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A SOLUTION TO HAMMER'S X-RAY RECONSTRUCTION PROBLEM

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Abstract. We propose algorithms for reconstructing a planar convex body K from possibly noisy measurements of either its parallel X-rays taken in a fixed finite set of directions or its point X-rays taken at a fixed finite set of points, in known situations that guarantee a unique solution when the data is exact. The algorithms construct a convex polygon P_k whose X-rays approximate (in the least squares sense) k equally spaced noisy X-ray measurements in each of the directions or at each of the points.

It is shown that these procedures are strongly consistent, meaning that, almost surely, P_k tends to K in the Hausdorff metric as $k \to \infty$. This solves, for the first time in the strongest sense, Hammer's X-ray problem published in 1963.

1. Introduction

In 1963, Hammer [14] published the following problem.

Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord along an X-ray line. (No diffusion, please.) How many pictures must be taken to permit exact reconstruction of the body if:
a. The X-rays issue from a finite point source?
b. The X-rays are assumed parallel?

From a modern perspective, Hammer's questions are clearly geometrical variants of the sort of problems considered in computerized tomography, the science behind the CAT scanner used in most major hospitals. Hammer's X-ray problem was a major inspiration for the development of geometric tomography, the area of mathematics dealing with the retrieval of information about a geometric object from data concerning its sections, or projections, or both. A full survey of geometric tomography is provided in [7], from Chapters 1 and 5 of which we present the following short summary of the contributions to Hammer's X-ray problem relevant for the present paper.

The earliest papers concern Hammer's question (b). The (parallel) X-ray of a convex body K in the direction u is the function giving the lengths of all the chords of K parallel to u. (See Section 2 for a formal definition and other notation and terminology.) The uniqueness aspect of question (b) is equivalent to asking which finite sets of directions are such that the corresponding X-rays distinguish between different convex bodies. Simple examples show that there are arbitrarily large sets of directions that do not have this desirable property and that no set of three directions does. A complete solution was found by Gardner and McMullen [10] (see also [7,

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Chapter 1]). A corollary of their result is that there are sets of four directions in S^1 such that the X-rays of any planar convex body in these directions determine it uniquely among all planar convex bodies. It was shown in [10] that a suitable set of four directions is one such that the corresponding set of slopes has a transcendental cross-ratio. Clearly this is an impractical choice of directions. However, Gardner and Gritzmann [8] showed that further suitable sets of four directions are those whose set of slopes, in increasing order, have a rational cross-ratio not equal to 3/2, 4/3, 2, 3, or 4. It follows that if $w_1 = (1, 0)$, $w_2 = (2, 1)$, $w_3 = (0, 1)$, and $w_4 = (-1, 2)$, for example, then the directions $u_i = w_i/\|w_i\|$, $i = 1, \dots, 4$ are such that X-rays in these directions determine planar convex bodies. (Many other practical choices are possible, of course.)

The corresponding uniqueness problem in higher dimensions can be solved by taking four directions, as specified above, all lying in the same 2-dimensional plane. Since the corresponding X-rays determine each 2-dimensional section of a convex body parallel to this plane, they determine the whole body.

The (point) X-ray of a convex body K at a point p is the function giving the lengths of all the chords of K lying on lines through p. The uniqueness aspect of Hammer's question (a) is not completely solved, but it is known that a planar convex body K is determined uniquely among all planar convex bodies by its X-rays taken at

(i) (Falconer [3], [4] and Gardner [5]) two points such that the line through them intersects K and it is known whether or not K lies between the two points (for a precise statement, see [7, Theorem 5.3.3]);

(ii) (Falconer and Gardner; see [7, Theorem 5.3.6]) three points such that K lies in the triangle with these points as vertices;

(iii) (Gardner [6]) any set of four collinear points whose cross ratio is restricted as in the parallel X-ray case above;

(iv) (Volčič [21]) any set of four points in general position.

Except in the case of (i), little is known about the uniqueness problem for point X-rays in higher dimensions.

It is clear from their phrasing that Hammer's questions are directed not just to issues of uniqueness, but also to the actual reconstruction of an unknown convex body from its X-rays taken in a finite set of directions or at a finite set of points that guarantees a unique solution. As far as we know, three such algorithms have been proposed for parallel X-rays. The first, due to Kölzow, Kuba, and Volčič [15], suffers from some serious deficiencies (see the discussion in [15] and [7, Note 1.2]). The second algorithm, proposed independently by Gardner and by Volčič (see [7, pp. 47–51]), makes some restrictive assumptions about the convex body and lacks a proof of convergence even under these assumptions. Finally, Brunetti and Daurat [2] suggest an algorithm that, like that in [15], is based on discretization, but they do not prove that it converges. For point X-rays, Falconer [3] (see also [7, Note 5.3]) gives a reconstruction algorithm for case (i) above. The only other algorithm seems to be that of Lam and Solmon [16] for the purpose of reconstructing a convex polygon from an X-ray at a single point, using a priori information about the location of the polygon relative to the point; the algorithm does not apply to general planar convex bodies.

The purpose of this paper is to present new algorithms for reconstructing planar convex bodies from their parallel or point X-rays, in situations that guarantee a unique solution when the data is exact. The algorithms are inspired by a least-squares optimization procedure used previously for reconstructing homogeneous objects from noisy X-ray data in a program developed by an electrical engineer, A. S. Willsky, and his students, from the early 1980's. This program was motivated by applications such as nondestructive testing (locating cracks in nuclear reactor cooling pipes, etc.), and medicine (nearly homogeneous regions such as kidneys and airspaces between organs). The articles [12] and [17] are representative of such work.

The mainly Fourier-transform-based algorithms of computerized tomography produce an approximate image of a density function. Of course, in practise one can only measure a finite number of values of each X-ray, and increasing this number will improve the image. However, for the class of density functions, there is a fundamental lack of uniqueness that in general also requires X-rays to be taken in more directions to enhance the image. This lack of uniqueness remains even in the class of compact sets. Algorithms in the just-cited papers arising from Willsky's program often reconstruct planar convex bodies, but they do not exploit the uniqueness results that hold for this restricted class.

One important feature of the present paper is that uniqueness results for convex bodies are utilized, so that *fixed* finite (and small) sets of X-rays are involved in the reconstruction, rather than the varying, and possibly large, number of X-rays employed in computerized tomography. Complete proofs of convergence are provided (a feature rather rare in the geometric tomography literature), solving, in a strong sense, Hammer's X-ray problem. Moreover, unlike all algorithms previously proposed for solving Hammer's problem, ours still work when the data is noisy, and our convergence proofs apply also in this case. To be more specific, the algorithms take as input k equally spaced noisy X-ray measurements of the unknown planar convex body K_0 in each of the fixed directions or at each of the fixed points, and produce a convex polygon P_k that, almost surely, converges in the Hausdorff metric to K_0 as $k \to \infty$. The noise is modeled in the traditional way by adding independent $N(0, \sigma^2)$ random variables.

In [9], the present authors and Milanfar apply techniques from the theory of empirical processes (see [20], for example) to obtain not only convergence proofs, but also rates of convergence, for algorithms for reconstructing convex bodies from data of different types, namely support and brightness functions. This application requires certain entropy estimates that, for now at least, seem very difficult to obtain for X-ray data, an extra obstacle being the lack of a suitable stability result (see, for example, [7, Problem 1.5]).

The parallel X-ray algorithm is presented in Section 3 and the proof of its convergence occupies Section 4. The point X-ray algorithm and the proof that it converges are the subject of Section 6. For simplicity, the latter algorithm is designed for the case (ii) of uniqueness discussed above, but it can easily be adapted for cases (i), (iii), and (iv) as well. In Section 5, we propose a modified parallel X-ray algorithm and prove that it also converges. This modified algorithm is expected to perform better in some situations, including when the data is exact. A similar modification to the point X-ray algorithm would be a routine matter. We stress, however, that this paper focuses on theory; a thorough investigation into the practical aspects, including implementation and testing, will be the subject of another article.

2. Definitions and notation

As usual, S^{n-1} denotes the unit sphere, B the unit ball, o the origin, and $\|\cdot\|$ the norm in Euclidean n-space $\mathbb{R}^n$. (We shall restrict to $n = 2$ after this section.) A *direction* is a unit vector, that is, an element of S^{n-1}. If u is a direction, then $u^\perp$ is the $(n-1)$-dimensional subspace orthogonal to u and l_u is the line through the origin parallel to u. If $x, y \in \mathbb{R}^n$, then $x \cdot y$ is the inner product of x and y, and $[x, y]$ is the line segment with endpoints x and y.

We denote by ∂A and $\operatorname{int} A$ the *boundary* and *interior* of a set A, respectively. If A is a Borel set in $\mathbb{R}^n$, then $V(A)$ is its n-dimensional Lebesgue measure.

If X is a metric space and $\varepsilon > 0$, a finite set $\{x_1, \dots, x_m\}$ is called an *ε-net* in X if for every point x in X, there is an $i \in \{1, \dots, m\}$ such that x is within a distance ε of x_i.

Let $\mathcal{K}^n$ be the family of compact convex sets in $\mathbb{R}^n$. A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Let $\mathcal{K}_o^n$ be the family of convex bodies in $\mathbb{R}^n$ and let $\mathcal{K}^n(A)$ (or $\mathcal{K}_o^n(A)$) be the family of compact convex sets (or convex bodies, respectively) contained in the subset A of $\mathbb{R}^n$.

If $K \in \mathcal{K}^n$, then

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, is its *support function.* Any $K \in \mathcal{K}^n$ is uniquely determined by its support function. We can regard h_K as a function on S^{n-1}, since $h_K(x) = \|x\| h_K(x/\|x\|)$ for $x \neq o$. The *Hausdorff distance* $\delta(K, L)$ between two sets $K, L \in \mathcal{K}^n$ can then be conveniently defined by

$$\delta(K, L) = \|h_K - h_L\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm on S^{n-1}.

The treatise of Schneider [18] is an excellent general reference for convex geometry.

Let K be a compact convex set in $\mathbb{R}^2$, let $u \in S^1$, and let $v \in S^1$ be orthogonal to u such that $\{u, v\}$ is oriented in the same way as the usual *orthonormal basis* $\{e_1, e_2\}$ for $\mathbb{R}^2$. The *(parallel) X-ray of* K in the direction u is the function $X_u K$ defined by

$$X_u K(t) = \int_{l_u + tv} 1_K(y)\,dy,$$

for $t \in \mathbb{R}$, where 1_K denotes the *characteristic function* of K. The *(point) X-ray of* K at a point $p \in \mathbb{R}^2$ is the function $X_p K$ defined by

$$X_p K(u) = \int_{l_u + p} 1_K(y)\,dy,$$

for $u \in S^1$. It will be convenient to identify a unit vector $u \in S^1$ with its polar angle θ, $0 \le \theta < 2\pi$, and regard $X_p K$ also as a function of θ.

3. The algorithm for parallel X-rays

For parallel X-rays, we shall assume throughout that the unknown convex body $K_0 \in \mathcal{K}_o^2(B)$, i.e. that K_0 is contained in the unit disk B in $\mathbb{R}^2$. This assumption can be justified on both purely theoretical and purely practical grounds. If the measurements are exact, then from the supports of the X-rays, a bounded polygon can be constructed that contains K_0. On the other hand, in practise, an unknown object whose X-rays are to be measured is contained in some known bounded region. In either case, one may as well suppose that K_0 is a subset of B.

Let u_i, $i = 1, \dots, 4$, be any fixed set of four directions in S^1 such that any planar convex body is determined by its X-rays in these directions. (We have given a specific example of a practical set of such directions in the introduction.) For $i = 1, \dots, 4$

and a given $k \in \mathbb{N}$, the X-rays $X_{u_i}K_0$ of K_0 are measured at equally spaced points $t_j \in [-1,1]$, $j = 1,\ldots,k$, with $t_1 = -1$ and $t_k = 1$. The measurements

$$M_j^{(i)} = X_{u_i}K_0(t_j) + N_j^{(i)}, \tag{1}$$

$i = 1,\ldots,4$, $j = 1,\ldots,k$ are noisy, the $N_j^{(i)}$'s being independent normal $N(0,\sigma^2)$ random variables.

Before describing our algorithm, we need some notation, and to keep this under control, we shall for a while regard k as fixed, even though later it will vary. For $i = 1,\ldots,4$, let v_i be orthogonal to u_i, where the orientation of $\{u_i, v_i\}$ is the same as $\{e_1, e_2\}$. For $i = 1,\ldots,4$, $j = 1,\ldots,k$, and $s \in \mathbb{R}$ define the point

$$q_{ij}(s) = su_i + t_jv_i$$

on the line $l_{u_i} + t_jv_i$. If

$$z_k = (x_{11}, y_{11}, x_{12}, y_{12}, \ldots, x_{1k}, y_{1k}, x_{21}, y_{21}, \ldots, x_{4k}, y_{4k}) \in [-1,1]^{8k}, \tag{2}$$

we define $P[z_k] = \operatorname{conv} T$, where

$$T = \bigcup_{i=1}^{4}\bigcup_{j=1}^{k}\{q_{ij}(x_{ij}), q_{ij}(y_{ij}) \,:\, q_{ij}(x_{ij}) \neq q_{ij}(y_{ij})\}\,. \tag{3}$$

Then $P[z_k]$ is a convex polygon with at most $8k$ vertices, each lying on one or more lines corresponding to the beams of the X-rays along which measurements are taken.

In the definition of T, pairs of points $q_{ij}(x_{ij})$ and $q_{ij}(y_{ij})$ are omitted if they coincide. The reason for this is that if a measurement line $l_{u_i} + t_jv_i$ does not intersect K_0, then it is desirable that it also does not intersect $P[z_k] = \operatorname{conv} T$. Since this feature may introduce some instability in practise, we propose in Section 5 a modified algorithm from which it has been removed. This modified algorithm is certainly applicable when the data is exact and very likely in other situations as well.

Algorithm NoisyXrayLSQ

Input: $k \in \mathbb{N}$; vectors $u_i \in S^{n-1}$, $i = 1,\ldots,4$ and reals $t_j \in [-1,1]$, $j = 1,\ldots,k$, as specified above; noisy X-ray measurements

$$M_j^{(i)} = X_{u_i}K_0(t_j) + N_j^{(i)},$$

for $i = 1,\ldots,4$ and $j = 1,\ldots,k$, of an unknown planar convex body $K_0 \in \mathcal{K}_o^2(B)$, where the $N_j^{(i)}$'s are independent normal $N(0,\sigma^2)$ random variables.

Task: Construct a convex polygon P_k that approximates K_0.

Action: Solve the following least squares problem:

$$\min_{z_k \in [-1,1]^{8k}} \sum_{i=1}^{4}\sum_{j=1}^{k}\left(M_j^{(i)} - X_{u_i}P[z_k](t_j)\right)^2. \tag{4}$$

If $\hat{z}_k$ is a solution of (4), let $P_k = P[\hat{z}_k]$ be the output.

By construction, P_k is contained in the convex hull of the union of four centered squares S_i, $i = 1,\ldots,4$, where S_i has center at o, side length 2, and one side parallel to u_i. Hence $P_k \subset \sqrt{2}B$, but in general, we do not have $P_k \in \mathcal{K}^2(B)$. Using the assumption $K \subset B$, one could also consider $P[\hat{z}_k] \cap B$ as an output of the algorithm.

We shall not do so, as this would lead to a non-polygonal approximation. Alternatively, it would be easy to modify the constraint $z_k \in [-1,1]^{8k}$ in (4) so as to ensure that $P[\hat{z}_k] \subset B$. Neither modification would affect the main result.

In general, (4) may not have a solution. To avoid unnecessary complications, we postpone a detailed discussion of this issue until Remark 4.7 at the end of the next section.

The justification of our formulation of the optimization problem comes from the next lemma, where it is shown that any output of Algorithm NoisyXRayLSQ best approximates the given X-ray measurements among *all* compact convex sets in $\sqrt{2}B$. The (infinite-dimensional) optimization problem (5) below corresponds to finding the maximum likelihood estimator in the parameter space $\mathcal{K}^2(\sqrt{2}B)$ for the unknown convex body K_0 under the given assumptions.

Lemma 3.1. *Let $k \in \mathbb{N}$ and let $K_0 \in \mathcal{K}^2_o(B)$. If $\hat{z}_k$ is any solution of (4), then $P_k = P[\hat{z}_k]$ is a solution of the problem*

$$\min_{K \in \mathcal{K}^2(\sqrt{2}B)} \sum_{i=1}^{4} \sum_{j=1}^{k} \left(M_j^{(i)} - X_{u_i}K(t_j) \right)^2. \tag{5}$$

Proof. Assume that P_k is not a solution of (5). Then there is a $K \in \mathcal{K}^2(\sqrt{2}B)$ giving a strictly smaller objective function value in (5). For $i = 1, \ldots, 4$ and $j = 1, \ldots, k$ let $Z_{ij} = (l_{u_i} + t_j v_i) \cap \partial K$. If Z_{ij} is empty, let $x_{ij} = y_{ij} = 0$. Otherwise, we have either $Z_{ij} = \{q_{ij}(x_{ij}), q_{ij}(y_{iy})\}$ or $Z_{ij} = [q_{ij}(x_{ij}), q_{ij}(y_{ij})]$ for some (possibly identical) $x_{ij}, y_{ij} \in \mathbb{R}$. Define $z_k \in [-1,1]^{8k}$ as in (2), T as in (3), and $P[z_k] = \operatorname{conv} T$. Then $P[z_k] \subset K$ and

$$(l_{u_i} + t_j v_i) \cap K = (l_{u_i} + t_j v_i) \cap P[z_k],$$

whenever the left-hand side is not a singleton, in which case the right-hand side is empty. It follows that

$$X_{u_i}K(t_j) = X_{u_i}P_k[z_k](t_j),$$

for $i = 1, \ldots, 4$ and $j = 1, \ldots, k$. Therefore the objective function value in (5) is unchanged if we replace K by $P[z_k]$. But then the objection function value in (4) is strictly smaller for this $P[z_k]$ than for $P[\hat{z}_k]$, a contradiction to the definition of $P[\hat{z}_k]$. $\square$

4. Proof of convergence

Let $f_0(t) = 2\sqrt{1-t^2}$, $-1 \le t \le 1$, and let $\mathcal{G}$ be the class of all nonnegative functions g on $[-1,1]$ that are concave on their supports and such that $g \le f_0$. Note that for each $u \in S^1$, we have

$$\mathcal{G} = \{X_u K : K \in \mathcal{K}^2(B)\}. \tag{6}$$

For each $k \in \mathbb{N}$, define a pseudonorm $|\cdot|_k$ on $\mathcal{G}$ by

$$|g|_k = \left(\frac{1}{k} \sum_{j=1}^{k} g(t_j)^2 \right)^{1/2}, \qquad g \in \mathcal{G}. \tag{7}$$

For $K \in \mathcal{K}^2_o(B)$, $u \in S^1$, and a vector $\mathbf{N} = (N_1, \ldots, N_k)$ of independent $N(0,\sigma^2)$ random variables, let

$$\Psi_k(K, u, \mathbf{N}) = \frac{1}{k} \sum_{j=1}^{k} X_u K(t_j) N_j. \tag{8}$$

Lemma 4.1. *Let $k \in \mathbb{N}$ and let $K_0 \in \mathcal{K}_o^2(B)$. If P_k is an output from Algorithm NoisyXrayLSQ as stated above, then*

$$\frac{1}{2}\sum_{i=1}^{4}|X_{u_i}K_0 - X_{u_i}P_k|_k^2 \le \sum_{i=1}^{4}\Psi_k\left(P_k, u_i, \mathbf{N}^{(i)}\right) - \sum_{i=1}^{4}\Psi_k\left(K_0, u_i, \mathbf{N}^{(i)}\right), \tag{9}$$

where $\mathbf{N}^{(i)} = (N_1^{(i)}, \dots, N_k^{(i)})$, $i = 1, \dots, 4$.

Proof. By Lemma 3.1, P_k is a solution of (5). Therefore

$$\sum_{i=1}^{4}\sum_{j=1}^{k}\left(M_j^{(i)} - X_{u_i}P_k(t_j)\right)^2 \le \sum_{i=1}^{4}\sum_{j=1}^{k}\left(M_j^{(i)} - X_{u_i}K_0(t_j)\right)^2.$$

Substituting for $M_j^{(i)}$ from (1) and rearranging, we obtain

$$\sum_{i=1}^{4}\sum_{j=1}^{k}\left(X_{u_i}K_0(t_j) - X_{u_i}P_k(t_j)\right)^2 \le 2\sum_{i=1}^{4}\sum_{j=1}^{k}\left(X_{u_i}P_k(t_j) - X_{u_i}K_0(t_j)\right)N_j^{(i)}.$$

In view of (7) and (8), this is the required inequality. □

The main part of the proof of convergence consists in showing that, almost surely, the term on the left-hand side of the inequality in the statement of the previous lemma converges to zero as $k \to \infty$. Some technical lemmas are required.

Let K be any convex body in $\mathbb{R}^2$ and let $\varepsilon > 0$. Define the *inner parallel body* $K \ominus \varepsilon B$ by

$$K \ominus \varepsilon B = \{x \in \mathbb{R}^2 : x + \varepsilon B \subset K\}.$$

Then

$$K \ominus \varepsilon B = \bigcap_{y \in \varepsilon B}(K - y),$$

so the inner parallel body is convex. For further properties, see [18, pp. 133–137].

Lemma 4.2. *If $K \in \mathcal{K}_o^2(B)$ and $0 < \varepsilon < 1$, then*

$$V(K) - V(K \ominus \varepsilon B) < 4\pi\varepsilon.$$

Proof. Suppose first that $K \ominus \varepsilon B^2$ contains at most one point. Then $V(K \ominus \varepsilon B^2) = 0$ and any ball contained in K has a radius at most ε, i.e., the inradius of K is less or equal to ε. For an arbitrary convex body L in $\mathbb{R}^2$ with inradius r, Bonnesen proved that

$$V(L) + \pi r^2 \le rS(L), \tag{10}$$

where $S(L)$ denotes the perimeter of L. (Hadwiger [13] gives a short proof of (10) and Bokowski [1] provides a generalization to n dimensions.) Since $K \subset B$ implies $S(K) \le 2\pi$, we obtain

$$V(K) - V(K \ominus \varepsilon B) = V(K) < 2\pi\varepsilon. \tag{11}$$

Now suppose that $K \ominus \varepsilon B$ contains at least two points, and let $K_\varepsilon = (K \ominus \varepsilon B) + \varepsilon B$. If it is not empty, the interior of the set $K \setminus K_\varepsilon$ consists of at most countably many disjoint components, whose closures we may label C_i, $i \in \mathbb{N}$. (There may only be finitely many.) Note that K is the union of K_ε and C_i, $i \in \mathbb{N}$, and all these sets have disjoint interiors.

For each i, there is a point $x_i \in \partial(K \ominus \varepsilon B)$ such that the boundary of C_i consists of two arcs, one contained in ∂K and one, A_i say, in $\partial(\varepsilon B + x_i)$. Then the arcs $A_i - x_i \subset \partial(\varepsilon B)$, $i \in \mathbb{N}$, have disjoint relative interiors. Let

$$Q = \varepsilon B \cup \bigcup_{i=1}^{\infty}(C_i - x_i),$$

and note that the sets in this union have disjoint interiors and that Q is convex. Since $K \subset B$, a translate of Q is contained in B. By (10) with L and r replaced by this translate of Q and ε, respectively, we obtain

$$V(Q) < \varepsilon S(Q) - \pi\varepsilon^2 < 2\pi\varepsilon$$

and hence

$$V(K) = V(K_\varepsilon) + \sum_{i=1}^{\infty} V(C_i) = V(K_\varepsilon) + V(Q) - V(\varepsilon B) < V(K_\varepsilon) + 2\pi\varepsilon - \varepsilon^2 V(B). \tag{12}$$

Using (12) together with the concept of mixed volumes and their monotonicity (see, for example, [7, Section A.3]), we conclude that

$$\begin{aligned} V(K) - V(K \ominus \varepsilon B) &< V(K_\varepsilon) - V(K \ominus \varepsilon B) - \varepsilon^2 V(B) + 2\pi\varepsilon \\ &= 2\varepsilon V(K \ominus \varepsilon B, B) + 2\pi\varepsilon \\ &\le 2\varepsilon V(B) + 2\pi\varepsilon = 4\pi\varepsilon. \end{aligned}$$

□

Lemma 4.3. *Let $0 < \varepsilon < 1$ be given. Then there is a finite set $\{(g_i^L, g_i^U) \mid i = 1, \dots, m\}$ of pairs of functions in $\mathcal{G}$ such that*

(i) *$\|g_i^U - g_i^L\|_1 \le \varepsilon$ for $i = 1, \dots, m$ and*

(ii) *for each $g \in \mathcal{G}$, there is an $i \in \{1, \dots, m\}$ such that $g_i^L \le g \le g_i^U$.*

Proof. Let $0 < \varepsilon < 1$ and $u \in S^1$, and note that by (6) we have $\mathcal{G} \subset L_1([-1, 1])$. Since $\mathcal{K}^2(B)$ with the Hausdorff metric is compact, there is an $\varepsilon/(7\pi)$-net $\{K_1, \dots, K_m\}$ in $\mathcal{K}^2(B)$. For each $i = 1, \dots, m$, let $K_i^U = (K_i + (\varepsilon/(7\pi))B) \cap B$ and $K_i^L = K_i \ominus (\varepsilon/(7\pi))B$. Define $g_i^U = X_u K_i^U$ and $g_i^L = X_u K_i^L$, $i = 1, \dots, m$. By (6), both g_i^U and g_i^L belong to $\mathcal{G}$, $i = 1, \dots, m$.

We first prove (ii). Let $g \in \mathcal{G}$. By (6), there is a $K \in \mathcal{K}^2(B)$ such that $g = X_u K$. Choose $i \in \{1, \dots, m\}$ such that $\delta(K, K_i) \le \varepsilon/(7\pi)$. Since $K \subset B$ and $K \subset K_i + (\varepsilon/(7\pi))B$, we have $K \subset (K_i + (\varepsilon/(7\pi))B) \cap B = K_i^U$. Also, we have

$$(K_i \ominus (\varepsilon/(7\pi))B) + (\varepsilon/(7\pi))B \subset K_i \subset K + (\varepsilon/(7\pi))B,$$

yielding $K_i^L = K_i \ominus (\varepsilon/(7\pi))B \subset K$. These facts imply that $g_i^L \le g \le g_i^U$, as required.

It remains to prove (i). Applying Lemma 4.2 with ε replaced by $\varepsilon/(7\pi)$, and using again mixed volumes and their monotonicity, we obtain

$$\begin{aligned} \|g_i^U - g_i^L\|_1 &= \int_{-1}^{1} \left(g_i^U(t) - g_i^L(t)\right) dt \\ &\le V\left(K_i + \frac{\varepsilon}{7\pi}B\right) - V(K_i \ominus \frac{\varepsilon}{7\pi}B) \\ &= V(K_i) + \frac{2\varepsilon}{7\pi}V(K_i, B) + \left(\frac{\varepsilon}{7\pi}\right)^2 V(B) - V\left(K_i \ominus \frac{\varepsilon}{7\pi}B\right) \\ &< 4\pi\frac{\varepsilon}{7\pi} + \frac{2\varepsilon}{7\pi}\pi + \left(\frac{\varepsilon}{7\pi}\right)^2 \pi < \varepsilon. \end{aligned}$$

This proves the claim. □

By analogy with [20, Definition 2.2], we refer to a finite set $\{(g_i^L, g_i^U) \,|\, i = 1, \dots, m\}$ of pairs of functions in $\mathcal{G}$ satisfying (i) and (ii) of Lemma 4.3 as an *ε-net with bracketing* for the class $\mathcal{G}$.

The following lemma is a version of the strong law of large numbers that applies to a triangular family, rather than a sequence, of random variables. While its proof follows standard arguments (see, for example, [19, Theorem 1, p. 388]), we are unable to find it explicitly stated in the literature and so provide the details.

Lemma 4.4. *Let X_{jk}, $k \in \mathbb{N}$, $j = 1, \dots, k$, be an independent family of random variables, each with zero mean. If there is a constant C such that*

$$E\left(X_{jk}^4\right) \le C, \qquad k \in \mathbb{N},\ j = 1, \dots, k, \tag{13}$$

then, almost surely, we have

$$\frac{1}{k} \sum_{j=1}^{k} X_{jk} \to 0 \tag{14}$$

as $k \to \infty$.

Proof. Let $\varepsilon > 0$, let $k \in \mathbb{N}$, and let $A_k(\varepsilon)$ be the event

$$A_k(\varepsilon) = \left\{ \left| \frac{1}{k} \sum_{j=1}^{k} X_{jk} \right| > \varepsilon \right\}.$$

By Markov's inequality,

$$P\left(A_k(\varepsilon)\right) \le \frac{E\left[\left(\sum_{j=1}^{k} X_{jk}\right)^4\right]}{(k\varepsilon)^4} = \frac{\alpha_k}{(k\varepsilon)^4}, \tag{15}$$

say. By Jensen's inequality for integrals (see, for example, [7, (B.8), p. 367]) and (13), we have

$$E\left(X_{jk}^2\right) \le \left(E\left(X_{jk}^4\right)\right)^{1/2} \le C^{1/2},$$

for $j = 1 \dots, k$. Using this, independence, the fact that the random variables have zero mean, and (13), we obtain

$$\begin{aligned} \alpha_k = E\left[\left(\sum_{j=1}^{k} X_{jk}\right)^4\right] &= \sum_{i=0}^{4} \binom{4}{i} E\left[\left(\sum_{j=1}^{k-1} X_{jk}\right)^{4-i}\right] E\big(X_{kk}^i\big) \\ &= \alpha_{k-1} + 6E\left[\left(\sum_{j=1}^{k-1} X_{jk}\right)^2\right] E\big(X_{kk}^2\big) + E\big(X_{kk}^4\big) \\ &= \alpha_{k-1} + 6E\big(X_{kk}^2\big) \sum_{j=1}^{k-1} E\big(X_{jk}^2\big) + E\big(X_{kk}^4\big) \\ &\le \alpha_{k-1} + 6(k-1)C + C. \end{aligned}$$

Together with $\alpha_1 \le C$, induction gives $\alpha_k \le 6Ck^2$. Now (15) implies

$$P\left(A_k(\varepsilon)\right) \le \frac{6C}{\varepsilon^4} k^{-2},$$

yielding

$$\sum_{k=1}^{\infty} P\left(A_k(\varepsilon)\right) < \infty.$$

The Borel-Cantelli lemma gives $P(\limsup_{k\to\infty} A_k(\varepsilon)) = 0$, and this is equivalent to the almost sure convergence in (14). □

We shall need to examine Algorithm NoisyXRayLSQ for varying $k \in \mathbb{N}$. In view of this, we relabel the measurements from (1) as

$$M_{jk}^{(i)} = X_{u_i} K_0(t_{jk}) + N_{jk}^{(i)}, \tag{16}$$

for $i = 1, \dots, 4$, $j = 1, \dots, k$, and $k \in \mathbb{N}$, where the t_{jk}'s, $j = 1, \dots, k$, are equally spaced points in $[-1, 1]$ with $t_{1k} = -1$ and $t_{kk} = 1$, and the $N_{jk}^{(i)}$'s are independent normal $N(0, \sigma^2)$ random variables.

Lemma 4.5. *Let $\rho > 0$, let $u \in S^1$, and let N_{jk}, $k \in \mathbb{N}, j = 1, \dots, k$, be independent $N(0, \sigma^2)$ random variables. Then, almost surely,*

$$\sup_{K \in \mathcal{K}^2(\rho B)} \Psi_k(K, u, \mathbf{N}_k) \to 0$$

as $k \to \infty$, where for each $k \in \mathbb{N}$, $\Psi_k(K, u, \mathbf{N}_k)$ is defined by (8) with $\mathbf{N} = \mathbf{N}_k = (N_{1k}, \dots, N_{kk})$ and $t_j = t_{jk}$.

Proof. We may assume without loss of generality that $\rho \ge 1$. Let $0 < \varepsilon < 1$ and let $\{(g_i^L, g_i^U) : i = 1, \dots, m\}$ be an ε-net with bracketing for $\mathcal{G}$, as provided by Lemma 4.3. Let $u \in S^1$, let $K \in \mathcal{K}^2(\rho B)$, and define $g = X_u((1/\rho)K) \in \mathcal{G}$. Then $\rho g(t/\rho) = X_u K(t)$ for all $t \in [-1, 1]$. Choose $i \in \{1, \dots, m\}$ such that $g_i^L \le g \le g_i^U$. Define $N_{jk}^+ = \max\{N_{jk}, 0\}$ and $N_{jk}^- = N_{jk}^+ - N_{jk}$ for $k \in \mathbb{N}$ and $j = 1, \dots, k$. Then for $k \in \mathbb{N}$, we have

$$\begin{aligned}
\Psi_k(K, u, \mathbf{N}_k) &= \frac{\rho}{k} \sum_{j=1}^{k} g(t_{jk}/\rho) N_{jk}^+ - \frac{\rho}{k} \sum_{j=1}^{k} g(t_{jk}/\rho) N_{jk}^- \\
&\le \frac{\rho}{k} \sum_{j=1}^{k} g_i^U(t_{jk}/\rho) N_{jk}^+ - \frac{\rho}{k} \sum_{j=1}^{k} g_i^L(t_{jk}/\rho) N_{jk}^- \\
&\le \rho W_k(\varepsilon),
\end{aligned}$$

where

$$W_k(\varepsilon) = \max_{i=1,\dots,m} \left\{ \frac{1}{k} \sum_{j=1}^{k} g_i^U(t_{jk}/\rho) N_{jk}^+ - \frac{1}{k} \sum_{j=1}^{k} g_i^L(t_{jk}/\rho) N_{jk}^- \right\}$$

is independent of K. Consequently,

$$\sup_{K \in \mathcal{K}^2(\rho B)} \Psi_k(K, u, \mathbf{N}_k) \le \rho W_k(\varepsilon), \tag{17}$$

for all $0 < \varepsilon < 1$.

Fix $i \in \{1, \dots, m\}$, and let

$$X_{jk} = g_i^U(t_{jk}/\rho) N_{jk}^+ - g_i^U(t_{jk}/\rho) E(N_{jk}^+),$$

for $k \in \mathbb{N}$ and $j = 1, \dots, k$. Since $g_i^U(t_{jk}/\rho) \le f_0(t_{jk}/\rho) \le 2$, it is easy to check that the random variables X_{jk} satisfy the hypotheses of Lemma 4.4. By (14), we obtain, almost surely,

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} g_i^U(t_{jk}/\rho)N_{jk}^+ &= \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} g_i^U(t_{jk}/\rho)E(N_{jk}^+)\\ &= \frac{1}{2}\int_{-1}^{1} g_i^U(t/\rho)\,dt\ E(N_{11}^+).\\ &= \frac{\rho}{2}\int_{-1/\rho}^{1/\rho} g_i^U(t)\,dt\ E(N_{11}^+).\end{aligned}$$

The same argument applies when X_{jk} is defined by $X_{jk} = g_i^L(t_{jk}/\rho)N_{jk}^- - g_i^L(t_{jk}/\rho)E(N_{jk}^-)$. Therefore, almost surely,

$$\lim_{k\to\infty} W_k(\varepsilon) = \frac{\rho}{2}\max_{i=1,\dots,m}\left\{\int_{-1/\rho}^{1/\rho} g_i^U(t)\,dt\,E(N_{11}^+) - \int_{-1/\rho}^{1/\rho} g_i^L(t)\,dt\,E(N_{11}^-)\right\}.$$

Since the variable N_{11} has zero mean, $E(N_{11}^-) = E(N_{11}^+)$. Also, by Lemma 4.3(i) we have $\|g_i^U - g_i^L\|_1 \le \varepsilon$ and by Lemma 4.3(ii) we may assume that $g_i^U - g_i^L \ge 0$, for $i = 1,\dots,m$. We conclude that

$$\lim_{k\to\infty} W_k(\varepsilon) \le \frac{\rho}{2}\max_{i=1,\dots,m}\int_{-1}^{1}(g_i^U(t) - g_i^L(t))\,dt\,E(N_{11}^+) \le \frac{\rho}{2}E(N_{11}^+)\varepsilon,$$

almost surely. Therefore, almost surely, for each $s \in \mathbb{N}$ we have

$$\lim_{k\to\infty} W_k(1/s) \le \rho E(N_{11}^+)/(2s). \tag{18}$$

Let $\varepsilon_0 > 0$, and let $s \in \mathbb{N}$ be such that $\rho^2 E(N_{11}^+)/(2s) < \varepsilon_0$. Then, by (17) and (18), almost surely, there is a $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, we have

$$\sup_{K\in\mathcal{K}^2(\rho B)} \Psi_k(K, u, \mathbf{N}_k) \le \rho^2 E(N_{11}^+)/(2s) < \varepsilon_0.$$

This proves the lemma. $\square$

Theorem 4.6. *Let $k \in \mathbb{N}$ and let $K_0 \in \mathcal{K}_o^2(B)$. If P_k is an output from Algorithm NoisyXrayLSQ as stated above, then, almost surely,*

$$\lim_{k\to\infty} \delta(K_0, P_k) \to 0, \tag{19}$$

as $k \to \infty$.

Proof. Let $i \in \{1,\dots,4\}$, and relabel the measurements in Algorithm NoisyXrayLSQ as in (16). By Lemma 4.1 (with $\mathbf{N}^{(i)} = \mathbf{N}_k^{(i)} = (N_{1k}^{(i)},\dots,N_{kk}^{(i)})$) and Lemma 4.5 (with $u = u_i$, $\rho = \sqrt{2}$ and $\mathbf{N}_k = \mathbf{N}_k^{(i)}$), we obtain, almost surely,

$$|X_{u_i}K_0 - X_{u_i}P_k|_k \to 0, \tag{20}$$

as $k \to \infty$. Fix a realization for which (20) holds for $i = 1,\dots,4$. As $P_k \subset \sqrt{2}B$ for each k, Blaschke's selection theorem implies that P_k converges to K_0 in the Hausdorff metric if and only if the only accumulation point of the sequence (P_k) is K_0. Let $Q \subset \sqrt{2}B$ be an arbitrary accumulation point of (P_k) and let $(P_{k'})$ be a subsequence converging to Q. If we can show that for $i = 1,\dots,4$,

$$|X_{u_i}K_0 - X_{u_i}P_{k'}|_{k'}^2 \to \frac{1}{2}\int_{-1}^{1}(X_{u_i}K_0(t) - X_{u_i}Q(t))^2\,dt \tag{21}$$

as $k' \to \infty$, then (20) implies

$$\|X_{u_i}K_0 - X_{u_i}Q\|_2 = 0,$$

and hence

$$X_{u_i}K_0 = X_{u_i}Q,$$

for $i = 1, \ldots, 4$. Our choice of the directions u_i then ensures that $K_0 = Q$, as required.

It remains to prove (21). Let $i \in \{1, \ldots, 4\}$. Define $F_{k'} = X_{u_i}K_0 - X_{u_i}P_{k'}$ and $F = X_{u_i}K_0 - X_{u_i}Q$. Since $P_{k'}$ converges to Q as $k' \to \infty$, it is easy to show that $F_{k'}$ converges uniformly to F on any compact subset of $[-1, 1]$ not containing the two points in the boundary Z of the support of $X_{u_i}Q$. Fix $\varepsilon > 0$ and let Z_ε be the set of all $t \in [-1, 1]$ having distance less than ε from Z. Then

$$|F_{k'}|_{k'}^2 = \frac{1}{k'} \sum_{j=1}^{k'} F_{k'}^2(t_{jk'}) 1_{[-1,1]\setminus Z_\varepsilon}(t_{jk'}) + \frac{1}{k'} \sum_{j=1}^{k'} F_{k'}^2(t_{jk'}) 1_{Z_\varepsilon}(t_{jk'}),$$

where the first term converges to

$$\frac{1}{2} \int_{[-1,1]\setminus Z_\varepsilon} F^2(t)\, dt$$

by the uniform convergence of $F_{k'}$ to F. Noting that Z_ε is the union of two intervals of length 2ε and that $|F_{k'}| \le 2\sqrt{2}$, we see that the second term is asymptotically bounded by 16ε. Hence

$$\limsup_{k' \to \infty} \left| |F_{k'}|_{k'}^2 - \frac{1}{2} \int_{-1}^{1} F^2(t) dt \right| \le \frac{1}{2} \int_{Z_\varepsilon} F^2(t) dt + 16\varepsilon \le 32\varepsilon.$$

This implies (21) and the proof is complete. □

Remark 4.7. When the X-ray measurements are exact, the existence of a solution $\hat{z}_k$ of (4) and hence an output P_k of Algorithm NoisyXrayLSQ is guaranteed. This is a consequence of the proof of Lemma 3.1, when K is replaced by K_0. When the measurements are noisy, however, there may not be a solution of (4), as the following example shows.

For simplicity consider just two directions, $e_1 = (1, 0)$ and $e_2 = (0, 1)$; the example can easily be modified for the four directions above. Let $k = 5$ and let $M_3^{(1)} = 1$, $M_j^{(1)} = 0$ for $j = 1, 2, 4, 5$, $M_j^{(2)} = 1$ for $j = 2, 3, 4$, and $M_j^{(2)} = 0$ for $j = 1, 5$. Let $0 < \varepsilon < 1/2$ and let K_ε be the rectangle $[-1/2, 1/2] \times [-1/2 + \varepsilon, 1/2 - \varepsilon]$. Then

$$\sum_{i=1}^{2} \sum_{j=1}^{5} \left(M_j^{(i)} - X_{e_i}K_\varepsilon(t_j) \right)^2 = 6\varepsilon^2.$$

The minimum in the corresponding version of (4) could therefore only be zero, so if there were a solution, the argument of Lemma 3.1 shows that there would be a convex body K such that

$$\sum_{i=1}^{2} \sum_{j=1}^{5} \left(M_j^{(i)} - X_{e_i}K(t_j) \right)^2 = 0.$$

But then K meets the three vertical lines l_{e_2} and $l_{e_2} \pm e_1/2$ in line segments of length 1 and meets each of the two horizontal lines $l_{e_1} \pm e_2/2$, if at all, in a single point. This is clearly impossible.

Note however that it is sufficient for our convergence proof that in (4) there is a $z_k^* \in [-1, 1]^{8k}$ that yields an objective function value close to the infimum. Specifically, if $f(z_k)$ is the objective function in (4), suppose that for sufficiently large k, there is a $z_k^* \in [-1,1]^{8k}$ such that

$$f(z_k^*) \le \inf_{z_k \in [-1,1]^{8k}} f(z_k) + \varepsilon_k,$$

for some $\varepsilon_k > 0$. Then Lemma 4.1 holds for $P_k = P[z_k^*]$ when ε_k is added to the right-hand side of (9). If $\varepsilon_k \to 0$ as $k \to \infty$, then Theorem 4.6 still holds for $P_k = P[z_k^*]$.

In practise, of course, solving (4) would require nonlinear optimization software that already implement procedures for stopping when close to a minimum.

5. A modified parallel X-ray algorithm

We again suppose that $K_0 \in \mathcal{K}_o^2(B)$ and the directions u_i, $i = 1, \dots, 4$ are as before. Let the support of the X-ray $X_{u_i}K_0$ be $[a_i, b_i]$, $i = 1, \dots, 4$. In this section we assume that it is possible to find, from the X-ray measurements (16), intervals $[a_{ik}, b_{ik}] \subset [-1,1]$ such that $a_{ik} \to a_i$ and $b_{ik} \to b_i$ as $k \to \infty$, $i = 1, \dots, 4$. Note that this assumption is certainly fulfilled if the data is exact, because in this case one can take a_{ik} and b_{ik} to be the smallest and largest values, respectively, of t_{jk} such that $X_{u_i}K_0(t_{jk}) > 0$. It will very likely also be satisfied even when the data is noisy, via suitable averaging and thresholding of the measurements, and we plan to investigate this in a future study.

Let H_{ik} be the strip bounded by the parallel lines $l_{u_i} + a_{ik}v_i$ and $l_{u_i} + b_{ik}v_i$, for $i = 1, \dots, 4$ and $k \in \mathbb{N}$. Then $H_{ik} \to H_i$, where H_i is the strip containing K_0 and bounded by lines parallel to u_i supporting K_0, $i = 1, \dots, 4$. Let

$$R_k = \bigcap_{i=1}^{4} H_{ik},$$

for $k \in \mathbb{N}$. Then the R_k's are (possibly empty or degenerate) convex polygons that converge to the convex polygon $\cap_{i=1}^{k} H_i$ circumscribing K_0.

For each $i = 1, \dots, 4$ and $k \in \mathbb{N}$, let J_{ik} denote the set of contiguous indices j such that the line $l_{u_i} + t_{jk}v_i$ meets R_k. We consider the modified version of Algorithm NoisyXRayLSQ in which the measurements $M_{jk}^{(i)}$ are restricted to $j \in J_{ik}$. This simply means that we are only using, for each $i = 1, \dots, 4$, the X-ray $X_{u_i}K_0$ on those equally spaced lines parallel to u_i that meet the convex polygon R_k.

Fix a $k \in \mathbb{N}$. For $i = 1, \dots, 4$ and $j \in J_{ik}$, $L_{jk}^{(i)} = (l_{u_i} + t_{jk}v_i) \cap R_k$ is a (possibly degenerate, but nonempty) chord of R_k. Let

$$S_{jk}^{(i)} = \{s \in \mathbb{R} : q_{ij}(s) = su_i + t_{jk}v_i \in L_{jk}^{(i)}\} \subset [-1,1].$$

The vector z_k is defined as in (2), except that only double indices ij with $j \in J_{ik}$ are used, and the values of x_{ij} and y_{ij} (which, as earlier, actually depend on k) are restricted to $S_{jk}^{(i)}$. Thus the total number of components of z_k is now

$$c_k = 2\sum_{i=1}^{4} |J_{ik}|,$$

which is no more than $8k$ and in general less, and z_k belongs in general to a strict subset F_k of $[-1,1]^{c_k}$. Then let $P[z_k] = \operatorname{conv} T$, where instead of (3) we define

$$T = \bigcup_{i=1}^{4} \bigcup_{j \in J_{ik}} \{q_{ij}(x_{ij}), q_{ij}(y_{ij})\}. \tag{22}$$

A crucial difference here is that we no longer omit pairs of points $q_{ij}(x_{ij})$ and $q_{ij}(y_{ij})$ (which, as earlier, also actually depend on k) if they coincide. As before, $P[z_k]$ is a convex polygon with at most c_k vertices, each lying on one or more lines corresponding to the reduced set of beams of the X-rays along which measurements are taken into consideration.

The task and action of the modified algorithm is the same as before, with the corresponding modified version of (4), namely,

$$\min_{z_k \in F_k} \sum_{i=1}^{4} \sum_{j \in J_{ik}} \left(M_{jk}^{(i)} - X_{u_i} P[z_k](t_{jk}) \right)^2 . \tag{23}$$

If $\hat{z}_k$ is a solution of (23), we denote by $P_k = P[\hat{z}_k]$ the output of this modified algorithm.

For each $k \in \mathbb{N}$, let $\mathcal{H}_k$ be the set of compact convex sets contained in R_k and meeting each of the lines $l_{u_i} + t_{jk} v_i$, $i = 1, \dots, 4$ and $j \in J_{ik}$. Note that by its construction, we have $P[z_k] \in \mathcal{H}_k$ and hence $P_k \in \mathcal{H}_k$ for each k. The following lemma corresponds to Lemma 3.1.

Lemma 5.1. *Let $k \in \mathbb{N}$ and let $K_0 \in \mathcal{K}_o^2(B)$. If $\hat{z}_k$ is any solution of (23), then $P_k = P[\hat{z}_k]$ is a solution of the problem*

$$\min_{K \in \mathcal{H}_k} \sum_{i=1}^{4} \sum_{j \in J_{ik}} \left(M_{jk}^{(i)} - X_{u_i} K(t_j) \right)^2 . \tag{24}$$

Proof. Assume that P_k is not a solution of (24). Then there is a $K \in \mathcal{H}_k$ giving a strictly smaller objective function value in (24). For $i = 1, \dots, 4$ and $j \in J_{ik}$, let $Z_{ij} = (l_{u_i} + t_{jk} v_i) \cap \partial K$. By the definition of $\mathcal{H}_k$, Z_{ij} cannot be empty, so either $Z_{ij} = \{q_{ij}(x_{ij}), q_{ij}(y_{iy})\}$ or $Z_{ij} = [q_{ij}(x_{ij}), q_{ij}(y_{ij})]$ for some (possibly identical) $x_{ij}, y_{ij} \in S_{jk}^{(i)}$. Use these values of x_{ij} and y_{ij} to define $z_k \in F_k$, T as in (22), and $P[z_k] = \operatorname{conv} T$. Then

$$(l_{u_i} + t_{jk} v_i) \cap K = (l_{u_i} + t_j v_i) \cap P[z_k].$$

It follows that

$$X_{u_i} K(t_j) = X_{u_i} P_k[z_k](t_j),$$

for $i = 1, \dots, 4$ and $j \in J_{ik}$. Therefore the objective function value in (24) is unchanged if we replace K by $P[z_k]$. But then the objection function value in (23) is strictly smaller for this $P[z_k]$ than for $P[\hat{z}_k]$, a contradiction to the definition of $P[\hat{z}_k]$. □

Let $k \in \mathbb{N}$. On each chord $L_{jk}^{(i)}$, $i = 1, \dots, 4$, $j \in J_{ik}$, of R_k defined above that is disjoint from K_0, choose the point nearest to K_0, and let K_k be the convex hull of $K_0 \cap R_k$ and all the points chosen in this way. Clearly $K_k \in \mathcal{H}_k$ for each k. We also have $K_k \to K_0$ as $k \to \infty$, because the distance from $L_{ij}^{(i)}$ to K_0 converges to zero as $k \to \infty$, for $i = 1, \dots, 4$ and $j \in J_{ik}$.

By Lemma 5.1, P_k is a solution of (24). Since $K_k \in \mathcal{H}_k$, we have

$$\sum_{i=1}^{4} \sum_{j \in J_{ik}} \left(M_{jk}^{(i)} - X_{u_i} P_k(t_{jk}) \right)^2 \le \sum_{i=1}^{4} \sum_{j \in J_{ik}} \left(M_{jk}^{(i)} - X_{u_i} K_k(t_{jk}) \right)^2 .$$

Substituting for $M_{jk}^{(i)}$ from (16) and rearranging, we obtain, in notation used earlier,

$$\frac{1}{2}\sum_{i=1}^{4}|X_{u_i}K_0 - X_{u_i}P_k|_k^2 \le \sum_{i=1}^{4}\Psi_k\left(P_k, u_i, \mathbf{N}_k^{(i)}\right) - \sum_{i=1}^{4}\Psi_k\left(K_k, u_i, \mathbf{N}_k^{(i)}\right) + \frac{1}{2}\sum_{i=1}^{4}|X_{u_i}K_0 - X_{u_i}K_k|_k^2 . \tag{25}$$

The bound (25) replaces (9) in Lemma 4.1.

Theorem 5.2. *Let $k \in \mathbb{N}$ and let $K_0 \in \mathcal{K}_o^2(B)$. If P_k is an output from the modified version of Algorithm NoisyXrayLSQ, then, almost surely,*

$$\lim_{k\to\infty}\delta(K_0, P_k) \to 0,$$

as $k \to \infty$.

Proof. Let $i \in \{1,\dots,4\}$. Since $K_k \to K_0$, we have

$$|X_{u_i}K_0 - X_{u_i}K_k|_k \to 0,$$

as $k \to \infty$. By (25) and Lemma 4.5 (with $u = u_i$, $\rho = \sqrt{2}$ and $\mathbf{N}_k = \mathbf{N}_k^{(i)}$), we obtain, almost surely,

$$|X_{u_i}K_0 - X_{u_i}P_k|_k \to 0,$$

as $k \to \infty$. The rest of the proof is exactly the same as that of Theorem 4.6. ☐

Of course, Remark 4.7 also applies to the above modified algorithm.

6. The algorithm for point X-rays

In this section, we give an algorithm similar to that in Section 3, but designed for the purpose of reconstructing convex bodies from point X-rays. The basic idea is quite similar to the parallel X-ray case, so rather than giving full details, we supply only those that are necessary once Section 3 has been understood.

We shall assume throughout that the unknown convex body $K_0 \in \mathcal{K}_o^2(E)$, where E is a fixed equilateral triangle inscribed in B. Let p_i, $i = 1, 2, 3$, be the vertices of E, so that by [7, Theorem 5.3.6], any planar convex body contained in E is determined by its X-rays at these points (case (ii) of the introduction). For $i = 1, 2, 3$ and $k \in \mathbb{N}$, the X-rays $X_{p_i}K_0$ of K_0 are measured at equally spaced angles θ_{jk}, $j = 1, \dots, k$, where by rotating the domains of these functions, we may assume that $\theta_{jk} \in [0, \pi/3]$, $\theta_{1k} = 0$, and $\theta_{kk} = \pi/3$. The measurements

$$M_{jk}^{(i)} = X_{p_i}K_0(\theta_{jk}) + N_{jk}^{(i)}, \tag{26}$$

for $i = 1, 2, 3$ and $j = 1, \dots, k$, are noisy, the $N_{jk}^{(i)}$'s being independent normal $N(0, \sigma^2)$ random variables.

A less formal description of the algorithm should suffice. Similarly to the parallel X-ray case, a vector z_k, now with $3k$ pairs of nonnegative real components x_{ij} and y_{ij}, is used to define pairs of points $q_{ij}(x_{ij})$ and $q_{ij}(y_{ij})$ (which, as earlier, actually depend on k) lying on the line $l_{\theta_{jk}} + p_i$, where we are identifying the angle θ_{jk} with the unit vector in this direction. By restricting x_{ij} and y_{ij} suitably, we can further ensure that all these points lie in E; let F_k be the set of corresponding vectors z_k. The set T is the union of the $q_{ij}(x_{ij})$'s and $q_{ij}(y_{ij})$'s except for those such that $q_{ij}(x_{ij}) = q_{ij}(y_{ij})$, and

$P[z_k] = \operatorname{conv} T$. The algorithm is the same as Algorithm NoisyXrayLSQ except that input consists of the point X-ray measurements (26) and the least squares problem is

$$\min_{z_k \in F_k} \sum_{i=1}^{3} \sum_{j=1}^{k} \left(M_{jk}^{(i)} - X_{p_i} P[z_k](\theta_{jk}) \right)^2. \tag{27}$$

If $\hat{z}_k$ is a solution of (27), let $P_k = P[\hat{z}_k]$ be the output of the algorithm.

With a proof essentially the same as that of Lemma 3.1, we have that if $\hat{z}_k$ is any solution of (27), then $P_k = P[\hat{z}_k]$ is a solution of the problem

$$\min_{K \in \mathcal{K}^2(E)} \sum_{i=1}^{3} \sum_{j=1}^{k} \left(M_{jk}^{(i)} - X_{p_i} K(\theta_{jk}) \right)^2. \tag{28}$$

Let $\mathcal{G}$ be the class of all functions on $[0, \pi/3]$ that are X-rays $X_o K(\theta)$ of convex bodies K contained in the triangle E_0 with vertices o, $(\sqrt{3}, 0)$, and $(\sqrt{3}/2, 3/2)$ congruent to E. Note that if $K \in \mathcal{K}_o^2(E)$, we can regard $X_{p_i} K$, $i = 1, 2, 3$, as members of $\mathcal{G}$ by rotating their domains as above.

For each $k \in \mathbb{N}$, define the pseudonorm $|\cdot|_k$ on $\mathcal{G}$ by

$$|g|_k = \left(\frac{1}{k} \sum_{j=1}^{k} g(\theta_{jk})^2 \right)^{1/2}, \qquad g \in \mathcal{G}.$$

The functions $\Psi_k(K, p, \mathbf{N})$, $p \in \mathbb{R}^2$, are defined analogously to (8). With a proof similar to that of Lemma 4.1, using the fact that P_k is a solution of (28), we obtain the corresponding bound,

$$\frac{1}{2} \sum_{i=1}^{3} |X_{p_i} K_0 - X_{p_i} P_k|_k^2 \le \sum_{i=1}^{3} \Psi_k \left(P_k, p_i, \mathbf{N}^{(i)} \right) - \sum_{i=1}^{3} \Psi_k \left(K_0, p_i, \mathbf{N}^{(i)} \right). \tag{29}$$

Lemma 4.3 still holds, exactly as stated there with the new class $\mathcal{G}$ just defined, but for the proof the following lemma is needed.

Lemma 6.1. *Let H_1 and H_2 be convex bodies with $H_1 \subset \operatorname{int} H_2 \subset E_0$. Then*

$$\int_0^{\pi/3} \left(X_o H_2(\theta) - X_o H_1(\theta) \right) d\theta \le 2 \left(\frac{2\pi}{3} \left(V(H_2) - V(H_1) \right) \right)^{1/2}.$$

Proof. It is easy to see that the annular region $A = H_2 \setminus \operatorname{int} H_1$ can be expressed as the union of two closed regions A_1 and A_2, disjoint except on their boundaries, such that

$$A_j = \{ (r, \theta) \in \mathbb{R}^2 : s_j(\theta) \le r \le t_j(\theta), 0 \le \theta \le \pi/3 \},$$

for some (at least piecewise) continuous functions s_j and t_j, $j = 1, 2$. Now for $j = 1, 2$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \int_0^{\pi/3} \left(t_j(\theta) - s_j(\theta) \right) d\theta &\le \left(\int_0^{\pi/3} \left(t_j(\theta) - s_j(\theta) \right)^2 d\theta \right)^{1/2} \left(\int_0^{\pi/3} 1^2 \, d\theta \right)^{1/2} \\ &\le \left(\frac{\pi}{3} \int_0^{\pi/3} \left(t_j(\theta)^2 - s_j(\theta)^2 \right) d\theta \right)^{1/2} \\ &= \left(\frac{2\pi V(A_j)}{3} \right)^{1/2} \le \left(\frac{2\pi V(A)}{3} \right)^{1/2}. \end{aligned}$$

The inequality in the statement of the lemma follows immediately. $\square$

The proof of the new version of Lemma 4.3 runs as follows. Let $0 < \varepsilon < 1$ and note that $\mathcal{G} \subset L_1([0,\pi/3])$. Since $\mathcal{K}^2(E_0)$ with the Hausdorff metric is compact, for any $c > 0$ there is an $c\varepsilon^2$-net $\{K_1,\dots,K_m\}$ in $\mathcal{K}^2(E_0)$. For each $i = 1,\dots,m$, let $K_i^U = (K_i + c\varepsilon^2 B) \cap E_0$ and $K_i^L = K_i \ominus c\varepsilon^2 B$. Define $g_i^U(\theta) = X_o K_i^U(\theta)$ and $g_i^L(\theta) = X_o K_i^L(\theta)$, for $0 \le \theta \le \pi/3$ and $i = 1,\dots,m$. Then both g_i^U and g_i^L belong to $\mathcal{G}$, $i = 1,\dots,m$. The proof of (ii) of the new version Lemma 4.3 is as before, so it remains to prove (i). Using Lemma 6.1 with $H_1 = K_i^L$ and $H_2 = K_i^U$, and the estimates employing mixed volumes at the end of Lemma 4.3, we obtain

$$\begin{aligned}
\|g_i^U - g_i^L\|_1 &= \int_0^{\pi/3} \left(g_i^U(\theta) - g_i^L(\theta)\right)\, d\theta \\
&\le 2\left(\frac{2\pi}{3}\left(V(K_i + c\varepsilon^2 B) - V(K_i \ominus c\varepsilon^2 B)\right)\right)^{1/2} \\
&< 2\left(\frac{2\pi}{3} 7\pi c\varepsilon^2\right)^{1/2} < \varepsilon,
\end{aligned}$$

for sufficiently small c.

The rest of the proof of convergence for the point X-ray algorithm now routinely follows that for Algorithm NoisyXrayLSQ in Section 4. We remark only that the proof of Lemma 4.5 uses a homogeneity property for the parallel X-ray of ρK, which does not hold for the point X-ray. However, as K_0 and P_k are subsets of $E \subset B$, the point X-ray version of Lemma 4.5 is only needed for $\rho = 1$, and in this case the proof is essentially the same. We obtain the following result.

Theorem 6.2. *Let $k \in \mathbb{N}$ and let $K_0 \in \mathcal{K}_o^2(E)$. If P_k is an output from the point X-ray version of Algorithm NoisyXrayLSQ outlined above, then, almost surely,*

$$\lim_{k\to\infty} \delta(K_0, P_k) \to 0,$$

as $k \to \infty$.

Again, Remark 4.7 also applies to the point X-ray algorithm.

The algorithm and proof of convergence would be easy to adapt to the other cases (i), (iii), and (iv) in the introduction for which uniqueness results for point X-rays are available. For these it should be assumed that the unknown convex body $K_0 \in \mathcal{K}_o^2(B)$, as for parallel X-rays, and the points in T should be restricted to B to avoid use of the homogeneity property of Lemma 4.5 mentioned above.

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