more particularly, how to determine the probability density $p(x;t)$ of P_t from the Lévy density u of U, when these densities exist.

As part of the problem, we discuss conditions (for $d = 1$) ensuring that $p(x;t)$ possesses a power series expansion in t:

$$p(x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} u_n(x). \tag{2}$$

This issue is of some independent interest. In (2), necessarily, $u_1 = u$ and the question is how the further coefficients u_n may be calculated from u, using possibly also properties of the cumulant function of X_1 (which, of course, is essentially determined by u).

We also consider the measure version of (2), i.e.

$$P(\mathrm{d}x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} U_n(\mathrm{d}x) \tag{3}$$

where the $U_n(\mathrm{d}x)$ are, in general signed, measures and with U_1 being equal to U, the Lévy measure of $P(\mathrm{d}x;t)$.

When both $p(x;t)$ and u are concentrated on the positive halfline $\mathbb{R}_{>0} = (0, \infty)$, it is convenient to give (3) the form

$$P^+(x;t) = \sum_{n=1}^{\infty} U_n^+(x) \frac{t^n}{n!} \tag{4}$$

with $P^+(x;t) = P([x, +\infty); t)$ and $U_n^+(x) = U_n([x, +\infty))$ denoting the upper tail integrals.

Except for the discussion in Section 2 and some remarks in Section 4, we only consider the case where the process X is a subordinator with infinite Lévy measure and without linear drift. In other words, X is an infinite activity pure jump subordinator. In the case of a finite Lévy measure X is a compound Poisson process, and validity of formulae (2) and (3), with straightforward modifications for the atom at zero, is easily established, see Section 4.1.

We shall discuss three methods for determining the coefficients $u_n(x)$: The first involves, as the final step, a limiting operation

$$u_n(x) = \lim_{\varepsilon \to 0} u_{n\varepsilon}(x), \tag{5}$$

where

$$u_{n\varepsilon}(x) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} c(\varepsilon)^{n-k} u_\varepsilon^{*k}(x). \tag{6}$$

Here $u_\varepsilon(x)$ is an approximation of the Lévy density $u(x)$, that corresponds to a compound Poisson process with intensity $c(\epsilon)$, and $*k$ indicates k-fold convolution. The second method uses derivatives of convolutions, namely,

$$u_n(x) = (-1)^n \frac{d^n}{dx^n} \left((U^+)^{*n} \right)(x). \tag{7}$$

The third method uses the complex contour integral

$$u_n(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} \kappa(\theta)^n e^{\theta x} d\theta, \tag{8}$$

where $\kappa(\theta)$ is the analytic continuation of the cumulant function to a complex cone containing $\Re\theta \geq 0$ and the contour is, roughly speaking, along the boundary of the cone. We will see that such an analytic continuation can be derived from an analytic continuation of the Lévy density $u(x)$ to a complex cone containing the positive real axis.

Formulae of the above type are of interest, in particular, in connection with recent work on stochastic modeling that seeks to capture observed distributional behavior by specification of Lévy densities rather than probability densities. For some case studies see Novikov (1994), Koponen (1995), Bouchaud et al. (1997), Carr et al. (2002), and Barndorff-Nielsen and Shephard (2001). Boyarchenko and Levendorskiĭ (2002) develop this approach considerably, including applications to option pricing. Another area of application is to statistical inference for discretely observed, continuous time models on, for instance, log prices of assets, see Woerner

(2001). The idea of Lévy copulas, see Cont and Tankov (2004) and Barndorff-Nielsen and Lindner (2006) is in the same line of reasoning.

We proceed to indicate other work related to the results of the present paper.

The comprehensive monograph by Sato (1999) contains many instances of the interesting relations between the probability distributions and the Lévy measures of infinitely divisible laws; cf. also Embrechts et al. (1979), Embrechts and Goldie (1981) and Sato and Steutel (1998). Some examples are the relation between unimodality properties of the two types of densities, Sato (1999, Section 52), and the behavior under exponential tilting (or Esscher transformation). See also Léandre (1987), Ishikawa (1994) and Picard (1997) who, partly in the wider setting of pure jump processes, study cases where the transition density exists and behaves as a power of t for $t \to 0$. Continuity of $P^+(x;t)$ at $t = 0$ is characterized in Doney (2004). In Rüschendorf and Woerner (2002), see also Woerner (2001), the authors have established the validity of expansions for the probability density or distribution function of X_t that are related to (2) and (4). More specifically, in the notation of the present paper, they show that under certain technical conditions — different from those considered here — one has, for $t \to 0$ and $x > \eta > 0$ and letting $u_{[\varepsilon,\infty)}(x) = 1_{[\varepsilon,\infty)}(x)u(x)$ and $U_\varepsilon^+(x) = U_\varepsilon([x,\infty))$, that

$$p(x;t) = e^{-tc(\varepsilon)} \sum_{k=1}^{n-1} u_\varepsilon^{*k}(x) \frac{t^k}{k!} + O_{\varepsilon,\eta}(t^n) \tag{9}$$

respectively

$$P^+(x;t) = \sum_{k=1}^{n-1} U_\varepsilon^{+*k}(x) \frac{t^k}{k!} + O_{\varepsilon,\eta}(t^n) \tag{10}$$

where $u_{[\varepsilon,\infty)}(x) = 1_{[\varepsilon,\infty)}(x)u(x)$ and $U_\varepsilon^+(x) = U_\varepsilon([x,\infty))$ is the tail mass of the approximating Lévy measure, and where for simplicity we have assumed that X_t is a subordinator. However, as these authors show, the same type of expansions hold for Lévy processes in general.

Burnaev (2006) gives two formulas for computing the Lévy measure from the corresponding cumulant function.

Section 2 below consists in an initial discussion of the problem of determining the coefficients u_n. This casts light on the nature of these coefficients and also leads to the idea behind the procedure for calculating the coefficients that will be established in Section 3, which contains our main mathematical results. Illustrative examples will be given in Section 4. Technical auxiliary material used in the proofs is in the Appendix.

The results in the present paper build partly on our previous, unpublished preliminary work Barndorff-Nielsen (2000) and Hubalek (2002).

2 Initial analysis

In Subsection 2.1 below we present a heuristic derivation of the following representation for the probability density of the Lévy process X at time t:

$$p(x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} u_n(x) \tag{11}$$

where

$$u_n(x) = \lim_{\varepsilon \to 0} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} c(\varepsilon)^{n-k} u_\varepsilon^{*k}(x) \tag{12}$$

and u_ε, with total mass $c(\varepsilon)$, is a (suitable) approximation to the Lévy density of X.

A rigorous derivation of the formula is given in Section 3 for the case of one-dimensional subordinators, under fairly strong assumptions. In subsequent work we hope to establish proofs for Lévy processes in $\mathbb{R}^d$.

Subsection 2.2 contains a general result for infinitely divisible distributions on $\mathbb{R}_{>0}$. In Subsection 2.3 the rather subtle nature of the limiting operation in (12) is discussed. The final Subsection 2.4 indicates the role of the tail behaviour of u.

2.1 Heuristic motivation

Let $p(x;t)$, and $u(x)$ be, respectively, the probability density, and the Lévy density (assumed to exist) of an infinite activity Lévy process X on $\mathbb{R}^d$. Let $u_\varepsilon(x), \varepsilon > 0$ be integrable Lévy densities, that we think of as approximations to $u(x)$ for $\varepsilon \to 0$, and set

$$c(\varepsilon) = \int_{\mathbb{R}^d} u_\varepsilon(x)dx, \tag{13}$$

where we suppose $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$, in accordance with X being of infinite activity. In the following we give the heuristic calculation, that led us to the formula (2), with $u_n(x)$ given by (5), and

$$u_{n\varepsilon}(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} c(\varepsilon)^{n-k} u_\varepsilon^{*k}(x). \tag{14}$$

The function $u_\varepsilon(x)$ determines a compound Poisson process with intensity $c(\varepsilon)$ and jumps from the distribution with density

$$a_\varepsilon(x) = c(\varepsilon)^{-1} u_\varepsilon(x). \tag{15}$$

Let $p_\varepsilon(x;t)$ be the density of the absolutely continuous part of the associated probability distribution, and assume $x \in \mathbb{R}^d \setminus \{0\}$. Then

$$p_\varepsilon(x;t) = \sum_{n\geq 1} e^{-c(\varepsilon)t} \frac{(c(\varepsilon)t)^n}{n!} a_\varepsilon^{*n}(x) \tag{16}$$

$$= \left(\sum_{n\geq 0} (-1)^n c(\varepsilon)^n \frac{t^n}{n!} \right) \left(\sum_{n\geq 1} c(\varepsilon)^n a_\varepsilon^{*n}(x) \frac{t^n}{n!} \right). \tag{17}$$

Computing the product of the two series as Cauchy product and using (15) yields

$$p_\varepsilon(x;t) = \sum_{n\geq 1}\left(\sum_{k=1}^{n}\binom{n}{k} a_\varepsilon^{*k}(x)(-1)^{n-k}c(\varepsilon)^n\right)\frac{t^n}{n!} \tag{18}$$

$$= \sum_{n\geq 1}\left(\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}c(\varepsilon)^{n-k}u_\varepsilon^{*k}(x)\right)\frac{t^n}{n!}. \tag{19}$$

By (14) we obtain

$$p_\varepsilon(x;t) = \sum_{n\geq 1} u_{n\varepsilon}(x)\frac{t^n}{n!}. \tag{20}$$

In many examples we can verify that the limit (5) exists, that $\lim_{\varepsilon\to 0} p_\varepsilon(x;t) = p(x;t)$, and that we can interchange the limit and the summation in (20), and thus obtain (2).

Let us emphasize that the problem is not only to give a justification of interchanging limits. Each summand in (14) diverges as $\varepsilon \to 0$, yet 'magically', by massive cancellation, the sum converges. This behaviour is investigated in some more detail in Section 2.3.

By a similar heuristic calculation we expect

$$P(dx;t) = \sum_{n\geq 1} U_n(dx)\frac{t^n}{n!} \tag{21}$$

to hold on $\mathbb{R}^d \setminus \{0\}$, with

$$U_n(dx) = \lim_{\varepsilon\to 0} U_{n\varepsilon}(dx), \tag{22}$$

where

$$U_{n\varepsilon}(x) = \sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}c(\varepsilon)^{n-k}U_\varepsilon^{*k}(dx). \tag{23}$$

Obviously, if we can integrate (2) term by term, then

$$U_n(dx) = u_n(x)dx \tag{24}$$

on $\mathbb{R}^d \setminus \{0\}$.

Remark 1. *Suppose $p(x;0) = 0$ and*

$$\lim_{t\to 0} t^{-1}p(x,t) = u(x) \tag{25}$$

and let

$$u_\varepsilon(x) = \varepsilon^{-1}p(x;\varepsilon), \tag{26}$$

in which case $c(\varepsilon) = \varepsilon^{-1}$. With u_ε thus defined, formula (6) takes the form

$$u_{n\varepsilon}(x) = \varepsilon^{-n}\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}p^{*k}(x;\varepsilon) \tag{27}$$

$$= \varepsilon^{-n}\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}p(x;k\varepsilon). \tag{28}$$

Now, the right hand side is, in fact, an n-th order difference quotient of $p(x;t)$ so that provided $p(x;t)$ is n times differentiable from the right at $t=0$ we have

$$\lim_{\varepsilon\to 0} u_{n\varepsilon}(x) = \frac{\partial^n}{\partial t^n} p(x;0). \tag{29}$$

Thinking of $u_\varepsilon(x)$ as an approximation to $\varepsilon^{-1}p(x;\varepsilon)$, as well as an approximation to $u(x)$, this then is a further indication that in considerable generality $p(x;t)$ may be calculated via (6) in the manner discussed above. (Of course, in practice, choosing $u_\varepsilon(x) = \varepsilon^{-1}p(x;\varepsilon)$ is not an option since the point is to determine $p(x;t)$ in terms of the Lévy density $u(x)$.)

Remark 2. *It may also be noted that subject to the other assumptions of the Theorem, condition (35) is satisfied in particular if there exists an integrable function v on $\mathbb{R}_{>0}$ such that*

$$(x \wedge 1)u_\varepsilon(x) \le v(x) \tag{30}$$

for all $x \in \mathbb{R}_{>0}$ and all ε. Some candidates for u_ε are

$$u_\varepsilon(x) = \mathbf{1}_{[\varepsilon,\infty)}(x)u(x) \tag{31}$$

or

$$u_\varepsilon(x) = u(x)e^{-x/\varepsilon}. \tag{32}$$

2.2 A simple, general result for $\mathbb{R}_{>0}$

Let $P(x;t)$ and $u(x)$ be, respectively, the cumulative distribution function and the Lévy density (assumed to exist) of an infinite activity Lévy process on $\mathbb{R}_{>0}$. Let the $u_\varepsilon(x)$ be integrable Lévy densities, that we think of as approximations to $u(x)$. Let us define $c(\varepsilon)$ and $u_{n\varepsilon}(x)$ as in (13) and (14) above, and set

$$U_{0\varepsilon}(x) = 1, \qquad U_{n\varepsilon}(x) = -\int_x^\infty u_{n\varepsilon}(y)dy \quad (n \ge 1). \tag{33}$$

Theorem 3. *Suppose*

$$\lim_{\varepsilon\to 0} c(\varepsilon) = \infty, \tag{34}$$

and

$$\lim_{\varepsilon\to 0} \int (1 \wedge x)|u_\varepsilon(x) - u(x)|dx = 0. \tag{35}$$

Then

$$P(x;t) = \lim_{\varepsilon\to 0} \sum_{n\ge 0} U_{n\varepsilon}(x)\frac{t^n}{n!} \tag{36}$$

pointwise for each $x \in \mathbb{R}_{>0}$ and $t > 0$.

Proof. Let $P_\varepsilon(x;t)$ the distribution functions of the approximating compound Poisson processes with Lévy density $u_\varepsilon(x)$. Then we can justify term by term integration of (20) by dominated convergence and obtain

$$P_\varepsilon(x;t) = e^{-c(\varepsilon)t} + \sum_{n\ge 0} U_{n\varepsilon}(x)\frac{t^n}{n!}. \tag{37}$$

The assumption (35) implies that the Fourier cumulant function of $P_\varepsilon(\cdot,t)$, converges to the Fourier cumulant function of $P(\cdot,t)$, and thus the characteristic functions of $P_\varepsilon(\cdot,t)$ converges to the characteristic functions of $P(\cdot,t)$. The continuity theorem then yields $P_\varepsilon(x,t)\to P(x,t)$ when x is a continuity point. But (Sato, 1999, Theorem 27.4) shows, that $P(x;t)$ is continuous in $x>0$. Since the first term on the right hand side of (37) trivially tends to zero, we have (36). □

Remark 4. *The approximations* (31) *satisfy the assumptions of Theorem 3.*

2.3 Nature of limit relations

We proceed to discuss the nature of the main limit relations of the foregoing. The first four of the functions $u_{n\varepsilon}$ of (14) are

$$u_{1\varepsilon}(x)=u_\varepsilon(x) \tag{38}$$
$$u_{2\varepsilon}(x)=u_\varepsilon^{*2}(x)-2c(\varepsilon)u_\varepsilon(x) \tag{39}$$
$$u_{3\varepsilon}(x)=u_\varepsilon^{*3}(x)-3c(\varepsilon)u_\varepsilon^{*2}(x)+3c(\varepsilon)^2u_\varepsilon(x) \tag{40}$$
$$u_{4\varepsilon}(x)=u_\varepsilon^{*4}-4c(\varepsilon)u_\varepsilon^{*3}+6c(\varepsilon)^2u_\varepsilon^{*2}(x)-4c(\varepsilon)^3u_\varepsilon(x) \tag{41}$$

Further, it follows from the well-known inverse relations

$$a_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k b_k \iff b_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k, \tag{42}$$

see, for example, (Riordan, 1968, Sec.2.1, p.43), applied to $a_0=0$, $b_0=0$ and

$$a_k=(-1)^k\frac{u_{k\varepsilon}(x)}{c(\varepsilon)^k},\quad b_k=\frac{u_\varepsilon^{*k}(x)}{c(\varepsilon)^k},\quad (1\le k\le n) \tag{43}$$

that formula (14) may be reexpressed as

$$u_\varepsilon^{*n}(x)=\sum_{k=1}^{n}\binom{n}{k}c(\varepsilon)^{n-k}u_{k\varepsilon}(x). \tag{44}$$

Another useful variant of (14) and (44) is

$$u_{n\varepsilon}(x)=u_\varepsilon^{*n}(x)-\sum_{k=1}^{n-1}\binom{n}{k}c(\varepsilon)^{n-k}u_{s\varepsilon}(x). \tag{45}$$

In particular, we have

$$u_{1\varepsilon}(x)=u_\varepsilon(x) \tag{46}$$
$$u_{2\varepsilon}(x)=u_\varepsilon^{*2}(x)-2c(\varepsilon)u_{1\varepsilon}(x) \tag{47}$$
$$u_{3\varepsilon}(x)=u_\varepsilon^{*3}(x)-3c(\varepsilon)u_{2\varepsilon}(x)-3c(\varepsilon)^2u_\varepsilon(x) \tag{48}$$
$$u_{4\varepsilon}(x)=u_\varepsilon^{*4}-4c(\varepsilon)u_{3\varepsilon}-6c(\varepsilon)^2u_{2\varepsilon}(x)-4c(\varepsilon)^3u_\varepsilon(x). \tag{49}$$

Section 2.3 exemplifies the convergence of $u_{n\varepsilon}(x)$ to a function $u_n(x)$. Note that such convergence implies a subtle cancellation of singularities, cf. formula (14); see the first few instances of that formula, listed at the beginning of this subsection and recall that $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

To gain an understanding of how this cancellation occurs, note first that for $n = 0, 1, 2, \dots$ we have

$$u_\varepsilon^{*(n+1)}(x) = (n+1)x^{-1} \int_0^x \bar{u}_\varepsilon(x-y)u_\varepsilon^{*n}(y)\mathrm{d}y \tag{50}$$

where $u_\varepsilon^{*0}(y)\mathrm{d}y$ is interpreted as δ-measure at the origin. This may be shown by induction. In fact, for $n = 0$ the statement is trivial and assuming validity up till $n-1$ we find

$$\begin{aligned}
xu_\varepsilon^{*(n+1)}(x) &= (x-y+y)\int_0^x u_\varepsilon^{*n}(x-y)u_\varepsilon(y)\mathrm{d}y \\
&= \int_0^x (x-y)u_\varepsilon^{*n}(x-y)u_\varepsilon(y)\mathrm{d}y + \int_0^x u_\varepsilon^{*n}(x-y)yu_\varepsilon(y)\mathrm{d}y \\
&= \int_0^x yu_\varepsilon^{*n}(y)u_\varepsilon(x-y)\mathrm{d}y + \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= n\int_0^x u_\varepsilon(x-y)\int_0^y u_\varepsilon^{*(n-1)}(y-z)\bar{u}_\varepsilon(z)\mathrm{d}z\mathrm{d}y + \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= n\int_0^x \bar{u}_\varepsilon(z)\int_z^x u_\varepsilon(x-y)u_\varepsilon^{*(n-1)}(y-z)\mathrm{d}y\mathrm{d}z + \int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= (n+1)\int_0^x u_\varepsilon^{*n}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y
\end{aligned}$$

Furthermore,

$$u_{n+1\varepsilon}(x) = (n+1)x^{-1}\left\{\int_0^x u_{n\varepsilon}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y + (-1)^n c(\varepsilon)^n \bar{u}_\varepsilon(x)\right\} \tag{51}$$

as follows by the calculation

$$\begin{aligned}
\int_0^x u_{n\varepsilon}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y &= c(\varepsilon)^n \sum_{\nu=1}^n (-1)^{n-\nu}\binom{n}{\nu}\int_0^x a_\varepsilon^{*\nu}(x-y)\bar{u}_\varepsilon(y)\mathrm{d}y \\
&= c(\varepsilon)^{n+1} x \sum_{\nu=1}^n (-1)^{n-\nu}\binom{n}{\nu}(\nu+1)^{-1}a_\varepsilon^{*(\nu+1)}(x) \\
&= \frac{1}{n+1} x c(\varepsilon)^{n+1} \sum_{\nu=1}^n (-1)^{n+1-(\nu+1)}\binom{n+1}{\nu+1} a_\varepsilon^{*(\nu+1)}(x) \\
&= \frac{1}{n+1}\bar{u}_{n+1\varepsilon}(x) - (-1)^n c(\varepsilon)^n \bar{u}_\varepsilon(x)
\end{aligned}$$

Next we discuss the limiting behavior of $u_{n\varepsilon}(x)$ as $\varepsilon \to 0$. Consider first the case $n = 2$, and let

$$U_\varepsilon^+(x) = \int_x^\infty u_\varepsilon(y)\mathrm{d}y, \qquad U^+(x) = \int_x^\infty u(y)\mathrm{d}y. \tag{52}$$

Using (51) and noting that

$$c(\varepsilon) = \int_0^x u_\varepsilon(y)\mathrm{d}y + U_\varepsilon^+(x) \tag{53}$$

we may rewrite $u_{2\varepsilon}(x)$ as

$$u_{2\varepsilon}(x) = 2x^{-1}\left\{\int_0^x u_\varepsilon(y)\{\bar{u}_\varepsilon(x-y) - \bar{u}_\varepsilon(x)\}\mathrm{d}y - \bar{u}_\varepsilon(x)U_\varepsilon^+(x)\right\}. \tag{54}$$

Hence, by letting $\varepsilon \to 0$ and invoking condition (30), we obtain

Proposition 5. *Suppose the Lévy density u is differentiable. Then*

$$u_2(x) = 2x^{-1}\left\{\int_0^x u(y)\{\bar{u}(x-y) - \bar{u}(x)\}\mathrm{d}y - \bar{u}(x)U^+(x)\right\} \tag{55}$$

with the integral existing and being finite.

We note that (55) may be reexpressed as

$$\frac{1}{2}\bar{u}_2(x) = \int_0^x u(y)\{\bar{u}(x-y) - \bar{u}(x)\}\mathrm{d}y - \bar{u}(x)U^+(x). \tag{56}$$

Remark 6. *Formula (54), first given in Barndorff-Nielsen (2000), has been generalized by Woerner (2001) to*

$$\frac{1}{n+1}\bar{u}_{n+1\varepsilon}(x) = \frac{1}{n}\left[\int_0^x u_\varepsilon(y)\left\{\bar{u}_{n\varepsilon}(x-y) - \bar{u}_{n\varepsilon}(x)\right\}\mathrm{d}y - \bar{u}_{n\varepsilon}(x)U^+(x)\right]. \tag{57}$$

Typically though, this cannot be used to pass to the limit $\varepsilon \to 0$ for $n > 1$.

2.4 A note on the tail behaviour

Suppose $r > 0$ and $u(x)$ is an infinite activity Lévy density on $\mathbb{R}_{>0}$. Then

$$v(x) = e^{-rx}u(x) \tag{58}$$

is also an infinite activity Lévy density on $\mathbb{R}_{>0}$. This transformation is well-known as Esscher transform or exponential tilting. Let $\kappa(\theta)$ and $\lambda(\theta)$ be the Laplace cumulant functions corresponding to $u(x)$ and $v(x)$. Then

$$\lambda(\theta) = \kappa(\theta + r) - \kappa(r). \tag{59}$$

Let $p(x,t)$ and $q(x,t)$ be the probability densities corresponding to $u(x)$ and $v(x)$. Then

$$q(x,t) = e^{-rx-\kappa(r)t}p(x,t). \tag{60}$$

Thus $p(x,t)$ admits a series representation (2) if and only if $q(x,t)$ admits a series representation

$$q(x,t) = \sum_{n\geq 1} v_n(x)\frac{t^n}{n!}, \tag{61}$$

where we have

$$v_n(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \kappa(r)^{n-k} e^{-rx} u_n(x) \tag{62}$$

and

$$u_n(x) = \sum_{k=1}^{n} \binom{n}{k} \kappa(r)^{n-k} e^{rx} v_n(x). \tag{63}$$

This indicates that heavy tails do not matter much in the problems studied in the present context, as we can deal with them using the Esscher transform. Rather it is the behaviour of small jumps that can cause difficulties, as reflected in the assumptions of the theorems given below.

3 Main results

In this section we analyze the following issues:

- Does $u_{n\varepsilon}(x)$ converge as $\varepsilon \to 0$? If so, can we find a more direct method to compute the limit $u_n(x)$ from $u(x)$ and so avoid the difficult cancellations in $u_{n\varepsilon}(x)$ as $\varepsilon \to 0$?
- If we have convergence, is $p(x;t)$ in fact n-times differentiable (from the right) at $t = 0$ and, if so, is $u_n(x)$ the n-th derivative?
- If the answer to the previous question is yes for all $n \geq 1$, do we have a convergent Taylor expansion of $p(x;t)$ at $t = 0$? Is $p(x;t)$ in fact an entire function in $t \in \mathbb{C}$?

In the proofs we refer to several technical estimates that are provided in the appendix. Recall that we are assuming that the process X is an infinite activity subordinator.

3.1 Pointwise convergence of the coefficient functions

In this subsection we investigate the limiting behavior of $u_{n\varepsilon}(x)$ as $\varepsilon \to 0$ for the particular choice

$$u_\varepsilon(x) = e^{-\varepsilon/x} u(x). \tag{64}$$

This approximation is simple, it is always feasible in the sense that it implies

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x) \qquad \forall x > 0, \tag{65}$$

$$\lim_{\varepsilon \to 0} \int_x^\infty u_\varepsilon(y) dy = \int_x^\infty u(y) dy \qquad \forall x > 0, \tag{66}$$

and $u_\varepsilon(x)$ will be smooth if $u(x)$ is, a property exploited below. We provide conditions on $u(x)$ that imply the convergence of $u_{n\varepsilon}(x)$ and obtain an expression for the limit $u_n(x)$, namely

$$u_n(x) = (-1)^n \frac{d^n}{dx^n} (U^+)^{*n}(x). \tag{67}$$

In words: we can obtain $u_n(x)$ as the n-th derivative of the n-th convolution power of the upper tail integral of the Lévy density $u(x)$.

Theorem 7. *Suppose* $n \in \mathbb{N}$ *and*

$$\int_0^\infty e^{-rx} x^{k+1} |u^{(k)}(x)| dx < \infty, \quad k = 0, 1, \ldots, m. \tag{68}$$

holds for some $m \geq n + 2$ *and* $r > 0$.

(i) Let

$$U^+(x) = \int_x^\infty u(y) dy, \quad x > 0. \tag{69}$$

Then the n*-th convolution power of* $U^+(x)$ *is well-defined for* $x > 0$*, and* $(U^+)^{*n}(x)$ *is* n*-times continuously differentiable.*

(ii) If we set for $\varepsilon > 0$

$$u_\varepsilon(x) = e^{-\varepsilon/x} u(x), \quad x > 0, \tag{70}$$

and define

$$u_{n\varepsilon}(x) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} u_\varepsilon^{*k}(x) c(\varepsilon)^{n-k}, \tag{71}$$

where

$$c(\varepsilon) = \int_0^\infty u_\varepsilon(x) dx, \tag{72}$$

then

$$\lim_{\varepsilon \to 0} u_{n\varepsilon}(x) = u_n(x), \quad x > 0, \tag{73}$$

where

$$u_n(x) = (-1)^n \frac{d^n}{dx^n} (U^+)^{*n}(x). \tag{74}$$

Proof. In Lemma 22 we show that $(U^+)^{*n}(x)$ exists, in Lemma 23, that it is n-times differentiable. Both facts are less obvious, than it seems at first sight, cf. Doetsch (1950, Section 2.14, p.104ff) and Uludağ (1998).

Let $\lambda_{n\varepsilon}(\theta)$ denote the Laplace transform of $u_{n\varepsilon}(x)$. It is given by

$$\lambda_{n\varepsilon}(\theta) = \kappa_\varepsilon(\theta)^n - (-1)^n c(\varepsilon)^n. \tag{75}$$

Let

$$\lambda_n(\theta) = \kappa(\theta)^n. \tag{76}$$

Note that $\lambda_n(\theta)$ is *not* the Laplace transform of $u_n(x)$; the Laplace transform of $u_n(x)$ does not exist. But using the estimates from Lemma 18, namely

$$|\lambda_{n\varepsilon}^{(m)}(\theta)| \leq E_{mn}/|\theta|^{m-n}, \qquad |\lambda_n^{(m)}(\theta)| \leq E_{mn}/|\theta|^{m-n} \tag{77}$$

for some constants E_{mn}, we see, that both $x^m u_{n\varepsilon}(x)$ and $x^m u_n(x)$ have integrable Laplace transforms, namely $\lambda_{n\varepsilon}^{(m)}(\theta)$ and $\lambda_n^{(m)}(\theta)$. According to Lemma 24 we can write the inversion integrals

$$u_{n\varepsilon}(x) = \frac{1}{2\pi i x^m} \int_{r-i\infty}^{r+i\infty} \lambda_{n\varepsilon}^{(m)}(\theta) e^{\theta x} d\theta, \qquad u_n(x) = \frac{1}{2\pi i x^m} \int_{r-i\infty}^{r+i\infty} \lambda_n^{(m)}(\theta) e^{\theta x} d\theta, \tag{78}$$

and obtain

$$|u_{n\varepsilon}(x) - u_n(x)| \le \frac{e^{rx}}{2\pi x^m} \int_{-\infty}^{+\infty} \left|\lambda_{n\varepsilon}^{(m)}(r+iy) - \lambda_n^{(m)}(r+iy)\right| dy. \tag{79}$$

We have

$$0 \le u_\varepsilon(x) \le u(x), \quad x > 0, \varepsilon > 0 \tag{80}$$

and $u_\varepsilon(x) \to u(x)$ for $\varepsilon \to 0$. Thus looking at

$$\kappa_\varepsilon(\theta) = \int_0^\infty (e^{-\theta x} - 1) u_\varepsilon(x) dx, \qquad \kappa(\theta) = \int_0^\infty (e^{-\theta x} - 1) u(x) dx, \tag{81}$$

and

$$\kappa_\varepsilon^{(k)}(\theta) = (-1)^k \int_0^\infty e^{-\theta x} x^k u_\varepsilon(x) dx, \qquad \kappa^{(k)}(\theta) = (-1)^k \int_0^\infty e^{-\theta x} x^k u(x) dx, \tag{82}$$

for $k = 1, \ldots, n$, we see by dominated (or monotone) convergence, that

$$\lim_{\varepsilon\to 0} \kappa_\varepsilon(\theta) = \kappa(\theta), \qquad \lim_{\varepsilon \to 0} \kappa_\varepsilon^{(k)}(\theta) = \kappa^{(k)}(\theta). \tag{83}$$

The functions $\lambda_{n\varepsilon}^{(m)}(\theta)$ and $\lambda_n^{(m)}(\theta)$ are polynomials in $\kappa_\varepsilon(\theta)$, $\kappa_\varepsilon'(\theta)$, $\ldots$, $\kappa_\varepsilon^{(m)}(\theta)$, respectively in $\kappa(\theta)$, $\kappa'(\theta)$, $\ldots$, $\kappa^{(m)}(\theta)$, thus

$$\lim_{\varepsilon \to 0} \lambda_{n\varepsilon}^{(m)}(\theta) = \lambda_n^{(m)}(\theta). \tag{84}$$

Moreover, they are dominated by the integrable function $E_{mn}/|\theta|^{m-n}$ and we have by dominated convergence in (79) the desired result. □

An interesting class of infinitely divisible distributions on $\mathbb{R}_{>0}$ is the family of generalized gamma convolutions. This class is characterized by having absolutely continuous Lévy measures with densities $u(x)$, such that $\bar{u}(x) = xu(x)$ are completely monotone functions, (Bondesson, 1992, Theorem 3.1.1).

Theorem 8. *If $u(x)$ is a Lévy density, such that $\bar{u}(x) = xu(x)$ is completely monotone, then the integrability assumptions* (68) *in Theorem 7 hold for all $n \in \mathbb{N}$ with arbitrary $r > 0$.*

Proof. By the Bernstein-Widder representation for completely monotone functions we know $\bar{u}(x)$ is holomorphic in $\Re x > 0$, thus the Taylor series at $x > 0$ has radius of convergence x and

$$\bar{u}(x/2) = \bar{u}(x - x/2) = \sum_{n=0}^{\infty} (-1)^n \bar{u}^{(n)}(x) \frac{x^n}{2^n n!}. \tag{85}$$

In our setting, $\int_0^\infty e^{-\theta x} \bar{u}(x/2) dx < \infty$ for $\theta > 0$ and as $(-1)^n \bar{u}^{(n)}(x) \ge 0$ we can integrate the series term by term. As $\bar{u}^{(n)}(x) = xu^{(n)}(x) + nu^{(n-1)}(x)$ for $n \ge 1$ and we know that $\int e^{-rx} xu(x) dx < \infty$ we obtain inductively the result. □

Remark 9. *An example where Theorem 7 applies, but the corresponding distribution is not a generalized gamma convolution is given by*

$$u(x) = x^{-3/2} e^{\sin(x)}. \tag{86}$$

An example where the integrability conditions (68) *do not hold for* $n \geq 1$ *and any* $r > 0$ *is given by*

$$u(x) = x^{-3/2} \sin(x^{-3})^2. \tag{87}$$

We do not know, whether the conclusion of the theorem is nevertheless true in this case.

3.2 Differentiability in time

For the proof of differentiability properties of the probability densities $p(x,t)$ with respect to $t \geq 0$ we need slightly different integrability properties of the cumulant function $\kappa(\theta)$ and its derivatives. Sufficient conditions to guarantee those from assumptions on the Lévy density $u(x)$ are conveniently formulated in terms of the *integral modulus of continuity.*

Definition 10. *If* $f(x)$ *is an integrable function and* $\delta > 0$ *a real number, then the integral modulus of continuity* $\omega^{(1)}(\delta; f)$ *is defined by*

$$\omega^{(1)}(\delta; f) = \sup_{0<|h|\leq\delta} \int_{-\infty}^{+\infty} |f(x+h) - f(x)| dx. \tag{88}$$

□

We note, that

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x)| dx = \int_{-\infty}^{+\infty} |f(x-h) - f(x)| dx \tag{89}$$

and thus it is sufficient to consider $0 < h \leq \delta$ in (88).

We use the above definition for functions $f(x)$, that are a priori defined for $x > 0$ with the understanding that $f(x) = 0$ if $x \leq 0$. In that case, the actual (and less elegant) expression for the integral modulus of continuity is

$$\omega^{(1)}(\delta; f) = \sup_{0<h\leq\delta} \left[\int_0^h |f(x)| dx + \int_0^{+\infty} |f(x+h) - f(x)| dx \right]. \tag{90}$$

Theorem 11. *Suppose* $m \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha \in (0,1)$, $r \in [0,\infty)$, *and* $u(x)$ *is the Lévy density of an infinite-activity subordinator. Suppose*

$$m > \frac{1+n\alpha}{1-\alpha}, \tag{91}$$

$u(x)$ *is* m*-times differentiable in* $x > 0$, *the functions*

$$v_\ell(x) = (-1)^\ell e^{-rx} x^{\ell+1} u^{(\ell)}(x) \quad (\ell = 0, \ldots, m) \tag{92}$$

are integrable, and their integral modulus of continuity satisfies

$$\omega^{(1)}(\delta; v_\ell) = \mathcal{O}(\delta^{1-\alpha}) \quad (\delta \to 0). \tag{93}$$

Let $p(x,t)$ *denote the probability densities corresponding to* $u(x)$. *Then* $p(x,t)$ *is for all* $x > 0$ n*-times differentiable in* $t \geq 0$*; furthermore,*

$$u_k(x) = (-1)^k \frac{\partial^k}{\partial x^k} (U^+)^{*k}(x), \quad (k = 1, \dots, n), \tag{94}$$

is well-defined, and

$$\frac{\partial^k}{\partial t^k} p(x, 0) = u_k(x), \quad (k = 1, \dots, n). \tag{95}$$

Proof. Let

$$\lambda_n^{(m)}(\theta; t) = \frac{\partial^{m+n}}{\partial \theta^m \partial t^n} e^{\kappa(\theta) t}, \tag{96}$$

and, for brevity of notation,

$$p_n(x; t) = \frac{\partial^n}{\partial t^n} p(x, t). \tag{97}$$

We will show inductively for $n' = 1, \dots, n$ the following statement: We have for all $x > 0$, that $p(x,t)$ is n'-times differentiable in $t \geq 0$, and that

$$p_{n'}(x; t) = \frac{(-1)^m}{2\pi i x^m} \int_{c-i\infty}^{c+i\infty} \lambda_{n'}^{(m)}(\theta; t) e^{\theta x} d\theta. \tag{98}$$

First, with $n' = 0$ we observe that $(-1)^m x^m p(x;t)$ has Laplace transform $\lambda_0^{(m)}(\theta; t)$. In Lemma 21 below we show that the assumptions on the integral modulus of continuity imply $\lambda_0^{(m)}(\theta; t) = \mathcal{O}\left(|\theta|^{-m(1-\alpha)}\right)$ as $\Im(\theta) \to \pm\infty$. Since (91) implies $m > 1/(1-\alpha)$ we have that $\lambda_0^{(m)}(\theta; t)$ is integrable and we can apply the inversion formula. This is all that was to be shown for $n' = 0$. Suppose now we have shown the claim for some $n' - 1$ and want to show it for n'. We can write

$$\begin{aligned} &h^{-1}\left(p_{n'-1}(x; t+h) - p_{n'-1}(x; t)\right) \\ &\quad = \frac{(-1)^m}{2\pi i x^m} \int_{c-i\infty}^{c+i\infty} h^{-1}\left(\lambda_{n'-1}^{(m)}(\theta; t+h) - \lambda_{n'-1}^{(m)}(\theta; t)\right) e^{\theta x} d\theta. \end{aligned} \tag{99}$$

In Lemma 21 below we show that the assumptions on the integral modulus of continuity imply $h^{-1}\left(\lambda_{n'-1}^{(m)}(\theta; t+h) - \lambda_{n'-1}^{(m)}(\theta; t)\right) = \mathcal{O}\left(|\theta|^{\alpha(m+n')-m}\right)$ as $\Im(\theta) \to \pm\infty$. Now (91) implies that the integrand in (99) is dominated by an integrable function and we can apply dominated convergence as $h \to 0$. This shows that $p_{n'-1}(x;t)$ is differentiable with respect to t and its derivative is given by (98). This finishes the induction. To complete the proof we observe that

$$\lambda_n(\theta) = \kappa(\theta)^n = \frac{\partial^n}{\partial t^n}\left[e^{\kappa(\theta) t}\right]_{t=0} = \lambda_n^{(0)}(\theta; 0), \tag{100}$$

and in view of (96), (98), and (78) we obtain indeed $p_n(x; 0) = u_n(x)$. □

3.3 Power series representation in time

The purpose of this section is to show that, subject to some regularity conditions, the probability densities $p(x;t)$ are *analytic functions* in t, represented by a power series

$$p(x;t) = \sum_{n\geq 1} u_n(x)\frac{t^n}{n!}. \tag{101}$$

To be able to do so, we assume that the Lévy density $u(x)$ is an analytic function satisfying some growth condition.

Theorem 12. *Suppose*

$$a > 0, \quad 0 < \alpha < 1, \quad \beta > -1, \quad \gamma > 0, \quad 0 < \psi < \frac{\pi}{2}, \tag{102}$$

and the Lévy density $u(z)$ is an analytic function, in a domain containing

$$W = \{z \in \mathbb{C} : z \neq 0, |\arg(z)| \leq \psi\}. \tag{103}$$

Assume moreover that

$$u(z) = az^{-1-\alpha} + \mathcal{O}\left(|z|^\beta\right) \quad \text{as } z \to 0 \text{ in } W, \tag{104}$$

and

$$u(z) = \mathcal{O}\left(e^{\gamma\cdot\Re z}\right) \quad \text{as } z \to \infty \text{ in } W. \tag{105}$$

Then the cumulant function

$$\kappa(\theta) = \int_0^\infty (e^{-\theta x} - 1)u(x)dx \tag{106}$$

admits an analytic continuation from $\{\theta \in \mathbb{C} : \Re\theta > \gamma\}$ to $\{\theta \in \mathbb{C} : \theta \neq \gamma, |\arg(\theta - \gamma)| < \frac{\pi}{2} + \psi\}$, that goes uniformly to 0 as $\theta \to \infty$ in $\{\theta \in \mathbb{C} : \theta \neq \gamma, |\arg(\theta - c)| \leq \frac{\pi}{2} + \psi\}$, where $c > \gamma$ is arbitrary, but fixed.

Furthermore $p(x;t)$ is for all $x > 0$ an entire function in $t \in \mathbb{C}$, and we have the power series expansion

$$p(x;t) = \sum_{n\geq 1} u_n(x)\frac{t^n}{n!} \tag{107}$$

where

$$u_n(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} \kappa(\theta)^n e^{\theta x} d\theta, \tag{108}$$

with $\mathcal{C}$ the contour $|\arg(\theta - c)| = \psi$, with $\theta = 0$ being passed on the left.

Proof. Let $v(z) = u(z) - az^{-1-\alpha}$ and

$$\lambda(\theta) = \int_0^\infty e^{-\theta x} v(x)dx. \tag{109}$$

Using

$$\int_0^\infty (e^{-\theta x} - 1)x^{-1-\alpha}dx = \Gamma(-\alpha)\cdot\theta^\alpha \tag{110}$$

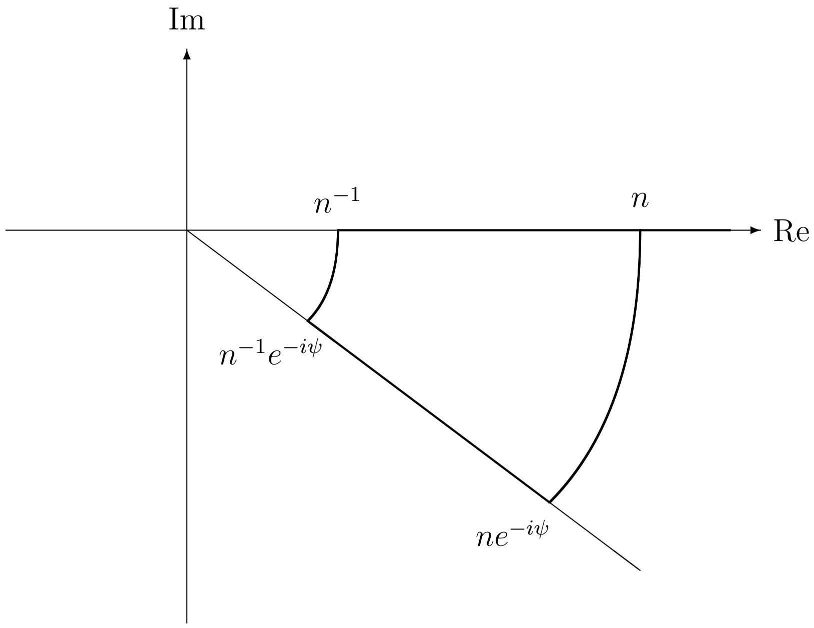

Figure 1: Integration contour for the analytic continuation of $\kappa(\theta)$

we have

$$\kappa(\theta) = a\Gamma(-\alpha)\cdot\theta^{\alpha} + \lambda(\theta) - \lambda(0), \tag{111}$$

valid for $\Re\theta \geq 0$.

Let $v_{-}(x) = v(e^{-i\psi}x)e^{-i\psi}$, then we have the growth estimates $v_{-}(x) = \mathcal{O}(|x|^{\beta})$ as $x \to 0$, and $v_{-}(x) = \mathcal{O}(\exp((\gamma\cos\psi)x))$ as $x \to \infty$. Thus the Laplace transform $\lambda_{-}(\theta) = \int_0^\infty e^{-\theta x}v_{-}(x)dx$ is absolutely convergent for $\Re\theta > \gamma\cos\psi$ and $\lambda_{-}(\theta) \to 0$ uniformly as $\theta \to \infty$ in $\Re\theta \geq c\cos\psi$, see Doetsch (1950, Satz 4, p.142 and Satz 7, p.171).

Next we show that $\lambda(\theta) = \lambda_{-}(\theta e^{-i\psi})$ for real $\theta > \gamma$: Suppose $n \geq 1$ and let us integrate $e^{-\theta z}v(z)$ over the closed contour consisting of a straight line from n^{-1} to n, a circular arc from n to $ne^{-i\psi}$, a straight line from $ne^{-i\psi}$ to $n^{-1}e^{-i\psi}$, and a circular arc from $n^{-1}e^{-i\psi}$ to n^{-1}, see Figure 1. By Cauchy's Theorem this integral is zero. The estimates (104) and (105) show that the contributions from the circular arcs vanish as $n \to \infty$, and we obtain

$$\lambda(\theta) = \int_0^\infty e^{-\theta x}v(x)dx = \int_0^{e^{-i\psi}\cdot\infty} e^{-\theta z}v(z)dz \tag{112}$$

$$= \int_0^\infty e^{-\theta e^{-i\psi}x}v(e^{-i\psi})e^{-i\psi}dx = \lambda_{-}(\theta e^{-i\psi}). \tag{113}$$

A similar argument shows that the function $v_{+}(x) = v(e^{i\psi}x)e^{i\psi}$ has Laplace transform $\lambda_{+}(\theta) = \int_0^\infty e^{-\theta x}v_{+}(x)dx$, which is absolutely convergent for $\Re\theta > \gamma\cos\psi$, and satisfies $\lambda_{+}(\theta) \to 0$ uniformly as $\theta \to \infty$ in $\Re\theta \geq c\cos\psi$, and $\lambda(\theta) = \lambda_{+}(\theta e^{i\psi})$ for real $\theta > \gamma$.

Looking at (111) reveals that the Laplace transform of $p(x;t)$, namely $e^{\kappa(\theta)t}$, is

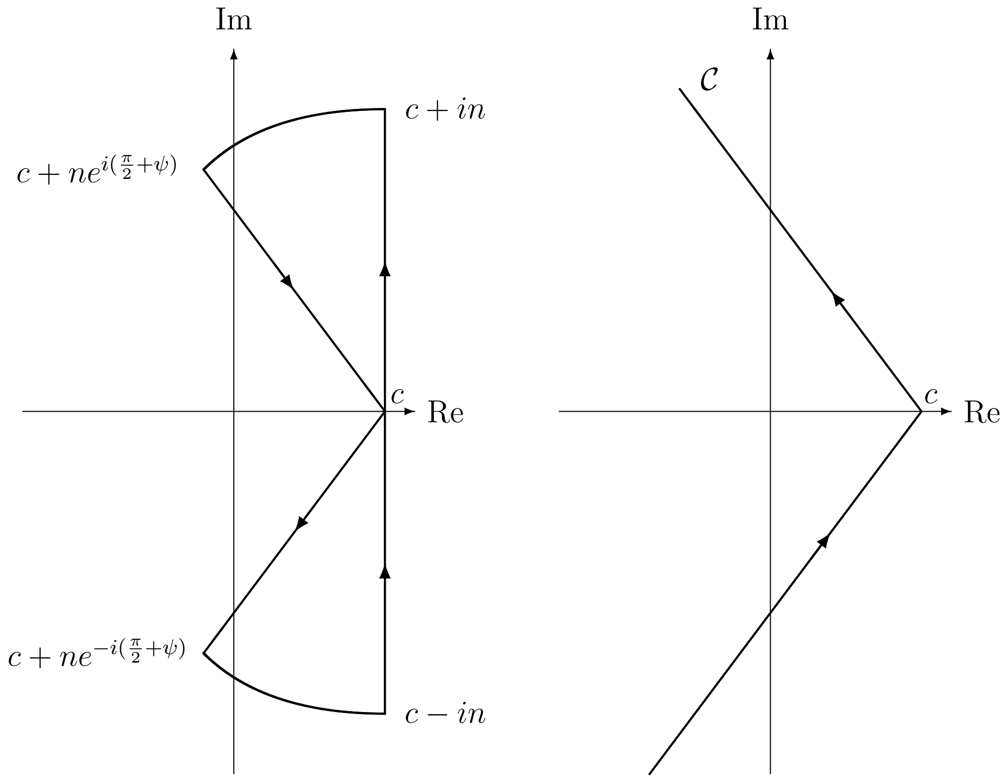

Figure 2: The contour used to derive (115) and the final contour $\mathcal{C}$

integrable on the vertical line $\Re\theta = c$. Thus we can use the inversion integral

$$p(x;t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\kappa(\theta)t+\theta x} d\theta. \tag{114}$$

Let $n \geq 1$ and consider the integrand $e^{\kappa(\theta)t+\theta x}$ on the closed contour consisting of the vertical line connecting $c - i \cdot n$ and $c + i \cdot n$, the circular arc with center c and radius n going from $c + i \cdot n$ to $c + e^{i(\pi/2+\psi)} \cdot n$, the straight lines connecting $c + e^{i(\pi/2+\psi)} \cdot n$, c, and $c + e^{-i(\pi/2+\psi)} \cdot n$, and finally, the circular arc from $c + e^{-i(\pi/2+\psi)} \cdot n$ to $c - i \cdot n$, see Figure 2. By Cauchy's theorem the integral is zero. Again, looking at (111) and the properties of the analytical continuation $\lambda(\theta)$ reveals that the integrand vanishes uniformly on the circular arcs as $n \to \infty$, and by Jordan's Lemma we conclude

$$p(x;t) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\kappa(\theta)t+\theta x} d\theta. \tag{115}$$

On $\mathcal{C}$ the linear term θx dominates $\kappa(\theta)t$ as $\theta \to \infty$. Consequently (115) makes sense for any $t \in \mathbb{C}$, in contrast to (114) where $t > 0$ is required for convergence. We observe that taking $t = 0$ yields

$$\frac{1}{2\pi i} \int_{\mathcal{C}} e^{\theta x} d\theta = 0; \tag{116}$$

thus $p(x;0) = 0$, according to our convention above. We can differentiate (115) under the integral: Let us consider $h \in \mathbb{C}$ with $|h| \leq 1$ and $t \in \mathbb{C}$ arbitrary. Using (115) we can write the complex difference quotient

$$\frac{p(x;t+h) - p(x;t)}{h} = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{e^{\kappa(\theta)h} - 1}{h} \cdot e^{\kappa(\theta)t + \theta x} d\theta. \tag{117}$$

Invoking again the asymptotic behavior of $\kappa(\theta)$ as $\theta \to \infty$ on $\mathcal{C}$ we can, by dominated convergence, prove the existence of the complex derivative $\partial p(x;t)/\partial t$ for all $t \in \mathbb{C}$ and the formula

$$\frac{\partial}{\partial t} p(x;t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \kappa(\theta) \cdot e^{\kappa(\theta)t + \theta x} d\theta. \tag{118}$$

It follows, that $p(x;t)$ is an entire function in t and

$$\frac{\partial^n}{\partial t^n} p(x;t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \kappa(\theta)^n \cdot e^{\kappa(\theta)t + \theta x} d\theta. \tag{119}$$

□

4 Examples

4.1 Compound Poisson case

In this example we deviate from the standing assumptions in the rest of the paper. Suppose X is a compound Poisson process with intensity $c > 0$ and jumps from a distribution $A(dx)$ on $\mathbb{R}^d$. Then it is well-known that

$$P(dx,t) = \sum_{n \geq 0} e^{-ct} \frac{(ct)^n}{n!} A^{*n}(dx). \tag{120}$$

In this case the Lévy measure is $U(dx) = cA(dx)$. Using $U^{*n}(dx) = c^n A^{*n}(dx)$, expanding the exponential at $t = 0$, and collecting powers of t yields

$$P(dx,t) = \sum_{n \geq 0} U_n(dx) \frac{t^n}{n!} \tag{121}$$

with

$$U_n(dx) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} c^{n-k} U^{*k}(dx). \tag{122}$$

Note, that in contrast to the infinite activity case studied in the rest of the paper, these formulae involve a term for $n = 0$ resp. $k = 0$ due to the atom at zero.

4.2 The positive α-stable law

We consider the positive α-stable distribution with Lévy density

$$u(x) = -\frac{x^{-1-\alpha}}{\Gamma(-\alpha)} \tag{123}$$

where $0 < \alpha < 1$. Note, that we interpret $\Gamma(s)^{-1}$ as an entire function with zeroes at the nonpositive integers. Alternatively we could use the functional equations $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(s)\Gamma(1-s) = \pi \csc(\pi s)$ to rewrite expressions in a more familiar form.

In general there is no closed form expression for $p(x,t)$ in terms of elementary functions, but it is well known, that the series representation (2) holds true with

$$u_n(x) = \frac{(-1)^n}{\Gamma(-n\alpha)} x^{-1-n\alpha}. \tag{124}$$

Let us first illustrate the convergence of $u_{2\varepsilon}(x)$ to $u_2(x)$ for the approximation (31) by a direct calculation. We have

$$c(\varepsilon) = \frac{\varepsilon^{-\alpha}}{\Gamma(1-\alpha)}. \tag{125}$$

For $x > 2\varepsilon$ we obtain by symmetry and partial integration

$$\begin{aligned} u_\varepsilon^{*2}(x) = \frac{2}{\alpha\Gamma(-\alpha)^2} \Bigg[& \varepsilon^{-\alpha}(x-\varepsilon)^{-1-\alpha} \\ & - \left(\frac{x}{2}\right)^{-1-2\alpha} + (1+\alpha) \int_\varepsilon^{x/2} y^{-\alpha}(x-y)^{-2-\alpha} dy \Bigg]. \end{aligned} \tag{126}$$

As $u_{2\varepsilon}(x) = u_\varepsilon^{*2}(x) - 2c(\varepsilon)u_\varepsilon(x)$ we obtain in the limit

$$u_2(x) = \frac{2}{\alpha\Gamma(-\alpha)^2} \left[-\left(\frac{x}{2}\right)^{-1-2\alpha} + (1+\alpha) \int_0^{x/2} y^{-\alpha}(x-y)^{-2-\alpha} dy \right]. \tag{127}$$

The integral on the right hand side can be expressed in terms of the incomplete beta function, and, in this particular case, reduced to integrals for the complete beta function by elementary substitutions. This yields finally agreement with (124) for $n = 2$.

In principle, though less explicit and more cumbersome, the method can be used for $n > 2$. Instead, let us illustrate the application of Theorem 7: Obviously $u(x)$ is arbitrarily often differentiable, and

$$u^{(n)}(x) = -\frac{x^{-n-1-\alpha}}{\Gamma(-n-\alpha)}. \tag{128}$$

Thus the assumption (68) of Theorem 7 is satisfied for all $m \geq n+2$ and all $r > 0$. The tail integral is

$$U^+(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \tag{129}$$

and by induction, or quicker, by looking at the Laplace transforms, we see that

$$(U^+)^{*n}(x) = \frac{x^{n-n\alpha}}{\Gamma(n-n\alpha)}. \tag{130}$$

Differentiating this equation n times and applying the functional equation of the gamma function to simplify the expression we obtain (124).

Next let us illustrate Theorem 11. The elementary integral

$$\int_0^h x^{-\alpha}dx + \int_0^\infty \left(x^{-\alpha} - (x+h)^{-\alpha}\right) dx = \frac{2h^{1-\alpha}}{1-\alpha} \tag{131}$$

shows, that the auxiliary functions $v_\ell(x)$ satisfy the assumptions (93) with $r = 0$ for all $n \in \mathbb{N}$ and all m satisfying (91). We can conclude, that $p(x;t)$ is in fact arbitrarily many times differentiable at $t = 0$, and with $u_n(x)$ being the n-the derivative.

Finally let us illustrate that we can use Theorem 12 to show that the desired power series expansion actually holds. Clearly the assumptions are satisfied with $a = -1/\Gamma(-\alpha)$ and arbitrary other constants in (102). The Laplace cumulant function is

$$\kappa(\theta) = -\theta^\alpha, \tag{132}$$

and we obtain

$$u_n(x) = \frac{(-1)^n}{2\pi i}\int_C \theta^{n\alpha}e^{\theta x}d\theta. \tag{133}$$

To see that this gives in fact the explicit expression (124), we have to substitute $\theta \mapsto \theta/x$ and recognize the resulting integral as a variant of the Hankel contour integral for $\Gamma(-n\alpha)^{-1}$.

4.3 The gamma distribution

Suppose X is the gamma process, for which X_1 has the law $\Gamma(\nu,\alpha)$ with parameters $\nu = 1$ and $\alpha = 1$. The Lévy density is

$$u(x) = x^{-1}e^{-x}. \tag{134}$$

and the probability density is

$$p(x,t) = \frac{1}{\Gamma(t)}x^{t-1}e^{-x}. \tag{135}$$

To illustrate our results we choose the approximation

$$u_\varepsilon(x) = x^\varepsilon u(x). \tag{136}$$

We note, that $u_\varepsilon(x) = \Gamma(\varepsilon)p(x,\varepsilon)$, and thus

$$u_{n\varepsilon}(x) = \Gamma(1+\varepsilon)^n \cdot \varepsilon^{-n}\sum_{k=1}^n (-1)^{n-k}\binom{n}{k}p(x;k\varepsilon). \tag{137}$$

So we are basically in the situation of Remark 1 and the convergence of $u_{n\varepsilon}(x)$ to $u_n(x)$ is equivalent to the convergence of the n-th order difference quotient of (135) at $t = 0$ to the n-th derivative from the right. The gamma probability density is for any $x > 0$ an entire function in t and the coefficients in the series expansion (2) are given by

$$u_n(x) = x^{-1}e^{-x}\sum_{k=0}^{n-1}\binom{n}{k}k!c_k \ln^{n-k-1} x.$$

The numbers c_k arise in the expansion

$$\frac{1}{\Gamma(1+z)} = \sum_{n\geq 0} c_n z^n.$$

They can be expressed explicitly as

$$c_n = \frac{1}{(n-1)!}Y_{n-1}\left(\gamma, -\zeta(2), 2\zeta(3), \dots, (-1)^{n-2}(n-2)!\zeta(n-1)\right)$$

with Y_n the complete exponential Bell polynomials, see Appendix B, γ the Euler-Masceroni constant, and ζ the Riemann Zeta function.

Let us now pretend, we did not know (135). The function $xu(x)$ is obviously completely monotone, and Theorem 8, and so Theorem 7 applies. Let us illustrate the calculation of $u_2(x)$ by formula (74). The tail integral is

$$U^+(x) = E_1(x), \tag{138}$$

where $E_1(x)$ denotes the *exponential integral*, see (Abramowitz and Stegun, 1992, 5.1.1, p.228). A direct calculation of $(U^+)^{*2}(x)$ is not very explicit. Let us write

$$U^+(x) = V(x) - L(x), \tag{139}$$

where

$$L(x) = \ln x, \qquad V(x) = \ln x + E_1(x). \tag{140}$$

This decomposition is useful, because $L(x)$ is simple, while $V(x)$ and its derivatives are integrable at zero. We have

$$V(0) = -\gamma, \qquad V'(0) = 1. \tag{141}$$

Equation (139) implies

$$(U^+)^{*2}(x) = L^{*2}(x) - 2(L * V)(x) + V^{*2}(x). \tag{142}$$

Next we observe

$$L^{*2}(x) = \left(\ln^2 x - 2\ln x + 2 - \frac{\pi^2}{6}\right) x \tag{143}$$

and thus

$$\left[L^{*2}(x)\right]'' = \frac{2\ln x}{x}. \tag{144}$$

To compute the second derivatives of $(L * V)(x)$ and $V^{*2}(x)$, we can interchange differentiation and convolution by the usual formulas, see (Doetsch, 1950, 2.14.5, p.115ff). Namely, we use

$$\left[V^{*2}(x)\right]'' = (V')^{*2}(x) + 2V(0)V'(x) \tag{145}$$

and

$$[(V * L)(x)]'' = (V'' * L)(x) + V'(0)L(x) + V(0)L'(x). \tag{146}$$

The convolution integrals on the right hand side of (145) and (146) can be computed in terms of the exponential integral. Combining the three contributions those terms cancel, and we obtain

$$u_2(x) = 2x^{-1}e^{-x}(\ln x + \gamma), \tag{147}$$

in agreement with (137) above.

Finally, what can we say about Theorem 12 in this case? In its present form it does not apply, since (104) is not satisfied, though formula (108) is correct. The cumulant function is

$$\kappa(\theta) = -\ln(1+\theta) \tag{148}$$

and

$$u_n(x) = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \ln(1+\theta)^n e^{\theta x} d\theta, \tag{149}$$

Agreement of this formula with (137) can be established by referring to the Hankel contour integrals for the derivatives of $\Gamma(z)^{-1}$ at $z = 1$.

4.4 The inverse Gaussian distribution

The Inverse Gaussian distribution $IG(\delta, \gamma)$ with $\delta = 1$ and $\gamma = 1$ has a Lévy density of the form

$$u(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-x/2}, \tag{150}$$

and the probability density is

$$p(x,t) = \frac{1}{\sqrt{2\pi}} t e^t x^{-3/2} e^{-(t^2 x^{-1} + x)/2}. \tag{151}$$

Using the generating function for the Hermite polynomials $H_n(x)$, namely,

$$e^{2xt - t^2} = \sum_{n \geq 0} H_n(x) \frac{t^n}{n!}, \tag{152}$$

we find

$$u_n(x) = \frac{n}{\sqrt{\pi}} 2^{-n/2} x^{-1-n/2} e^{-x/2} H_{n-1}\left(\sqrt{\frac{x}{2}}\right). \tag{153}$$

Let us choose the approximation

$$u_\varepsilon(x) = \exp\left(-\frac{\varepsilon^2}{2x}\right) u(x). \tag{154}$$

We recognize that this is a multiple of $p(x, \varepsilon)$ and, again, showing the convergence of $u_{n\varepsilon}(x)$ to $u_n(x)$ reduces essentially to a study of the n-the order difference quotient of $p(x;t)$ at $t = 0$.

Let us look at the second approach, based on the tail integral. Again, $xu(x)$ is completely monotone. We have

$$U^+(x) = \sqrt{\frac{2}{\pi x}} e^{-x/2} - \operatorname{erfc}\left(\sqrt{\frac{x}{2}}\right), \tag{155}$$

where $\operatorname{erfc}(x)$ is the *complementary error function*, see (Abramowitz and Stegun, 1992, 7.1.2, p.297). Let us illustrate the computation of $u_3(x)$. By looking at Laplace transforms we establish

$$(U^+)^{*3}(x) = 2\sqrt{\frac{2x}{\pi}} e^{-x/2}(2+x) - (2x^2+6x)\operatorname{erfc}\left(\sqrt{\frac{x}{2}}\right), \tag{156}$$

and differentiating $-(U^+)^{*3}(x)$ three times we obtain

$$u_3(x) = \frac{3}{\sqrt{2\pi}} x^{-5/2}(x-1)e^{-x/2}, \tag{157}$$

in agreement with (153) above. Finally Theorem 12 applies, the cumulant function is

$$\kappa(\theta) = 1 - \sqrt{1+2\theta} \tag{158}$$

and we get

$$u_n(x) = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} (1-\sqrt{1+2\theta})^n e^{\theta x} d\theta. \tag{159}$$

Agreement of this formula with (153) can be established as follows: First we substitute $\theta \mapsto (\theta-1)/2$ and expanding the integrand by the binomial theorem we obtain a sum of Hankel integrals of the form (133) with $\alpha = 1/2$, producing a sum of powers of x. Using the well-known explicit form of the coefficients of the Hermite polynomials shows (153).

4.5 An example on $\mathbb{R}$: The Meixner distribution

This example is not covered by the standing assumptions in this paper, as the Meixner distribution, see Schoutens (2003, Sec.5.3.10, p.62), is an infinitely divisible distribution on $\mathbb{R}$, not on $\mathbb{R}_{>0}$.

Let us consider the Meixner distribution with parameters $\mu = 0$, $\delta = 1$, $\alpha = 1$, and $\beta = 0$. It has the density

$$p(x,t) = \frac{1}{\pi} 2^{2t-1} \frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(2t)}.$$

This expression can be expanded in a series

$$p(x,t) = \sum_{n\geq 1} u_n(x) \frac{t^n}{n!}, \qquad |t| < |x|$$

with

$$u_n(x) = \frac{n}{x\sinh(\pi x)} Y_{n-1}\left(a_1(x), \ldots, a_{n-1}(x)\right)$$

where

$$a_1(x) = \psi(ix) + \psi(-ix) + 2\ln 2 + 2\gamma$$

and

$$a_n(x) = \psi^{(n)}(ix) + \psi^{(n)}(-ix) - (-1)^n 2^n (n-1)!\zeta(n) \quad (n \geq 2).$$

Here Y denotes again the complete exponential Bell polynomials, γ is the Euler-Masceroni constant, ψ is the digamma function, and ζ the Riemann zeta function. Note, however, that here $p(x,t)$ is not an entire function in t, due to the poles of the gamma function. Thus we have to expect qualitative differences to the cases studied in the present paper.

4.6 A bivariate example: The Inverse Gaussian-Normal Inverse Gaussian law

Again this example is not covered by the assumptions of the rest of this paper, as it deals with a bivariate distribution on $\mathbb{R}_{>0} \times \mathbb{R}$. We consider the probability densities

$$p(x,y;t) = \frac{t}{2\pi} e^t x^{-2} \exp\left[-\frac{1}{2}\left(\frac{t^2+y^2}{x} + x\right)\right] \tag{160}$$

on $\mathbb{R}_{>0} \times \mathbb{R}$. They correspond to an Inverse Gaussian-Normal Inverse Gaussian, or IG-NIG, Lévy process. For properties of this type of law and its origin in a first passage time problem for a bivariate Brownian motion see (Barndorff-Nielsen and Blæsild, 1983, Example 4.1) and (Barndorff-Nielsen and Shephard, 2001, Example 4.3).

The associated Lévy measure is

$$u(x,y) = \frac{1}{2\pi} x^{-2} \exp\left[-\frac{1}{2}\left(\frac{y^2}{x} + x\right)\right], \tag{161}$$

and the Laplace cumulant function is

$$\kappa(\theta,\eta) = 1 - \sqrt{1 + 2\theta - \eta^2}. \tag{162}$$

Looking again at the generating function for the Hermite polynomials, see (152), we obtain from (160) the series representation

$$p(x,y;t) = \sum_{n\geq 1} u_n(x,y)\frac{t^n}{n!} \tag{163}$$

with

$$u_n(x,y) = n 2^{-\frac{n-1}{2}} x^{-\frac{n-1}{2}} u(x,y) H_{n-1}\left(\sqrt{\frac{x}{2}}\right). \tag{164}$$

For this example we can verify the validity of the bivariate extension of our Method 1 along the lines of Remark 1: A convenient approximation to the Lévy density is

$$u_\varepsilon(x,y) = \varepsilon^{-1} p(x,y;\varepsilon) \tag{165}$$

and we obtain

$$u_n(x,y) = \lim_{\varepsilon\to 0} \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \varepsilon^{n-k} u_\varepsilon^{*k}(x,y), \tag{166}$$

since the sum on the right hand side can be written as n-th order difference quotient of $p(x,y;t)$ at $t=0$, and $p(x,y,t)$ is n-times differentiable at $t=0$. Obviously the caveat of Remark 1 applies here as well.

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A Auxiliary results

This section provides technical estimates used in the proofs of Theorem 7 and Theorem 11. We use Faa di Bruno's formula for derivatives for composite functions, expressed in terms of partial Bell polynomials B_{nk}. Our main reference is Comtet (1970, Section 3, p.144ff), but for the reader's convenience we have collected definition and properties used in this paper here.

A.1 Bell polynomials

The partial Bell polynomials, see Comtet (1970, Section 3, p.144ff), are defined as

$$B_{nk}(x_1,\dots,x_{n-k+1}) = \sum_{a\in_{nk}} b_{nk}(a) x_1^{a_1} \dots x_{n-k+1}^{a_{n-k+1}} \tag{167}$$

with

$$\mathcal{A}_{nk} = \left\{ a \in \mathbb{R}_{\geq 0}^{n-k+1} : \sum_{\ell=1}^{n-k+1} \ell a_\ell = n, \sum_{\ell=1}^{n-k+1} a_\ell = k \right\} \tag{168}$$

and

$$b_{nk}(a) = \frac{n!}{\prod_{\ell=1}^{n-k+1} a_\ell! \ell!^{a_\ell}}. \tag{169}$$

They satisfy the recurrence relation

$$B_{n0} = \delta_{n0}, \quad B_{nk} = \frac{1}{k} \sum_{\ell=k-1}^{n-1} \binom{n}{\ell} x_{n-\ell} B_{\ell,k-1}. \tag{170}$$

The first few polynomials are given by

$$B_{11} = x_1 \tag{171}$$
$$B_{21} = x_2, \qquad B_{22} = x_1^2 \tag{172}$$
$$B_{31} = x_3, \qquad B_{32} = 3x_1x_2, \qquad B_{33} = x_1^3 \tag{173}$$
$$B_{41} = x_4, \qquad B_{42} = 4x_1x_3 + 3x_2^2, \qquad B_{43} = 6x_1^2x_2, \qquad B_{44} = x_1^4. \tag{174}$$

We will need also that

$$B_{n1} = x_n, \qquad B_{n2} = \frac{1}{2}\sum_{\ell=1}^{n-1}\binom{n}{\ell} x_\ell x_{n-\ell}, \tag{175}$$

and

$$B_{n,n-1} = \frac{n(n-1)}{2} x_1^{n-2} x_2, \qquad B_{nn} = x_1^n. \tag{176}$$

Furthermore we have the useful property

$$B_{nk}(ax_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{nk}(x_1, x_2, \ldots, x_{n-k+1}). \tag{177}$$

Theorem 13 (Faa di Bruno's formula)**.** *Suppose $f(x)$ and $g(x)$ are n-times differentiable functions. Then*

$$h(x) = f(g(x)) \tag{178}$$

is n-times differentiable and

$$h^{(n)}(x) = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{nk}(g'(x), \ldots, g^{(n-k+1)}(x)). \tag{179}$$

The complete Bell polynomials are defined as

$$Y_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{nk}(x_1, \ldots, x_{n-k+1}). \tag{180}$$

The first few are

$$Y_1 = x_1 \tag{181}$$
$$Y_2 = x_1^2 + x_2 \tag{182}$$
$$Y_3 = x_1^3 + 3x_1x_2 + x_3 \tag{183}$$
$$Y_4 = x_1^4 + 6x_1^2x_2 + 3x_2^2 + 4x_1x_3 + x_4 \tag{184}$$

For brevity it is convenient to introduce the following quantities, defined in terms of the partial Bell polynomials and the cumulant function $\kappa(\theta)$:

$$\beta_{nk}(\theta) = B_{nk}(\kappa'(\theta), \ldots, \kappa^{(n-k+1)}(\theta)) \tag{185}$$

and

$$\beta^*_{nk}(\theta) = B_{nk}(|\kappa'(\theta)|, \ldots, |\kappa^{(n-k+1)}(\theta)|). \tag{186}$$

We recall, that the functions $\lambda_{n\varepsilon}(\theta)$ and $\lambda_n(\theta)$, which are crucial to the proof of Theorem 7, are given by (75) resp. (76).

A.2 Auxiliary estimates for the cumulant function

The structure of this subsection is as follows: In Lemma 15 we show that the integrability assumptions (68) of Theorem 7 imply a certain asymptotic behavior of the derivatives of $u(x)$ as $x \to 0$ and as $x \to \infty$. This is used in Lemma 16 to derive estimates of the derivatives of the cumulant function $\kappa(\theta)$ as $\theta \to \infty$, by partial integration. Those estimates are plugged into the Bell polynomials $\beta_{nk}(\theta)$ in Lemma 17. Using the latter we obtain estimates for the derivatives of $\lambda_n(\theta)$ as $\theta \to \infty$ in Lemma 18. Next, Lemma 19 shows that $u_\varepsilon(x)$ satisfies the assumptions (68) uniformly for $0 < \varepsilon \leq 1$, and thus, applying Lemma 15–18 to $u_\varepsilon(x)$, gives uniform estimates for the derivatives of $\lambda_{n\varepsilon}(\theta)$ as $\theta \to \infty$. Finally, Lemma 20 provides a refined estimate for $\beta_{nk}(\theta)$ from the slightly stronger assumptions of Theorem 11.

We are considering distributions on $\mathbb{R}_{>0}$, thus we have

$$|e^{\kappa(\theta)}| \leq 1 \qquad \forall \Re(\theta) \geq 0. \tag{187}$$

Moreover $\kappa(\theta)$ is analytic for $\Re(\theta) > 0$ and

$$\kappa^{(n)}(\theta) = (-1)^n \int_0^\infty e^{-\theta x} x^n u(x) dx, \quad \Re(\theta) > 0, n \geq 1. \tag{188}$$

Definition 14. *Suppose $n \in \mathbb{N}$ and $c > 0$. Then we say that assumption $\mathcal{A}_n(c)$ holds for $u(x)$ if $u(x)$ is n-times continuously differentiable and*

$$\int_0^\infty e^{-cx} x^{k+1} |u^{(k)}(x)| dx < \infty, \quad k = 0, 1, \ldots, n. \tag{189}$$

A simple consequence of assumption $\mathcal{A}_n(c)$ is, that the Laplace transform

$$\int_0^\infty e^{-\theta x} x^m u^{(n)}(x) dx \tag{190}$$

exists for any $m \geq n+1$ and $\Re(\theta) > c$.

Lemma 15. *Suppose $n \geq 2$ and $c > 0$. If assumption $\mathcal{A}_n(c)$ holds then*

$$\lim_{x \to 0} e^{-\theta x} x^n u^{(n-2)}(x) = 0 \tag{191}$$

and

$$\lim_{x \to \infty} e^{-\theta x} x^n u^{(n-2)}(x) = 0 \tag{192}$$

for $\Re(\theta) > c$.

Proof. Let $0 < a < b$. Partial integration gives

$$\int_a^b e^{-\theta x} (x^n u^{(n-2)}(x))' dx = e^{-\theta x} x^n u^{(n-2)}(x) \Big|_a^b + \theta \int_a^b e^{-\theta x} x^n u^{(n-2)}(x) dx. \tag{193}$$

We have

$$(x^n u^{(n-2)}(x))' = n x^{n-1} u^{(n-2)}(x) + x^n u^{(n-1)}(x) \tag{194}$$

and see from the integrability assumptions $\mathcal{A}_n(c)$ and $\Re(\theta) > c$ that both integrals in (193) converge to a finite value as $a \to 0$ and $b \to \infty$ separately. Thus

$$\lim_{x\to 0} e^{-\theta x} x^n u^{(n-2)}(x) = \alpha, \qquad \lim_{x\to\infty} e^{-\theta x} x^n u^{(n-2)}(x) = \omega \tag{195}$$

exist with finite α and ω. But $\alpha \neq 0$ or $\omega \neq 0$ would imply

$$e^{-\theta x} x^{n-1} u^{(n-2)}(x) \sim \frac{\alpha}{x}, \quad x \to 0 \tag{196}$$

respectively

$$e^{-\theta x} x^{n-1} u^{(n-2)}(x) \sim \frac{\omega}{x}, \quad x \to \infty. \tag{197}$$

Both properties would contradict the integrability of $e^{-\theta x} x^{n-1} u^{(n-2)}(x)$, that follows, again, from assumption $\mathcal{A}_n(c)$. Thus we must have $\alpha = 0$ and $\omega = 0$. □

The following lemma is essentially a reformulation of the well-known fact that a function $f(x)$, that is n-times differentiable with $f^{(k)}(x)$ integrable for $0 \leq k \leq n$, has a Fourier transform $\hat{f}(y)$, that satisfies $\hat{f}(y) = \mathcal{O}(|y|^{-n})$ as $|y| \to \infty$. As we will need uniform growth estimates later, we provide a more detailed statement with explicit bounds.

Lemma 16. *Suppose $n \geq 0$ and $c > 0$. If assumption $\mathcal{A}_n(c)$ holds and*

$$L_k(c) = \int_0^\infty e^{-cx} x^{k+1} |u^{(k)}(x)| dx, \quad k = 0, 1, \ldots, n, \tag{198}$$

then

$$|\kappa^{(n)}(\theta)| \leq \frac{M_n(c)}{|\theta|^{n-1}}, \quad \Re(\theta) > c, \tag{199}$$

where

$$M_0(c) = L_0(c), \qquad M_n(c) = \sum_{k=0}^{n-1} \binom{n-1}{k} (n)_{n-1-k} L_k(c), \quad (n \geq 1). \tag{200}$$

Proof. For $n = 1$ we have

$$\kappa'(\theta) = -\int_0^\infty e^{-\theta x} x u(x) dx \tag{201}$$

and the assertion of the lemma is obvious, namely, $|\kappa'(\theta)| \leq L_0$. For $n = 0$ we can write

$$\kappa(\theta) = \int_0^\theta \kappa'(\zeta) d\zeta \tag{202}$$

and the assertion follows, namely $|\kappa(\theta)| \leq L_0 |\theta|$. For $n \geq 2$ we recall

$$\kappa^{(n)}(\theta) = (-1)^n \int_0^\infty e^{-\theta x} x^n u(x) dx. \tag{203}$$

Let $0 < a < b$. Repeated partial integration gives

$$\int_a^b e^{-\theta x} x^n u(x) dx$$
$$= -\sum_{k=1}^{n-1} \frac{1}{\theta^k} e^{-\theta x} (x^n u(x))^{(k-1)} \Big|_a^b + \frac{1}{\theta^{n-1}} \int_a^b e^{-\theta x} (x^n u(x))^{(n-1)} dx, \tag{204}$$

and by Leibniz' rule we obtain

$$(x^n u(x))^{(k-1)} = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (n)_{k-1-\ell} x^{n-1-k} x^{\ell+2} u^{(\ell)}(x). \tag{205}$$

From assumption $\mathcal{A}_n(c)$ and Lemma 15 we conclude, letting $a \to 0$ and $b \to \infty$, that

$$\kappa^{(n)}(\theta) = \frac{(-1)^n}{\theta^{n-1}} \int_0^\infty e^{-\theta x} \left(x^n u(x)\right)^{(n-1)} dx. \tag{206}$$

Using (205), this time with $k = n$, we get

$$\left(x^n u(x)\right)^{(n-1)} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (n)_{n-1-\ell} x^{\ell+1} u^{(\ell)}(x). \tag{207}$$

This shows that the integral in (206) is bounded by $M_n(c)$. □

Lemma 17. *Suppose $n \geq 0$ and $c > 0$. If assumption $\mathcal{A}_n(c)$ holds then*

$$\beta^*_{nk}(\theta) \leq \frac{M_{nk}}{|\theta|^{n-k}}, \quad k = 1, \ldots, n, \tag{208}$$

where

$$M_{nk} = B_{nk}(M_1, \ldots, M_{n-k+1}) \tag{209}$$

and the constants $M_1, \ldots, M_n$ are as in Lemma 16 above.

Proof. The Bell polynomials B_{nk} have nonnegative coefficients, and are therefore increasing functions of each argument. Using the bounds from Lemma 16 we have

$$\beta^*_{nk}(\theta) = B_{nk}(|\kappa'(\theta)|, \ldots, |\kappa^{(n-k+1)}|) \leq B_{nk}\left(M_1, \frac{M_2}{|\theta|}, \ldots, \frac{M_{n-k+1}}{|\theta|^{n-k}}\right). \tag{210}$$

Using (177) with $a = |\theta|$ and $b = |\theta^{-1}|$ we obtain the desired result. □

Lemma 18. *Suppose $m \geq 1$, $n \geq 1$, and that assumption $\mathcal{A}_n(c)$ holds for $u(x)$ with some $c > 0$, and let*

$$\lambda_n(\theta) = \kappa(\theta)^n. \tag{211}$$

Then

$$|\lambda_n^{(m)}(\theta)| \leq \frac{E_{mn}}{|\theta|^{m-n}} \tag{212}$$

where

$$E_{mn} = \sum_{j=1}^{m \wedge n} (n)_j L_0^{n-j} M_{mj}. \tag{213}$$

Proof. From Faa di Bruno's formula we get

$$\lambda_n^{(m)}(\theta) = \sum_{j=1}^{m\wedge n} (n)_j \kappa(\theta)^{n-j} \beta_{mj}(\theta). \tag{214}$$

Using the estimates from Lemma 17 we obtain the result. □

Lemma 19. *Suppose $n \geq 0$, assumption $\mathcal{A}_n(c)$ holds for $u(x)$ with some $c > 0$ and*

$$L_k(c) = \int_0^\infty e^{-cx} x^{k+1} |u^{(k)}(x)| dx, \quad k = 0, 1, \ldots, n. \tag{215}$$

If we set for $\varepsilon > 0$

$$u_\varepsilon(x) = e^{-\varepsilon/x} u(x), \quad x > 0, \tag{216}$$

then assumption $\mathcal{A}_n(c)$ holds for $u_\varepsilon(x)$ and we have for any $\varepsilon > 0$ the uniform bound

$$\int_0^\infty e^{-cx} x^{n+1} |u_\varepsilon^{(n)}(x)| dx \leq \bar{L}_n(c), \tag{217}$$

where

$$\bar{L}_n(c) = \sum_{k=0}^{n} \sum_{\ell=1}^{k} \binom{n}{k} \binom{k-1}{\ell-1} \frac{k!}{\ell!} \ell^\ell e^{-\ell} L_{n-k}(c) \tag{218}$$

Proof. This follows from

$$\left[e^{-\varepsilon/x}\right]^{(k)} = \sum_{\ell=1}^{k} (-1)^{\ell+k} \binom{k-1}{\ell-1} \frac{k!}{\ell!} x^{-k} \left(\frac{\varepsilon}{x}\right)^\ell e^{-\varepsilon/x} \tag{219}$$

and

$$0 \leq x^\ell e^{-x} \leq \ell^\ell e^{-\ell}, \quad x \geq 0. \tag{220}$$

□

Lemma 20. *Suppose $m \in \mathbb{N}$, $\alpha \in (0,1)$, $c \in (0,\infty)$, and $u(x)$ is the Lévy density of an infinite activity subordinator, that is m-times differentiable and such that the functions*

$$v_\ell(x) = e^{-cx} x^{\ell+1} u^{(\ell)}(x) \quad (\ell = 0, \ldots, m) \tag{221}$$

are integrable, and their integral modulus of continuity satisfies

$$\omega^{(1)}(\delta; v_\ell) = \mathcal{O}(\delta^{1-\alpha}) \quad (\delta \to 0). \tag{222}$$

Then we have for $n = 0, \ldots, m$

$$\kappa^{(n)}(\theta) = \mathcal{O}(|\theta|^{\alpha-n}), \quad (\Im(\theta) \to \pm\infty) \tag{223}$$

and for $\ell = 1, \ldots, m$

$$\beta_{m\ell}(\theta) = \mathcal{O}(|\theta|^{\ell\alpha-m}), \quad (\Im(\theta) \to \pm\infty). \tag{224}$$

Proof. From the proof of Lemma 16 above we know that

$$\kappa^{(n)}(\theta) = \frac{(-1)^n}{\theta^{n-1}} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (n)_{n-1-\ell} \cdot \int_0^\infty e^{-\theta x} x^{\ell+1} u^{(\ell)}(x) dx \tag{225}$$

for $n = 1, \ldots, m$. Using the assumptions (222) in Lemma 26 yields (223) for $n = 1, \ldots, m$. The case $n = 0$ follows immediately by using the estimate for $\kappa'(\theta)$ in

$$\kappa(\theta) = \int_0^\theta \kappa'(\zeta) d\zeta. \tag{226}$$

Plugging these estimates into (185) and looking at the explicit formula for the Bell polynomials in Subsection A.1 shows

$$\beta_{m\ell}(\theta) = \mathcal{O}\left(\sum |\theta|^{a_1(\alpha-1)+\ldots+a_m(\alpha-m)}\right) = \mathcal{O}\left(|\theta|^{\ell\alpha-m}\right). \tag{227}$$

□

Lemma 21. *Suppose the assumptions for Lemma 20 hold, and*

$$\lambda_n^{(m)}(\theta; t) = \frac{\partial^{m+n}}{\partial \theta^m \partial t^n} e^{\kappa(\theta) t}. \tag{228}$$

Then

$$\lambda_n^{(m)}(\theta; t) = \mathcal{O}(|\theta|^{(m+n)\alpha - m}), \quad (\Im(\theta) \to \pm\infty). \tag{229}$$

Proof. First we have

$$\lambda_n^{(0)}(\theta; t) = \kappa(\theta)^n e^{\kappa(\theta) t}. \tag{230}$$

Using (188) and (223) with $n = 0$, shows the claim for $m = 0$. Next, by differentiating (230) $m \geq 1$ times according to Faa di Bruno's formula, we get

$$\lambda_n^{(m)}(\theta; t) = \sum_{\ell=1}^{m} \sum_{j=0}^{\ell} \binom{\ell}{j} (n)_j \kappa(\theta)^{n-j} t^{\ell-j} e^{\kappa(\theta) t} \beta_{m\ell}(\theta) \kappa(\theta)^n e^{\kappa(\theta) t}. \tag{231}$$

Using (188), (223) with $n = 0$, and (224) we obtain (229). □

A.3 Convolutions and Laplace transforms

In this subsection we provide further auxiliary results for the proof of Theorem 7. For notational convenience let us define

$$V(x) = U^+(x), \qquad V_n(x) = V^{*n}(x). \tag{232}$$

First it is shown in Lemma 22, that the convolution powers $V^{*n}(x)$ exist. Then, using Lemma 23 as intermediate step, we show that the $V^{*n}(x)$ are n-times differentiable and we provide an integral representation in Lemma 24. In this subsection we use estimates from the previous subsection.

Lemma 22. *Let*

$$V(x) = \int_x^\infty u(y)dy. \tag{233}$$

Then

$$V_n(x) = V^{*n}(x), \quad x > 0, \tag{234}$$

is well-defined for $n \geq 1$ and we have the Laplace transforms

$$\int_0^\infty e^{-\theta x} V_n(x)dx = (-1)^n \frac{\kappa(\theta)^n}{\theta^n}, \quad \Re(\theta) > 0. \tag{235}$$

Proof. We think it is instructive to give two proofs here. (a) The first proof uses the well-known theorem for the existence of convolutions of integrable functions. It will show existence only for almost all $x > 0$: Let $r > 0$ arbitrary. Then by the Fubini-Tonelli Theorem we have

$$\int_0^\infty e^{-rx} V(x)dx = \int_0^\infty \int_x^\infty e^{-rx} u(y)dydx \tag{236}$$

$$= \int_0^\infty \int_0^y e^{-rx} u(y)dxdy \tag{237}$$

$$= \frac{1}{r} \int_0^\infty (1 - e^{-ry})u(y)dy \tag{238}$$

$$= -\frac{\kappa(r)}{r} < \infty. \tag{239}$$

This shows that $\tilde{V}(x) = e^{-rx}V(x)$ is integrable. Thus the convolution powers $\tilde{V}^{*n}(x)$ exist for almost all $x > 0$ and are integrable on $\mathbb{R}_{>0}$. We have

$$e^{rx}\tilde{V}^{*2}(x) = e^{rx} \int_0^x \tilde{V}(y)\tilde{V}(x-y)dy \tag{240}$$

$$= e^{rx} \int_0^x e^{-ry}V(y) \cdot e^{-r(x-y)}V(x-y)dy \tag{241}$$

$$= e^{rx} \int_0^x e^{-rx}V(y)V(x-y)dy \tag{242}$$

$$= \int_0^x V(y)V(x-y)dy. \tag{243}$$

Since all integrands are nonnegative this calculation shows that $V(y)V(x-y)$ is integrable on $(0, x)$ and $V^{*2}(x)$ exists for almost all $x > 0$. Repeating the argument shows that the higher convolution powers exist for almost all $x > 0$.

(b) The second proof is essentially using the theory of convolutions of functions of the class $\mathcal{J}_0$, as they are called in Doetsch (1950). We do not go into that, but rather focus on the present situation: The function $V(x)$ is nonnegative and decreasing.

We have again by Fubini-Tonelli

$$\int_0^x V(y)dy = \int_0^x \int_y^\infty u(z)dzdy \tag{244}$$
$$= \int_0^x \int_0^z u(z)dydz + \int_x^\infty \int_0^x u(z)dydz \tag{245}$$
$$= \int_0^x zu(z)dz + \int_x^\infty xu(z)dydz \tag{246}$$
$$= \int_0^x zu(z)dz + xV(x) < \infty. \tag{247}$$

We set

$$W(x) = \int_0^x V(y)dy, \qquad W_n(x) = \int_0^x V_n(y)dy. \tag{248}$$

Those functions are increasing. We will show by induction, that

$$V_n(x) \le nV\left(\frac{x}{2^{n-1}}\right) W\left(\frac{x}{2}\right)^{n-1}, \qquad W_n(x) \le W(x)^n. \tag{249}$$

For $n = 1$ nothing is to be shown. Suppose now the assertion is shown for n and we want to prove it for $n+1$. Using the monotonicity of $V(x)$ and $W(x)$ and the induction hypothesis we get first

$$V_{n+1}(x) = \int_0^x V_n(y)V(x-y)dy \tag{250}$$
$$= \int_0^{x/2} V_n(y)V(x-y)dy + \int_0^{x/2} V(y)V_n(x-y)dy \tag{251}$$
$$\le V\left(\frac{x}{2}\right) \int_0^{x/2} V_n(y) + \int_0^{x/2} V(y)nV\left(\frac{x-y}{2^{n-1}}\right) W\left(\frac{x-y}{2}\right)^{n-1} dy \tag{252}$$
$$\le V\left(\frac{x}{2^n}\right) W\left(\frac{x}{2}\right)^n + nV\left(\frac{x-y}{2^n}\right) W\left(\frac{x}{2}\right)^{n-1} W\left(\frac{x}{2}\right) \tag{253}$$
$$= (n+1)V\left(\frac{x}{2^n}\right) W\left(\frac{x}{2}\right)^n, \tag{254}$$

and then

$$W_n(x) = \int_0^x V_n(y)dy \tag{255}$$
$$= \int_0^x \int_{z=0}^y V_{n-1}(z)V(y-z)dzdy \tag{256}$$
$$= \int_0^x \int_0^{x-z} V_{n-1}(z)V(y)dydz \tag{257}$$
$$\le \int_0^x \int_0^x V_{n-1}(z)V(y)dydz \tag{258}$$
$$\le W_n(x)W(x) \le W(x)^n. \tag{259}$$

This shows that the convolution powers $V_n(x) = V^{*n}(x)$ exist for all $x > 0$. □

Lemma 23. *Suppose $n \in \mathbb{N}$ and assumption $\mathcal{A}_m(c)$ holds for some $m \geq n+2$ and $c > 0$. Then $V_n(x)$ is n-times differentiable and*

$$(x^m V_n(x))^{(k)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (-1)^{m+n} \left(\frac{\kappa(\theta)^n}{\theta^n} \right)^{(m)} \theta^k e^{\theta x} d\theta, \qquad x > 0, k = 0, 1, \ldots, n. \tag{260}$$

Proof. We have the Laplace transform

$$\mathcal{L}\left[\theta \ddagger x^m V_n(x)\right] = (-1)^{m+n} \left(\frac{\kappa(\theta)^n}{\theta^n} \right)^{(m)}. \tag{261}$$

Now by Leibniz's rule and Faa di Bruno's formula

$$\left(\frac{\kappa(\theta)^n}{\theta^n} \right)^{(m)} = \sum_{j=0}^{m} \binom{m}{j} \left(\kappa(\theta)^n\right)^{(j)} \left(\theta^{-n}\right)^{m-j} \tag{262}$$

$$= \sum_{j=0}^{m} \sum_{k=0}^{j \wedge n} \binom{m}{j} (n)_k (-n)_{m-j} \frac{\kappa(\theta)^{n-k} \beta_{jk}(\theta)}{\theta^{n+m-j}}. \tag{263}$$

Using the estimates from Lemma 16 for $n = 0$ and Lemma 17 we obtain

$$\left| \theta^k \left(\frac{\kappa(\theta)^n}{\theta^n} \right)^{(m)} \right| \leq \frac{D_{mn}}{|\theta|^{m-k}}, \tag{264}$$

where

$$D_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{j \wedge n} \binom{m}{j} (n)_k (-n)_{m-j} L_0^{n-k} M_{jk}. \tag{265}$$

Hence we can apply Lemma 25 to conclude that $(-1)^{m+n} x^m V_n(x)$, and thus also $V_n(x)$, is n-times continuously differentiable in $x > 0$. □

Lemma 24. *Suppose $n \in \mathbb{N}$ and assumption $\mathcal{A}_m(c)$ holds for some $m \geq n+2$ and $c > 0$. Then*

$$V_n^{(k)}(x) = \frac{(-1)^{m+n}}{2\pi i x^m} \int_{c-i\infty}^{c+i\infty} \left(\frac{\kappa(\theta)^n}{\theta^{n-k}} \right)^{(m)} e^{\theta x} d\theta, \quad x > 0, k = 0, 1, \ldots, n. \tag{266}$$

Proof. We do induction on k. Starting with $k = 0$ we have $(-1)^{m+1} x^m V_1'(x) = (-1)^m x^m u(x)$, which has the integrable Laplace transform $\kappa^{(m)}(\theta)$ and thus the assertion holds. Suppose now we have shown the assertion for $n-1$ and we want to prove it for n. Repeated application of Leibniz's' rule gives

$$x^m V_n^{(k)}(x) = (x^m V_n(x))^{(k)} + \sum_{j=0}^{k-1} \sum_{\ell=0}^{j} \binom{k}{j} \binom{j}{\ell} (-m)_{k-j} (m)_{j-\ell} x^{m-k-2} \cdot x^{\ell+2} V_n^{(\ell)}(x). \tag{267}$$

For the first term on the right hand side we can apply Lemma 23, for the terms in the sum the induction hypothesis. To finish the proof we have to show

$$\begin{aligned}(-1)^{n+m}\left(\frac{\kappa(\theta)^n}{\theta^{n-k}}\right)^{(m)} \\ &= (-1)^{m+n}\theta^k\left(\frac{\kappa(\theta)^n}{\theta^n}\right)^{(m)} \\ &+\sum_{j=0}^{k-1}\sum_{\ell=0}^{j}\binom{k}{j}\binom{j}{\ell}(-m)_{k-j}(m)_{j-\ell}(-1)^{m-k+\ell+n}\left(\frac{\kappa(\theta)^n}{\theta^{n-\ell}}\right)^{(m-k+\ell)}.\end{aligned} \tag{268}$$

Setting $\varphi(\theta)=\kappa(\theta)^n/\theta^n$ and multiplying the equation with $(-1)^{n+m}$ yields

$$\begin{aligned}\left(\varphi(\theta)\theta^k\right)^{(m)} &= \theta^k\varphi^{(m)}(\theta) \\ &+\sum_{j=0}^{k-1}\sum_{\ell=0}^{j}\binom{k}{j}\binom{j}{\ell}(-m)_{k-j}(m)_{j-\ell}(-1)^{k-\ell}\left(\varphi(\theta)\theta^{\ell}\right)^{(m-k+\ell)}.\end{aligned} \tag{269}$$

By interchanging the summation order and observing

$$\sum_{j=\ell}^{k-1}\binom{k}{j}\binom{j}{\ell}(-m)_{k-j}(m)_{j-\ell} = -\binom{k}{\ell}(m)_{k-\ell} \tag{270}$$

we obtain

$$\theta^k\varphi^{(m)}(\theta) = \sum_{\ell=0}^{k}\binom{k}{\ell}(m)_\ell(-1)^\ell\left(\varphi(\theta)\theta^{k-\ell}\right)^{(m-\ell)}. \tag{271}$$

Using Leibniz's' rule on the right hand side gives

$$\theta^k\varphi^{(m)}(\theta) = \sum_{\ell=0}^{k}\binom{k}{\ell}(m)_\ell(-1)^\ell\sum_{j=0}^{m-\ell}\binom{m-\ell}{j}\varphi^{(j)}(\theta)(k-\ell)_{m-\ell-j}\theta^{k+j-m}. \tag{272}$$

Changing the order of summation again gives

$$\theta^k\varphi^{(m)}(\theta) = \sum_{j=0}^{m}\left[\sum_{\ell=0}^{m-j}\binom{k}{\ell}(m)_\ell(-1)^\ell\binom{m-\ell}{j}(k-\ell)_{m-\ell-j}\right]\theta^{k+j-m}\varphi^{(j)}(\theta), \tag{273}$$

and this finally is true due to

$$\sum_{\ell=0}^{m-j}\binom{k}{\ell}(m)_\ell(-1)^\ell\binom{m-\ell}{j}(k-\ell)_{m-\ell-j} = \delta_{mj}. \tag{274}$$

□

A.4 On the derivatives of the inverse Laplace transform

Lemma 25. *Suppose $n \in \mathbb{N}$, and $\phi(\theta)$ is a function such that $(1 + |\theta|^n)|\phi(\theta)|$ is integrable on $c + i\mathbb{R}$ for some $c \in \mathbb{R}$. Then*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(\theta) e^{\theta x} d\theta, \quad x \in \mathbb{R}, \tag{275}$$

defines a function that is n-times continuously differentiable, with bounded derivatives that are given by

$$f^{(k)}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \theta^k \phi(\theta) e^{\theta x} d\theta, \quad x \in \mathbb{R}, k = 0, 1, \ldots, n. \tag{276}$$

Proof. For $n = 0$ dominated convergence and a crude estimate show that $f(x)$ is continuous and bounded. Let us prove the statement for $n = 1$. Using (275) yields

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(\theta) \left(\frac{e^{\theta h} - 1}{h} \right) e^{\theta x} d\theta. \tag{277}$$

We can find $K > 0$, such that

$$\left| \frac{e^{\theta h} - 1}{h} \right| \leq K|\theta| \tag{278}$$

for $\Re(\theta) = c$ and $0 < |h| \leq 1$. The integrand in (277) is dominated by the integrable function $Ke^{cx}|\theta||\phi(\theta)|$. We can let $h \to 0$ and obtain (276) for $n = 1$.

For $n > 1$ we note, that the integrability of $(1+|\theta|^n)|\phi(\theta)|$ implies that $(1+|\theta|^n) \cdot |\phi(\theta)|$ is integrable for $k = 1, \ldots, n$. So we can iterate the previous argument n times. □

A.5 On the integral modulus of continuity

Lemma 26. *Suppose $\alpha \in (0, 1)$, $c > 0$, $\Re(\theta) = c$, and*

$$f(x) = e^{-cx} g(x) I_{(x>0)} \tag{279}$$

is integrable and satisfies

$$\omega^{(1)}(f; \delta) = \mathcal{O}\left(\delta^{1-\alpha}\right), \quad (\delta \to 0). \tag{280}$$

Then

$$\int_0^\infty e^{-\theta x} g(x) dx = \mathcal{O}\left(|\theta|^{\alpha-1}\right), \quad (\Im(\theta) \to \pm\infty). \tag{281}$$

Proof. We have

$$\int_0^{\infty} e^{-\theta x} g(x) dx \tag{282}$$

$$= \int_{-\infty}^{+\infty} e^{-ixy} f(x) dx \tag{283}$$

$$= \int_{-\infty}^{+\infty} e^{-i(x+\pi/y)y} f(x+\pi/y) dx \tag{284}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \left(e^{-ixy} f(x) + e^{-i(x+\pi/y)y} f(x+\pi/y) \right) dx \tag{285}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-ixy} \left(f(x) - f(x+\pi/y) \right) dx. \tag{286}$$

Thus

$$\left| \int_0^{\infty} e^{-\theta x} g(x) dx \right| \leq \frac{1}{2} \int_{-\infty}^{+\infty} \left| f(x) - f(x+\pi/y) \right| dx \tag{287}$$

$$\leq \omega^{(1)} \left(\frac{\pi}{|y|}, f \right) = \mathcal{O} \left(\left| \frac{\pi}{y} \right|^{1-\alpha} \right) = \mathcal{O} \left(|\theta|^{\alpha-1} \right). \tag{288}$$

□

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