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On the classical, statistical, and stochastic approaches to the hydrodynamic turbulence

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On the classical, statistical, and stochastic approaches to the hydrodynamic turbulence (an overview for probabilists)

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Recently the International mathematical community celebrated the 70th anniversary of our distinguished colleague and friend Prof. Ole E. Barndorff-Nielsen. In March 20–24, 2006 the Centro de Investigación en Matemáticas (CIMAT) in Guanajuato, Mexico was host of the big International "Conference on Stochastics in Science" in Honor of Ole E. Barndorff-Nielsen. I was invited by the Organizing Committee to deliver a talk in a field where Ole worked and is working, as usual very efficient, now. Taking it into account, I decided that the right theme from Ole's interests should be *Turbulence*. As a result of this idea I have proposed the talk "On the classical, statistical, and stochastic approaches to the hydrodynamic turbulence", slides of which is presented in this booklet. So, this talk is for Ole to whom I have great respect and love.

Albert Shiryaev

§ 1. Hydrodynamics in XVIII century—an epoch of D. Bernoulli and L. Euler

Discover of differential and integral calculus by

I. Newton (1642–1727) and G. Leibniz (1646–1716)

offered an incentive for

D. Bernoulli (1700–1782) and L. Euler (1707–1793)

to create the theoretical hydrodynamics as a special science.

Newton, Leibniz: continuous motion of a single, discrete object (points, balls, planets,. . .)

Bernoulli, Euler: application of the "calculus" to description of dynamics of continuous medium (first of all to fluids)

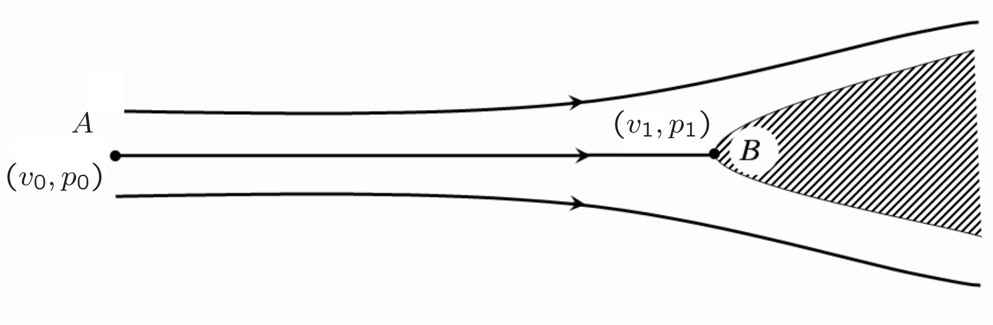
Bernoulli (1738)—treatise "Hydrodynamics" ("academic research which was done at the time of author's work in Petersbourg"; the term *hydrodynamics* has been introduced in this treatise)

D. Bernoulli proved a fundamental **Bernoulli theorem** that gives a formula of the relationship between

pressure (p), **level** (h), and **velocity** (v)

of the fluid with constant **density** (ρ) under **gravity force** (g):

$$\boxed{\frac{v^2}{2} + \frac{p}{\rho} + gh = \text{const}}\,. \qquad (*)$$

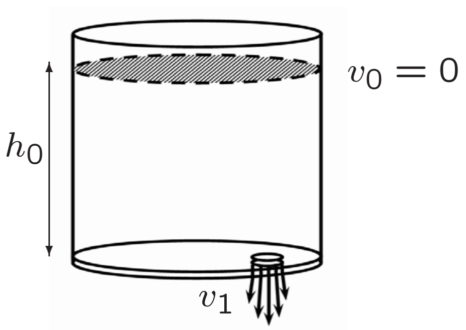


Application: At point B we have $v_1 = 0$ and from $(*)$ (with $\rho = 1$)

$$\boxed{p_1 = p_0 + \frac{1}{2} v_0^2}.$$

At first sight, it is surprising that pressure at point B increases as the initial velocity increases.

This seeming contradiction was explained by Euler who showed that Bernoulli's theorem holds not for the whole flow but only along narrow streams.



Equation $(*)$ gives the following result about the velocity v_1 of outflow from open container with an aperture in the bottom under gravity force:

$$\underset{(v_0=0)}{\frac{v_0^2}{2}} + gh_0 = \frac{v_1^2}{2} + \underset{(h_1=0)}{gh_1} \quad \Rightarrow \quad \boxed{v_1 = \sqrt{2gh_0}}$$

i.e., velocity of outflow is the same as the velocity under free downfall from the level h_0.

Very important novelty of D. Bernoulli in "Hydrodynamics" was an idea

> to use a **vector field** of the velocities for mathematical description of the motion of fluids.

This idea was used by L. Euler, who got a famous system of equations for ideal fluids, that was a fundament for analytic mechanics of continuous media.

There are two basic classical way (Lagrange and Bernoulli–Euler) for description of motion in the continuous media.

J. L. Lagrange (1736–1813) method:

a fixed particle occupies at time t_0 a position

$$\omega_0 = (x_0, y_0, z_0);$$

its evolution in time is given by

$$\omega_t = f(t; \omega_0)$$

where ω_t is the position of the particle at time t, *i. e.*,

$$\begin{aligned} x_t &= f_1(t; x_0, y_0, z_0), \\ y_t &= f_2(t; x_0, y_0, z_0), \\ z_t &= f_3(t; x_0, y_0, z_0). \end{aligned}$$

Bernoulli–Euler method

operates with the **field of velocities**

$$\mathbf{u} = (u_1, u_2, u_3);$$

where

$$\begin{aligned} u_1 &= u_1(t; x, y, z), \\ u_2 &= u_2(t; x, y, z), \\ u_3 &= u_3(t; x, y, z) \end{aligned}$$

are three velocities in the three axial directions.

Main difference between these descriptions:

In the **Lagrange** method points (x, y, z) are coordinates of a **moving** particle.

In the **Bernoulli–Euler** method (x, y, z) are coordinates of the **fixed point** and we are looking for speeds of the particles that pass (x, y, z) by.

L. Euler gave his famous equations for *ideal* (*i. e.,* without interior friction) incompressible fluid. These equations are good for description of many effects of the "flow round", percolation of fluids in channels. But they don't explain effects of resistance, warming of fluids as a result of dissipation of the mechanical energy in heat.

By applying Newton's law

Force $=$ **Mass** $\times$ **Acceleration**

to each point $P = P(x, y, z)$ in the ideal incompressible fluid, Euler got the following equations for the vector of the velocities $\mathbf{u} = (u_1(t; x, y, z), u_2(t; x, y, z), u_3(t; x, y, z))$:

$$\frac{d\mathbf{u}}{dt} = \mathbf{f} - \nabla p, \qquad (**)$$

where d/dt is a convective derivative,

$\mathbf{f} = \mathbf{f}(t; x, y, z)$ is a vector of external forces,

$p = p(t; x, y, z)$ is a (scalar) pressure and

$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$.

Since

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{u}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{u}}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\nabla)\mathbf{u},$$

where $\mathbf{u}\,.\nabla = u_1\,\partial/\partial x + u_2\,\partial/\partial y + u_3\,\partial/\partial z$, equation $(**)$ can be written in the form

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\nabla)\mathbf{u} = \mathbf{f} - \nabla p},$$

or, in more details,

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} = f_1 - \frac{\partial p}{\partial x_1},$$
$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3} = f_2 - \frac{\partial p}{\partial x_2},$$
$$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3} = f_3 - \frac{\partial p}{\partial x_3}.$$

Euler added also to these system the following equation:

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} = 0, \qquad \text{or} \qquad \operatorname{div}\mathbf{u} := \nabla\mathbf{u} = 0$$

which follows from assumption that the fluid is **incompressible**.

So, we have four equations for four unknown functions u_1, u_2, u_3, and p.

The *boundary* condition and (in case of nonstationary motion) the *initial* conditions should be added.

§ 2. THE NAVIER–STOKES EQUATION: A Millenium Problem— one of the greatest unsolved mathematical puzzle

Among 72 names of the XIX century French scientists listed on the four facade of the Eiffel Tower, together with Lagrange, Laplace, Legendre,. . . , we find name

Claude Louis Marie Henri Navier (1785–1836)

—known during his lifetime as one of France's most famous designer and builder of bridges.

In 1822 Navier made very essential step in hydrodynamics theory—he not only introduced an important characteristic of the real fluid

a **viscosity parameter** ν

as a measure of the friction between particles of fluid but also added the corresponding term to the Euler equation

$$\frac{d\mathbf{u}}{dt} + (\mathbf{u}\,.\nabla)\mathbf{u} = \mathbf{f} - \nabla p, \qquad (*)$$

thus obtaining a new equation

$$\boxed{\frac{d\mathbf{u}}{dt} + (\mathbf{u}\,.\nabla)\mathbf{u} = \nu\,\Delta\mathbf{u} + \mathbf{f} - \nabla p}\,. \qquad (**)$$

Note that ν is a *cinematic (molecular) viscosity* coefficient:

$$\boxed{\nu = \mu/\rho},$$

where μ is a coefficient of viscosity and
ρ is the density of the fluid.

Navier's reasoning (1822) for the validity of the equation $(**)$ was on engineering's level.

In 1845 Irish mathematician

George Gabriel Stokes (1819–1903)

rediscovered (in his case, with correct mathematical reasoning) the Navier equation. This is a reason why now we call equations $(**)$ the *Navier–Stokes equation*.

With these equations there is just one "small" problem—no one has been able to solve them and, in fact, no one has been able to show in principle whether a solution even exists!

The mathematics of fluid flow turned out extremely hard.

Taking into account these difficulties, the Clay Institute proposed to award the one-million dollars prize for the solution to any one of several variations of the problem.

The simplest version is the following:

> assuming $\mathbf{f} \equiv 0$ and $\nu > 0$, find functions $p = p(t; x, y, z)$ and $u_1(t; x, y, z)$, $u_2(t; x, y, z)$, $u_3(t; x, y, z)$ which satisfy the Navier–Stokes equations together with the incompressibility equation.

Let us mention that the analogous problem, where the viscosity $\nu = 0$ (*i. e.,* for the Euler equations) has neither been solved, but that version is not a Millennium Problem.

For the dimension $d = 2$ situation is better:

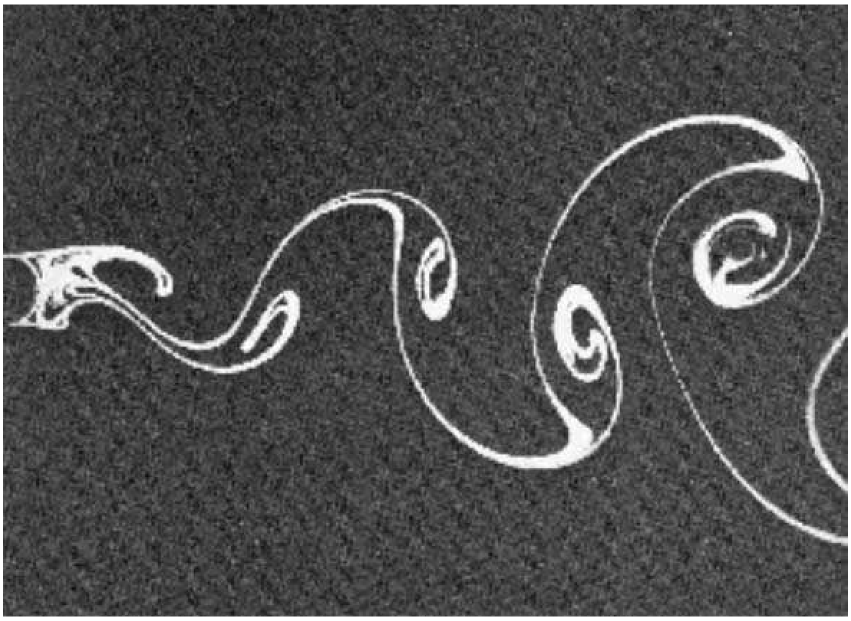
J. Leray (1933) demonstrated both existence and uniqueness for some of the plausible boundary conditions. We shall discuss these results a little bit later.

§ 3. Earlier theories of turbulence

It is known that the phenomena of turbulence (turbulenza) was described along time ago by **Leonardo da Vinci (1452–1519)** who placed obstructions in water and observed the result:

> "Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion."

Drawing of Leonardo da Vinci



Kármán vortex street (from the book of U. Frisch "Turbulence. The legacy of A. N. Kolmogorov")

The scientific study of turbulence was began in 1883 by

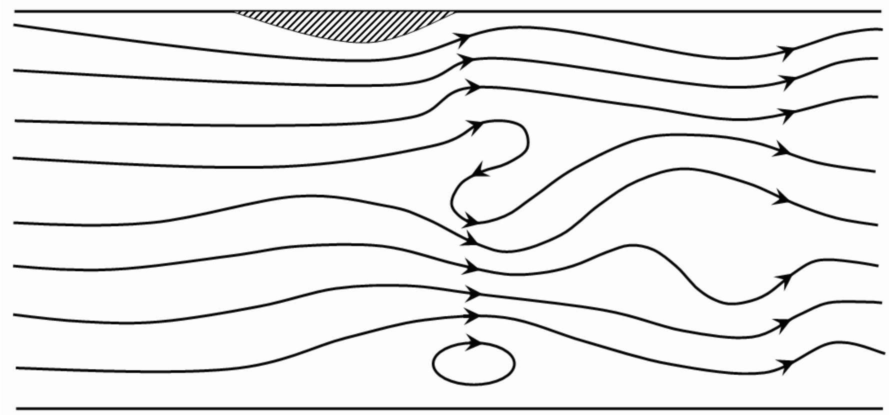
Osborne Reynolds (1842–1912)

when he observed the behaviour of water in a cylindrical pipe driven by a pressure-gradient and tried to find explanation of the strange dynamics of the fluid.

For small velocities the flows had laminar character—all lines of flow are parallel with the axis of the cylinder and all physical characteristics of the flow (pressure, velocity) depend only on the distance from axis.

Reynolds found that this type of flow prevailed, in every arrangement that he investigated, for velocities below a certain "critical" velocity (which depends on the conditions of the experiments).

When the critical velocity was exceeded, the flow became "turbulent", the stream-lines of it showed a highly irregular and rapidly changing pattern.



Beside of discovering of the described phenomena he also measured the relationship between

the *geometrical characteristics* of the pipe,

on the one hand, and

the *physical characteristics*

on the other. These latter being the pressure-gradient required to maintain the flow, the velocity of the resulting flow v and the (cinematic) viscosity coefficient ν.

He found that for **"laminar"** flow the pressure-gradient is proportional to v, as well as to ν.

In the **"turbulent"** domain he was led to an entirely different law of dependence for the pressure-gradient.

Indeed, in this domain the v-dependence is more like one of proportionality to v^2 rather than v, while the dependence on ν is less well developed.

He also found that

(a) v, ν, and the essential geometrical characteristics of the pipe (its diameter L) could be varied within very wide limits, and

(b) the only **combination** of these quantities which mattered in determining whether the "critical" point has been reached or not, *i. e.,* whether a "laminar" or a "turbulent" regime prevails, is the combination

$$R = \frac{Lv}{\nu}.$$

Now we call R **Reynolds number**.

By the way, emergence of the Reynolds number at this point can be justified by a simple **dimensional** consideration:

The dimensions of L, v, and ν are

$$[L] = \text{cm}, \qquad [v] = \frac{\text{cm}}{\text{sec}}, \qquad [\nu] = \frac{\text{cm}^2}{\text{sec}},$$

respectively. The number which determines whether a "laminar or a "turbulent" regime exists must obviously be **dimensionless**. The only dimensionless combination which can be formed out of L, v, and ν as indicated above is Lv/ν.

Reynolds found, furthermore, that the critical value of R depended essentially on how undisturbed the fluid was while the experiment was been conducted.

More specifically, in the experiment in which the fluid was least disturbed, where he merely injected some colored stream-lines the critical *R proved to be about 12,000*.

When the liquid was more seriously disturbed, namely, when the pressure measurements referred to above were effected, the critical *R was in the neighborhood of 2,000*.

2. The circumstances described above made it very plausible that turbulence is a phenomenon of *instability*. In the case of the cylindrical laminar flow pattern represents a *solution* of the Navier–Stokes equations.

But this laminar solution disappears for Reynolds' numbers in excess of some critical limit between 1,000 and 100,000. It is only reasonable to infer from this that the laminar flow, while still a solution ceases to be a stable one, or at least the most stable one.

It is plausible to conclude that the turbulent flow represents one or more solutions of a higher stability.

It is , by the way, obvious that one has to talk not of one turbulent solution, but of *many* turbulent solutions.

There is probably no such thing as a most favored or most relevant, turbulent solution. Instead, the turbulent solutions represent an *ensemble* of statistical properties, which they all share, and which alone constitute the essential and physically reproducible traits.

The Navier–Stokes equations describe evolution of vector field of the velocities in infinite-dimensional space. These equations are nonlinear that is a main reason for all mathematical difficulties and very complicated physical structure of the solutions and nature of the flows of fluids.

If viscosity ν is large (R is small) then vector field in infinite-dimensional space has an attracting point (**“attractor”**) that is a stationary solution which is called **“laminar” flow**. This motion is stable in the following sense:

> under small perturbation of the intial vector field of the velocities hydrodynamical evolution will automatically return flow to the nondisturbed laminar flow.

When R increases we will observe that "attracting point" (in infinite-dimensional space) loses stability.

We begin to observe periodical fluctuations of the flow.

But again this motion will be stable in the sense that influence of small perturbations will be damped with time.

So, we see that with growing of Reynolds a laminar flow (zero-dimensional "attractor") turns to a periodic motion (one-dimensional "attractor"). In both case we have stability.

With increase of R the periodic motion disappears. The "attractor" of larger dimensions will appear later on with growing of R and motions along attractors become less and less stable.

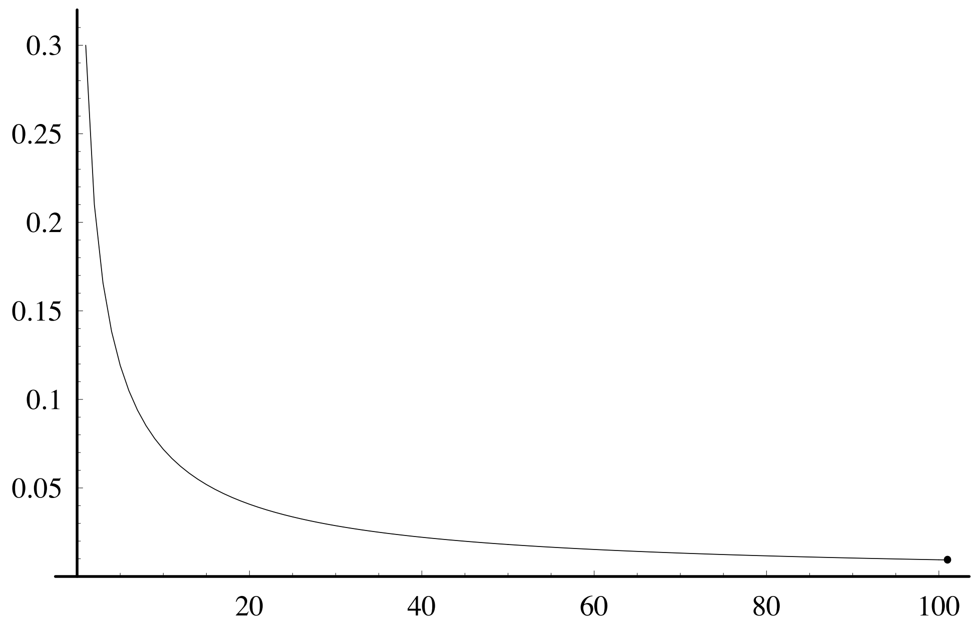
3. For the clarification of the described phenomena let us consider the following, very often cited example of the "discrete version of the Navier–Stokes system".

The question is the nonlinear dynamical system

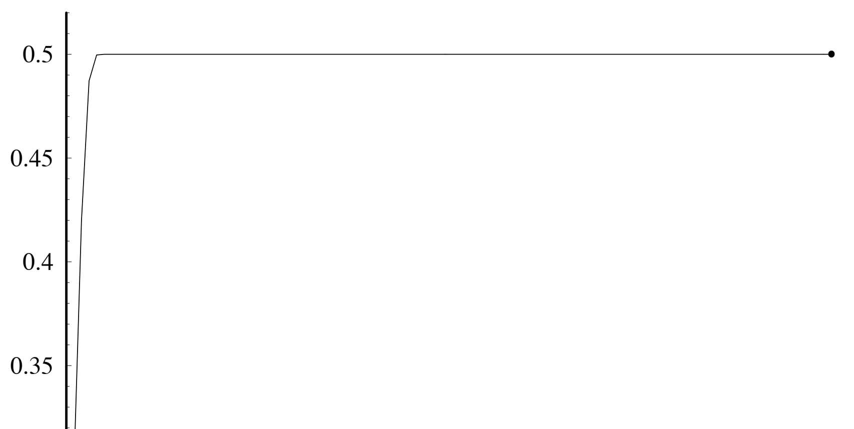
$$x_n = \lambda x_{n-1}(1 - x_{n-1}), \qquad n \geqslant 1, \quad x_0 \in (0,1). \qquad (*)$$

(Apparently, so-called *logistic* equation $(*)$ occurred first in the models of population dynamics that imposed constraints on the growth of a population.)

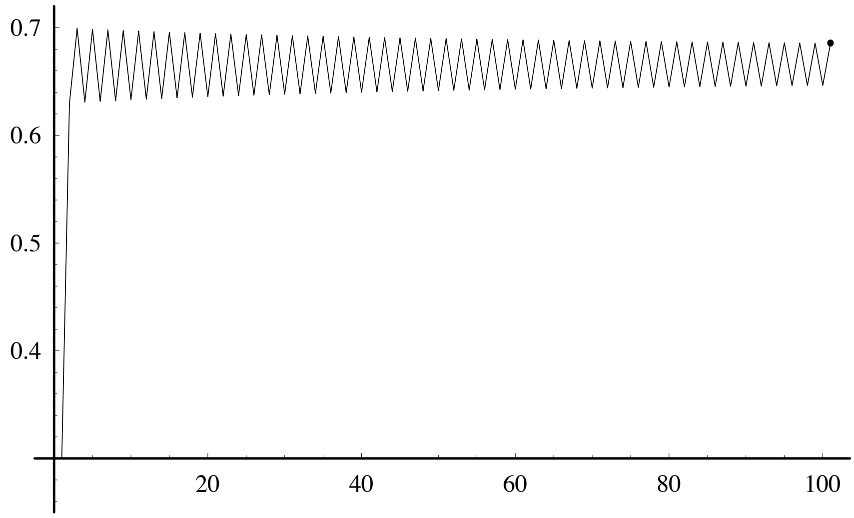
For $\lambda \leqslant 1$ the solutions $X_n = X_n(\lambda)$ converge monotonically to 0 as $n \to \infty$ for *all* $0 < x_0 < 1$. Thus, the stable state $x_\infty = 0$ is the *unique* *stable* state in this case, and it is the limit point of the x_n's as $n \to \infty$.



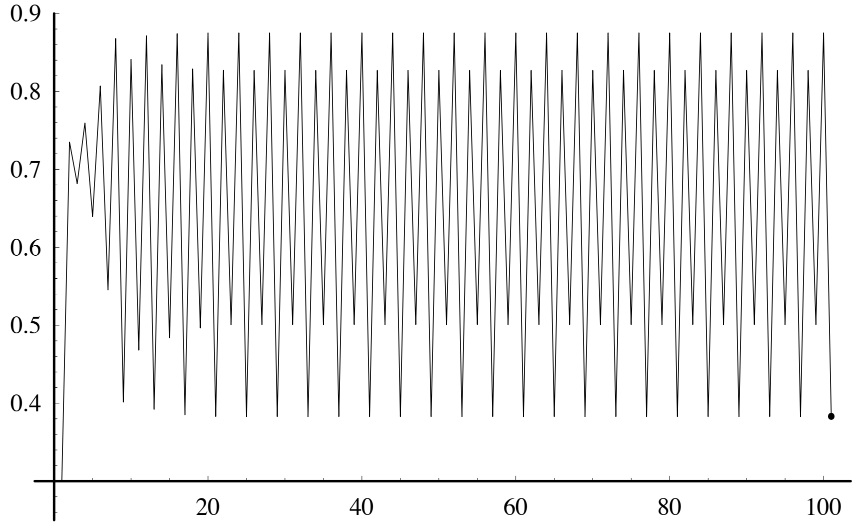
For $\lambda = 2$ we have $x_n \uparrow 1/2$. Hence there also exists in this case a unique stable state $x_n = 1/2$ attracting the x_n's as $n \to \infty$.



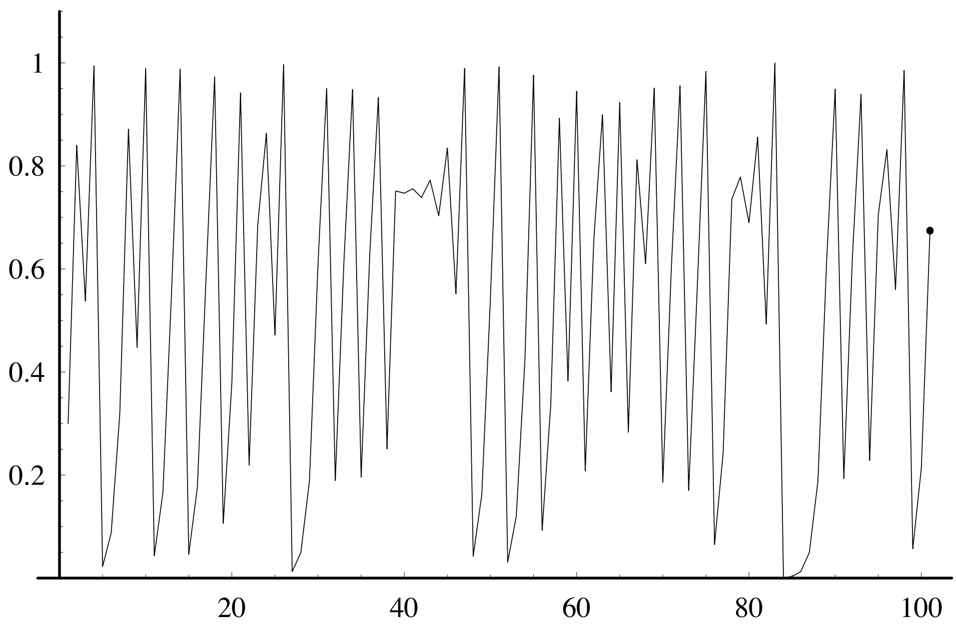
We now consider larger values of λ. For $\lambda < 3$ the system $(*)$ still has a unique stable state. However, an entirely new phenomenon occurs for $\lambda = 3$: as n grows, one can distinguish two states x_∞ and the system alternates between these states.

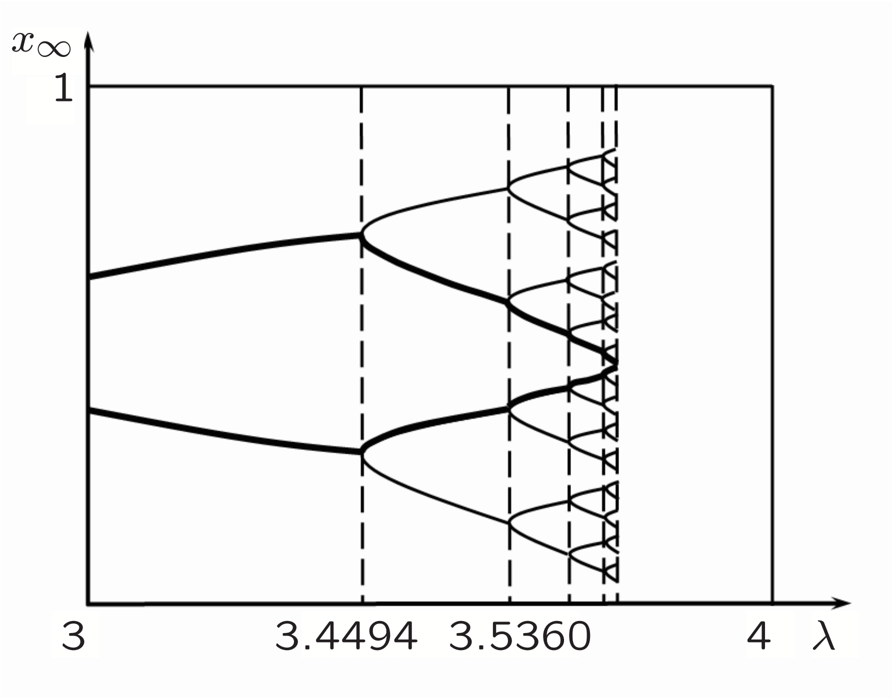


This pattern is retained as λ increases, until something new happens for $\lambda = 3.4494\ldots$: the system has now *four* different states x_∞ and leaps from one to another.



New distinguished states come into being with further increases in λ: there are 8 such states for $\lambda = 3.5360$, ..., 16 for $\lambda = 3.5644$, and so on. For $\lambda = 3.6$ there exists infinitely many such states, which is usually interpreted as a *loss of stability* and a transition into a chaotic state.





Now the periodic character of the movements between different states is completely lost; the system wanders over an infinite set of states jumping from one to another.

It should be pointed out that although our system is deterministic, it is impossible in practice to predict its position at some later time because

> *the limited precision in our knowledge about the values of the x_n's and λ can considerably influence the results.*

It is clear from this brief description already that the values λ_k of λ, at which the system "branches", "bifurcates", draw closer together in the process.

As conjectured by **M. Feigenbaum** and proved by **O. Lanford**,

for all parabolic systems,

$$\boxed{\frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} \to F, \qquad k \to \infty,}$$

where $F = 4.669201\ldots$ is a *universal* constant (Feigenbaum's constant).

The value $\lambda = 4$ is of particular importance for the dynamical system

$$x_n = \lambda x_{n-1}(1 - x_{n-1}), \qquad n \geqslant 1, \quad x_0 \in (0,1). \qquad (*)$$

It is for this value of the parameter that

the sequence of observations (x_n)
of our (**chaotic**) system

is similar to

a realization of a **stochastic** sequence of "white nose" type.

Indeed, let $x_0 = 0.1$. We calculate recursively the values of $x_1, x_2, \ldots, x_{1000}$ using $(*)$. The

(empirical) mean value and **the standard deviation**

evaluated on the basis of these 1000 numbers are

0.48887 and **0.35742**

respectively (up to the 5th digit).

The values of the (empirical) correlation function $\widehat{\rho}(k)$ calculated from $x_1, x_2, \ldots, x_{1000}$ show that in practice

for $\lambda = 4$ the values x_n can be assumed to be uncorrelated.

In this sense, (x_n) can be called **"chaotic white noise"**.

It is worth noting that the system

$$x_n = 4x_{n-1}(1 - x_{n-1}), \qquad n \geqslant 1, \quad x_0 \in (0,1),$$

has an **invariant distribution** P $\left[\text{i. e. } \mathrm{P}(T^{-1}A) = \mathrm{P}(A) \text{ for each Borel set } A \text{ of } (0,1) \text{ with } x \rightsquigarrow Tx = 4x(1-x)\right]$ **with density**

$$p(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \qquad x \in (0,1).$$

In this case

$$\boxed{\mathrm{E}x_0 = \tfrac{1}{2}, \quad \mathrm{E}x_0^2 = \tfrac{3}{8}, \quad \mathrm{D}x_0 = \tfrac{1}{8} = (0.35355)^2}$$

(cf. 0.48887 and 0.35742 above) and the correlation function

$$\boxed{\rho(k) \overset{\text{def}}{=} \frac{\mathrm{E}x_0 x_k - \mathrm{E}x_0 \mathrm{E}x_k}{\sqrt{\mathrm{D}x_0 \mathrm{D}x_k}} = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}}$$

4. Presented descriptions explain that, probably, to get an adequate model for the movement of the fluid we should use a statistical approach to the Navier–Stokes equations (with random initial velocities, random forces, etc.).

O. Reynolds had realized the need of the **stability theory** and the **statistical approach** to the turbulence.

We shall see that he had actually laid the foundations for the statistical theory. He used the decomposition of the velocity into two components:

slowly changing "mean value" + **the fluctuation part**.

Nevertheless, the early history of turbulence history evolves primary around **stability theories**.

These ones have actually dealt in the main with the following

4 CASES:

Couette flow	**Poiseuille flow**
Cylindrical Couette flow	**Boundary-layer flow**

(1) **Couette flow** which takes place between two parallel plates; the motion of the liquid is induced by these plates being in relative motion w.r.t. each other.

Laminal Couette flow is stable for all values of ν but this stability for small ν is very weak.

See details in only recently published paper by Kolmogorov about 2D turbulence:

> *Mathematical models of the turbulent motion of the incompressible viscous fluid*

(Russian Math. Surv. **59**:1 (2004), 5–10)].

(2) **Poiseuille flow**. This flow also takes place between two parallel plates. Here, however, the plates are fixed and the flow is driven by a pressure gradient.

This flow may thus be viewed as that in a two-dimensional pipe.

Although for some cases the motion is stable, the complete picture of stability for all Reynolds numbers, to our knowledge, is not clear.

(3) **Cylindrical Couette flow**. This flow also takes place between two concentric cylinders which rotate with different speeds.

G. I. Taylor (1922) has shown that, depending on the ratio of the radii and the values of the velocities, various stability and lability conditions may exist.

(4) **Boundary-layer flow**. This flow also takes place in the narrow layer close to a fixed wall in an otherwise uniformly flowing, low-viscosity liquid.

The mathematical concept of the boundary layer was developed by **L. Prandtle**.

The stability of the laminar flow in the boundary layer was exhaustively investigated by **W. Tollmien** (1929). He found that

the laminar boundary layer is stable for a certain distance (downstream) and then develops instability.

5. From the point of view of the mathematical approach to the Navier–Stokes equation, the work of **J. Leray** in the 1930s is very important. He considered the questions of existence and uniqueness for the solutions of the Navier–Stokes equations for any $\nu > 0$.

2D case: he demonstrated *both existence and uniqueness* (for some plausible boundary conditions).

3D case: he was able to prove the *uniqueness* of the solution provided that one existed, taking the stricter view of what constitutes a solution;

taking, alternatively, the wider view, he was able to demonstrate the *existence* of a solution, but not its uniqueness.

We don't go to the details. Only we want to say that the machinery of J. Leray uses different notions of the differentiability—everywhere, in a certain average sense only, *etc.*

In the case of wider view—where he proved existence of the solution—he gave also some heuristic reasons to expect that uniqueness might actually fail.

More details about recent approaches to, results on, and developments of the Navier–Stokes equation see in the paper by **E. C. Waymire**

Probability and incompressible Navier–Stokes equations: an overview of some recent developments

(Probab. Surv. **2** (2005), 1–32).

We conclude description of the classical approaches to the Navier–Stokes equation with the note that recently **Ya. G. Sinai** (using the Fourier transform method) got some interesting (local) results about the 3-dimensional Navier–Stokes equations. See his paper

On local and global existence and uniqueness of the 3d Navier–Stokes system on $\mathbb{R}^3$

(Perspectives in Analysis. Conference in honor of Lennart Carleson's 75th birthday, Springer).

Ya. G. Sinai wrote the Navier–Stokes system for four unknown functions

$$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \quad \text{and} \quad p(x,t),$$

where $x = (x_1, x_2, x_3)$, in the form

$$\boxed{\frac{\partial u(x,t)}{\partial t} + (u(x,t), \nabla)\, u(x,t) = \nu\, \Delta u(x,t) + \nabla p(x,t)}$$

(assuming the absence of an external force) with the *condition of the incompressibility*

$$\operatorname{div} u \stackrel{\text{def}}{=} \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0.$$

§ 4. Statistical approaches to the hydrodynamic turbulence

1. Already the founder of all turbulence O. Reynolds had realized that

- the study of the individual irregularly fluctuated hydrodynamic fields of the turbulent flows is impracticable and
- this study should use **statistical** description based on some smoothed characteristics, changing more smoothly and more regularly.

Keep in mind that ideas he proposed were to consider

the field of velocities $u = u(x,t)$ as a statistical object

($u(x,t) = u(x,t;\omega)$ using common probabilistic notation) splitting it into two components:

$$\boxed{u = \overline{u} + u'},$$

where $\overline{u}$ is a **mean velocity** and
u' is a **fluctuation (pulsating) velocity**.

O. Reynolds has studied mainly only *smooth* component $\overline{u}$, where he understood averaging as

- **averaging on a big interval** T:

$$\overline{u}(t,x) = \frac{1}{T}\int_{t-T/2}^{t+T/2} u(s,x)\,ds.$$

A. N. Kolmogorov, in his talk on the Conference of the Institute of Mechanics of the Moscow State University in December 1936, proposed to understand averaging $\overline{u}(t,x)$ as (usual now)

- **mathematical expectation** $\mathsf{E}u(t,x;\omega)$.

Write four Navier–Stokes equations (assuming $f \equiv 0$) in the form

$$\nu\,\Delta u_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = 0, \qquad i = 1,2,3,$$

$$\operatorname{div} u = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} = 0,$$

where $\nu = \frac{\mu}{p}$ is the "cinematic" (molecular) viscosity coefficient,
μ is a viscosity,
ρ is the density.

Then for any point $M = (x, y, z, t)$ we obtain for **ten** characteristics

$$\overline{u}_1, \quad \overline{u}_2, \quad \overline{u}_3, \quad \overline{p}, \qquad \text{and}$$
$$b_{ij}(M) = \mathsf{E}(u_i(M) - \overline{u}_i(M))(u_j(M) - \overline{u}_j(M)), \quad i, j = 1, 2, 3,$$

only **four** equations

$$\begin{cases} \nu\, \Delta \overline{u}_i - \dfrac{1}{\rho} \dfrac{\partial \overline{u}}{\partial x_i} - \dfrac{\partial \overline{u}_i}{\partial t} - \displaystyle\sum_{j=1}^{3} \overline{u}_j \dfrac{\partial \overline{u}_i}{\partial x_j} - \displaystyle\sum_{j=1}^{3} \dfrac{\partial \overline{b}_{ij}}{\partial x_j} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, 3, \\ \operatorname{div} u = \displaystyle\sum_{j=1}^{3} \dfrac{\partial u_i}{\partial x_j} = 0, \end{cases} \qquad (*)$$

How to close this system of 4 equations for 10 unknown functions **?**

Russian scientists

A. A. Friedmann and **L. Keller**

in their talk on the

First International Congress for Applied Mechanics
in Delft (Holland) in 1925

proposed to close systems of the above type an idea of attracting **additional** characteristics of the fluctuations of velocities and pressures in two points of the flow of the fluid.

Let

$$M' = (x_1', x_2', x_3'; t'), \quad M'' = (x_1'', x_2'', x_3''; t''), \quad M''' = (x_1''', x_2''', x_3'''; t''')$$

be three points and

$$B_{\alpha\beta}(M', M'') = \mathsf{E}[u_\alpha(M') - \overline{u}_\alpha(M')][u_\beta(M'') - \overline{u}_\beta(M'')],$$

$$B_{\alpha\beta\gamma}(M', M'', M''') = \mathsf{E}[u_\alpha(M') - \overline{u}_\alpha(M')][u_\beta(M'') - \overline{u}_\beta(M'')] \times [u_\gamma(M''') - \overline{u}_\gamma(M''')].$$

Then with the neglect of these latter moments $B_{\alpha\beta\gamma}$ of order three **Kolmogorov (1936)** and **Millionshchikov (1938)** obtained (additionally to the previous 4 equations) the new equations (for point M').

In case when $\overline{u}_i = 0$, $i = 1, 2, 3$, the systems for $B_{ik}(M', M'')$ have the following form:

$$\begin{cases} \nu\, \Delta' B_{ik}(M', M'') - \dfrac{1}{\rho} \dfrac{\partial B_{ik}(M', M'')}{\partial x_i'} - \dfrac{\partial B_{ik}(M', M'')}{\partial t'} = 0, \\ \qquad \displaystyle\sum_{j=1}^{3} \dfrac{\partial B_{ik}(M', M'')}{\partial x_j'} = 0; \\ \nu\, \Delta'' B_{ik}(M'', M') - \dfrac{1}{\rho} \dfrac{\partial B_{ik}(M'', M')}{\partial x_i''} - \dfrac{\partial B_{ik}(M'', M')}{\partial t''} = 0, \\ \qquad \displaystyle\sum_{j=1}^{3} \dfrac{\partial B_{ik}(M'', M')}{\partial x_j''} = 0. \end{cases}$$

We will return later to the problem of solving these equations—it is important for getting the results about rate of **degeneracy** of the turbulence (Kármán, Howarth, Millionshchikov, et al.)

Now we want to consider some general aspects about a statistical solution of the Navier–Stokes equation.

Essentially, here there are the following two models:

IMPLICIT statistical models	and	**EXPLICIT statistical models**

Papers related with the models of the first type (implicit statistical models):

Reynolds (1880)
Friedmann and Keller (1924)
Taylor (1935)
Kármán and Howarth (1938)
Millionshchikov (1938)
Hopf (1952)
Foias and Temam (1970, 1980)
Vishik and Fursikov (1980), . . .

The general statistical problem for the Navier–Stokes equation can be formulated as follows.

Suppose that

$$\Omega = \{\omega : \omega = u(x,t),\ x \in \mathbb{R}^3,\ t \geq 0\}$$

is a space of all possible solutions to the Navier–Stokes equation.

A **probability-statistical solution** of the Navier–Stokes equation is a probability measure P whose restriction onto the velocities at time $t = 0$ coincides with *a priori* given measure P_0.

The **main difficulty** for construction of such measure P:

> there is no operator S_t that transfer initial values $u(x,0)$ to solutions $u(x,t)$ of the Navier–Stokes equation at any time t.

One way to avoid this difficulty consists in

construction of Galerkin's approximations

for which we have the corresponding operators $S_t^{(n)}$ that give a possibility to construct step-by-step the corresponding measures $\mathsf{P}_t^{(n)}$ and measures $\mathsf{P}^{(n)}$ on $\mathbb{R}^3 \times \mathbb{R}_+$.

Using some energy estimates it is possible to show that the family $\{P^{(n)}\}$ is tight and so by Prokhorov's theorem this family is relatively compact:

$$\boxed{\text{the family } \{P^{(n)}\} \text{ is tight}} \qquad (energy\ \ estimates)$$

$$\Downarrow \qquad (Prokhorov's\ \ theorem)$$

$$\boxed{\text{the family } \{P^{(n)}\} \text{ is relatively compact}}$$

Any limit point of the family $\{P^{(n)}\}$ is a **statistical solution**. (**Uniqueness** is still an **open problem**.)

Very last results about

> homogeneous and isotropic probability measures supported by weak solution of the Navier–Stokes system

were obtained in the preprint (dated November 13, 2005) by **S. Dostoglou**, **A. V. Fursikov**, and **J. D. Kahl**

> *Homogeneous and isotropic solutions of the Navier–Stokes equation.*

Important characteristics of the statistical solution $\mathsf{P}(d\omega)$ are moments

$$m_k = m_k(t, x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_k) = \mathsf{E} u^{\lambda_1}(t, x_1) \cdots u^{\lambda_k}(t, x_k),$$

where $x_i \in \mathbb{R}^3$ and $\lambda_i = 1, 2 \ldots$.

Above-mentioned infinite chains of the Friedmann–Keller equation can be written in the following short form:

$$\frac{\partial m_k}{\partial t} + A_k m_k = B_k m_{k+1}, \qquad (*)$$

where

A_k is a second-order differential operator on $\Omega^k = \Omega \times \cdots \times \Omega$,

B_k is an operator which transfers function m_{k+1} given on Ω^{k+1} to a function given on Ω^k.

Emphasize again that it is clear from $(*)$ that

equation for moments m_k of order k contains moments m_{k+1} of order $k+1$.

(This is a consequence of the presence of the quadratic nonlinear terms in the Navier–Stokes equation.)

Let us mention also results of **H. Hopf (1952)** who gave an equation in variational derivatives for the characteristic functionals of the measures P_t supported by the solutions of the Navier–Stokes equations.

Very important contributions in the turbulence theories have been done in 1935 **by G. I. Taylor** in the series of papers

Statistical theory of turbulence. I–IV.

In his theory, as well as in most of the subsequent statistical turbulence theories, the turbulent phenomena under consideration are assumed to be

- (statistically) **homogeneous** and **isotropic** in appropriately delimited, relevant parts of space and
- (statistically) **stationary** in time.

The **homogeneity assumption** means that, *e.g.*, second moments $B_{ij}(M^{(1)}, M^{(2)})$ depend only on distance between point $M^{(1)}$ and $M^{(2)}$. This assumption requires, of course, that the flow be not considered too close to the boundaries.

The **isotropy assumption** means that these characteristics are invariant with respect

- to the rotation of the system of coordinates around its axes and
- to the reflection of the coordinate system in the planes pass through $(0, 0, 0)$-point.

Of course, in the general setting the corresponding notion are formulated not only for second moments $B_{ij}(M^{(1)}, M^{(2)})$ but for all n-dimensional laws F_n for velocities in n point $M^{(1)}, \dots, M^{(n)}$.

Let us illustrate how these concepts permit to obtain results concerning laws of **decay** of the turbulent perturbation (**Kármán**, **Howarth**, **Millionshchikov**).

Under assumptions that one may neglect third moments (it is reasonable at the last steps of decay of turbulence then only slow and big vortices are preserved) we have already given above the system

$$\begin{cases} \nu\,\Delta' B_{ik}(M', M'') - \dfrac{1}{\rho}\dfrac{\partial B_{ik}(M', M'')}{\partial x'_i} - \dfrac{\partial B_{ik}(M', M'')}{\partial t'} = 0, \\ \qquad \displaystyle\sum_{j=1}^{3} \dfrac{\partial B_{ik}(M', M'')}{\partial x'_j} = 0; \\ \nu\,\Delta'' B_{ik}(M'', M') - \dfrac{1}{\rho}\dfrac{\partial B_{ik}(M'', M')}{\partial x''_i} - \dfrac{\partial B_{ik}(M'', M')}{\partial t''} = 0, \\ \qquad \displaystyle\sum_{j=1}^{3} \dfrac{\partial B_{ik}(M'', M')}{\partial x''_j} = 0. \end{cases}$$

From isotropy assumption it follows that

longitudinal (along the coordinate axis x_1)

and

transversal (along the coordinate axis x_2)

covariance functions $B_{dd}(r, t', t'')$ and $B_{nn}(r, t', t'')$ ($B_{dd} = B_{11}$, $B_{nn} = B_{22} = B_{33}$) for small values of density ρ are given by

$$B_{dd}(r, t', t'') = \frac{k}{[2\nu(t' + t'')]^{5/2}} \exp\left\{ -\frac{r^2}{4\nu(t' + t'')} \right\},$$

$$B_{nn}(r, t', t'') = B_{dd}(r, t', t'')\left(1 - \frac{r^2}{4\nu(t' + t'')}\right).$$

In particular, for the coefficients of correlations $R_{dd}(r, t, t)$ and $R_{nn}(r, t, t)$ of the longitudinal and transversal components of velocities in two points disposed at the distance r, we have:

$$R_{dd}(r, t, t) = \exp\left\{ -\frac{r^2}{8\nu t} \right\},$$

$$R_{nn}(r, t, t) = \left(1 - \frac{r^2}{8\nu t}\right) \exp\left\{ -\frac{r^2}{8\nu t} \right\}.$$

It is interesting that

$R_{dd} > 0$ **for all** r, but
$R_{nn} > 0$ **only for** $r^2 < 8\nu t$.

(The experiments of **H. L. Dryden**'s group confirms this behaviour of R_{dd} and R_{nn}.)

These results have been obtained a long time ago, in the 1930s. Of course, since that time many new results about the behaviour of decay were obtained. We bring these (old) results to demonstrate

- the beginning of the statistical theories of turbulence,
- character of the practically interesting problems (type of decay), and
- used methods (neglecting the third moments; Kármán and Howarth had tried also to use third moments neglecting the fourth moments and Millionshchikov later in 1941 neglected third moments but for the fourth moments he used Gaussian approximation, *i.e.*, assumed that semiinvariants of order 4 are equal to zero as for Gaussian case).

§ 5. Kolmogorov's theory of the local isotropic turbulence

1. Before 1941 when Kolmogorov published his paper on turbulence, nobody suspected that chaotic turbulent pulsations were subjected to the simple quantitative regularities for very well developed turbulence, *i.e.*, for the case where Reynolds number is much larger than the critical threshold which characterizes transfer of laminar flow to the turbulent motion.

The key idea of Kolmogorov was to introduce the notions of the

locally homogeneous and **locally isotropic**

turbulence.

In connection with Taylor's definition of the isotropic turbulence Kolmogorov wrote that his definition is **narrower** in the sense that distributions of the increments of velocities $u_i(M) - u_i(M^\circ)$ are stationary in time but at the same time his definition is **wider** because restrictions lay only on distributions of the difference of velocities but not on velocities themselves.

The main characteristics of Kolmogorov's theory are longitudinal and transversal values

$$B_{dd}(r) = \overline{(u_d(M') - u_d(M))^2},$$
$$B_{nn}(r) = \overline{(u_n(M') - u_n(M))^2},$$

where

- r is distance between points M and M',
- $u_d(M)$ and $u_d(M')$ are components of velocities at points M and M' in the direction $\overline{MM'}$,
- $u_n(M)$ and $u_n(M')$ are components of velocities at points M and M' in a direction perpendicular to MM'-direction.

The basic result of the first Kolmogorov's paper *The local structure of turbulence in an incompressible field at very high Reynolds number* (1941) is so-called law of two-thirds that is, essentially, a fundamental law of the nature of the turbulence.

it is interesting to note that this paper is written in purely physical style and it is a magnificent piece of applied mathematical analysis.

Kolmogorov has considered turbulent velocities as a random fields for probability-statistical behavior to which he formulated two physically natural hypotheses. (He said later that it is "difficult to prove them but they are correct (!)".)

For reasoning of these two hypotheses about behavior of turbulence Kolmogorov first of all remarks (in the implicit form) that Navier–Stokes equations for the field of the velocities has a certain property of *invariancy*.

More exactly, for these equations there is invariance under the following transformations:

spatial translations: $x \to x + r$,

spatial rotations: $(x, v) \to (Ax, Av)$ with $A \in SO(3)$,

scale transforms: $(t, x, v) \to (\lambda^{1-h} t, \lambda x, \lambda^h v)$ with $h > 0$, $\lambda > 0$.

Taking into account these properties and assuming that there exists a probability measure on the functional space of the velocities, Kolmogorov postulated that this measure is such that the following laws of similarities hold (we use Kolmogorov's handwritten notes dated 28 April 1943; in his 1941 paper they were formulated in a little bit different form).

Let $(V_0, V_1, \dots, V_n)$ be the vector of the velocities at points $M_0, M_1, \dots, M_n$; the number $n \geq 1$ is arbitrary.

Denote

$$a^2 = |V_1 - V_0|^2 + \cdots + |V_n - V_0|^2,$$
$$q^2 = \rho^2(M_1, M_0) + \cdots + \rho^2(M_n, M_0),$$

where ρ is the distance between points.

The FIRST distributional law of similarity:

For the small size of the groups of points $M_0, M_1, \dots, M_n$, the distributional law $F_R^{(n)}$ of the values

$$\frac{V_1 - V_0}{a}, \quad \frac{V_2 - V_0}{a}, \quad \dots, \quad \frac{V_n - V_0}{a}$$

for given Reynolds' number $R = qa/\nu$ is invariant with w.r.t. the transformations listed above.

The SECOND distributional law of similarity:

The laws $F_R^{(n)}$, as R increases, converges to a law $F_\infty^{(n)}$ which depends only on the "form" of the group of points $M_0, M_1, \dots, M_n$.

Kolmogorov emphasizes that his

> "...hypotheses concerning the local structure of turbulence at high Reynolds number...were based physically on **Richardson's idea** of the existence in the turbulent flow
>
> - of vortices on all possible scale $l < r < L$ between the 'external scale' L and the 'internal scale' l and
> - of a certain *uniform* mechanism of energy transfer from the coarser-scaled vortices to the finer..."

Big whirls have little whirls
That heed on their velocity,
And little whirls have littler whirls
And so on to viscosity.

L. F. Richardson

Quite soon after Kolmogorov's hypotheses were originated,

> "**D. Landau** noticed that they did take into account a circumstance which arises directly from the assumption of the essentially accidental and random character of the mechanism of energy transfer from the coarser vortices to the finer:
>
> *with increase of the ratio $L : l$ the variation of the dissipation of energy*
>
> $$\mathcal{E} = \frac{1}{2}\nu \sum_{\alpha} \sum_{\beta} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2$$
>
> *should increase without limit.*"

This remark led Kolmogorov to introduce the third hypothesis on the behavior of the velocity increments, from which he concludes that, for large L/l, the dispersion $\sigma^2_{\log \mathcal{E}}$ has the following form:

$$\boxed{\sigma^2_{\log \mathcal{E}} \sim A + K \log \frac{L}{l}}.$$

From his two first hypotheses Kolmogorov concludes that the quantities

$$B_{dd}(r) = \Big[V_1(x_1 + r, x_2, x_3) - V_1(x_1, x_2, x_3)\Big]^2$$

(**longitudinal** characteristic),

$$B_{nn}(r) = \Big[V_2(x_1 + r, x_2, x_3) - V_2(x_1, x_2, x_3)\Big]^2$$

(**transversal** characteristic)

must be functions of r and average energy $\overline{\mathcal{E}}$ of dissipation of the fluid per unit mass and unit time, only.

It is remarkable that this character of dependence of $B(r)$ ($= B_{dd}(r)$, $B_{nn}(r)$) can be derived from the

dimensional considerations

alone.

Indeed, the quantities $B(r)$, $\overline{\mathcal{E}}$, and r have the following dimensions:

$$[B(r)] = \frac{\text{cm}^2}{\text{sec}^2}, \qquad [\overline{\mathcal{E}}] = \frac{\text{cm}^2}{\text{sec}^3}, \qquad [r] = \text{cm}.$$

Hence, there exists one and only one combination of $\overline{\mathcal{E}}$ and r which agrees dimensionally with $B(r)$:

$$\boxed{B(r) = c\,\overline{\mathcal{E}}^{2/3} r^{2/3}},$$

where c is a dimensionless absolute constant.

The longitudinal and transversal characteristics $B_{dd}(r)$ and $B_{nn}(r)$ are measured repeatedly in sea and in atmosphere.

It is remarkable that in the experiments both functions $B_{dd}(r)$ and $B_{nn}(r)$ were proportional to $r^{2/3}$ on the significant interval of the values r.

There exists also **spectral analogue** of the "2/3-law". It is so-called **5/3-law**, which says that, e. g.,

the spectral measure $\Phi_{dd}(k) = E_{dd}(k)\,dk$ *is such that*

$$\boxed{E_{dd}(k) \sim k^{-5/3}}.$$

[This follows from: Let $B(r) = \int_{-\infty}^{\infty}(1 - e^{ikr})E(k)\,dk$ be a spectral representation of $B(r)$; then for $E(k) = |k|^{-\alpha}$

$$B(r) = cr^{\alpha-1} \quad \text{with} \quad c = \int_{\infty}^{\infty}(1 - e^{ik})|k|^{-\alpha}\,dk.$$

Thus, if $B(r) \sim r^{-2/3}$, then $\alpha - 1 = 2/3$ and $\alpha = 5/3$, i. e., $E(k) = |k|^{-5/3}$.]

We conclude the description of Kolmogorov's 1941 results in the theory of the local isotropic turbulence by two remarks:

(a) the similar results were later obtained by **L. Onsager** (1945), **C. F. von Weizsäcker** (1948), and **W. Heisenberg** (1948);

(b) Kolmogorov, in his 1962 paper

A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number

obtained a new formula for $B_{dd}(r)$:

$$\boxed{B_{dd}(r) \sim \overline{\mathcal{E}}^{2/3} r^{2/3} \left(\frac{L}{r}\right)^{-k}},$$

where k is a constant.

In Comments (1985) to his work in the theory of turbulence, A. N. Kolmogorov wrote that it was clear for him (already in the end of 1930s) that

> the **main mathematical tool** for investigation of turbulence should be the theory of random functions of many parameters (time, coordinates, *etc.*), *i.e.*, the **random field**,

(at that time this theory was only at its beginning).

In 1940 Kolmogorov published two papers (*Curves in a Hilbert space that are invariant under the one-parameter group of motions* and *Wienersche Spiralen und einige andere interessante Kurven in Hilbertschen Raum*) that are important for both the turbulence and the general theory of stochastic processes and fields.

It is reasonable to recall now some definitions and facts concerning **homogeneous random processes** and **fields**.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $V(s) = V(s, \omega)$, $\omega \in \Omega$, a family of (complex-valued) random variables, where $s \in S$ and S is a homogeneous space of points with a transitive group $G = \{g\}$ of transformations, mapping the space S into itself ($S \to gS = S$).

The **random field** $V = V(s)$, $s \in S$, is called **homogeneous** (in a wide sense) if

$$\mathsf{E}|V(s)|^2 < \infty, \quad \mathsf{E}V(s) = \mathsf{E}V(gs), \quad \mathsf{E}V(s)\overline{V(t)} = \mathsf{E}V(gs)\overline{V(gt)}$$

for all $s, t \in S$ and $g \in G$.

IMPORTANT CASES:

$S = \mathbb{R}^k$ and G is the group of parallel shifts;

$S = \mathbb{R}^k$ and G is the group of isometric transformations on S generated by the parallel shifts, rotations and reflections (V is thus called the homogeneous **isotropic** random field).

The special case of the homogeneous fields is a (wide-sense) stationary process $V = V(s)$, $s \in \mathbb{R}$, where

$$\mathsf{E}|V(s)|^2 < \infty, \qquad \mathsf{E}V(s) = \text{const},$$

$$\mathsf{E}V(s)\overline{V(t)} \quad \text{depends only on the difference } t - s.$$

Assume $m = 0$ and denote by $R(t) = \mathsf{E}V(s+t)\overline{V(s)}$ the correlation function of the process V.

For the **mean-square continuous** process V this function $R(t)$ admits the spectral representation (**BOCHNER–KHINCHIN**)

$$\boxed{R(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, F(d\lambda)},$$

where the **spectral measure** $F = F(A)$ is a finite measure on the Borel sets $A \in \mathcal{B}(\mathbb{R})$.

For **real-valued** process X we have similar representation:

$$\boxed{R(t) = \int_{0}^{\infty} \cos \lambda t\; G(d\lambda)}.$$

IMPORTANT:

$$\mathsf{E}|V(t) - V(s)|^2 = 2\int_{-\infty}^{\infty} (1 - e^{i\lambda(t-s)})\, F(d\lambda)$$

and for a **real-valued** process V

$$\mathsf{E}|V(t) - V(s)|^2 = 2\int_0^{\infty} (1 - \cos\lambda(t-s))\, G(d\lambda).$$

Essentially, the same is true for the homogeneous one-dimensional random real-valued field $V = V(s)$, $s \in \mathbb{R}^k$.

In particular, for the homogeneous isotropic fields the covariance function $R(t)$, $t = (t_1, \ldots, t_k)$, is defined as

a function of $\|t\| = \sqrt{t_1^2 + \cdots + t_k^2}$: $\quad R(t) = \mathbf{R}(\|t\|)$.

In this case

$$\boxed{\mathbf{R}(u) = 2^{(k-2)/2}\, \Gamma\Big(\frac{k}{2}\Big) \int_{-\infty}^{\infty} \frac{I_{(k-2)/2}(\lambda u)}{(\lambda u)^{(k-2)/2}}\, Q(d\lambda)},$$

where $I_\nu(x)$ is the Bessel function of index ν and
Q is a non-negative random measure on $\mathcal{B}(\mathbb{R}_+)$ such that $Q(\mathbb{R}_+) = G(\mathbb{R}^k)$.

Recall also spectral representation for the stationary (in a wide sense) mean-square continuous process $V = V(t)$, $t \in \mathbb{R}$:

$$\boxed{V(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, Z(d\lambda)}$$

(**Cramér**, **Karhunen**, **Kolmogorov**), where $Z = Z(A)$ is a complex-valued random measure with $\mathsf{E}|Z(A)|^2 = F(A)$.

The similar representation is valid for the **homogeneous random fields** on $\mathbb{R}^k$ with values in $\mathbb{R}^l$.

The first Kolmogorov work on the turbulence (1941) was preceded by his two papers mentioned above, in which he made the first steps towards the notion of **local** isotropy that became the main mathematical means in analyzing the turbulence phenomena (especially in the case of high Reynolds numbers).

For a $\mathbb{R}^1$-valued random process $V = V(s)$, $s \in \mathbb{R}^1$, with stationary increments he considered the increments

$$\Delta_r V(t) = V(t) - V(t-r)$$

and supposed that

- **(a)** $\mathsf{E}\Delta_r V(t)$ depends only on r and
- **(b)** $\mathsf{E}\Delta_{r_1} V(t+s)\,\Delta_{r_2} V(s)$ does not depend on s for any s, t, r_1, r_2.

This will be stressed by the notation

$$B(t, r_1, r_2) = \mathsf{E}\Delta_{r_1} V(t+s)\,\Delta_{r_2} V(s).$$

The function $B(t, r_1, r_2)$ is called the **structural function** of the process B.

Due to the equality

$$(a-b)(c-d) \;=\; \tfrac{1}{2}\Big\{(a-d)^2 + (b-c)^2 - (a-c)^2 - (b-d)^2\Big\}$$

the structural function $B(t, r_1, r_2)$ can be represented as a function of one variable:

$$B(r) = B(0, r, r) = \mathsf{E}|\Delta V_r(t)|^2$$

which is called the structural function as well.

Kolmogorov obtained the spectral representation for $B(r)$:

$$B(r) = 2\int_{\mathbb{R}\setminus\{0\}} (1 - \cos\,\lambda r)\,\Phi(d\lambda) + ar^2$$

and the spectral representation for the process V:

$$V(t) = \int_{-\infty}^{\infty} (e^{i\lambda t} - 1)\,Z(d\lambda) + ut + V \qquad (*)$$

with $\mathsf{E}|Z(A)|^2 = \Phi(A)$, $A \in \mathcal{B}(\mathbb{R})$. If the process V is stationary in itself (in a wide sense) then its spectral representation $V(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\,Z(d\lambda)$ can be derived from $(*)$.

Similarly, Kolmogorov obtained the corresponding representation also for the *locally homogeneous and locally isotropic* $\mathbb{R}^l$-valued vector fields $V(M) = (V_1(M), \dots, V_l(M))$, $M \in \mathbb{R}^k$.

In 1968 **R. H. Kraichnan**—in his paper *"Small-scale structure of a scalar field convected by turbulence"*—proposed, following to the described mathematics, the **Gaussian model** for a random field.

This random field $V_{\text{kra}}(t,x)$ is a generalized Gaussian field with zero mean and covariance

$$K(x-y, t-s) = C(x-y)\,\delta(t-s)$$

such that the spatial part

- is of the form

  $$C^{ij}(x-y) = A^{ij} + D^{ij}|x-y|^{\varkappa}$$

  for $|x-y| \ll 1$, where $\varkappa \in (0,2)$, and
- decays rapidly as $|x-y| \to \infty$.

This velocity field $V_{\mathsf{kra}}(t,x)$ can be realized by its identification with a random field of the form

$$\sigma(x)\,\dot W(t), \quad \text{where} \;\; \sigma(x) = \sqrt{C(x)}$$

(**Y. Le Yan**, **O. Raimond**. Integration of Brownian vector fields, *Ann. Probab.* **30** (2002), 826–873).

It could be shown that if $\varkappa = 2/3$ the fluid particles diverge at the velocity which is characteristic for Kolmogorov's 1/3 law.

Therefore, Kraichnan's velocity field with $\varkappa = 2/3$ can be considered as an approximation of Kolmogorov's velocity.

Other explicit models that capture the salient features of turbulence were developed in the last time.

It is necessary to say that one of the great mysteries of the turbulence theory, as we know it, is how (if at all) the Kolmogorov turbulence theory is related to the Navier–Stokes equations.

Particularly discouraging is that no relation of Kolmogorov's or Kraichnan's theory to Newtonian mechanics has been established yet.

We want to speak now only about

THREE DIFFERENT STOCHASTIC APPROACHES TO THE TURBULENCE

that give some strategies to link the two considered theories.

Ya. G. Sinai
and his collaborators

B. Rozovsky, **R. Mikulevicius**, **S. Lototsky**,
and their collaborators

O. E. Barndorff-Nielsen, **J. Schmiegel**,
and their collaborators

One approach (Ya. G. Sinai and his collaborators) is to investigate the **spectrum** of invariant measure of the **stochastic** Navier–Stokes equation

$$\boxed{\partial_t u = \nu \, \Delta u - (u, \nabla) u - \nabla p + \sqrt{Q} \, \dot{W}(t)},$$

$$u(0) = u_0, \quad \operatorname{div} u = 0,$$

$W(t)$ is a Hilbert-space-valued Brownian motion and
Q is an operator determining the spatial covariance.

The aim is to compare the asymptotics of the energy spectrum of this invariant measure to the ones characteristic for Kolmogorov theory.

Some preliminary results regarding existence and uniqueness of the invariant measures for the stochastic Navier–Stokes equation in dimension 2 have been obtained by

S. B. Kuksin, **A. Shirikan**, **J. Mattingly**, **M. Hairer**, **Ya. Sinai**.

Of course, there is no guarantee that the Brownian motion is the correct perturbation; other types of perturbations are investigated by the group of **O. E. Barndorff-Nielsen** (we shall see it below).

Another approach is being explored by

B. Rozovsky, **R. Mikulevicius**, **S. Lototsky** (USC)

and their collaborators.

The idea of their approach is to consider the velocity vector field

$$\boxed{\mathbf{v}(t,x) = \mathbf{u}(t,x) + \mathbf{V}_{\text{kol}}(t,x)},$$

where $\mathbf{V}_{\text{kol}}(t,x)$ is the Kolmogorov velocity field and $\mathbf{u}(t,x)$ is a more regular function.

The function $\mathbf{u}(t,x)$ is selected in such a way that the total velocity $\mathbf{v}(t,x)$ satisfies the second Newton law.

In fact, this idea is a generalization of the Reynolds method of splitting up the velocity field into a sum of slow oscillating and fast oscillating stochastic components.

So far this idea has been implemented only for Kraichnan's velocity field that can be considered as approximation of Kolmogorov velocity.

The basic result of B. Rozovsky and R. Mikulevicius is the following:

> the regular component $\mathbf{u}(t,x)$ is a solution to the stochastic Navier–Stokes equation

$$\partial_t \mathbf{u} = \nu \, \Delta u - (\mathbf{u}, \nabla)\mathbf{u} - \nabla p + f + ((\sigma, \nabla)\mathbf{u} - \nabla \tilde{\mathbf{u}} + g) \circ \dot{W},$$
$$\operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0.$$

Special cases of this equation include the standard deterministic Navier–Stokes and Euler, randomly forced Navier–Stokes equation

$$\partial_t \mathbf{u} = \nu \, \Delta \mathbf{u} - (\mathbf{u}, \nabla)\mathbf{u} - \nabla p + \sqrt{Q}\, \dot{W}(t),$$
$$u(0) = u_0, \quad \operatorname{div} u = 0.$$

B. Rozovky and R. Mikulevicius developed also the Lagrangian approach which postulates that the dynamics of a fluid particle is given by the stochastic diffeomorphism

$$\begin{aligned} dX(t) &= \mathbf{u}(t, X(t))\,dt + V(t, X(t))\,dt \\ &= \mathbf{u}(t, X(t))\,dt + \sigma(t, X(t)) \circ dW, \end{aligned}$$

(where $\circ$ denotes Stratonovich integration)

$$\operatorname{div} \sigma = 0, \quad \operatorname{div} \mathbf{u} = 0.$$

For simplicity assume that the fields is non-viscous.

Also assume that $\mathbf{u}(t, x)$ is a Brownian semimartingale:

$$d\mathbf{u}(t, x) = a(t, x)\,dt + b(t, x) \circ dW.$$

By Itô's formula

$$\begin{aligned} d\mathbf{u}(t, X(t)) = a(t, X(t))\,dt &+ b(t, X(t)) \circ dW \\ &+ u^j(t, X(t))\,\partial_j \mathbf{u}(t, X(t)) \\ &+ \sigma^j(t, X(t))\,\partial_j \mathbf{u}(t, X(t)) \circ dW_t. \end{aligned}$$

One now assumes that the force acting on the fluid particles has a slowly varying (adjective) part, as well as an irregular (diffusive) part, each of them split into a pressure and a bulk force term:

$$dF(t, X(t)) = \overbrace{(\underbrace{-\nabla p}_{\text{pressure}} + \underbrace{f}_{\text{bulk}})\, dt}^{\text{adjective part}} + \overbrace{(\underbrace{-\nabla \tilde{p}}_{\text{pressure}} + \underbrace{g}_{\text{bulk}}) \circ dW_t}^{\text{diffusive part}}.$$

Using Newton's second law $d\mathbf{u}(t, X(t)) = dF(t, X(t))$ and matching similar terms, we arrive at the **stochastic Euler equation**:

$$\partial_t \mathbf{u}(t, x) = -u^j(t, x)\, \partial_j \mathbf{u}(t, x) - \nabla p(t, x) + \mathbf{f}$$
$$+ [-\sigma^j \partial_j \mathbf{u} + \mathbf{g} - \nabla \bar{p}] \circ dW_t.$$

(For case $d = 2$ Rozovsky and Mikulevicius gave proof of existence of strong path-wise unique and strongly continuous solution of this equation and the corresponding stochastic Navier–Stokes equation with an additional term $\nu \Delta \mathbf{u}$.)

The approach of

O. E. Barndorff-Nielsen, **J. Schmiegel**

and their collaborators to the dynamics of turbulent velocities is based on the following idea.

Suppose that $u_t = u_t(\sigma; \omega)$ is the main component of the velocity (in the direction of the mean flow, *i.e.*, in the longitudinal direction), where σ is a fixed position.

Let us assume that u_t has the form

$$\boxed{u_t = \bar{u} + \int_{-\infty}^{\infty} g(t-s)\, dY_s},$$

where

- $\bar{u}$ is a constant,
- g is a deterministic kernel, and
- the process Y has the differential

  $$dY_t = \beta \mathcal{E}_t\, dt + \sqrt{\mathcal{E}_t}\, dW_t,$$

  where
  - β is a constant,
  - $(\mathcal{E}_t)$ is a positive stationary process, and
  - W is a Wiener process (Brownian motion).

The authors of this model show that suitable choices of g and $(\mathcal{E}_t)$ can reproduce key stylized features of the time-wise behavior of the velocity.

From given representations for u_t and Y_t we see that

$$u_t = \bar{u} + \beta \int_{-\infty}^{t} g(t-s)\,\mathcal{E}_s\,ds + \int_{-\infty}^{t} g(t-s)\,\sqrt{\mathcal{E}_s}\,dW_s.$$

This is again a **Reynolds-type decomposition** with

$\bar{u} + \beta \int_{-\infty}^{t} g(t-s)\,\mathcal{E}_s\,ds$	playing the role of the **slowly** varying component,
$\int_{-\infty}^{t} g(t-s)\,\sqrt{\mathcal{E}_s}\,dW_s$	reflecting the **strongly** varying component.

In other words, u_t has a decomposition of the semimartingale type.

The authors of this promising approach demonstrated that their model is capable to capture many basic stylized facts of the statistics of temporary velocity increments.

In particular, they show that the probability density of velocity increments is characterized by

- a normal inverse Gaussian shape with heavy tails for **small** scales and
- approximately Gaussian tails for **large** scales.

They show also that their model is in accordance with the experimental verification of three refined Kolmogorov's similarity hypotheses.

As the authors say, their approach is

> *a step in a project to formulate a full-fledged tempo-spatial stochastic process model for the three-dimensional velocity field.*