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Abstract

In the present paper, we give a condensed review for the non-specialist reader of a new modelling framework for spatio-temporal processes, based on Lévy theory. We show the potential of the approach in stochastic geometry and spatial statistics by studying Lévy based growth modelling of planar objects. The growth models considered are spatio-temporal stochastic processes on the circle. As a by-product, new flexible models for space-time covariance functions on the circle are provided. An application of the Lévy based growth models to tumour growth is discussed.

KEYWORDS: growth models, Lévy basis, spatio-temporal modelling, tumour growth.

1 Introduction

Stochastic spatio-temporal modelling is of great importance in a variety of disciplines of natural science, including biology (Cantalapiedra et al. (2001), Brix and Chadoeuf (2002), Fewster (2003), Gratzer et al. (2004)), image analysis (Feideropoulou and Pesquet-Popescu (2004)), geophysics (Calder (1986), Lovejoy et al. (1992), Sornette and Ouillon (2005)) and turbulence (Schmiegel et al. (2004), Schmiegel et al. (2005)), just to name a few. In particular, modelling of tumour growth dynamics has been a very active research area in recent years (Delsanto et al. (2000), Peirolo and Scalerandi (2004), Pang and Tzeng (2004), Schmiegel (2006)). In most of the above cited works, the model is given implicitly and the resulting dynamics are difficult to control explicitly. However, for applications and for the theoretical understanding of the employed modelling framework it is essential to connect the ingredients of the model with dynamical and spatial properties of the system in focus. Furthermore, for a parsimonious description of systems which are different with respect to the dynamics and physical mechanisms underlying the dynamics, it is desirable to have at hand a flexible and, at the same time, mathematically tractable modelling framework.

Lévy based models are a promising modelling framework to meet these requirements concerning flexibility and dynamical control. Until now, Lévy based models have mainly been used for describing turbulent flows (Barndorff-Nielsen and Schmiegel (2004), Schmiegel et al. (2004), Schmiegel et al. (2005)). In the present paper, we show that Lévy based spatio-temporal modelling also have important applications in stochastic geometry and spatial statistics. The focus is on Lévy based growth models but we will also briefly touch upon another emerging area of application, viz. the Lévy driven Cox processes. We expect that the Lévy based approach will have many applications in stochastic geometry and spatial statistics.

Lévy based spatio-temporal models are constructed from Lévy bases, i.e. infinitely divisible and independently scattered random measures. This terminology was recently introduced in Barndorff-Nielsen and Schmiegel (2004). One example of such a model describes the growth of a planar star-shaped object, using its radial function $R_t(\phi)$ at time t and angle ϕ. Here, $R_t(\phi)$ is the distance from a reference point z to the boundary of the object at time t in direction $\phi \in [-\pi, \pi)$, see also Figure 2 below. The time derivative of the radial function is of the following form

$$\frac{\partial}{\partial t} R_t(\phi) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi; \phi) Z(d\xi), \ \phi \in [-\pi, \pi),$$

where Z is a Lévy basis, $A_t(\phi) \subseteq [-\pi, \pi) \times (-\infty, t]$ is a subset of the past of time t, a so-called ambit set[1], cf. Barndorff-Nielsen and Schmiegel (2004), $f_t(\cdot; \phi)$ is a deterministic weight function and μ_t a deterministic function. The induced model for the radial function is of the same type. An important advantage of these models is that explicit expressions for

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$$

can be derived in terms of the components of the model. This part of the paper is a natural continuation of the work initiated in Schmiegel (2006) which was mainly directed towards an audience of physicists. An introduction to different growth modelling approaches, including a short treatment of the Lévy based approach adopted in the present paper, may be found in the SemStat contribution Jensen et al. (2006).

The organisation of the paper is as follows. Section 2 provides some background on Lévy bases which is the essential component of the modelling approach. In Section 3, Lévy based spatio-temporal models are reviewed while Lévy based growth models are studied in Section 4. Section 5 contains explicit results for the covariance functions. In Section 6, an application of the Lévy based growth models to tumour growth is discussed while problems for future research are collected in Section 7.

2 Lévy bases

This section gives a brief overview of the general theory of Lévy bases, in particular, the theory of integration with respect to a Lévy basis. For a more detailed exposition, see Barndorff-Nielsen and Schmiegel (2004). As mentioned in the introduction, a Lévy basis is an infinitely divisible and independently scattered random measure. Comprehensive accounts of the theory of independently scattered random measures may be found in Kallenberg (1989), Rajput and Rosinski (1989) and Kwapien and Woyczynski (1992).

Let $\mathcal{R}$ be a Borel subset of $\mathbb{R}^d$, let $\mathcal{B} = \mathcal{B}(\mathcal{R})$ be the Borel subsets of $\mathcal{R}$ and let $\mathcal{B}_b = \mathcal{B}_b(\mathcal{R})$ denote the class of bounded elements of $\mathcal{B}$. A collection of random variables $Z = \{Z(A) : A \in \mathcal{B}\}$ or $Z = \{Z(A) : A \in \mathcal{B}_b\}$ is said to be an *independently scattered random measure*, if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{B}$, respectively $\mathcal{B}_b$, the random variables $Z(A_n)$, $n = 1, 2, \ldots$, are independent and $Z(\bigcup A_n) = \sum Z(A_n)$ a.s.,

[1]Latin: *ambitus*. 1. The bounds or limits of a place or district. 2. A sphere of action, expression or influence.

where in the case $Z = \{Z(A) : A \in \mathcal{B}_b\}$ we further require $\cup A_n \in \mathcal{B}_b$. We need to distinguish between the two cases $\mathcal{B}$ and $\mathcal{B}_b$ because the sums $\sum Z(A_n)$ also have to be controlled in case Z can take both positive and negative values. If, moreover, $Z(A)$ is *infinitely divisible* for all $A \in \mathcal{B}$ or $\mathcal{B}_b$, Z is called a *Lévy basis*, cf. Barndorff-Nielsen and Schmiegel (2004).

For a random variable X, let us denote the cumulant function $\log \mathbb{E}(e^{i\lambda X})$ by $C(\lambda \ddagger X)$. When Z is a Lévy basis, the cumulant function of $Z(A)$ can, by the famous Lévy-Khintchine representation, be written as

$$C(\lambda \ddagger Z(A)) = i\lambda a(A) - \frac{1}{2}\lambda^2 b(A) + \int_{\mathbb{R}} (e^{i\lambda u} - 1 - i\lambda u \mathbf{1}_{[-1,1]}(u)) U(du, A), \qquad (1)$$

where a is a signed measure on $\mathcal{B}$ or $\mathcal{B}_b$, b is a positive measure on $\mathcal{B}$ or $\mathcal{B}_b$, $U(du, A)$ is a Lévy measure on $\mathbb{R}$ for fixed A and a measure on $\mathcal{B}$ or $\mathcal{B}_b$ for fixed du. The measure U will be referred to as the generalised Lévy measure. The Lévy basis Z is said to have characteristics (a, b, U). If $b = 0$ then L is called a Lévy jump basis, if $U = 0$ then L is a Gaussian basis, see the examples below. It follows from (1) that any Lévy basis Z can be expressed as the sum of a Lévy jump basis Z_1 and an independent zero mean Gaussian basis Z_2.

Without loss of generality (for details, see Rajput and Rosinski (1989)) we can assume that there exists a measure μ such that the generalised Lévy measure factorises as

$$U(du, d\xi) = V(du, \xi)\mu(d\xi),$$

where $V(du, \xi)$ is a Lévy measure for fixed ξ. Furthermore, the measures a and b are absolutely continuous with respect to the measure μ, i.e.

$$a(d\xi) = \tilde{a}(\xi)\mu(d\xi), \quad b(d\xi) = \tilde{b}(\xi)\mu(d\xi),$$

and $\tilde{a}$ and $\tilde{b}$ are uniformly bounded by some constant $C > 0$. One possible choice (however not the only one - see Rajput and Rosinski (1989)) for μ is

$$\mu(A) = |a|(A) + b(A) + \int_{\mathbb{R}} (1 \wedge r^2) U(dr, A),$$

where $|a|$ denotes the total variation measure of a and $\wedge$ denotes minimum.

Let $Z'(\xi)$ be a random variable with the cumulant function

$$C(\lambda \ddagger Z'(\xi)) = i\lambda \tilde{a}(\xi) - \frac{1}{2}\lambda^2 \tilde{b}(\xi) + \int_{\mathbb{R}} (e^{i\lambda u} - 1 - i\lambda u \mathbf{1}_{[-1,1]}(u)) V(du, \xi).$$

Then,

$$C(\lambda \ddagger Z(d\xi)) = C(\lambda \ddagger Z'(\xi))\mu(d\xi). \qquad (2)$$

If $\tilde{a}(\xi)$, $\tilde{b}(\xi)$ and the Lévy measure $V(\cdot; \xi)$ do not depend on ξ, we call Z a *factorisable* Lévy basis and then $Z'(\xi) = Z'$ does also not depend on ξ. If, moreover, μ is proportional to the Lebesgue measure, then Z is called a *homogeneous* Lévy basis and all finite dimensional distributions of Z are translation invariant.

The usefulness of the definitions above becomes clear in connection with the integration of a measurable function f on $\mathcal{R}$ with respect to a Lévy basis Z. For simplicity, we denote this integral by $f \bullet Z$. Important for many calculations are the following equation for the cumulant function of the stochastic integral $f \bullet Z$ (subject to minor regularity conditions, cf., for instance, Barndorff-Nielsen and Thorbjørnsen (2003))

$$C(\lambda \ddagger f \bullet Z) = \int C(\lambda f(\xi) \ddagger Z'(\xi))\mu(d\xi). \tag{3}$$

The result (3) can heuristically be derived from (2). A similar result holds for the logarithm of the Laplace transform of $f \bullet Z$ (assumed to be finite),

$$K(\lambda \ddagger f \bullet Z) = \int K(\lambda f(\xi) \ddagger Z'(\xi))\mu(d\xi). \tag{4}$$

The function K will in the following be called the kumulant function.

We will now give a few examples of Lévy bases.

Example 1. (Gaussian Lévy basis) If Z is a Lévy basis with $Z(A) \sim N(a(A), b(A))$, where a is a signed measure and b is a positive measure, we call Z a Gaussian Lévy basis. The Gaussian Lévy basis has characteristics $(a, b, 0)$ and the cumulant function is

$$C(\lambda \ddagger Z(A)) = i\lambda a(A) - \tfrac{1}{2}\lambda^2 b(A).$$

We have that $Z'(\xi) \sim N(\tilde{a}(\xi), \tilde{b}(\xi))$, i.e. $C(\lambda \ddagger Z'(\xi)) = i\lambda\tilde{a}(\xi) - \frac{1}{2}\lambda^2\tilde{b}(\xi)$. Furthermore,

$$C(\lambda \ddagger f \bullet Z) = \int C(\lambda f(\xi) \ddagger Z'(\xi))\mu(d\xi) = i\lambda(f \bullet a) - \frac{1}{2}\lambda^2(f^2 \bullet b). \tag{5}$$

Note that $f \bullet Z \sim N(f \bullet a, f^2 \bullet b)$. □

Example 2. (Lévy jump basis) A Lévy basis is called a Lévy jump basis if the characteristics of the basis is $(a, 0, U)$. In Table 1 we specify the functions V and $\tilde{a}$ for three important examples of Lévy jump bases, the Poisson basis, the Gamma basis and the inverse Gaussian basis. We also list the distribution of the random variable $Z'(\xi)$, its cumulant function, mean and variance. All parameters are positive.

	Poisson	Gamma	Inverse Gaussian
$V(du, \xi)$	$\delta_1(du)$	$\mathbf{1}_{\mathbb{R}_+}(u)\beta u^{-1}e^{-\alpha(\xi)u}du$	$\frac{\eta}{\sqrt{2\pi}}\mathbf{1}_{\mathbb{R}_+}(u)u^{-\frac{3}{2}}e^{-\frac{1}{2}\gamma^2(\xi)u}du$
$\tilde{a}(\xi)$	1	$\beta\left(\frac{1-e^{-\alpha(\xi)}}{\alpha(\xi)}\right)$	$\frac{\eta}{\sqrt{2\pi}}\int_0^1 u^{-\frac{1}{2}}e^{-\frac{1}{2}\gamma^2(\xi)u}du$
$Z'(\xi)$	$\mathrm{Po}(1)$	$\Gamma(\beta, \alpha(\xi))$	$\mathrm{IG}(\eta, \gamma(\xi))$
$C(\lambda \ddagger Z'(\xi))$	$e^{i\lambda} - 1$	$-\beta\log\left(1 - \frac{i\lambda}{\alpha(\xi)}\right)$	$\eta\gamma(\xi)\left(1 - \sqrt{1 - \frac{2i\lambda}{\gamma^2(\xi)}}\right)$
$\mathbb{E}(Z'(\xi))$	1	$\frac{\beta}{\alpha(\xi)}$	$\frac{\eta}{\gamma(\xi)}$
$\mathbb{V}(Z'(\xi))$	1	$\frac{\beta}{\alpha^2(\xi)}$	$\frac{\eta}{\gamma^3(\xi)}$

Table 1: The definition of three Lévy jump bases, the Poisson basis, the Gamma basis and the inverse Gaussian basis, and the distribution of $Z'(\xi)$, with the corresponding cumulant function, mean and variance.

Note that if Z is a Poisson basis, then $Z(A) \sim \mathrm{Po}(\mu(A))$ with probability function

$$\frac{e^{-\mu(A)}\mu(A)^x}{x!}, \; x = 0, 1, 2, \ldots.$$

If Z is a Gamma basis with $\alpha(\xi) \equiv \alpha$, then $Z(A) \sim \Gamma(\beta\mu(A), \alpha)$ with density

$$\frac{\alpha^{\beta\mu(A)}}{\Gamma(\beta\mu(A))} x^{\beta\mu(A)-1} e^{-\alpha x}, \quad x > 0,$$

while if Z is an inverse Gaussian basis with $\gamma(\xi) \equiv \gamma$, then $Z(A) \sim \mathrm{IG}(\eta\mu(\mathrm{A}), \gamma)$ with density

$$\frac{\eta\mu(A) e^{\eta\mu(A)\gamma}}{\sqrt{2\pi}} x^{-3/2} \exp\left\{ -\frac{1}{2} \left((\eta\mu(A))^2 x^{-1} + \gamma^2 x \right) \right\}, \quad x > 0.$$

□

The Poisson, Gamma and inverse Gaussian Lévy bases are examples of the random $G-$ measures introduced in Brix (1999). These measures are purely discrete and can be written as (the Lévy-Ito representation)

$$Z(A) = a_0(A) + \int_{\mathbb{R}_+} x N(dx, A), \tag{6}$$

where N is a Poisson basis on $\mathbb{R}_+ \times \mathcal{R}$ with intensity measure U and

$$a_0(A) = a(A) - \int_0^1 x U(dx, A).$$

Note that equation (6) can also be written as

$$Z(A) = a_0(A) + \sum_{(u,\xi) \in \Phi} u \mathbf{1}_A(\xi), \tag{7}$$

where Φ is a Poisson point process on $\mathbb{R}_+ \times \mathcal{R}$ with intensity function U. If f is a measurable function on $\mathcal{R}$, we get that

$$f \bullet Z = f \bullet a_0 + \sum_{(u,\xi) \in \Phi} u f(\xi).$$

Finally, it should be noted that any Lévy process $\{Z_t\}_{t \in \mathbb{R}}$ induces a Lévy basis Z on $\mathcal{R}$ by

$$Z((a, b]) = Z_b - Z_a, \quad a, b \in \mathbb{R}.$$

3 Lévy based spatio-temporal modelling

Let us consider a random variable $X_t(\sigma)$ depending on time t and a position σ in space. In the following, we will assume that $(\sigma, t) \in \mathcal{R} = \mathcal{S} \times \mathbb{R}$, where $\mathcal{S} \subseteq \mathbb{R}^n$, say. A Lévy based spatio-temporal model for $X = \{X_t(\sigma) : (\sigma, t) \in \mathcal{R}\}$ is based on the intuitive picture of an ambit set $A_t(\sigma)$ associated with each point $(\sigma, t) \in \mathcal{R}$, which defines the dependency on the past at time t and position σ. The ambit set $A_t(\sigma)$ will always satisfy

$$(\sigma, t) \in A_t(\sigma)$$
$$A_t(\sigma) \subseteq \mathcal{S} \times (-\infty, t].$$

An illustration is shown in Figure 1. The *linear spatio-temporal Lévy model* for $X = \{X_t(\sigma) : (\sigma, t) \in \mathcal{R}\}$ is then defined as

$$X_t(\sigma) = \int_{A_t(\sigma)} f_t(\xi;\sigma) Z(d\xi), \tag{8}$$

where Z is a Lévy basis and $f_t(\xi;\sigma)$ is a deterministic weight function, which is assumed to be suitable for the integral to exist. The process

$$\tilde{X} = \{\exp(X_t(\sigma)) : (\sigma, t) \in \mathcal{R}\}$$

is said to follow an *exponential spatio-temporal Lévy model.*

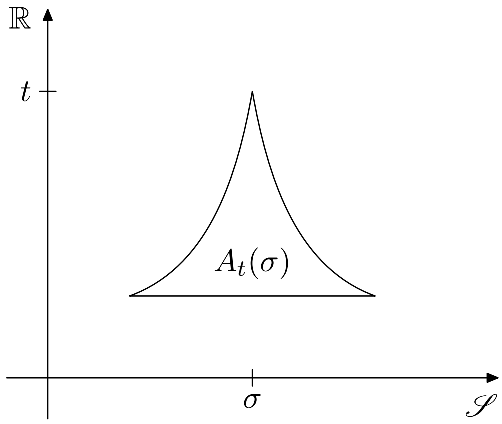

Figure 1: The ambit set $A_t(\sigma)$.

Using the key relation (3), we can derive expressions for moments in the linear spatio-temporal Lévy model. Thus, we find

$$\mathbb{E}(X_t(\sigma)) = \int_{A_t(\sigma)} f_t(\xi;\sigma)\mathbb{E}(Z'(\xi))\mu(d\xi) \tag{9}$$

$$\mathbb{V}(X_t(\sigma)) = \int_{A_t(\sigma)} f_t^2(\xi;\sigma)\mathbb{V}(Z'(\xi))\mu(d\xi), \tag{10}$$

where $\mathbb{V}$ is the notation used for variance. The covariances are of the form

$$\mathrm{Cov}(X_t(\sigma), X_{t'}(\sigma')) = \int_{A_t(\sigma)\cap A_{t'}(\sigma')} f_t(\xi;\sigma) f_{t'}(\xi;\sigma')\mathbb{V}(Z'(\xi))\mu(d\xi). \tag{11}$$

If the weight function is constant, $f_t(\xi;\sigma) \equiv f$, and if the Lévy basis Z is factorisable, then (11) reduces to

$$\mathrm{Cov}(X_t(\sigma), X_{t'}(\sigma')) = f^2\mathbb{V}(Z')\mu(A_t(\sigma)\cap A_{t'}(\sigma')). \tag{12}$$

In this case, the covariance structure only depends on the μ-measure of the intersection of the two ambit sets.

Equation (4) enables us to calculate arbitrary mixed n-order moments of $\tilde{X}_t(\sigma) = \exp(X_t(\sigma))$. If the moments are finite, then

$$\mathbb{E}\big(\tilde{X}_{t_1}(\sigma_1)\cdot\ldots\cdot\tilde{X}_{t_n}(\sigma_n)\big) = \exp\bigg(\int_{\mathcal{R}} K\bigg(\sum_{j=1}^{n} f_{t_j}(\xi;\sigma_j)\mathbf{1}_{A_{t_j}(\sigma_j)}(\xi) \ddagger Z'(\xi)\bigg)\mu(d\xi)\bigg). \tag{13}$$

The corresponding expressions for the mixed n-order moments of $X_t(\sigma)$ are obtained from

$$\mathbb{E}(X_{t_1}(\sigma_1)\cdot\ldots\cdot X_{t_n}(\sigma_n)) = \frac{\partial^n}{\partial\lambda_1\cdot\ldots\cdot\partial\lambda_n}\mathbb{E}\big(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\cdot\ldots\cdot\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\big)\Big|_{\lambda_1=\cdots=\lambda_n=0} \tag{14}$$

where

$$\begin{aligned}&\mathbb{E}\big(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\cdot\ldots\cdot\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\big)\\ &\qquad = \exp\Bigg(\int_{\mathcal{R}} K\Bigg(\sum_{j=1}^{n}\lambda_j f_{t_j}(\xi;\sigma_j)\mathbf{1}_{A_{t_j}(\sigma_j)}(\xi)\ddagger Z'(\xi)\Bigg)\mu(d\xi)\Bigg).\end{aligned} \tag{15}$$

The relative second order moments of $\tilde{X}_t(\sigma)$ has a particularly attractive form

$$\frac{\mathbb{E}(\tilde{X}_t(\sigma)\tilde{X}_{t'}(\sigma'))}{\mathbb{E}(\tilde{X}_t(\sigma))\mathbb{E}(\tilde{X}_{t'}(\sigma'))} = \exp\Bigg(\int_{A_t(\sigma)\cap A_{t'}(\sigma')} g(\xi;t,t',\sigma,\sigma')\mu(d\xi)\Bigg), \tag{16}$$

where

$$\begin{aligned}g(\xi;t,t',\sigma,\sigma') &= K((f_t(\xi;\sigma)+f_{t'}(\xi;\sigma'))\ddagger Z'(\xi))\\ &\quad - K(f_t(\xi;\sigma)\ddagger Z'(\xi)) - K(f_{t'}(\xi;\sigma')\ddagger Z'(\xi)).\end{aligned}$$

In the simple case where the weight functions are constant, i.e. $f_t(\xi;\sigma)\equiv f$ for all $(\sigma,t)\in\mathcal{R}$ and $\xi\in\mathcal{R}$, and where the underlying Lévy basis is factorisable, $Z'(\xi)=Z'$, (16) reduces to

$$\exp\left(\overline{C}\mu(A_t(\sigma)\cap A_{t'}(\sigma'))\right), \tag{17}$$

where $\overline{C} = K(2f\ddagger Z') - 2K(f\ddagger Z')$. For a factorisable Lévy basis Z and a constant weight function, one can express

$$\frac{\mathbb{E}\big(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\cdot\ldots\cdot\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\big)}{\mathbb{E}\big(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\big)\cdot\ldots\cdot\mathbb{E}\big(\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\big)} \tag{18}$$

in terms of different overlaps of the corresponding ambit sets (Schmiegel et al. (2005)).

4 Lévy based growth models

In this section we exemplify the potential of the Lévy setup in stochastic geometry and spatial statistics by constructing Lévy based stochastic models for growing objects. We focus on planar objects but generalisations to higher dimensions are straightforward. We denote the planar object at time t by $Y_t\subset\mathbb{R}^2$ and we will assume that Y_t is compact and star-shaped with respect to a point $z\in Y_t$ for all t. The boundary of the star-shaped object Y_t can be determined by its radial function $R_t=\{R_t(\phi):\phi\in[-\pi,\pi)\}$, where

$$R_t(\phi) = \max\{r : z + r(\cos\phi,\sin\phi)\in Y_t\},\ \ \phi\in[-\pi,\pi),$$

cf. Figure 2.

The growth rate will be described by the equation

$$\frac{\partial}{\partial t}R_t(\phi) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi;\phi)Z(d\xi). \tag{19}$$

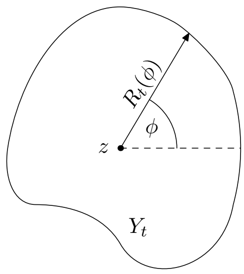

Figure 2: The star-shaped object Y_t is determined by its radial function $R_t(\phi)$ at time t and angle ϕ.

Here, the deterministic function $\mu_t : [-\pi, \pi) \to \mathbb{R}$ contributes to the overall growth pattern while the stochastic integral determines the dependence structure in the growth process. The ambit set $A_t(\phi) \subseteq [-\pi, \pi) \times (-\infty, t]$ relates to past events, $f_t(\cdot; \phi) : [-\pi, \pi) \times \mathbb{R} \to \mathbb{R}$ is a deterministic weight function (assumed to be suitable for the integral to exist) and Z is a Lévy basis on $[-\pi, \pi) \times \mathbb{R}$. The weight functions and the ambit sets must be defined cyclically in the angle such that the radial function $R_t(\phi)$ becomes cyclic. In the following, all angular calculations are regarded as cyclic.

Using (9), the mean growth rate becomes

$$\mathbb{E}\left(\frac{\partial}{\partial t} R_t(\phi)\right) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi; \phi) \mathbb{E}(Z'(\xi)) \mu(d\xi).$$

In the special case where Z is a zero mean Gaussian Lévy basis, $\mu_t(\phi)$ is indeed the mean growth rate at time t in direction ϕ. In other cases, $\mu_t(\phi)$ must be chosen such that the mean growth rate becomes as desired. There is a large literature on deterministic modelling of growth. A classical example is Gompertz growth rate specified by

$$\mathbb{E}\left(\frac{\partial}{\partial t} R_t(\phi)\right) = \mu_t = \kappa_0 \exp\left[\frac{\eta}{\gamma}(1 - \exp(-\gamma t))\right] \eta \exp(-\gamma t)$$

cf. e.g. Steel (1977).

The ambit set $A_t(\phi)$ plays an important role in this modelling approach and affects the degree of dependence on the past. The extent of the dependence on the past may be specified by the minimal time lag $T(t)$ such that

$$A_t(\phi) \subseteq [-\pi, \pi) \times [t - T(t), t], \quad \phi \in [-\pi, \pi).$$

For an illustration, see Figure 3. Note that it follows from the fact that Z is independently scattered that the random growth rates at time t and t' are independent if $\min(t, t') < \max(t - T(t), t' - T(t'))$. The actual form of the ambit set $A_t(\phi)$ will depend on the specific growth process to be modelled. A number of examples are given below. The induced correlation structure will be discussed in more detail in Section 5. A discrete version of (19) with a Gaussian Lévy basis has earlier been discussed in Jónsdóttir and Jensen (2005).

For the interpretation of (19) as a growth model, it is illuminating to represent the ambit set as a stochastic subset of the growing object. This is possible if the stochastic time transformation $t \to R_t(\phi)$ is non-decreasing for each $\phi \in [-\pi, \pi)$. We can then represent the ambit set $A_t(\phi)$ as a subset of Y_t

$$\tilde{A}_t(\phi) = \{(R_s(\theta)\cos\theta, R_s(\theta)\sin\theta) : (\theta, s) \in A_t(\phi)\}.$$

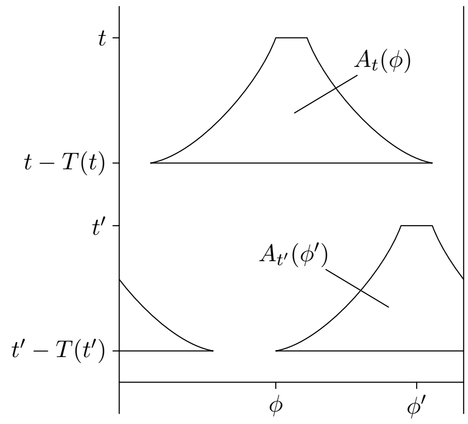

Figure 3: Two ambit sets $A_t(\phi)$ and $A_{t'}(\phi')$. Note the cyclic definition in the angle. The vertical lines are $\phi = -\pi$ and $\phi = \pi$, respectively.

It follows from the fact that $A_t(\phi) \subseteq [-\pi, \pi) \times (-\infty, t]$ that $\tilde{A}_t(\phi)$ is actually a subset of Y_t. Furthermore, since $(\phi, t) \in A_t(\phi)$, the set $\tilde{A}_t(\phi)$ touches the boundary of Y_t at the point $(R_t(\phi)\cos\phi, R_t(\phi)\sin\phi)$. It is the 'events' in $\tilde{A}_t(\phi)$ that influence the growth rate at time t in direction ϕ. Figure 4 illustrates the set $\tilde{A}_t(\phi)$.

In the particular case where Z is a Poisson basis and Ψ the associated Poisson point process on $[-\pi, \pi) \times \mathbb{R}$, we can represent the part of the spatio-temporal point process Ψ, arrived before time t,

$$\Psi_t = \{(\theta_i, t_i) : t_i \leq t\},$$

as a subset of Y_t

$$\tilde{\Psi}_t = \{(R_{t_i}(\theta_i)\cos\theta_i, R_{t_i}(\theta_i)\sin\theta_i) : t_i \leq t\}.$$

We can think of $\tilde{\Psi}_t$ as locations of outbursts at time points before t. Finally, if we let

$$\tilde{f}_t((s\cos\theta, s\sin\theta); \phi) = f_t((\theta, s); \phi),$$

the fundamental equation (19) can be written as

$$\frac{\partial}{\partial t} R_t(\phi) = \mu_t(\phi) + \sum_{\tilde{\xi} \in \tilde{\Psi}_t \cap \tilde{A}_t(\phi)} \tilde{f}_t(\tilde{\xi}; \phi). \tag{20}$$

According to (20), the growth rate at time t in direction ϕ depends on the outbursts at time points before t, lying in the stochastic neighbourhood $\tilde{A}_t(\phi)$. This Poisson model is closely related to other recently suggested growth models, cf. Section 7.1.

Under (19), the induced model for $R_t(\phi)$ will be of the same linear form since

$$\begin{aligned} R_t(\phi) &= R_0(\phi) + \bar{\mu}_t(\phi) + \int_0^t \int_{A_s(\phi)} f_s(\xi; \phi) Z(d\xi) ds \\ &= R_0(\phi) + \bar{\mu}_t(\phi) + \int_{\bar{A}_t(\phi)} \bar{f}_t(\xi; \phi) Z(d\xi), \end{aligned} \tag{21}$$

where R_0 is the radial function at time $t = 0$,

$$\bar{\mu}_t(\phi) = \int_0^t \mu_s(\phi) ds,$$

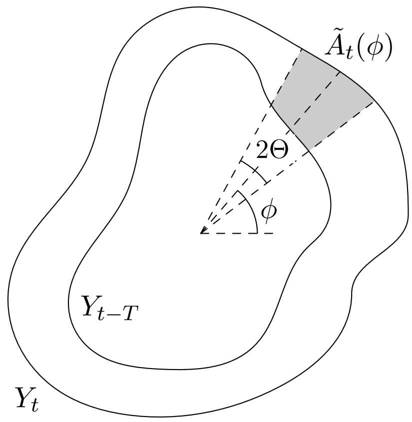

Figure 4: Illustration of the stochastic representation $\tilde{A}_t(\phi)$ (shown hatched) of the ambit set $A_t(\phi) = \{(\theta, s) : |\theta - \phi| \leq \Theta, t - T \leq s \leq t\}$.

$$\bar{A}_t(\phi) = \bigcup_{0 \leq s \leq t} A_s(\phi),$$

and

$$\bar{f}_t(\xi; \phi) = \int_0^t \mathbf{1}_{A_s(\phi)}(\xi) f_s(\xi; \phi) ds. \tag{22}$$

Note that the ambit sets associated with the radial function itself are increasing

$$t \leq t' \Rightarrow \bar{A}_t(\phi) \subseteq \bar{A}_{t'}(\phi).$$

If $t - T(t)$ is a non-decreasing function of t, then, because Z is independently scattered,

$$R_{\leq t-T(t)} = \{R_{t'}(\phi) : t' \leq t - T(t), \phi \in [-\pi, \pi)\}$$

and

$$R_{>t} - R_t = \{R_{t'}(\phi) - R_t(\phi) : t' > t, \phi \in [-\pi, \pi)\},$$

are independent.

The representation (21) is, of course, not unique. If, in particular,

$$A_t(\phi) = B_t \cap C_\phi \tag{23}$$

then

$$\bar{A}_t(\phi) = \bar{B}_t \cap C_\phi,$$

where

$$\bar{B}_t = \bigcup_{0 \leq s \leq t} B_s,$$

and we may choose, instead of (22),

$$\bar{f}_t(\xi; \phi) = \int_0^t \mathbf{1}_{B_s}(\xi) f_s(\xi; \phi) ds.$$

In some cases it might be more natural to formulate the model in terms of the time derivative of $\ln(R_t(\phi))$,

$$\frac{\partial}{\partial t}(\ln(R_t(\phi)) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi; \phi) Z(d\xi).$$

In this case the induced model is an exponential spatio-temporal Lévy model

$$R_t(\phi) = R_0(\phi) \exp\left(\bar{\mu}_t(\phi) + \int_{\bar{A}_t(\phi)} \bar{f}_t(\xi;\phi) Z(d\xi) \right).$$

Mixed moments of $R_t(\phi)$ can be derived, using the results in Section 3.

The choice of the Lévy basis Z, the ambit sets $A_t(\phi)$, the weight functions $f_t(\xi;\phi)$ and $\mu_t(\phi)$ completely determines the growth dynamics. These four ingredients can be chosen arbitrarily and independently which results in a great variety of different growth dynamics. We will now give a number of examples.

Example 3. Consider a Lévy growth model for the time derivative of the radial function

$$\frac{\partial}{\partial t} R_t(\phi) = Z(A_t(\phi)), \tag{24}$$

where Z is a Poisson Lévy basis with intensity measure concentrated on $[-\pi, \pi) \times \mathbb{R}_+$ of the form

$$\mu(d\xi) = g(s) ds d\theta, \quad \xi = (\theta, s).$$

Note that the corresponding point process in the Euclidean plane

$$\{(s \cos\theta, s \sin\theta) : \ (\theta, s) \text{ is a support point of } Z\}$$

constitutes a Poisson point process with intensity measure

$$\tilde{\mu}(dx) = \frac{g(\|x\|)}{\|x\|} dx, \ x \in \mathbb{R}^2.$$

In particular, if $g(s) = as$, $a > 0$, then the Poisson point process in the plane is homogeneous.

The ambit sets considered in this example are of the form

$$A_t(\phi) = \{(\theta, s) : |\theta - \phi| \leq \frac{\Theta}{s}, \max(0, t-T) \leq s \leq t\}.$$

Represented as subsets of the Euclidean plane, they will as $t \to \infty$ approach rectangles of side lengths 2Θ and T. Note that we can write the ambit set as

$$A_t(\phi) = B_t \cap C_\phi,$$

where

$$B_t = \{(\theta, s) : \max(0, t-T) \leq s \leq t\},$$

and

$$C_\phi = \left\{ (\theta, s) : s \geq \frac{\Theta}{\pi}, |\theta - \phi| \leq \frac{\Theta}{s} \right\} \cup \left\{ (\theta, s) : 0 \leq s \leq \frac{\Theta}{\pi} \right\}.$$

The mean growth rate at time t and in direction ϕ is for $t > T + \frac{\Theta}{\pi}$

$$\mu(A_t(\phi)) = 2\Theta \int_{t-T}^{t} \frac{g(s)}{s} ds.$$

If $g(s) = as$, $a > 0$, the mean growth rate is constant. Figure 5 shows simulations of this model with constant mean growth rate. □

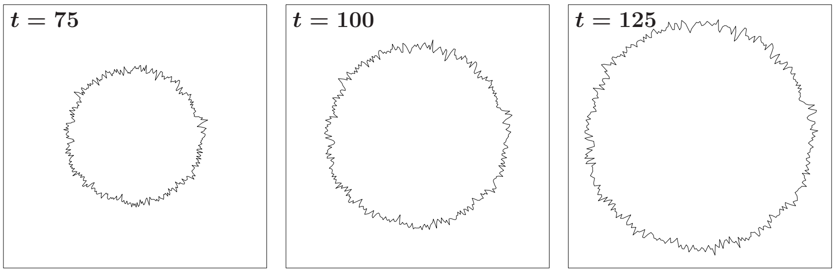

Figure 5: Simulation of the Lévy growth model (24) for the derivative of the radial function at time points $t = 75$, 100, 125, using a Poisson Lévy basis. The parameters of the simulation are $g(s) = 10s$, $T = 1$ and $\Theta = 1/2$.

Example 4. The size of the ambit sets plays an important role in the control of the local and global fluctuations of the boundary of the object Y_t. As an example, let us consider a Lévy growth model of the form

$$R_t(\phi) = \mu_t + Z(A_t(\phi)), \tag{25}$$

where

$$A_t(\phi) = \{(\theta, s) : \ |\theta - \phi| \leq \Theta(s), t - T(t) \leq s \leq t\}.$$

In Figure 6, simulations are shown under this model, using a normal Lévy basis with

$$Z(A) \sim N(0, \sigma^2 \mu(A)),$$

and μ equal to the Lebesgue measure on $\mathcal{R}$. Note that $\mu(A_t(\phi))$ does not depend on ϕ. The simulations are based on a discretisation of Z on a grid with $\Delta t = 1$ and $\Delta\phi = \frac{2\pi}{1000}$. The upper and lower row of Figure 6 show simulations for two choices of angular extension of the ambit set at three different time points. The angular extension of the ambit set is $\Theta(s) = \frac{\pi}{100}$ for the upper row, while $\Theta(s) = \frac{\pi}{5}$ for the lower row. For the smaller angular extension we observe localised fluctuations of the profiles, but the global appearance is circular. For the larger angular extension the fluctuations are on a much larger scale and the global appearance is more variable.

□

Example 5. In this example, we study a model as the one described in Example 4, but now with a Gamma Lévy basis. The model equation is

$$R_t(\phi) = \tilde{\mu}_t + Z(A_t(\phi)), \tag{26}$$

where $A_t(\phi)$ is defined as in Example 4,

$$Z(A) \sim \Gamma(\beta\mu(A), \alpha),$$

and μ is the Lebesgue measure on $\mathcal{R}$. The parameters α, β and $\tilde{\mu}_t$ are chosen such that $\mathbb{E}(R_t(\phi))$ and $\mathbb{V}(R_t(\phi))$ are the same as in the previous example. Here, we have used Table 1 together with (9) and (10). Accordingly, the parameters are chosen such that

$$\tilde{\mu}_t = \mu_t - \sigma\sqrt{\beta}\mu(A_t(0)),$$

$$\alpha = \sqrt{\frac{\beta}{\sigma^2}}.$$

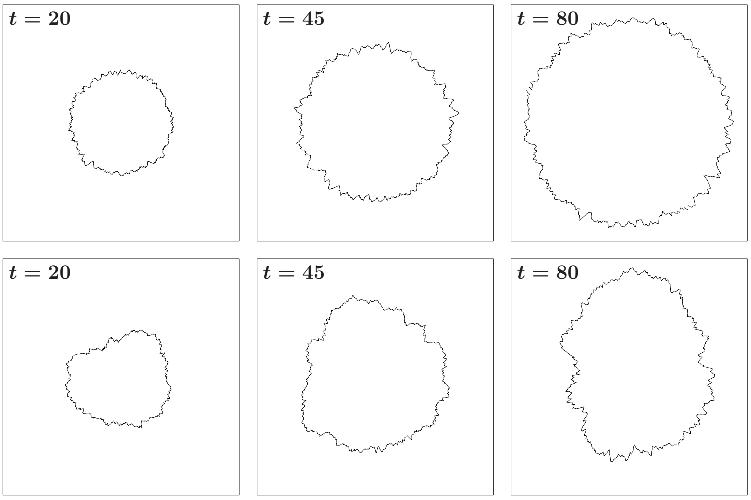

Figure 6: Simulation of the Lévy growth model (25) at time points $t = 20, 45, 80$, using a Gaussian Lévy basis. The upper row and lower row show simulations of two choices of the angular extension of the ambit set $\Theta(s) = \frac{\pi}{100}$ and $\Theta(s) = \frac{\pi}{5}$, respectively. Otherwise the parameters of the simulation are $\mu_{20} = 16$, $\mu_{45} = 24$, $\mu_{80} = 32$, $\sigma^2 = 1$ and $T(t) = t/5$.

The only free parameter is $\beta > 0$, determining the skewness of the Gamma distribution of $Z(A_t(\phi))$. For large values of β, the distribution will resemble the Gaussian distribution.

The resulting simulations for $\beta = 1$ are shown in the upper and lower row of Figure 7 for two choices of angular extension of the ambit set, $\Theta(s) = \frac{\pi}{100}$ and $\Theta(s) = \frac{\pi}{5}$, respectively. Note that more sudden outbursts are seen compared to the previous example. □

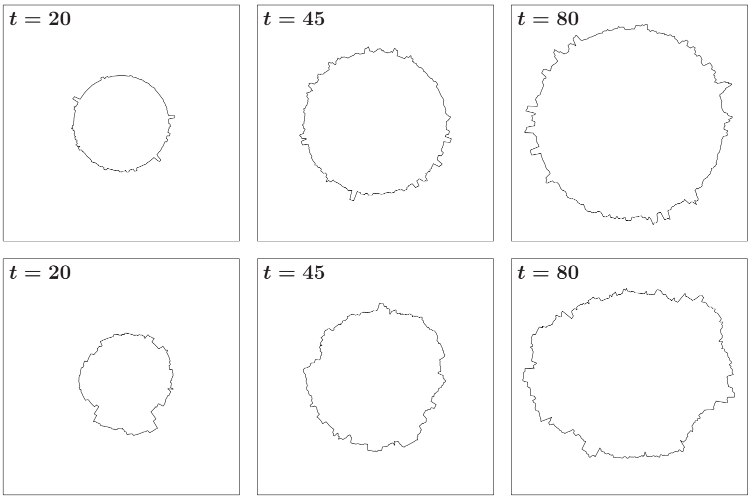

Figure 7: Simulation of the Lévy growth model (26) at time points $t = 20, 45, 80$, using a Gamma Lévy basis. The upper row and lower row show simulations of two choices of the angular extension of the ambit set $\Theta(s) = \frac{\pi}{100}$ and $\Theta(s) = \frac{\pi}{5}$, respectively. Otherwise, $\beta = 1$ and the remaining parameters are determined by the parameters used in Example 4.

Example 6. In Figure 8, we show simulations from the Lévy growth model

$$R_t(\phi) = f(\phi)\big(\mu_t + Z(A_t(\phi))\big), \tag{27}$$

where μ_t, $A_t(\phi)$ and Z are specified as in Example 4 and

$$f_t(\phi) = 0.35 \exp\left(\frac{|\phi - \pi|}{\pi}\right).$$

Clearly the growth of the object is asymmetric. The weight function $f_t(\phi)$ puts more weight on the angle $\phi_0 = 0$. □

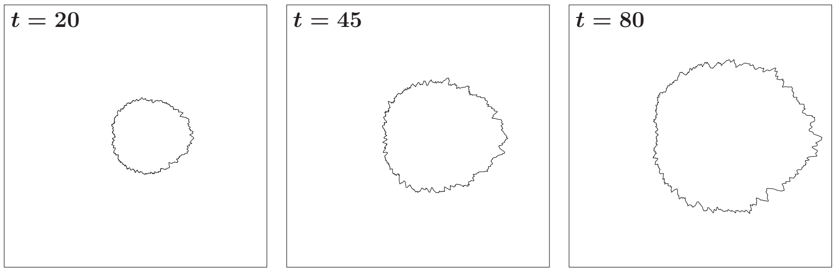

Figure 8: Simulation of the model (27) at time points $t = 20, 45, 80$, using a Gaussian Lévy basis, with parameters as specified in Example 4. The weight function is given by $f_t(\phi) = 0.35 \exp(\frac{|\phi-\pi|}{\pi})$.

5 The induced covariance structure

The Lévy based growth models induce new flexible models for space-time covariance functions on the circle, as we shall see in this section. The results presented here are of general interest for spatio-temporal processes on the circle.

We will derive expressions for $\text{Cov}(R_t(\phi), R_{t'}(\phi'))$ under various assumptions on the Lévy basis Z, the ambit sets $A_t(\phi)$ and the weight functions $f_t(\xi; \phi)$. We will concentrate on the Lévy growth model (21) of linear form for R_t. Since we now are interested in covariances it suffices to look at the following model equation

$$R_t(\phi) = \int_{A_t(\phi)} f_t(\xi; \phi) Z(d\xi),$$

where we, for simplicity, have omitted the bar on the ambit set and weight function. The covariance structure of $R_t(\phi)$ is then given by, cf. (11),

$$\text{Cov}(R_t(\phi), R_{t'}(\phi')) = \int_{A_t(\phi)\cap A_{t'}(\phi')} f_t(\xi; \phi) f_{t'}(\xi; \phi') \mathbb{V}(Z'(\xi))\mu(d\xi). \tag{28}$$

Throughout this section, we will assume that

$$\begin{aligned} A_t(\phi) &= (\phi, 0) + A_t(0), \\ f_t(\xi; \phi) &= f_t((|\theta - \phi|, s); 0), \\ \mathbb{V}(Z'(\xi))\mu(d\xi) &= g(s) ds d\theta \end{aligned} \tag{29}$$

for all $\xi = (\theta, s) \in \mathcal{R}$ and $(\phi, t) \in \mathcal{R}$. These conditions ensure that $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ only depends on the cyclic difference between ϕ and ϕ'. Accordingly, the spatio-temporal process

$$\{R_t(\phi) : t \in \mathbb{R}, \phi \in [-\pi, \pi)\}$$

will be second-order stationary in the space coordinate but not necessarily in the time coordinate.

We will first consider the case where the angular extension of the ambit set is the full angular space but the weight functions are arbitrary. Secondly, we consider the case of constant weight functions but quite arbitrary ambit sets.

5.1 Ambit sets with full angular range

In this subsection we consider ambit sets of the form

$$A_t(\phi) = [-\pi, \pi) \times [t - T(t), t].$$

In order to express the formulae as compactly as possible, we use in the proposition below the notation $t \cap t'$ for the time points shared by $A_t(\cdot)$ and $A_{t'}(\cdot)$, i.e.

$$t \cap t' = \begin{cases} [\tilde{t}_1, \tilde{t}_2] & \text{if } \tilde{t}_1 \leq \tilde{t}_2 \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\tilde{t}_1 = \max(t - T(t), t' - T(t')) \text{ and } \tilde{t}_2 = \min(t, t').$$

Using this notation we can derive the following convenient and general expression for the covariances.

Proposition 7. Let us assume that the ambit set is of the form $A_t(\phi) = [-\pi, \pi) \times [t - T(t), t]$ for all $(\phi, t) \in \mathcal{R}$ and let

$$f_t(\xi; \phi) = a_0^t(s) + \sum_{k=1}^{\infty} a_k^t(s) \cos(k(\theta - \phi)), \tag{30}$$

$\xi = (\theta, s)$, be the Fourier expansion of the weight function. Then, the spatio-temporal covariances are

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = 2\tau_0(t, t') + \sum_{k=1}^{\infty} \tau_k(t, t') \cos(k(\phi - \phi')), \tag{31}$$

where

$$\tau_k(t, t') = \pi \int_{t \cap t'} a_k^t(s) a_k^{t'}(s) g(s) ds.$$

□

Proof. The proof is straightforward. First note that the actual form (30) of the Fourier expansion of the weight function is a consequence of (29). We get that

$$\begin{aligned}
&\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) \\
&= \int_{A_t(\phi) \cap A_{t'}(\phi')} f_t(\xi; \phi) f_{t'}(\xi; \phi') \mathbb{V}(Z'(\xi)) \mu(d\xi) \\
&= \pi \left[2 \int_{t \cap t'} a_0^t(s) a_0^{t'}(s) g(s) ds + \sum_{k=1}^{\infty} \left(\int_{t \cap t'} a_k^t(s) a_k^{t'}(s) g(s) ds \right) \cos(k(\phi - \phi')) \right].
\end{aligned}$$

□

Example 8. Suppose that the weight function is of the form (30) with $a_k^t(s) \equiv 0$ if $k \neq 1$. Then,

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = \pi \cos(\phi - \phi') \int_{t \cap t'} a_1^t(s) a_1^{t'}(s) g(s) ds.$$

Since the covariance is a product of a spatial term and a temporal term, this model is separable, cf. Stein (2005) and references therein. The sign of the covariance may be positive or negative. □

Note that according to (31) the covariance $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ depends on ϕ and ϕ' only via $|\phi - \phi'|$. For some choices of model parameters, the covariance also becomes stationary in the time coordinate. For instance, if $g(s) = 1$, $T(t) = T$ and $a_k^t(s) = b_k(t - s)$, we have

$$\tau_k(t, t') = \pi \int_{\max(t-t',0)}^{T+\min(t-t',0)} b_k(u) b_k(t' - t + u) du.$$

The induced model (31) for the covariance function is not in general separable in the sense that the covariance function can be written as a product of a term depending only on t and t' and a term depending only on ϕ and ϕ'. This may be regarded as a strength of the model because separable covariance functions are often believed to give a too simplistic description of spatio-temporal data, cf. e.g. Stein (2005). If, nevertheless, such simplifying assumptions are made, we get the following results.

Corollary 9. Let the assumptions be as in Proposition 7. Assume that $a_k^t(s) = a_k^t$. Then, the spatial correlations are determined by the Fourier coefficients of the weight function f

$$\rho(R_t(\phi), R_t(\phi')) := \frac{\mathrm{Cov}(R_t(\phi), R_t(\phi'))}{\sqrt{\mathbb{V}(R_t(\phi))\mathbb{V}(R_t(\phi'))}} = \frac{2(a_0^t)^2 + \sum_{k=1}^{\infty}(a_k^t)^2 \cos(k(\phi - \phi'))}{2(a_0^t)^2 + \sum_{k=1}^{\infty}(a_k^t)^2}.$$

If, in addition, $a_k^t = b_t c_k$, then the covariance model (31) is separable. Furthermore, the spatial correlations $\rho(R_t(\phi), R_t(\phi'))$ do not depend on t, while the temporal correlations are determined by $T(t)$ and the function g,

$$\rho(R_t(\phi), R_{t'}(\phi)) = \frac{\int_{t \cap t'} g(s) ds}{\left[\int_{t-T(t)}^{t} g(s) ds \cdot \int_{t'-T(t')}^{t'} g(s) ds\right]^{1/2}}.$$

□

The covariance model (31) provides a possibility for extending stationary covariance functions on the circle (spatial covariance functions) to a spatio-temporal context. When R_t is a stationary process on the circle, its covariance function can be expressed as

$$\mathrm{Cov}(R_t(\phi), R_t(\phi')) = 2\lambda_0^t + \sum_{k=1}^{\infty} \lambda_k^t \cos(k(\phi - \phi')). \tag{32}$$

Such a covariance function can be obtained by choosing the Fourier coefficients of the weight function as

$$a_k^t(s) = a_k^t = \frac{1}{\sqrt{\pi}} \left[\lambda_k^t / \int_{t-T(t)}^{t} g(s) ds\right]^{1/2}.$$

Note that there is still freedom in the modelling by choosing an arbitrary time lag $T(t)$ and function g.

Example 10. The p-order model for a stationary covariance function on the circle, described in Hobolth et al. (2003), has

$$\lambda_0^t = \lambda_1^t = 0,\ \lambda_k^t = \left[\alpha_t + \beta_t\left(k^{2p} - 2^{2p}\right)\right]^{-1},\ k = 2, 3, \dots .$$

The model is called a $p-$order model because it can be derived as a limit of discrete $p-$order Markov models defined on a finite, systematic set of angles, cf. Hobolth and Jensen (2000). This covariance structure is obtained by choosing

$$\begin{aligned} a_0^t(s) &= a_1^t(s) = 0, \\ a_k^t(s) &= \left[\pi \int_{t-T(t)}^{t} g(s)ds\right]^{-1/2} \left[\alpha_t + \beta_t\left(k^{2p} - 2^{2p}\right)\right]^{-1/2},\ k = 2, 3, \dots \end{aligned}$$

If α_t and β_t are proportional, the simplifying assumptions of Corollary 9 are fulfilled. In Jónsdóttir and Jensen (2005), this model has been used for the time derivative of the radial function. Only Gaussian Lévy bases are considered and neighbour time points are assumed to be so far apart that the increments can be regarded as independent. The more general approach of the present paper allows for temporal correlations. Under the assumption $a_k^t = b_t c_k$, the temporal correlations are particularly simple. For instance, suppose that $T(t) \equiv 1$ and $t' - 1 \le t \le t'$. Then, we get for $g(s) = ae^{-bs}$, $a, b > 0$,

$$\rho(R_t(\phi), R_{t'}(\phi)) = \frac{1}{e^b - 1}\left[e^{\frac{1}{2}b(t-t')+b} - e^{-\frac{1}{2}b(t-t')}\right],$$

while for $g(s) = as^\alpha$, $a > 0$, $\alpha \ge 1$,

$$\rho(R_t(\phi), R_{t'}(\phi)) = \frac{t^{\alpha+1} - (t'-1)^{\alpha+1}}{\left[(t^{\alpha+1} - (t-1)^{\alpha+1})((t')^{\alpha+1} - (t'-1)^{\alpha+1})\right]^{1/2}}.$$

Only in the first case, the temporal correlations are always stationary. □

5.2 Constant weight functions

In this subsection, we consider the case of constant weight functions. Without loss of generality, we assume that $f_t(\xi;\phi) \equiv 1$ and (28) reduces to

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = \int_{A_t(\phi)\cap A_{t'}(\phi')} \mathbb{V}(Z'(\xi))\mu(d\xi) = \mathbb{V}(Z')\mu(A_t(\phi) \cap A_{t'}(\phi')), \quad (33)$$

where the last equality holds if the Lévy basis is factorisable.

It is not difficult (but sometimes tedious) to find explicit expressions for $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ for specific choices of ambit sets. One simplifying assumption is to focus on ambit sets of the form

$$A_t(\phi) = B_t \cap C_\phi,$$

where

$$\begin{aligned} B_t &= \{(\theta, s) : \max(0, t - T(t)) \le s \le t\} \\ C_\phi &= \{(\theta, s) : |\phi - \theta| \le \Theta(s)\}. \end{aligned}$$

Usually, it is simpler to find expressions for the temporal covariances than for the spatial covariances.

Evidently, (33) implies that $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) \geq 0$ which may be a severe restriction for the spatial covariances. In the proposition below, the spatial covariances are expressed in terms of the function deliminating the ambit set. The proposition gives insight into the class of spatial covariances that can be modelled using this approach.

Proposition 11. Let $\mu(d\xi) = g(s)dsd\theta$ for $\xi = (\theta, s)$. Let us suppose that there exists a continuous function $h_t : [-\pi, \pi) \to \mathbb{R}$ with the properties

$$\begin{gathered} h_t(\theta) = h_t(-\theta) \\ h_t \text{ is decreasing on } [0, \pi) \\ h_t(0) = t \end{gathered} \tag{34}$$

such that

$$A_t(0) = \{(\theta, s) : h_t(\pi) \leq s \leq h_t(\theta)\},$$

cf. Figure 9. Let

$$\bar{h}_t(\phi) = \int_0^{h_t(\phi)} g(s)ds.$$

Then, if the Fourier expansion of $\bar{h}_t$ is ($\bar{h}_t(\phi) = \bar{h}_t(-\phi)$)

$$\bar{h}_t(\phi) = \sum_{k=0}^{\infty} \gamma_k^t \cos(k\phi), \tag{35}$$

then

$$\mu(A_t(0) \cap A_t(\phi)) = \sum_{k=0}^{\infty} \lambda_k^t \cos(k\phi), \tag{36}$$

where

$$\begin{aligned} \lambda_0^t &= \sum_{k \text{ odd}} \left[2\pi - \frac{16}{\pi k^2}\right] \gamma_k^t - 2\pi \sum_{k \text{ even}} \gamma_k^t \\ \lambda_j^t &= \frac{16}{\pi} \sum_{k \text{ odd}} \frac{1}{(2j)^2 - k^2} \gamma_k^t, \quad j = 1, 2, \ldots. \end{aligned}$$

□

Proof. It is not difficult to show that

$$\mu(A_t(0) \cap A_t(\phi)) = 2\int_{-\pi}^{-\pi+\frac{\phi}{2}} \bar{h}_t(\theta)d\theta + 2\int_{\frac{\phi}{2}}^{\pi} \bar{h}_t(\theta)d\theta - 2\pi\bar{h}_t(\pi), \quad \phi \in [0, \pi). \tag{37}$$

Using (35), we find

$$\begin{aligned} &\mu(A_t(0) \cap A_t(\phi)) \\ &\quad = \begin{cases} -4\sum_{k \text{ odd}} \frac{\gamma_k^t}{k} \sin(k\frac{\phi}{2}) + 2\pi \sum_{k \text{ odd}} \gamma_k^t - 2\pi \sum_{k \text{ even}} \gamma_k^t & \text{if } \phi \in [0, \pi] \\ 4\sum_{k \text{ odd}} \frac{\gamma_k^t}{k} \sin(k\frac{\phi}{2}) + 2\pi \sum_{k \text{ odd}} \gamma_k^t - 2\pi \sum_{k \text{ even}} \gamma_k^t & \text{if } \phi \in [-\pi, 0]. \end{cases} \end{aligned}$$

The result is now obtained by deriving a Fourier expansion of the latter expression and comparing with (36). □

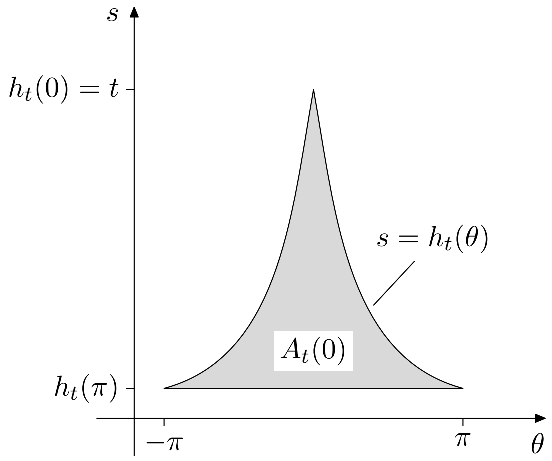

Figure 9: Illustration of the ambit set $A_t(0)$ bounded by the function h_t, cf. (34).

Example 12. In the particular case where $g(s) = 1$ and

$$\bar{h}_t(\phi) = h_t(\phi) = \gamma_0^t + \gamma_1^t \cos\phi$$

we find

$$\lambda_0^t = \left[2\pi - \frac{16}{\pi}\right]\gamma_1^t - 2\pi\gamma_0^t$$

$$\lambda_j^t = \frac{16}{\pi}\frac{1}{(2j)^2 - 1}\gamma_1^t, \quad j = 1, 2, \ldots$$

It follows that

$$(\lambda_j^t)^{-1} = \alpha_t + \beta_t j^2, \quad j = 1, 2, \ldots, \tag{38}$$

where $\alpha_t = -\pi/(16\gamma_1^t)$ and $\beta_t = \pi/(4\gamma_1^t)$. Under the assumption of a normal Lévy basis, (38) is a special case of the $p-$order model considered in Jónsdóttir and Jensen (2005) with $p = 1$ and α proportional to β. Note that the requirements (34) implies that $\gamma_0^t = t - \gamma_1^t$ and $\gamma_1^t > 0$. It does not seem to be possible to obtain p-order models with $p > 1$, using this approach. □

6 An application to tumour growth

In Schmiegel (2006), snapshots of a growing brain tumour in vitro have been analysed, using the approach described in this paper, see Figure 10. The data were first studied in Brú et al. (1998).

A detailed initial analysis of the covariance structure showed negative spatial covariances and a need for modelling both small and large scale fluctuations in the growth process. The model used was an exponential spatio-temporal Lévy model of the form

$$\begin{aligned} R_t(\phi) = \exp\Big\{ & \mu_t + \alpha(t)\int_{t-T(t)}^{t-t_0(t)}\int_{-\pi}^{\pi}\cos(\phi-\theta)Z(dsd\theta) \\ & + \beta(t)\int_{t-t_0(t)}^{t}\int_{\phi-h_t(s-t+t_0(t))}^{\phi+h_t(s-t+t_0(t))} Z(dsd\theta)\Big\}. \end{aligned} \tag{39}$$

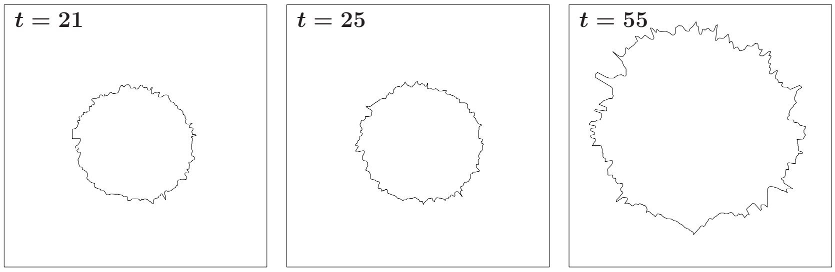

Figure 10: Profiles of a growing brain tumour in vitro at time points $t = 21, 25, 55$.

Here h_t is a deterministic and monotonically decreasing function defined on $[0, t_0(t)]$, satisfying $h_t(t_0(t)) = 0$ and $h_t(0) = \phi_0(t)/2$. Accordingly, the weight function is on the form

$$f_t(\xi; \phi) = \alpha(t) \cos(\phi - \theta) \mathbf{1}_{[t-T(t), t-t_0(t)]}(s) + \beta(t) \mathbf{1}_{[t-t_0(t), t]}(s) \mathbf{1}_{[0, h_t(s-t+t_0(t))]}(|\phi - \theta|).$$

The associated ambit set is shown in Figure 11. In Schmiegel (2006), a Gaussian Lévy

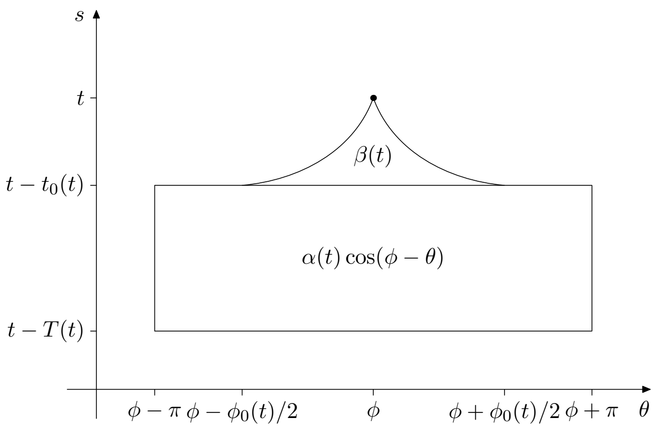

Figure 11: The ambit set $A_t(\phi)$ for the model defined by (39).

basis has been used and the function h_t was assumed to be of the form

$$h_t(s) = \frac{\phi_0(t)}{2} - \frac{\phi_0(t)}{2t_0(t)} s, \quad s \in [0, t_0(t)].$$

The parameters of the model (39) were estimated by the method of moments, using the results given in Section 3. The estimated parameters are given in Table 2 and a simulation under the model with a Gaussian Lévy basis is shown in Figure 12.

t	$T(t)$	$t_0(t)$	$\alpha(t)$	$\beta(t)$	$\phi_0(t)$
21	21	19	0.04	−0.033	0.19
25	25	17	0.02	−0.033	0.19
55	18	4	0.01	−0.067	0.23

Table 2: The estimated parameters for the model (39), using a Gaussian Lévy basis.

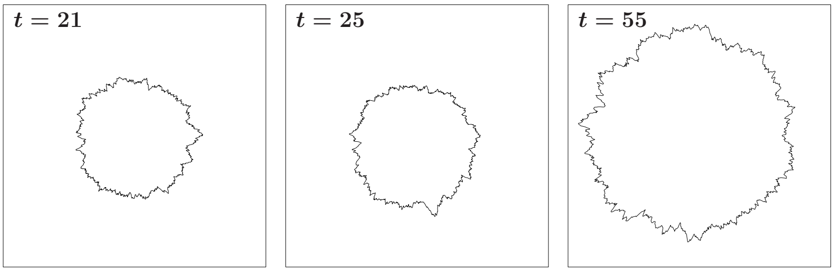

Figure 12: Simulation of the model (39) for time points $t = 21, 25, 55$, using a Gaussian Lévy basis.

Here we will study the use of Gamma and inverse Gaussian Lévy bases. Simulations under the latter basis are shown in Figure 13. The inverse Gaussian Lévy basis is chosen such that $\mathbb{E}(R_t(\phi))$ and $\mathbb{V}(R_t(\phi)$ are the same as in the case where a Gaussian basis is used. The upper row of Figure 13 shows simulations where $\eta = 316$ and the lower row shows a simulation where $\eta = 5$. For $\eta = 316$ the inverse Gaussian Lévy basis provides fits of a similar quality as the normal basis but for $\eta = 5$, more outburst are observed as is the case for the data. The difference is due to the fact that the inverse Gaussian distribution has heavier right tails for the latter choice of parameters. It should be noted that all the profiles simulated under the model (39) using the Lévy basis mentioned in this section show somewhat more fluctuations on a local scale than the observed profiles. At present, we do not know whether this feature is caused by non-perfect model selection and estimation of parameters or artefacts due to the discretisation in the simulation procedure.

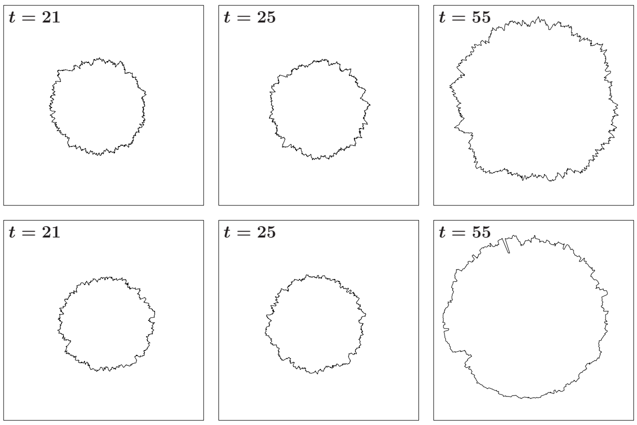

Figure 13: Simulations of the model (39) for time points $t = 21, 25, 55$, using an inverse Gaussian Lévy basis with $\eta = 316$ (upper row) and $\eta = 5$ (lower row).

7 Discussion

In the present paper, we have given a condensed review of Lévy based spatio-temporal modelling and shown its potential use in stochastic geometry and spatial statistics by developing Lévy based growth models and space-time covariances on the circle. Below, we discuss further perspectives and topics for future research.

7.1 Related growth models

In the growth literature, there is a variety of growth models for objects in discrete space, cf. e.g. Bramson and Griffeath (1981), Qi et al. (1993), Lee and Cowan (1994), Kansal et al. (2000) and references therein. An important early example is the Richardson model, introduced in Richardson (1973). Here, the growth is described by a Markov process. For a growing object in the plane, the state at time t is a random subset Y_t of $\mathbb{Z}^2$ consisting of the 'infected sites'. An uninfected site is transferred to an infected site with a rate proportional to the number of infected nearest neighbours. It can be shown that if Y_0 consists of a single site, Y_t/t has a non random shape as $t \to \infty$. Note that the growth model described in the present paper in (20) may be regarded as a continuous analogue of the Richardson model.

A related growth model in continuous space has recently been discussed in Deijfen (2003). For planar objects, the model is constructed from a spatio-temporal Poisson point process on $\mathbb{R}^3$

$$\Psi = \{(x_i, t_i)\}.$$

The random growing object $Y_t \subset \mathbb{R}^2$ is a subset of

$$\bigcup_{\{i: t_i \leq t\}} B(x_i, r),$$

constructed such that Y_t is always connected. Here, $B(x, r)$ is a circular disc with centre x and radius r. In this model, t_i is thought of as a time point of outburst and x_i is the location of the outburst in the tumour, say. A closely related discrete time Markov growth model has been discussed in detail in Cressie and Hulting (1992). This model can be characterised as a sequence of Boolean models

$$Y_{t+1} = \bigcup \{B(x_i, r) : x_i \in Y_t\},$$

where $\{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^2$, see also Cressie and Laslett (1987) and Cressie (1991a,b).

An issue of interest in growth modelling is the asymptotic shape of the growing object, cf., e.g., Durrett and Liggett (1981) and Deijfen (2003). It is expected that it is also possible to obtain asymptotic results for Lévy based growth models, using that Lévy bases are independently scattered random measures.

7.2 Lévy driven Cox processes

Another interesting application of spatio-temporal Lévy models in spatial statistics are Lévy driven Cox processes, cf. Hellmund (2006), Hellmund et al. (2006) and Prokešová et al. (2006). As an example, we may use $\exp(X_t(\sigma))$, where $X_t(\sigma)$ is given in (8), as driving random intensity for a spatio-temporal Cox process. If Z is a Gaussian Lévy basis, then the resulting Cox process is log Gaussian. Another example is obtained by assuming that Z is a positive Lévy basis, then the Cox process driven by $X_t(\sigma)$ is a spatio-temporal shot-noise Cox process. Cox processes have been studied intensively in recent years, cf. Brix (1998), Wolpert and Ickstadt (1998), Brix (1999), Brix and Diggle (2001), Brix and Møller (2001), Brix and Chadoeuf (2002), Møller (2003).

7.3 Estimation of model parameters

It still remains to develop inference procedures for Lévy based growth models (and more generally for Lévy based spatio-temporal models). There are a number of interesting problems here, including non-parametric estimation of the ambit sets.

A Fourier expansion of the radial function may be useful when making inference about the shape of the growing object, cf. e.g. Alt (1999) and Jónsdóttir and Jensen (2005). Let us consider the Fourier coefficients of $R_t(\phi)$,

$$A_k^t = \frac{1}{\pi}\int_{-\pi}^{\pi} R_t(\phi)\cos(k\phi)d\phi, \quad B_k^t = \frac{1}{\pi}\int_{-\pi}^{\pi} R_t(\phi)\sin(k\phi)d\phi$$

$k = 0, 1, \ldots$ Under the assumptions of Proposition 7 it can be shown that

$$A_k^t = \int_{-\pi}^{\pi}\int_{t-T(t)}^{t} a_k^t(s)\cos(k\theta)Z(d\theta ds), \quad B_k^t = \int_{-\pi}^{\pi}\int_{t-T(t)}^{t} a_k^t(s)\sin(k\theta)Z(d\theta ds),$$

so the Fourier coefficients also follow a linear spatio-temporal Lévy model. It can be shown that for $k \neq j$, $t, t' \geq 0$,

$$\mathrm{Cov}(A_k^t, A_j^{t'}) = \mathrm{Cov}(B_k^t, B_j^{t'}) = \mathrm{Cov}(A_k^t, B_j^{t'}) = 0,$$

and

$$\mathrm{Cov}(A_k^t, A_k^{t'}) = \mathrm{Cov}(B_k^t, B_k^{t'}) = \tau_k(t, t'),$$

where $\tau_k(t, t')$ is given in Proposition 7.

In the case where Z is a Gaussian Lévy basis, this means that $\{A_k^t\}_{t\in\mathbb{R}}$ and $\{B_k^t\}_{t\in\mathbb{R}}$ $k = 0, 1, \ldots,$ are independent Gaussian stochastic processes with covariance functions $\tau_k(t, t')$. If one observes A_k^t and B_k^t, for some time points $t = t_1, \ldots, t_n$, and some orders $k = 1, \ldots, K_t$, the likelihood function is tractable.

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References

Alt, W. (1999). Statistics and dynamics of cellular shape changes. In Chaplain, M. A. J., Singh, G. D., and McLachlan, J. C., editors, *On Growth and Form: Spatio-temporal Pattern Formation in Biology*, pages 287–307. Wiley, Chichester.

Barndorff-Nielsen, O. and Schmiegel, J. (2004). Lévy based spatial-temporal modelling, with applications to turbulence. *Russian Math. Surveys*, 59(1):65–90.

Barndorff-Nielsen, O. and Thorbjørnsen, S. (2003). A connection between classical and free infinite divisibility. Technical report, 2003-7, MaPhySto, University of Aarhus, Denmark.

Bramson, M. and Griffeath, D. (1981). On the Williams-Bjerknes tumour growth model, I. *Ann. Prob.*, 9:173–185.

Brix, A. (1998). *Spatial and Spatio-temporal Models for Weed Abundance.* Phd thesis, Royal Veterinary and Agricultural University, Copenhagen.

Brix, A. (1999). Generalized Gamma measures and shot-noise Cox processes. *Adv. Appl. Prob.*, 31:929–953.

Brix, A. and Chadoeuf, J. (2002). Spatio-temporal modelling of weeds by shot-noise G Cox processes. *Biom. J.*, 44:83–99.

Brix, A. and Diggle, P. J. (2001). Spatiotemporal prediction for log-Gaussian Cox processes. *J. R. Statist. Soc. B*, 63:823–841.

Brix, A. and Møller, J. (2001). Space-time multitype log Gaussian Cox processes with a view to modelling weeds. *Scand. J. Statist.*, 28:471–488.

Brú, A., Pastor, J. M., Fernaud, I., Brú, I., Melle, S., and Berenguer, C. (1998). Super-rough dynamics on tumour growth. *Physical Review Letters*, 81:4008–4011.

Calder, I. (1986). A stochastic model of rainfall interception. *J. of Hydrology*, 89:65–71.

Cantalapiedra, I., Lacasta, A., Auguet, C., Peñaranda, A., and Ramirez-Piscina, L. (2001). Pattern formation modelling of bacterial colonies. In *Branching in Nature*, pages 359–364. EDP Sciences, Springer.

Cressie, N. (1991a). Modelling growth with random sets. In Possolo, A. and Hayward, C. A., editors, *Spatial Statistics and Imaging*, pages 31–45. IMS Lecture Notes, Proceedings of the 1988 AMS-IMS-SIAM Joint Summer Research Conference.

Cressie, N. (1991b). *Statistics for Spatial Data.* Wiley, New York.

Cressie, N. and Hulting, F. (1992). A spatial statistical analysis of tumor growth. *J. Amer. Statist. Assoc.*, 87:272–283.

Cressie, N. and Laslett, G. M. (1987). Random set theory and problems of modelling. *SIAM Review*, 29:557–574.

Deijfen, M. (2003). Asymptotic shape in a continuum growth model. *Adv. Appl. Prob. (SGSA)*, 35:303–318.

Delsanto, P., Romano, A., Scalerandi, M., and Pescarmona, G. (2000). Analysis of a "phase transition" from tumor growth to latency. *Phys. Rev. E*, 62:2547–2554.

Durrett, R. and Liggett, T. (1981). The shape of the limit set in Richardsons growth model. *Ann. Prob.*, 9:186–193.

Feideropoulou, G. and Pesquet-Popescu, B. (2004). Stochastic modelling of the spatio-temporal wavelet coefficients and applications to quality enhancement and error concealment. *EURASIP JASP*, 12:1931–1942.

Fewster, R. (2003). A spatiotemporal stochastic process model for species spread. *Biometrics*, 59:640–649.

Gratzer, G., Canham, C., Dieckmann, U., Fischer, A., Iwasa, Y., Law, R., Lexer, M., Sandman, H., Spies, T., Splechtna, B., and Szwagrzyk, L. (2004). Spatio-temporal development of forests – current trends in field methods and models. *Oikos*, 107:3–15.

Hellmund, G. (2006). *Lévy driven Cox processes with a view to modelling tropical rain forests.* Master Thesis, Department of Mathematical Sciences, University of Aarhus.

Hellmund, G., Prokešová, M., and Jensen, E. B. V. (2006). Spatial and spatio-temporal Lévy based Cox point processes. In preparation.

Hobolth, A. and Jensen, E. (2000). Modelling stochastic changes in curve shape, with an application to cancer diagnostics. *Adv. Appl. Prob. (SGSA)*, 32:344–362.

Hobolth, A., Pedersen, J., and Jensen, E. B. V. (2003). A continuous parametric shape model. *Ann. Inst. Statist. Math.*, 55:227–242.

Jensen, E. B. V., Jónsdóttir, K. Y., Schmiegel, J., and Barndorff-Nielsen, O. E. (2006). Spatio-temporal modelling - with a view to biological growth. In Finkenstadt, B. and Isham, V., editors, *Statistical Methods of Spatio-Temporal Systems*, pages 45–73. Chapman & Hall/CRC, Boca Raton.

Jónsdóttir, K. Ý. and Jensen, E. B. V. (2005). Gaussian radial growth. *Image Analysis & Stereology*, 24:117–16.

Kallenberg, O. (1989). *Random Measures.* Akademie Verlag, Berlin, 4 edition.

Kansal, A. R., Torquato, S., Harsh, G. R., Chiocca, E. A., and Deisboeck, T. S. (2000). Simulated brain tumor growth dynamics using a three-dimensional cellular automaton. *J. Theor. Biol.*, 203:367–382.

Kwapien, S. and Woyczynski, W. (1992). *Random Series and Stochastic Integrals: Single and Multiple.* Birkhäuser, Basel.

Lee, T. and Cowan, R. (1994). A stochastic tessellation of digital space. In Serra, J., editor, *Mathematical Morphology and Its Applications to Image Processing*, pages 217–224. Kluwer, Dordrecht.

Lovejoy, S., Schertzer, D., and Watson, B. (1992). Radiative transfer and multifractal clouds: theory and applications. *I.R.S.*, 92:108–111.

Møller, J. (2003). Shot noise Cox processes. *Adv. Appl. Prob. (SGSA)*, 35:614–640.

Pang, N. and Tzeng, W. (2004). Anomalous scaling of superrough growing surfaces: From correlation functions to residual local interfacial widths and scaling exponents. *Phys. Rev. E*, 70(036115).

Peirolo, R. and Scalerandi, M. (2004). Markovian model of growth and histologic progression in prostate cancer. *Phys. Rev. E*, 70(011902).

Prokešová, M., Hellmund, G., and Jensen, E. B. V. (2006). On spatio-temporal Lévy based Cox processes. In *Proceedings of S4G, International Conference on Stereology, Spatial Statistics and Stochastic Geometry*, pages 111–116. Union of Czech Mathematics and Physicists.

Qi, A. S., Zheng, X., Du, C. Y., and An, B. S. (1993). A cellular automaton model of cancerous growth. *J. Theor. Biol.*, 161:1–12.

Rajput, B. and Rosinski, J. (1989). Spectral representations of divisible processes. *Probability Theory and Related fields*, 89:451–487.

Richardson, D. (1973). Random growth in a tessellation. *Proc. Camb. Phil. Soc.*, 74:515–528.

Schmiegel, J. (2006). Self-scaling tumor growth. *Physica A*, 367:509–524.

Schmiegel, J., Barndorff-Nielsen, O., and Eggers, H. (2005). A class of spatio-temporal and causal stochastic processes, with application to multiscaling and multifractality. *South African Journal of Science*, 101:513–519.

Schmiegel, J., Cleve, J., Eggers, H., Pearson, B., and Greiner, M. (2004). Stochastic energy-cascade model for 1+1 dimensional fully developed turbulence. *Phys. Lett. A*, 320:247–253.

Sornette, D. and Ouillon, G. (2005). Multifractal scaling of thermally activated rupture processes. *Phys. Rev. Lett.*, 94(038501).

Steel, G. G. (1977). *Growth kinetics of tumours*. Clarendon Press, Oxford.

Stein, M. (2005). Space-time covariance functions. *J. Amer. Statist. Assoc.*, 100:310–321.

Wolpert, R. L. and Ickstadt, K. (1998). Poisson/Gamma random field models for spatial statistics. *Biometrika*, 85:251–267.