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Efficient simulation of finite horizon problems in queueing and insurance risk

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Abstract

Let $\psi(u,t)$ be the probability that the workload in an initially empty M/G/1 queue exceeds u at time $t < \infty$, or, equivalently, the ruin probability in the classical Crámer-Lundberg model. Assuming service times/claim sizes to be subexponential, various Monte Carlo estimators for $\psi(u,t)$ are suggested. A key idea behind the estimators is conditional Monte Carlo. Variance estimates are derived in the regularly varying case, the efficiencies are compared numerically and also one of the estimators is shown to have bounded relative error. In part, also extensions to general Lévy processes are treated.

Key words: Bounded relative error, complexity, conditional Monte Carlo, Lévy process, regularly varying distribution, finite horizon ruin function, M/G/1 queue

1 Introduction

Stochastic models with heavy tails have been the subject of intense interest during recent years, not least in areas such as insurance risk and queueing theory (and its telecommunications applications). See, e.g., Adler, Feldman and Taqqu (1998); Embrechts, Klüppelberg and Mikosch (1997); and the Notes in Asmussen (2003) pp. 300–301. The purpose of this paper is to develop some new simulation algorithms in some heavy-tailed settings, where crude Monte Carlo simulation sofar has been the only possibility.

Our primary object of interest is the transient workload process $\{V(t)\}$ in the M/G/1 queue with $V(0) = 0$ and

$$\psi(u,t) \;=\; \mathbb{P}(V(t) > u)\,.$$

Another standard interpretation of $\psi(u,t)$ is as the ruin probability in the classical Cramér-Lundberg risk model with Poisson arrivals of claims and constant premium inflow at unit rate. The netput process has the form $X(t) = \sum_1^{N(t)} U_k - ct$ in both cases where the U_k are the service times/claim sizes and $\{N(s)\}$ an independent Poisson process at rate β. In queueing, c is the service rate and in insurance risk,

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c is the premium rate. A simple time change argument shows that we may take $c = 1$, which is done throughout in the following.

Recently there has, however, been substantial interest in this simple model perturbed by Brownian motion or an infinite activity jump part, and therefore we also partially deal with such extensions.

In the light-tailed case, it has long been understood how to perform efficient simulation of $\psi(u) = \lim_{t\to\infty} \psi(u,t)$ (the tail of the stationary workload or the infinite horizon ruin probability) as well as of $\psi(u,t)$. The basic tool is importance sampling, see Asmussen (2000) pp. 287–292 (the algorithms described there exploit variants of large deviations ideas, see in particular Siegmund (1976); Bucklew, Ney and Sadowsky (1990); Anatharam (1998)). In the heavy-tailed case, all algorithms has sofar only dealt with the infinite horizon case and the Cramér-Lundberg setting. Here efficient simulation algorithms for $\psi(u)$ have been developed in a number of recent papers, in particular Asmussen and Binswanger (1997); Asmussen, Binswanger and Højgaard (2000); Juneja and Shahabuddin (2002); Asmussen and Kroese (2006). All of these heavily rely on the Pollaczek-Khinchine formula, expressing $\psi(u)$ as the tail probability of a geometric sum. For the finite horizon problem, the only reference we know of is Boots and Shahabuddin (2001). For the algorithm in that paper to be efficient, it is, however, needed that u and t vary together in a specific manner, and that F is not too far from the heavy-tailed Weibull.

The contribution of the present paper is to suggest some completely different algorithms which appear to have a broader scope. In particular, for one of them (Algorithm B), we are able to theoretically verify good efficiency properties in the most important case of subexponentiality, regular variation. Algorithm B is, however restricted to the M/G/1—Cramér-Lundberg setting. The other idea we present, developed in Algorithm A, applies to general Lévy processes and we also empirically demonstrate that it has excellent efficiency, though at the moment we have not been able to come up with variance estimates quite as sharp as for Algorithm B.

2 Preliminaries

Going beyond the M/G/1—Cramér-Lundberg setting, we assume that the netput process X is a general Lévy process. As is well known, X can be written as the independent sum of a Brownian motion and a pure jump part, whose construction may involve compensation.

The traditional stability condition is $\rho < 1$ in the queuing setting where ρ denotes the traffic intensity. This means $\mu = \mathbb{E}\, X(1) < 0$, but we will only partially impose this assumption. All that will matter for us is the upper tail, and we use therefore the alternative decomposition $X = Y + Z$ where Y is compound Poisson with positive jumps, say the intensity is λ and the jump distribution is F. In the Cramér-Lundberg risk model, $Z(s) = -s$ (here the upward jumps correspond to claims), but it should be noted that the representation $X = Y + Z$ is completely general and can be achieved for any Lévy process by just letting Y be the sum of jumps of size at least 1 and $Z = X - Y$ (then Z is the sum of a Brownian component, possibly with drift, and a jump part with jumps bounded by 1). We write $\overline{X}(t_1, t_2) =$

$\sup_{t_1 \le s < t_2}\{X(s) - X(t_1)\}$, $\tau(u) = \inf\{s > 0 : X(s) > u\}$ and

$$\psi(u) \;=\; \mathbb{P}(\tau(u) < \infty), \quad \psi(u,t) \;=\; \mathbb{P}(\tau(u) < t) \;=\; \mathbb{P}(\overline{X}(0,t) > u).$$

In the M/G/1—Cramér-Lundberg setting, $\tau(u)$ is the probability that the steady-state workload exceeds u, or, equivalently, the infinite horizon ruin probability.

In the following, we will assume Y is compound Poisson with only positive jumps (not necessarily bounded below by 1) so that $Y(s) = \sum_1^{N(s)} U_i$, where $\{U_1 : i \in \mathbb{N}\}$ are i.i.d. with common distribution F supported by $\mathbb{R}^+$ and $\{N(s)\}$ is an independent Poisson process with rate λ and interarrival times $\{T_i : i \in \mathbb{N}\}$. We will further throughout assume that F is regularly varying with index $\alpha > 0$. Define the *integrated tail distribution*

$$F_I(x) = \mu_F^{-1} \int_0^x \overline{F}(y)dy$$

where $\overline{F}(x) := 1 - F(x)$ and $\mu_F := \mathbb{E}\, U_i < \infty$. We will assume that F is regularly varying with index $\alpha > 0$, that is, that $\overline{F}(x) = \ell(x)/x^\alpha$ with $\ell(x)$ slowly varying. If $\alpha > 1$, then $\overline{F}_I(x) = \ell_I(x)/x^{\alpha-1}$ where $\ell_I(x) \sim \ell(x)/(\alpha - 1)$. It is well known that if $\mu < 0$ and $\alpha > 1$, it holds for the Cramér-Lundberg model that

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_I(u)} \;=\; \frac{\rho}{|\mu|} \tag{2.1}$$

where $\rho = \lambda\mu_F$. In Asmussen and Klüppelberg (1997), asymptotics for $\psi(u,t)$ are given when t is of order $\mathbb{E}\,[U - u \,|\, U > u]$. This means in the regularly varying case that $t = t(u)$ depends on u in such a way that $t(u)/u \to k \in (0,\infty)$. Then, Corollary 1.6 from the cited reference states that under these assumptions

$$\lim_{u\to\infty} \frac{\psi(u,t(u))}{\psi(u)} \;=\; 1 - (1 + (1-\rho)k)^{-\alpha+1} \;>\; 0\;. \tag{2.2}$$

The same result holds in a general Lévy process, as can be seen from Klüppelberg, Kyprianou and Maller (2004) after some rewriting. If instead $t(u)/u \to 0$, then Foss, Palmowski and Zachary (2005); Tang (2004) imply

$$\lim_{u\to\infty} \frac{\psi(u,t(u))}{\lambda\, t(u)\overline{F}(u)} \;=\; 1 \tag{2.3}$$

(this result does not require $\mu < 0$ and holds under more general subexponential assumptions, as does (2.1); the given references only give the result for discrete-time random walks, but it is not difficult to extend it to the M/G/1—Cramér Lundberg setting and even to a general Lévy process by writing $X = Y + Z$ and treating Z as a light-tailed perturbation of Y).

By a *simulation estimator* $\widehat{\psi}(u,t)$ for $\psi(u,t)$, we understand a r.v. which can be generated by simulation and is unbiased, $\mathbb{E}\,\widehat{\psi}(u,t) \;=\; \psi(u,t)$. A family of such estimators has *bounded relative error* if $\operatorname{Var}\widehat{\psi}(u,t)/\psi(u,t)^2$ remains bounded as $u \to \infty$ (recall that t is considered a function of u), and is *logarithmically efficient* if $\operatorname{Var}\widehat{\psi}(u,t)/\psi(u,t)^{2-\epsilon}$ remains bounded for all $\epsilon > 0$. These definitions tacitly

assume that the time to generate one copy of $\widehat{\psi}(u,t)$ is of the same order t as for the crude Monte Carlo method (which is the case in the algorithms we consider). For further discussion of these concepts see Asmussen and Rubinstein (1995); Heidelberger (1995); Asmussen and Glynn (2007) Ch. VI.

When simulating X, the compound Poisson part Y is of course straightforward to generate. The remaining Lévy component Z may require more sophistication, and we refer to Asmussen and Glynn (2007) Ch. XII for a survey of methods. Note, however, that Y and Z need not be simulated separately, but X can be simulated and next Y extracted as the jumps > 1.

3 Algorithms

The cited references on simulation of $\psi(u)$ all use a representation of the form $\psi(u) = \mathbb{P}(S_N > u)$ where $S_N = V_1 + \cdots + V_N$ with $V_1, V_2, \ldots$ i.i.d. and non-negative with distribution say G, and N is an independent r.v. supported over $\mathbb{N}$. Our algorithms are further developments of the conditional Monte Carlo idea in Asmussen and Kroese (2006) which is as follows. Given N has been simulated, by exchangeability of the V_i's one has

$$\mathbb{P}(S_N > u) \;=\; \mathbb{E}\big[N;\, S_N > u, V_N > V_j, j < N\big]. \tag{3.1}$$

Then one simulates $V_1, \ldots, V_{N-1}$, forms the order statistics $V_{(1)} < \cdots < V_{(N-1)}$ and the conditional expectation of $[\ldots]$ is taken w.r.t. $\mathcal{F}_1 = \sigma\big(N, V_1, \ldots, V_{N-1}\big)$ resulting in the conditional Monte Carlo estimator

$$N\,\mathbb{P}\big(S_N > u, V_N > V_{(N-1)} \,\big|\, \mathcal{F}_1\big) \;=\; N\,\overline{G}\big((u - S_{N-1}) \vee V_{(N-1)}\big). \tag{3.2}$$

where $S_{N-1} = V_1 + \cdots + V_{N-1}$. The numerical evidence shows that the algorithm (3.2) is superior to all other algorithms suggested in the heavy-tailed $\mathbb{P}(S_n > u)$ setting.

In the present paper, the role of $V_1, V_2, \ldots$ is taken by the jumps of the compound Poisson part Y in Algorithm A and by the ladder heights in Algorithm B (see e.g. Asmussen, 2000, pp. 271-279). However, both the above algorithms needs substantial modification. In particular, the difficulty of adapting (3.2) is the presence of the Z component which destroys exchangeability. For example, if Z has negative drift, i.e., $\mathbb{E}\,Z(1) < 0$, then an early large jump of Y is more likely to cause ruin than a late one, because by the late time the negative drift is likely to have taken X to a smaller value.

3.1 Algorithm A

It follows from Asmussen and Klüppelberg (1997); Foss, Palmowski and Zachary (2005); Tang (2004) that ruin occurs with high probability as consequence of one big jump.

The idea here is to adapt the conditional algorithm in Asmussen and Kroese (2006) by first defining a r.v. J which given $\{X(s)\}_{s\le t}$ has a discrete uniform dis-

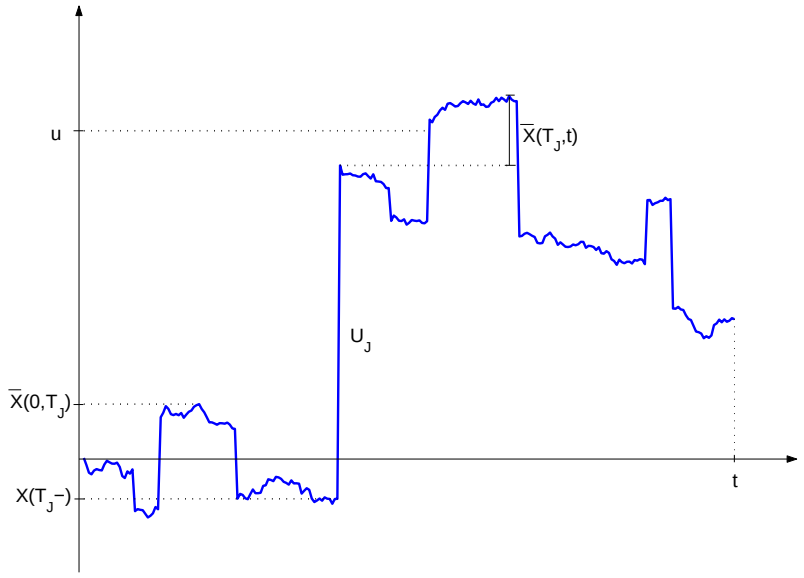

Figure 1: A sample path of X with a big jump U_J

tribution over $\{1,\dots,N(t)\}$. Then replace the symmetry argument in (3.1) by

$$\mathbb{P}(\tau(u)<t)=\mathbb{P}(\tau(u)<t,N(t)=0)+\mathbb{E}\left[N(t);\tau(u)<t,\right.$$
$$\left.U_J=\max\{U_i:i\le N(t)\},\ N(t)>0\right]$$

and compute the estimator as the conditional probability of $\{\tau(u)<t\}$ given

$$\mathcal{F}_{\mathrm{A}}\ =\ \sigma\big(J,\ \{N(s)\}_{s\le t},\ \{U_j:\ j\neq J\},\ \{Z(s)\}_{s\le t}\big),$$

the σ-field containing all information about the r.v. J and the process X except the size of jump J. This conditional probability comes out as

$$\mathbb{P}\big(\tau(u)<t\,\big|\,\mathcal{F}_{\mathrm{A}}\big)=\begin{cases}\mathbb{I}(\overline{Z}(0,t)>u) & N(t)=0\\ N(t)\,\mathbb{P}\big(\tau(u)\le t,\ U_J=\max\{U_i:i\le N(t)\}\,|\mathcal{F}_{\mathrm{A}}\big) & N(t)>0\end{cases}$$

If $N(t)>0$, ruin occurs typically as consequence of the big jump U_J or possibly if the value of the process after jump J exceeds u, i.e., if

$$X(T_J-)+U_J+\overline{X}(T_J,t)>u$$

where $X(T_J-)$ is the value of the process just before jump J. However, we also have to take into account the possibility that ruin occurs before the time of jump J (i.e., that $\tau(u)<T_J$). This argument is illustrated in Fig. 1. In conclusion,

$$\mathbb{P}\big(\tau(u)<t,\ U_J=\max\{U_i:i\le N(t)\}\,|\mathcal{F}_{\mathrm{A}}\big)=\mathbb{P}(U_J>W)$$

where

$$W\ :=\ \sup\{U_i:i\neq J\}\vee\big[\big(u-X(T_J-)-\overline{X}(T_J,t)\big)\mathbb{I}(\tau(u)\ge T_J)\big].$$

Here we used the $\mathcal{F}_{\mathrm{A}}$-measurability of $\mathbb{I}(\tau(u)\ge T_J)$, $\overline{X}(T_J,t)$ and $X(T_J-)$ to compute the conditional probability $\mathbb{P}(\tau(u)<t|\mathcal{F}_{\mathrm{A}})$. Algorithmically:

1. Simulate the Poisson process $\{N(s)\}_{s\le t}$ by generating $N(t) = N_t$ as Poisson(λt) and given that $N_t \ge 1$, the jump times $T_1 < T_2 < \cdots < T_{N_t}$ as the order statistics from the $(0,t)$-uniform distribution

2. Simulate the whole of $\{Z(s) : s \le t\}$ (we again refer to Asmussen and Glynn (2007) for this step). If $N_t = 0$ return $\mathbb{I}(\overline{Z}(0,t) > u)$. Else go to 3.

3. Simulate J as a discrete uniform r.v. over $\{1,\ldots,N_t\}$ and the U_i, $i \ne J$, from F.

4. Calculate W and return

$$\widehat{\psi}_A(u,t) \;=\; N_t\overline{F}(W)$$

Remark 1. Algorithm A can in a straightforward way be generalized to a discrete time random walk $X_N = Y_1 + \cdots + Y_N$ with increment distribution F. For example, one replaces N_t by t, T_i by i, $\overline{X}(t_1,t_2)$ by $\max_{i_1\le i<i_2} X_i$ and $\sup_{i\ne J} U_i$ by $\max_{j\le n, j\ne J} Y_j$. Also the derivation of the variance estimates of the next section is entirely similar. □

3.2 Algorithm B

We restrict here our attention to the M/G/1 queue or equivalently the Crámer-Lundberg risk model, which allow us to use the sample path decomposition of the process $\{X(s)\}$ according to ladder steps (e.g. Asmussen, 2000, pp. 271-279). Let the sequence of random vectors $\{(T_i^0, V_i, W_i) : i \in \mathbb{N}\}$ be the lengths of the ladder segments, the ladder heights, and the deficits before ladder epochs as illustrated in Fig. 2.

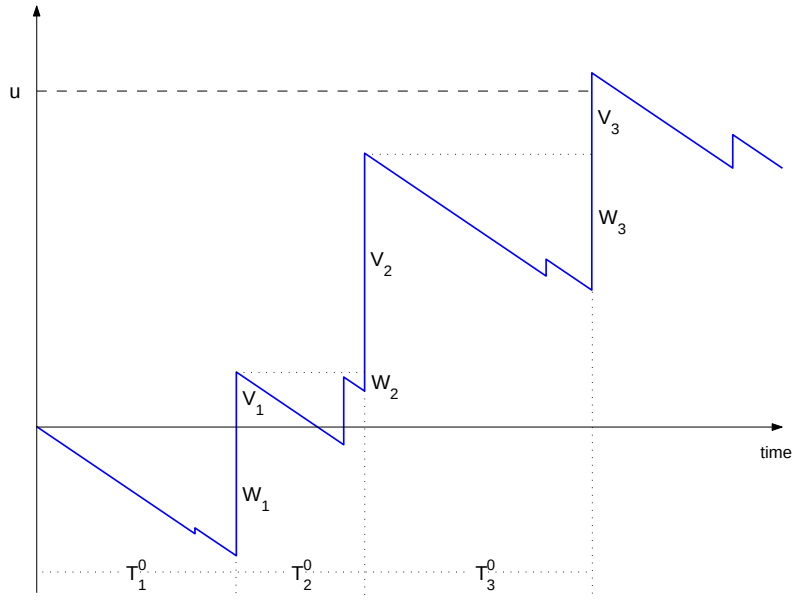

Figure 2: The ladder structure

Here, the number of ladder steps L is geometric(ρ), different ladder segments are independent and the joint distribution of (T_i^0, V_i, W_i) is as described in the following lemma, where parts (i), (ii) are classical and part (iii) from Asmussen and Klüppelberg (1997):

Lemma 1. *Under $\mathbb{P}^{(0)} = \mathbb{P}(\cdot|\tau(0) < \infty)$, V_i, W_i have the same joint distribution as the backward and forward recurrence time distribution in a renewal process with interarrival distribution F. That is,*

1. *The marginal distributions of V_i, W_i are both F_I.*
2. *The conditional distribution of $(W_i|V_i = v)$ is the same as the distribution of $(U - v|U > v)$ where U is a r.v. with distribution F.*

Moreover,

3. *The conditional distribution of $(T_i^0|W_i = w)$ is the same as the unconditional distribution of $\inf\{t > 0 : R(t) = w\}$, where the process R is given by $R(t) = t - \sum_{i=1}^{N^0(t)} U_i^0$ and is an independent copy of $-S$.*

In the Cramér-Lundberg model ruin occurs always at a ladder epoch. Therefore, since the probability of ruin given that no ladder steps occured (i.e., that $L = 0$) is 0, we can write

$$\mathbb{P}(\tau(u) < t) = \mathbb{P}(\tau(u) < t, L > 0) = (1 - \rho)\,\mathbb{P}(\tau(u) < t|L > 0).$$

It follows that $\tau(u) < t$ if and only if the sequence of random vectors $\{(V_i, T_i) : i \in \mathbb{N}\}$ is such that $V_1 + \cdots + V_L > u$ and $T_1^0 + \cdots + T_K^0 < t$ where $K := \inf\{k : V_1 + \cdots + V_k > u\}$ (observe that this two conditions together imply $K \leq L$). Thus

$$\{\tau(u) < t\} = \big\{V_1 + \cdots + V_L > u,\, T_1^0 + \cdots + T_K^0 < t\big\}.$$

In the implementation, it is convenient not to generate the T_i^0 separately from the representation in part (iii) of Lemma 1, but generate R in the whole of $[0, t]$, define $\overline{R}(t) = \overline{R}(0, t)$ and note that the upward skipfree property of R implies that the process of first passage time to different levels has stationary independent increments so that

$$\mathbb{P}(\tau(u) < t) \;=\; \mathbb{P}\big(V_1 + \cdots + V_L > u,\, W_1 + \cdots + W_K < \overline{R}(t)\big)\,. \tag{3.3}$$

The estimator $\mathbb{I}\big\{(V_1 + \cdots + V_L > u,\, W_1 + \cdots + W_K < \overline{R}(t)\big\}$ can ve viewed as the crude Monte Carlo estimator based on Lemma 1, and we proceed to develop some ideas that will reduce this estimator's variance.

We first adapt the argument in Asmussen and Kroese (2006) to the present setting as follows. We define J to be an independent random variable with a discrete uniform distribution on $\{1, \ldots, L\}$, replace the symmetry argument in (3.1) by

$$\mathbb{P}(\tau(u) < t) = \mathbb{E}\left[L; \tau(u) < t,\, V_J = \max\{V_i : i \leq L\},\, L > 0\right]$$

and compute our estimator as the conditional probability of (3.3) given

$$\mathcal{F}_B = \sigma\big(L, J, \{V_j : j \neq J\}, \{R(s) : s \leq t\}\big)\,,$$

the σ-field containing the information on the random variable J, the process $\{R(s)\}$ up to time t and the number and sizes of the ladder heights V_i except of the Jth. This conditional probability comes out as

$$\begin{aligned}\mathbb{P}\big(\tau(u) < t|\mathcal{F}_B\big) &= L\,\mathbb{P}\big(W_1 + \cdots + W_K < \overline{R}(t),\, V_J > H \,\big|\, \{V_j : j \neq J\}\big) \\ &= L\,\overline{F}_I(H)\mathbb{P}\big(W_1 + \cdots + W_K < \overline{R}(t) \,\big|\, \{V_j : j \neq J\},\, V_J > H\big) \quad (3.4)\end{aligned}$$

where

$$H = \Big(u - \sum_{j\neq J} V_j\Big) \vee \sup\big\{V_j : j \neq J\big\}.$$

However, K it is not measurable with respect the σ-field generated by $\big\{\{V_j : j \neq J\},\ \mathbb{I}(V_J > H)\big\}$ and therefore (3.4) should be calculated via conditional Monte Carlo. We let V_J^* be a r.v. having the conditional distribution of V_J given (H, ∞). Then

$$\begin{aligned}&\mathbb{P}\big(W_1 + \ldots + W_K < \overline{R}(t) \,\big|\, \{V_j : j \neq J\},\, V_J > H\big) \\ &\qquad = \mathbb{E}\left[\mathbb{P}\big(W_1 + \ldots + W_K < \overline{R}(t) \,\big|\, \{V_j : j \neq J\},\, V_J^*\big)\right]\end{aligned}$$

The last step of the algorithm is to reduce the variance coming from the set $\{W_i\}$ as follows: Since $\overline{R}(t) < t$, we have for $i \leq K$ that $\tau(u) \geq t$ if $W_i \geq t$. So, we let W_i^* be a r.v. having the conditional distribution of W_i given V_i and $W_i < t$. Then

$$\begin{aligned}&\mathbb{P}\big(W_1 + \cdots + W_K < \overline{R}(t) \,\big|\, \{V_j : j \neq J\},\, V_J^*\big) \\ &\qquad = \prod_{i=1}^{K} \mathbb{P}(W_i < t|V_i)\mathbb{P}(W_1^* + \cdots + W_K^* < \overline{R}_t)\end{aligned}$$

Combining the above steps leads to the following algorithm:

1. Simulate L, J, $\{V_i : i \neq J\}$ with distributions as above (in particular, $\mathbb{P}(L = n) = (1-\rho)\rho^{n-1}$, $n = 1, 2, \ldots$). To simulate R, generate $M = N_t^0$ as Poisson(λt), the jumps $U_1^0, \ldots, U_M^0$ as i.i.d. with distribution F, and the jump times $\pi_1, \ldots, \pi_M$ as the order statistics from the uniform distribution on $(0, t)$. Let $\pi_{M+1} = t$, $U_0 = 0$ and

$$\overline{R}(t) = \max_{1\leq k\leq M+1} \left\{\pi_k - \sum_{i=0}^{k-1} U_i^0\right\}$$

2. Calculate H and simulate the random variable V_J^*.

3. Identify K and simulate the random variables $\{W_i^* : i \leq K\}$.

4. Return the estimator

$$\overline{\psi}_B(u) = (1-\rho)L\overline{F}_I(H)\prod_{i=1}^{K} \mathbb{P}(W_i < t|V_i)\mathbb{I}(W_1^* + \cdots + W_K^* < \overline{R}(t))$$

4 The efficiency of the estimators

We can now state our main theoretical results on Algorithms A, B. They assume $t = t(u)$ to be a function of u and consider the limit $u \to \infty$.

Proposition 1. *Assume that $t/u \to 0$. Then for each $\epsilon > 0$, one has*

$$\mathbb{V}\mathrm{ar}\,\widehat{\psi}_A(u,t) \;=\; t^{2(\alpha+1)}\overline{F}^2(u)\mathrm{O}(\mathrm{e}^{\epsilon u})\,.$$

Note that this result does not require $\alpha > 1$ or $\mu < 0$. The result is, however, somewhat weaker than logarithmic efficiency: then the power of t should have been 2. The bound provided by Theorem 1 is sharpest when t grows relatively slowly with u. In particular, if $t = u^\beta$ with $0 \le \beta < 1$, then Proposition 1 guarantees that Algorithm A provides an improvement over the crude Monte Carlo algorithm only when $\beta < \alpha/(2\alpha+1)$ (but of course, the bound of Proposition 1 could be too rough; cf. the numerical results and discussion of Section 6!).

Theorem 1. *Assume $\alpha > 1$ and that*

$$\mathbb{P}(U - v < t | U > v) \;=\; \mathrm{O}(1)\frac{t}{v} \tag{4.1}$$

where the $\mathrm{O}(1)$ is uniform in $0 < t < t_0$ for any $t_0 < \infty$. Then the estimator $\widehat{\psi}_B(u,t)$ has bounded relative error when $t \to \infty$ with u in such a way that $t/u \to k \in [0,\infty)$.

Note that (4.1) is very weak. It holds, for example, if F has a density $f(x)$ satifying $f(x) \sim \ell_1(x)/x^\beta$ with ℓ_1 slowly varying (integration shows that one must have $\beta = \alpha + 1$, $\ell_1(x) \sim (\alpha+1)\ell(x)$, as is seen by straightforward calculus.

5 Proofs

The following Lemma will be needed in the proof of Theorem 1.

Lemma 2. *Let $\overline{Z}(0,t) = \sup\{Z(s) : s \le t\}$. Then $\mathbb{P}(\overline{Z}(0,t) \ge u) \le e^{-\gamma u}$ for some $\gamma > 0$.*

Proof. Let $Z_1(t) = Z(t) - bt$ where b is so large that $\mathbb{E}\,Z_1(1) < 0$. Then Lundberg's inequality (Asmussen, 2000) gives that $\mathbb{P}(\overline{Z}_1(0,\infty) \ge u) \le e^{-\gamma_1 u}$ for some $\gamma_1 > 0$ (note that all exponential moments of $Z_1(1)$ exist). It follows that

$$\mathbb{P}(\overline{Z}(0,t) \ge u) \;\le\; \mathbb{P}(\overline{Z}_1(0,\infty) \ge u - bt) \;\le\; e^{-\gamma_1(u-bt)}\,.$$

From this the result follows, since $t/u \to 0$. □

Proof of Theorem 1. We start with an upper bound for the variance of the estimator A.

$$\begin{aligned}\mathbb{V}\mathrm{ar}\,[\widehat{\psi}_A(u,t)] &\le \mathbb{E}\,[\widehat{\psi}_A^2(u,t)] \\ &= \mathbb{E}\left[\widehat{\psi}_A^2(u,t)\,\mathbb{I}(\overline{Z}(0,t) < \sqrt{u})\right] + \mathbb{E}\left[\widehat{\psi}_A^2(u,t)\,\mathbb{I}(\overline{Z}(0,t) \ge \sqrt{u})\right]\end{aligned}$$

Since $\widehat{\psi}_A(u,t) \leq 1$ a.s. the second term is smaller than $\mathbb{P}(\overline{Z}(0,t) \geq \sqrt{u})$ which is bounded above by $e^{-\gamma\sqrt{u}}$ for some $\gamma > 0$ by Lemma 2.

To get an upper bound for the first term it will be useful to rewrite Estimator A as follows

$$\begin{aligned}\widehat{\psi}_A(u,t) = \mathbb{I}(N_t = 0, \overline{Z}(0,t) > u) + \mathbb{I}(N_t > 0, \tau(u) < T_J)\, N_t\, \overline{F}(U_{\max}) \\ + \mathbb{I}(N_t > 0, \tau(u) \geq T_J)\, N_t\, \overline{F}\big(U_{\max} \vee (u - X(T_J-) - \overline{X}(T_J,t))\big)\end{aligned}$$

where $U_{\max} = \sup\{U_i : i \neq J\}$. The first indicator describes the event where the process $\{Y(s) : s < t\}$ has no jumps but the process $\{Z(s)\}$ reached u before time t. The second term corresponds to the case when ruin happens before time T_J, however we still require U_J to be larger than $U_{\max}$. The third term is the complement of the other two: ruin should happen after time T_J, so U_J is required both to be larger than $U_{\max}$ and to make the process reach the level u in the time interval $[T_J, t]$.

Since the events involved are disjoint, by taking the square of $\widehat{\psi}_A(u,t)$ we are left with the sum of the square of each term. Next we multiply by $\mathbb{I}(\overline{Z}(0,t) < \sqrt{u})$ and analyze each term separately:

$$\begin{aligned}&\widehat{\psi}_A^2(u,t)\, \mathbb{I}(\overline{Z}(0,t) < \sqrt{u}) \\ &\quad = \mathbb{I}(N_t = 0, \overline{Z}(0,t) > u,\, \overline{Z}(0,t) < \sqrt{u}) && (5.1) \\ &\qquad + \mathbb{I}(N_t > 0, \tau(u) < T_J,\, \overline{Z}(0,t) < \sqrt{u})\, N_t^2\, \overline{F}^2(U_{\max}) && (5.2) \\ &\qquad + \mathbb{I}(N_t > 0, \tau(u) \geq T_J,\, \overline{Z}(0,t) < \sqrt{u})\, N_t^2\, \overline{F}^2\big(U_{\max} \vee (u - X(T_J-) - \overline{X}(T_J,t))\big) && (5.3)\end{aligned}$$

When $u > 1$ the events $\{\overline{Z}(0,t) < \sqrt{u}\}$, $\{\overline{Z}(0,t) > u\}$ are disjoint and the term (5.1) is 0.

Next, consider the case where $J = 1$ and the process $\{X(s)\}$ reaches level u before the time of the first jump, then $\overline{Z}(0,t) > u$ and the corresponding term (5.2) is 0 when $u > 1$. Therefore

$$\begin{aligned}\mathbb{I}\big(N_t \geq 1, \tau(u) < T_J, \overline{Z}(0,t) < \sqrt{u}\big) &= \mathbb{I}\big(1 < J \leq N_t, \overline{X}(0,T_J) > u, \overline{Z}(0,t) < \sqrt{u}\big) \\ &\leq \mathbb{I}\big(1 < J \leq N_t, \overline{Y}(0,T_J) > u - \sqrt{u}\big) \qquad (5.4)\end{aligned}$$

Now, if $J > 1$ and $\overline{Y}(0,T_J) > u - \sqrt{u}$ there exists at least one jump larger than $(u - \sqrt{u})/J$ and the following relations remain true

$$U_{\max} > \frac{u - 2\sqrt{u}}{J} \geq \frac{u - 2\sqrt{u}}{N_t}.$$

Hence $\mathbb{I}\big(U_{\max} \geq (u - 2\sqrt{u})/N_t,\, N_t > 1\big)$ is an upper bound of (5.4) and it follows that (5.2) is smaller than

$$\mathbb{I}\big(1 < N_t, U_{\max} \geq (u - 2\sqrt{u})/N_t\big) N_t^2 \overline{F}^2(U_{\max}) \leq \mathbb{I}\big(1 < N_t\big) N_t^2 \overline{F}^2((u - 2\sqrt{u})/N_t).$$

We move to the term (5.3). Observe that if $\overline{Z}(0,t) < \sqrt{u}$ we have that

$$X(T_J-) + \overline{X}(T_J,t) \leq \overline{X}(0,T_J) + \overline{X}(T_J,t) < \sum_{i \neq J} U_i + 2\overline{Z}(0,t) < \sum_{i \neq J} U_i + 2\sqrt{u}$$

Thus we obtain the following upper bound for (5.3):

$$\mathbb{I}(N_t \geq 1) N_t^2 \overline{F}^2 \left(U_{\max} \vee \left(u - 2\sqrt{u} - \sum_{i \neq J} U_i \right) \right)$$

Now, if $U_{\max} < z/N_t$ then it follows that

$$z - \sum_{i \neq J_t} U_i \geq z - (N_t - 1) U_{\max} > z/N_t$$

for any positive value z. This relation implies that

$$U_{\max} \vee \left(u - 2\sqrt{u} - \sum_{i \neq J_t} U_i \right) > \frac{u - 2\sqrt{u}}{N_t}$$

So, the term (5.3) is bounded above by $\mathbb{I}(N_t \geq 1) N_t^2 \overline{F}^2\big((u - 2\sqrt{u})/N_t\big)$.

Taking expectation and putting all the terms together we have obtained that

$$\mathbb{V}\mathrm{ar}\, \widehat{\psi}_A(u,t) \leq e^{-\gamma\sqrt{u}} + 2\,\mathbb{E}\left[N_t^2 \overline{F}^2 \left(\frac{u - 2\sqrt{u}}{N_t} \right); N_t \geq 1 \right]$$

when $u > 1$. Divide the l.h.s. by $\overline{F}^2(u)$, take the limit as $u \to \infty$ and rewrite it as

$$\lim_{u\to\infty} \frac{\mathrm{e}^{-\gamma\sqrt{u}}}{\overline{F}^2(u)} + \frac{\overline{F}^2(u - 2\sqrt{u})}{\overline{F}^2(u)} 2\,\mathbb{E}\left[N_t^2 \frac{\overline{F}^2\big((u - 2\sqrt{u})/N_t\big)}{\overline{F}^2(u - 2\sqrt{u})}; N_t \geq 1 \right].$$

Since $\overline{F}$ is subexponential the first limit is 0. Recalling $\overline{F}(x) = \ell(x)/x^\alpha$, the second is the limit of

$$\mathbb{E}\left[N_t^{2+2\alpha} \frac{\ell(u/N_t)^2}{\ell(u)^2}; N_t \geq 1 \right].$$

We split this expectation into two parts corresponding to $1 \leq N_t \leq u$ or $N_t > u$. Since ℓ is slowly varying, $\ell(x)$ and $1/\ell(x)$ are both is $\mathrm{O}(x^{\epsilon/4})$, so the first part is bounded by

$$\mathbb{E}\left[N_t^{2+2\alpha} \frac{\sup_{1\leq v\leq u} \ell(v)^2}{\ell(u)^2}; 1 \leq N_t \leq u \right] \;=\; \mathrm{O}(u^\epsilon)\mathbb{E}\, N_t^{2+2\alpha} \;=\; \mathrm{O}(u^\epsilon)\mathrm{O}(t^{2+2\alpha})\,.$$

Using $\overline{F}(x) \leq 1$, the second part is bounded by

$$\mathbb{E}\left[N_t^2 \frac{u^{2\alpha}}{\ell(u)^2}; N_t > u \right] \;=\; \frac{u^{2\alpha}}{\ell(u)^2}\mathrm{O}(\mathrm{e}^{-\delta u}) \;=\; \mathrm{O}(\mathrm{e}^{-\delta u/2})$$

for some $\delta > 0$ where we used $t/u \to 0$ and easy tail estimates in the Poisson distribution. Putting these estimates together completes the proof. □

Proof of Theorem 1. For an upper bound of the variance of the estimator B we have

$$\begin{aligned}\mathbb{V}\mathrm{ar}\,[\overline{\psi}_B(u)] &\le \mathbb{E}\,[L^2\overline{F}_I^2(H)\mathbb{P}^2(W < t(u)|V = V_{\max})] \\ &= \mathbb{E}\,[L^2\overline{F}_I^2(H)\mathbb{P}^2(U - V_{\max} < t(u)|U > V_{\max})]\end{aligned}$$

where $V_{\max}$ is the the largest value among $V_1, \ldots, V_K$. Next observe that the process is simulated in such way that ruin occurs with probability 1, then it is not difficult to verify that $H > u/L$ and $V_{\max} > u/L$. Since $H > u/L$, the stated hypothesis on the overshoot distribution gives the following upper bound for $\mathbb{V}\mathrm{ar}\,\widehat{\psi}_B^2$:

$$\begin{aligned}&\mathbb{E}\,[L^2\overline{F}_I^2(u/L)\mathbb{P}^2(U - u/L < t(u)|U > u/L)] \\ &\qquad = \mathrm{O}(1)\frac{t^2}{u^2}\mathbb{E}\,[L^4\overline{F}_I^2(u/L);\, L \le u] \,+\, \mathbb{E}\,[L^2; L > u]\,. \quad (5.5)\end{aligned}$$

Now comparing (2.1) with (2.2) when $k > 0$ and with (2.3) when $k = 0$ shows that $\psi(u,t)$ is always of order $t\ell(u)/u^\alpha$. Since $k < \infty$, the second term in (5.5) is therefore $\mathrm{o}\big(\psi(u,t)^2\big)$. Dividing the first term by $\psi(u,t)^2$, we obtain

$$\mathrm{O}(1)u^{2\alpha-2}\mathbb{E}\left[L^4\frac{\overline{F}_I^2(u/L)}{\ell(u)^2};\, L \le u\right] \;=\; \mathrm{O}(1)\mathbb{E}\left[L^{2\alpha+4}\frac{\ell(u/L)^2}{\ell(u)^2};\, L \le u\right]. \quad (5.6)$$

The proof will therefore be completed if we can show that the r.h.s. of (5.6) remains bounded as $u \to \infty$. Less restrictively, it is easy to see from the above analysis that it suffices to show this assertion with the qualifier $L \le u$ replaced by $L \le au$ for some $a > 0$.

To this end, write

$$\frac{\ell(u/L)}{\ell(u)} \;=\; \frac{\ell(u/L)}{\ell(u/(L-1)}\cdot\frac{\ell(u/(L-1))}{\ell(u/(L-2)}\cdots\frac{\ell(u/2)}{\ell(u)}\,. \quad (5.7)$$

From the uniform convergence theorem for slowly varying functions (see the Appendix of Embrechts, Klüppelberg and Mikosch, 1997), it follows that given $\epsilon > 0$, there exists u_0 such that

$$\frac{\ell(ut)}{\ell(u)} \;\le\; 1+\epsilon \quad \text{for all } 1/2 \le t \le 1 \;\text{ and all } u \ge u_0\,. \quad (5.8)$$

For $u \ge u_0$, the r.v. $L^{2\alpha+4}\ell(u/L)^2/\ell(u)^2 \cdot \mathbb{I}\{L \le u/u_0\}$ is therefore bounded by $L^{2\alpha+4}(1+\epsilon)^{2L}$, which is integrable if ϵ is so small that $\rho(1+\epsilon)^2 < 1$. Since $L^{2\alpha+4}\ell(u/L)^2/\ell(u)^2 \cdot \mathbb{I}\{L \le u\} \;\to\; L^{2\alpha+4}$ a.s. by the definition of a slowly varying function, dominated convergence therefore gives

$$\mathbb{E}\left[L^{2\alpha+4}\frac{\ell(u/L)^2}{\ell(u)^2};\, L \le u/u_0\right] \;\to\; \mathbb{E}\,L^{2\alpha+4} \;<\; \infty\,,$$

and the proof is complete □

Remark 2. The efficiency result in Asmussen and Kroese (2006) on the estimator (3.2) includes a (weak) regularity condition on the slowly varying function ℓ. This can be removed by using the above dominated convergence argument based on (5.7), (5.8). □

6 Tables and figures

In this section, we give some numerical examples to illustrate the performance of our algorithms. We considered the M/G/1—Cramér-Lundberg setting, taking U to be Pareto(α) distributed with $\overline{F}(x) = (1+x)^{-\alpha}$. In the figures, the upper left panel shows the estimates $\widehat{\psi}_A(u,t)$, $\widehat{\psi}_B(u,t)$ of $\psi(u,t)$. The lower left panel gives the variance of $\widehat{\psi}_A(u,t)$, $\widehat{\psi}_A(u,t)$ and the lower right panel the elapsed computer time. Finally, the upper right panel gives variance times elapsed time (which is the most fair comparison measure of the algorithms, cf. Asmussen and Glynn (2007) Section III.10) normalized by $z^2 = \psi(u,t)^2$. $R = 1000$ replications were used in each case.

Example 1. Here we estimate $\psi(u,u)$ by implementing Algorithms A and B. The approximation given by (2.2) is included for comparison purposes. We have used $\alpha = 3/2$, $\beta = 1/3$ and a 'window' of $u = (100, 1000)$.

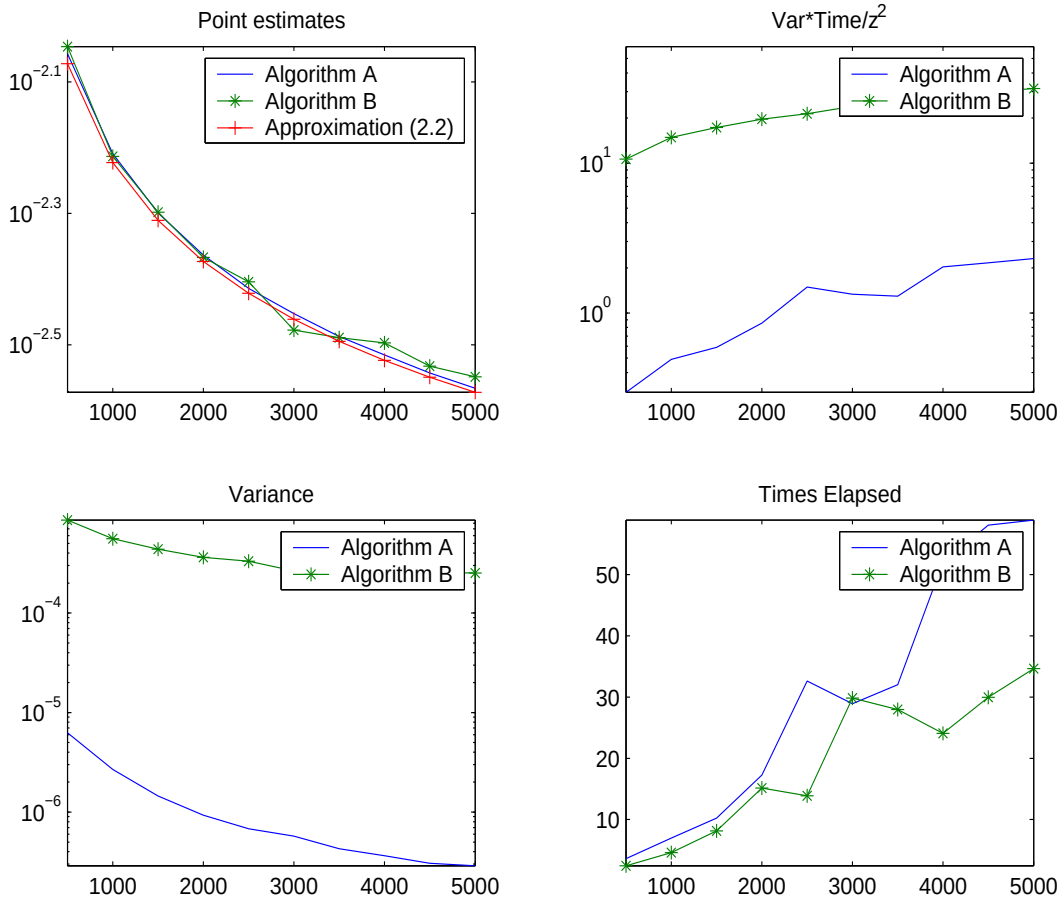

Figure 3: $\alpha = \frac{3}{2}, \beta = \frac{1}{3}$.

Example 2. Here we estimated $\psi(u, \sqrt{u})$, that is, considered a smaller time horizon than in Example 1.. The approximation given by (2.3) is included for comparison purposes. We have used $\alpha = 3/2$, $\beta = 1/3$ and a 'window' of $u = (100, 1000)$.

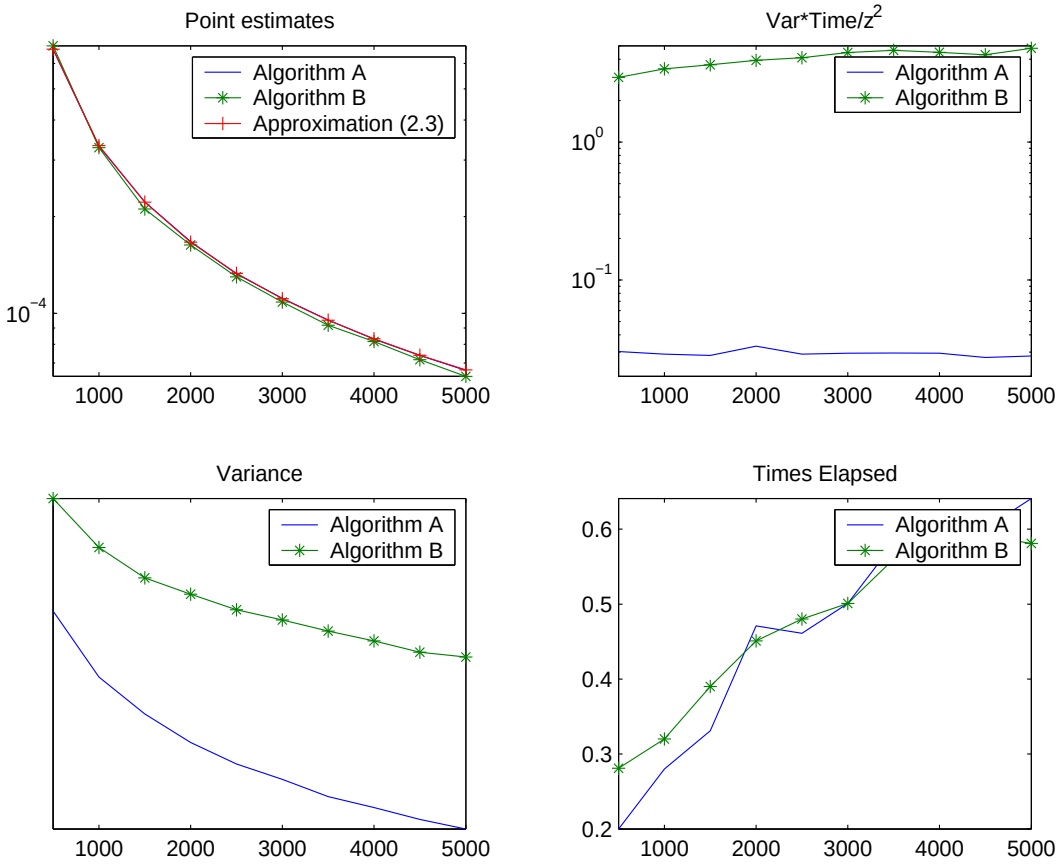

Figure 4: $\alpha = \frac{3}{2}, \beta = \frac{1}{3}$.

Conclusion

The conclusion of the numerical examples is that both algorithms appear to give excellent results. It is notable, however, that according to the variance times elapsed time criterion Algorithm A has a substantially better performance than Algorithm B. This could be an indication that the estimates of Proposition 1 are too rough, but we have not been able to come up with sharper bounds.

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