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ESTIMATION OF THE MEAN NORMAL MEASURE FROM FLAT SECTIONS

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Abstract

We discuss the determination of the mean normal measure of a stationary random set $Z \subset \mathbb{R}^d$ by measurements taken in intersections of Z with k-dimensional planes. We show that mean normal measures of sections with vertical planes determine the mean normal measure of Z, if $k \geq 3$ or $k = 2$ and an additional mild assumption holds. The mean normal measures of finitely many flat sections are not sufficient for this purpose. On the other hand, a discrete mean normal measure can be *verified* by mean normal measures of intersections with almost all m-tuples of planes, when $m \geq \lfloor d/k \rfloor + 1$. A consistent estimator for the mean normal measure of Z, based on stereological measurements in vertical sections, is also presented.

Keywords: random set; anisotropy; oriented mean normal measure; rose of normal directions; spherical projection; vertical sections; verification

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1. Introduction

Local first order properties of a stationary random surface are completely described by its surface area density and its rose of normal directions. The surface area density is a real valued quantity given as the mean surface area of the random surface per unit volume. The rose of normal directions is a measure on the unit sphere and can be interpreted as the distribution of the unit normal at the surface in a typical point. This distribution has been used to describe (average) anisotropy properties of the random surface and a number of stereological procedures has been suggested to estimate this measure from flat sections; see [28, Chapter 9], [1, Chapter 5] and [2] and the references therein. Actually, all estimation procedures first yield the *unoriented mean normal measure*, which is the product of the surface area density and the rose of normal directions. An estimator of the rose is then simply obtained by normalization. For sufficiently regular stationary random sets Z, the rose of normal directions is often defined as the rose of the boundary ∂Z of Z; see e.g. [18]. (By duality, also the roses of tangent directions of stationary fibre processes are covered by this theory.) However, not all kinds of anisotropy can be determined using the unoriented mean normal measure. Consider for example the random surface consisting of the union of small circles in the plane, arranged in horizontal rows. This random set clearly exhibits anisotropy, but its rose is uniform. Such kind of anisotropy, due to systematic spatial displacements, was formalized and studied in [27]. Even if the displacement is not systematic, the rose of normal directions can

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be uniform, although the process is intuitively not isotropic. Consider for example the boundary ∂Z of a stationary Boolean model Z in the plane whose typical particle is a deterministic Reuleaux triangle (an equilateral triangle T with its sides replaced by arcs with endpoints at two vertices of T and centered at the third). In this case, the rose of normal directions is uniform, as it does not distinguish between inner and outer unit normals in a boundary point.

Weil [31] introduced a variant of the unoriented mean normal measure, the *mean normal measure* $\overline{S}(Z,\cdot)$ of a stationary random set Z in the extended convex ring, taking only the outer unit normal into account. (Schneider [24] called $\overline{S}(Z,\cdot)$ the *oriented mean normal measure* of Z.) The mean normal measure was previously considered in [30], [17], [20] for Boolean models and further treated in [32], [33]. In the above example, where Z is a Boolean model with a Reuleaux triangle as typical particle, $\overline{S}(Z,\cdot)$ is not uniform. Figure 1 illustrates that the mean normal measure can also be used to distinguish between certain random sets, even if the unoriented mean normal measures coincide.

FIGURE 1: Two realizations of planar stationary random sets in a rectangular sampling window with different (oriented) mean normal measures, but coinciding unoriented mean normal measures.

The purpose of the present article is to discuss in how far $\overline{S}(Z,\cdot)$ is determined from information in lower dimensional sections and to suggest an estimator for $\overline{S}(Z,\cdot)$ using a stereological procedure based on vertical sections. Similar uniqueness questions were dealt with in [13], but there, the intersection planes L were considered to be uniform random and connections between certain means of the mean normal measures $\overline{S}'(Z\cap L,\cdot)$ of $Z\cap L$ (with respect to L) and $\overline{S}(Z,\cdot)$ were discussed. In contrast to the unoriented mean normal measure, $\overline{S}(Z,\cdot)$ cannot be recovered from one-dimensional sections alone, even if the distributions of the random sets $Z\cap g$ are known for all lines g; see [13]. Flat sections of dimension k are sufficient for this purpose, if k is at least two: It was shown in [7] that the mean normal measures $\overline{S}'(Z\cap L,\cdot)$ determine $\overline{S}(Z,\cdot)$, if L runs through the family of all k-dimensional planes. It will be shown in Corollary 3.1 in Section 3 that one can even restrict to planes containing a given direction u (vertical sections), if $k\geq 3$, and if $k=2$ and a mild additional assumption is satisfied. Finitely many mean normal measures of intersections are not sufficient to determine $\overline{S}(Z,\cdot)$, but if $\overline{S}(Z,\cdot)$ is discrete, it can be *verified by m intersections* for almost all m-tuples of k-dimensional planes, if $m\geq\lfloor d/k\rfloor+1$. This is made precise in Theorem 3.2.

In Section 4 we discuss a procedure to estimate the mean normal measure. In the planar case other approaches have been suggested. Weil [30] gave a method for sets with polygonal boundary. He also suggested an estimator that uses the additivity of the support function. Rataj [20] assumed the knowledge of the area dilation by suitable test sets and gives an estimator of certain (mean) mixed areas. From the latter one can derive an estimate of the mean normal measure. In some practical applications only digitizations of the random set Z are available. Estimators for the mean normal measure from digital images can be found in [14] for the planar case and in [9] and [34] for three-dimensional sets. The only other estimation procedure in higher dimensions was suggested by Schneider in [24]. He uses classical stereological information from intersections with pairs of parallel hyperplanes to estimate the mean normal measure of a stationary particle process. It relies, however, on the inversion of an integral transform, for which no discrete inversion algorithms exist. In Section 4 we will present an estimator which is based on the analysis of vertical sections of Z. For each vertical section plane L certain intersection points of $\partial(Z \cap L)$ with test lines in L are counted. A two-fold application of a known algorithm to invert the cosine transform then yields a discrete non-parametric estimator for $\overline{S}(Z, \cdot)$.

In the next section, we give a more formal definition of the mean normal measure and recall relevant integral formulae connecting it to flat sections of Z.

2. The mean normal measure

We introduce some notation. The Euclidean norm and the inner product in $\mathbb{R}^d$ are denoted by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, respectively, the unit ball by B^d, and the unit circle by $S^{d-1} = \partial B^d$. Let $u^\perp$ be the linear hyperplane with unit normal $u \in S^{d-1}$. We will often use the great circle $u^\circ := u^\perp \cap S^{d-1}$, orthogonal to u and the relative open half spheres generated by it. Put

$$u^\oplus := \{v \in S^{d-1} : \langle u, v\rangle > 0\},$$

and $u^\ominus := (-u)^\oplus$. Let $\mathcal{L}_k^d$ be the Grassmannian of all k-dimensional linear subspaces of $\mathbb{R}^d$, and ν_k be the rotation invariant probability measure on this space. We write

$$\mathcal{L}_k^d(u) := \{L \in \mathcal{L}_k^d : u \in L\}$$

for the sheaf of all of k-dimensional subspaces containing the direction u. For a measure μ, defined on a sigma-algebra $\mathcal{B}$, the restriction to the measurable sets in $B \in \mathcal{B}$ will be denoted by $\mu \llcorner B$. Note that there is a formal difference between $\mu \llcorner B$ and $\mu(\cdot \cap B)$, as the latter is a measure on $\mathcal{B}$ (concentrated on B). $\mathcal{H}^m$ will always denote the Hausdorff measure of dimension m in $\mathbb{R}^d$. A measure μ on S^{d-1} is *even*, if it is coinciding with its *even part* given by

$$A \mapsto \tfrac{1}{2}\left(\mu(A) + \mu(-A)\right)$$

for Borel sets $A \subset S^{d-1}$.

Let $\mathcal{K}$ be the set of convex bodies (compact convex sets) in $\mathbb{R}^d$. The *support measures* (generalized curvature measures) $\Theta_j(K, \cdot)$, $j = 0, \ldots, d-1$, of $K \in \mathcal{K}$ can be defined as coefficients of a Steiner-type formula for local parallel volumes; see [23]. As we will only use the support measure of order $j = d-1$, we give a more direct description, which applies to arbitrary closed sets [10], but will here only be used for sets in the extended convex ring. The convex ring $\mathcal{R}$ is the family of all finite unions of convex sets, and the extended convex ring

$$\mathcal{S} := \{A \subset \mathbb{R}^d : A \cap K \in \mathcal{R} \text{ for all } K \in \mathcal{K}\}$$

is the family of sets that can locally be written as finite unions of convex bodies. For $K \in \mathcal{S}$, let $\partial^+ K$ be the *positive boundary* consisting of all $x \in \partial K$ that have an outer normal $u \in S^{d-1}$, meaning that there is a positive ε such that x is the (unique) metric projection of $x + \varepsilon u$ on K. Equivalently, the intersection of the ball $(x + \varepsilon u) + \varepsilon B^d$ and K is the singleton $\{x\}$. The set $N(K, x)$ of all outer normals of K at $x \in \partial^+ K$ contains either one, two antipodal or infinitely many points. Set

$$\partial^i K := \{x \in \partial^+ K : \operatorname{card} N(K, x) = i\}, \quad i = 1, 2.$$

The unique normal in $x \in \partial^1 K$ at K is denoted by $u(K, x)$. For $x \in \partial^2 K$ we select $u(K, x)$ as one of the two possible normals in a measurable manner. As $K \in \mathcal{S}$, we have

$$\mathcal{H}^{d-1}(\partial K) = \mathcal{H}^{d-1}(\partial^+ K) = \mathcal{H}^{d-1}(\partial^1 K \cup \partial^2 K).$$

If $K \in \mathcal{S}$ is topologically regular (i.e. it is the closure of its interior), then $\partial^2 K = \emptyset$. The *support measure* (of order $d-1$) of K is the measure on $\mathbb{R}^d \times S^{d-1}$ given by

$$\Theta(K, \cdot) = \int_{\partial^1 K \cup \partial^2 K} 1_{(x, u(K,x)) \in \cdot} \, \mathcal{H}^{d-1}(dx) + \int_{\partial^2 K} 1_{(x, -u(K,x)) \in \cdot} \, \mathcal{H}^{d-1}(dx).$$

The projection $S(K, \cdot) = \Theta(K, \mathbb{R}^d \times \cdot)$ is the *surface area measure* (of order $d-1$) of K. If K is contained in a subspace $L \in \mathcal{L}_k^d$ with $k \le d-2$ the curvature measure $\Theta(K, \cdot)$ is trivial. We then often consider the curvature measure of K as a subset of L and denote it by $\Theta'(K, \cdot)$. The subspace L will be clear from the context. Note that $\Theta'(K, \cdot)$ is a measure on $L \times S^{k-1}(L)$, where $S^{k-1}(L) = S^{d-1} \cap L$ is the unit sphere in L. We denote the orthogonal projection of $x \in \mathbb{R}^d$ on L by $x|L$. Let $\operatorname{pr}_L(u) := (u|L)/\|u|L\|$ be the *spherical projection* of $u \in S^{d-1}$ on $S^{k-1}(L)$. As $\operatorname{pr}_L(u)$ is undefined when u is in the orthogonal complement $L^\perp$ of L, we set $\operatorname{pr}_L(u) := v_0$ for some fixed vector $v_0 \in S^{k-1}(L)$ in this case. Rataj [21] showed a translative Crofton-type formula for sets of positive reach, which, by additivity, also holds for sets $K \in \mathcal{R}$. For $L \in \mathcal{L}_k^d$, $k = 1, \dots, d-1$, and any measurable function $f : L^\perp \times L \times S^{k-1}(L) \to \mathbb{R}_+$ we have

$$\begin{aligned} \int_{L^\perp} \int_{L \times S^{k-1}(L)} & f(-z, x, u) \, \Theta'\left((K + z) \cap L, d(x, u)\right) \, \mathcal{H}^{d-k}(dz) \\ &= \int_{\mathbb{R}^d \times S^{d-1}} f(x|L^\perp, x|L, \operatorname{pr}_L(u)) \, \|u|L\| \, \Theta(K, d(x, u)). \end{aligned} \tag{1}$$

This relation can also be derived from [11, Satz 2.9 and Lemma 2.5] using support measures relative to a gauge body which is not necessarily the Euclidean ball; see also [15]. Specializing (1) to surface area measures, one obtains

$$\int_{L^\perp} S'((K + z) \cap L, \cdot) \, \mathcal{H}^{d-k}(dz) = \pi_L S(K, \cdot), \tag{2}$$

where π_L is a linear and weakly continuous operator from the space $\mathcal{M}$ of finite signed Borel measures on S^{d-1} to the space $\mathcal{M}(L)$ of finite signed Borel measures on $S^{k-1}(L)$. For $\mu \in \mathcal{M}$ the measure $\pi_L \mu$ is defined as the image measure of

$$\int_{(\cdot)} \|u|L\| \, \mu(du) \tag{3}$$

under the spherical projection pr_L. In particular, if $L = g = \mathrm{lin}\, v$ is a line with direction $v \in S^{d-1}$, the measure $\pi_g \mu$ is concentrated on $\{-v, v\}$ and has total mass

$$(\pi_g \mu)(\{-v, v\}) = C\mu(v), \tag{4}$$

where

$$C\mu = \int_{S^{d-1}} |\langle u, \cdot \rangle| \, \mu(du)$$

is the *cosine transform* of μ. For later use we note that

$$\pi_M^L \pi_L = \pi_M \tag{5}$$

for all non-trivial subspaces $M \subset L$, where π_M^L is the projection operator on the unit sphere in M, relative to L. The operator π_L is strongly related to the cosine transform and hence to projection bodies (see [29] and [3]). It is a special case of more general spherical projection operators $\pi_{L,m}$, where the integrand in (3) is taken to the power m (see [8] for their properties and use in geometric tomography, and a proof of (5) in terms of $\pi_{L,m}$). A generalization of (2) for mixed surface area measures was shown in [7].

For a definition and basic properties of random closed sets we refer the reader to the book of Schneider and Weil [26]. The mean normal measure $\overline{S}(Z, \cdot)$ can be defined for arbitrary closed sets in $\mathbb{R}^d$ without any further assumption, cf. [10, Section 7]. However, due to this generality even the intuitive property that its mean total mass for a topologically regular random set equals the surface area density does not hold. In the following we will only consider stationary random closed sets Z in the extended convex ring $\mathcal{S}$. Many of the results also hold for $\mathcal{U}_{PR}$-sets, that is sets that can be written as locally finite unions of sets of positive reach, which are such that any finite intersection of them has again positive reach. However, as we make extensive use of translative Crofton formulae for surface area measures, this would require additional assumptions on the relative positions of Z and the considered intersection plane; see [21]. We assume throughout that Z is a stationary set in $\mathcal{S}$ and has finite mean local surface area: We have

$$\mathbb{E}\mathcal{H}^{d-1}(\partial Z \cap K) < \infty \tag{6}$$

for one (and hence all) convex bodies K with interior points. In the literature, the stronger integrability condition

$$\mathbb{E}2^{N(Z \cap K)} < \infty \tag{7}$$

is sometimes assumed, where $N(M)$ is the minimal number of convex bodies needed to write the nonempty set $M \in \mathcal{R}$ as their union, and $N(\emptyset) = 0$. We will not require (7). For $K \in \mathcal{K}$ with the origin in its interior Weil [31] defines the *mean normal measure of* Z by

$$\overline{S}(Z, \cdot) := \lim_{r \to \infty} \frac{1}{\mathcal{H}^d(rK)} \mathbb{E} S(Z \cap rK, \cdot), \tag{8}$$

which is independent of K. Moreover,

$$\overline{S}(Z, \cdot) = \mathbb{E}\left(S(Z \cap C^d, \cdot) - S(Z \cap C_1^d, \cdot) \right), \tag{9}$$

where C^d is the unit cube in $\mathbb{R}^d$ and

$$C_1^d = \{x = (x_1, \dots, x_d) \in C^d : \max_{i=1,\dots,d} x_i = 1\}$$

is its "upper right boundary". A third representation of the mean normal measure is

$$\overline{S}(Z, \cdot) = \frac{1}{\mathcal{H}^d(K)} \mathbb{E}\Theta(Z; (\operatorname{int} K) \times \cdot), \tag{10}$$

where K is an arbitrary convex body with interior points. (10) follows from a special case of a translative integral formula for support measures in [22]. In [31] the condition (7) was assumed, but can be replaced by the weaker condition (6), as, for any Borel set $A \subset S^{d-1}$,

$$0 \le S(Z \cap K, A) \le 2\mathcal{H}^{d-1}(\partial Z \cap K),$$

where the right hand side is monotone in K with respect to set inclusion.

A combination of (10) and (1) implies

$$\overline{S}'(Z \cap L, \cdot) = \pi_L \overline{S}(Z, \cdot). \tag{11}$$

This was shown by Weil [31] (see also [32]) and is the starting point for our uniqueness results.

3. Determination of the mean normal measure

Fix $k \in \{2, \dots, d-1\}$. It was shown in [7] that a finite signed measure μ on the unit sphere is uniquely determined by the family of all its projections $\pi_L \mu$, $L \in \mathcal{L}_k^d$. In view of (11) and the linearity of π_L, this shows that the mean normal measure $\overline{S}(Z, \cdot)$ is uniquely determined by all mean normal measures of sections $\overline{S}'(Z \cap L, \cdot)$, $L \in \mathcal{L}_k^d$. Our first result states that the latter family actually contains considerable redundant information, as we can restrict to vertical planes, i.e. planes all containing a line with a given direction $u \in S^{d-1}$, at least when $k \ge 3$. Its proof is based on injectivity properties of the cosine transform and avoids the use of mixed volumes and spherical harmonics which are the basis of the uniqueness result in [7]. We will write $\mathbf{o}$ for the zero measure.

Theorem 3.1. *Let a finite signed measure μ on the unit sphere, $k \in \{2, \dots, d-1\}$, and $u \in S^{d-1}$ be given. If $k \ge 3$ then*

$$\pi_L \mu = \mathbf{o} \text{ for all } L \in \mathcal{L}_k^d(u) \tag{12}$$

implies $\mu = \mathbf{o}$. For $k = 2$ the same implication holds true when $\mu \llcorner u^\circ$ is even.

Proof. We first fix $v \in S^{d-1}$ and $m \ge 2$, and show an intermediate claim: If

$$\pi_M \mu = \mathbf{o} \text{ for all } M \in \mathcal{L}_m^d(v), \tag{13}$$

then $\mu \llcorner v^\oplus = \mathbf{o}$. In fact, as a unit vector is in $v^\oplus$ if and only if its spherical projection on M is in $v^\oplus \cap M$, (13) implies

$$\pi_M \left(\mu(\cdot \cap v^\oplus)\right) = (\pi_M \mu)(\cdot \cap v^\oplus) = \mathbf{o}.$$

Hence, if g is an arbitrary line in M and we abbreviate $\mu^\oplus := \mu(\cdot \cap v^\oplus)$, (5) gives

$$\pi_g(\mu^\oplus) = \pi_g^M\left(\pi_M \mu^\oplus\right) = \mathbf{o}.$$

As g was arbitrary in L, which in turn was arbitrary in $\mathcal{L}_m^d(v)$, (4) implies that the cosine transform $C\mu^\oplus$ of $\mu^\oplus$ vanishes on S^{d-1}. The cosine transform determines the even part of a measure; see [23, Theorem 3.5.3 and Note 3 on p. 192]. As $\mu^\oplus$ is concentrated on a relative-open half sphere, we find $\mu^\oplus = \mathbf{o}$ and the intermediate claim is shown.

Now let the assumptions of the theorem be satisfied with $k \geq 2$. The intermediate claim with $m = k$, $v = \pm u$ implies

$$\mu \llcorner (u^\oplus \cup u^\ominus) = \mathbf{o}, \tag{14}$$

and it is enough to show that $\mu^\circ := \mu \llcorner u^\circ$ vanishes. For $L \in \mathcal{L}_k^d(u)$ and $M = L \cap u^\perp$, we have $\mathrm{pr}_L = \mathrm{pr}_M$ on u°, so (14) and (12) imply

$$\pi_M^{u^\perp}(\mu^\circ) = \pi_L \mu = \mathbf{o}. \tag{15}$$

Consider the case $k \geq 3$. As L can be chosen arbitrarily, (15) is true for all subspaces M of $u^\perp$ of dimension $k-1 \geq 2$. Thus, the intermediate claim can be applied in $u^\perp$ with $m = k-1$, μ replaced by μ° and arbitrary $v \in u^\circ$. We obtain $\mu^\circ \llcorner (v^\oplus \cap u^\perp) = \mathbf{o}$ for all $v \in u^\circ$, which clearly gives the desired result $\mu = \mathbf{o}$.

For $k = 2$, M in (15) is a line and (4) applied in $u^\perp$ together with the injectivity of the cosine transform on even measures implies that the even part of μ° must be zero. As we assumed that μ° is even, this gives $\mu^\circ = \mathbf{o}$, as required.

Due to (11), we may rephrase this result in terms of the mean normal measure.

Corollary 3.1. *Let Z be a stationary random set in the extended convex ring in $\mathbb{R}^d$ such that* (6) *holds. For fixed $u \in S^{d-1}$, the mean normal measure $\overline{S}(Z, \cdot)$ is uniquely determined by the mean normal measures $\overline{S}'(Z \cap L, \cdot)$ where L runs through the sheaf of planes $\mathcal{L}_k^d(u)$, if $k \geq 3$.*

The same holds true if $k = 2$ and $\overline{S}(Z, \cdot) \llcorner u^\circ$ is even.

Some comments on the case $k = 2$ are in place here, as ordinary planes in $\mathbb{R}^3$ are the most important case for applications. Firstly, the additional condition for $k = 2$ is obviously fulfilled when $\overline{S}(Z, \cdot) \llcorner u^\circ = \mathbf{o}$, and this is true in particular if $\overline{S}(Z, \cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure. If this cannot be assumed, one can randomize the choice of u. If u is chosen as a random isotropic direction on S^{d-1} (independent of Z), then $\overline{S}(Z, \cdot) \llcorner u^\perp = \mathbf{o}$ holds almost surely. Thus, also the uniqueness result holds almost surely. If extra assumptions are to be avoided, one can consider one additional section with a hyperplane that is not containing the axis of the sheaf: With essentially the same arguments that led to Theorem 3.1 it can be shown that $\overline{S}(Z, \cdot)$ is uniquely determined by the collection of measures

$$\overline{S}'(Z \cap L, \cdot), \quad L \in \mathcal{L}_2^d(u) \cup \{L_0\},$$

where $L_0 \in \mathcal{L}_{d-1}^d$ satisfies $u \notin L_0$.

The question arises whether *finitely* many sections are enough to determine the mean normal measure. This is not the case, as we will show below. On the other hand, *discrete* mean normal measures can be *verified* by finitely many (actually very few) suitably chosen flat sections. To prove these results, we will describe the action of π_L on discrete measures in a more geometric way, apply tools from geometric tomography, and transfer the obtained statements afterwards to mean normal measures of random sets.

We call a measure *discrete*, if its support is at most countable. Let $\mathcal{M}_d$ be the space of all discrete finite signed measures and $\mathcal{M}_d^+$ be the sub-cone of positive measures in $\mathcal{M}_d$. Any measure $\mu \in \mathcal{M}_d^+$ is of the form $\mu = \sum_{i=1}^N \alpha_i \delta_{u_i}$ with positive masses α_i, pairwise different support points $u_1, u_2, \dots$ in S^{d-1} and $N \in \mathbb{N}$ or $N = \infty$. The idea that spherical projections of discrete measures are closely related to orthogonal projections of associated point sets in $\mathbb{R}^d$ was pointed out to us by W. Weil and is expressed formally by (17) below. For μ as above, we set

$$\mathcal{P}(\mu) := \bigcup_{i=1}^N \{\alpha_i u_i\}.$$

This gives rise to a bijection from $\mathcal{M}_d^+$ onto the family $\mathcal{F}$ of at most countable sets $F \subset \mathbb{R}^d \setminus \{0\}$ with the properties that

(a) any ray emanating from 0 hits F in at most one point, and

(b) $\sum_{x \in F} \|x\| < \infty$.

Let $L \in \mathcal{L}_k^d$, $1 \le k \le d-1$. The definition of π_L implies

$$\pi_L \mu = \sum_{x \in L \setminus \{0\}} \|x\| \cdot X_{\mathcal{P}(\mu)}(x)\, \delta_{\frac{x}{\|x\|}} \tag{16}$$

where

$$X_A(x) = \operatorname{card}\left(A \cap (x + L^\perp)\right), \qquad x \in L,$$

is the *(discrete) X-ray function* of $A \subset \mathbb{R}^d$ in direction $L^\perp$. Thus, $\pi_L \mu$ is uniquely determined by the X-ray function of $\mathcal{P}(\mu)$. The converse is true for L in

$$\mathcal{N}_k^d(\mu) := \big\{ M \in \mathcal{L}_k^d : \operatorname{pr}_M(u_i), \dots, \operatorname{pr}_M(u_N) \text{ are pairwise different and } M^\perp \cap \{u_1, \dots, u_N\} = \emptyset \big\}.$$

For $L \in \mathcal{N}_k^d(\mu)$, we have $X_{\mathcal{P}(\mu)} = 1_{\mathcal{P}(\mu)|L}$ on L and

$$\mathcal{P}(\pi_L \mu) = \mathcal{P}(\mu)|L. \tag{17}$$

Note that

$$\nu_k(\mathcal{N}_k^d(\mu)) = 1 \quad \text{if} \quad 2 \le k \le d-1$$

for all $\mu \in \mathcal{M}_d^+$, whereas $\mathcal{N}_1^d(\mu) = \emptyset$ if the support of μ consists of at least three points. Surface area measures of convex bodies with interior points have this property for $d \ge 2$, which is the reason that (17) is of interest only for $k \ge 2$.

We now ask whether a discrete measure μ is determined by spherical projections on subspaces in a given finite set $\mathcal{L} \subset \mathcal{L}_k^d$. Due to (17) and the injectivity of $\mathcal{P}$, this question can be translated to the problem, whether a discrete set in $\mathbb{R}^d$ is determined by all X-ray functions on planes in $\mathcal{L}$. Discrete tomography yields an answer to this question.

Proposition 3.1. *Let $1 \le k \le d-1$ and $\mathcal{L} \subset \mathcal{L}_k^d$ be finite. Then there are two different finite measures μ and μ', indistinguishable by their spherical projections on the members of $\mathcal{L}$:*

$$\pi_L \mu = \pi_L \mu'$$

for all $L \in \mathcal{L}$. These measures can be chosen even and with finite supports.

Proof. Due to (5) it is enough to consider hyperplanes L. Let $\mathcal{L} = \{L_1, \dots, L_m\} \subset \mathcal{L}_{d-1}^d$. According to [4, Lemma 2.3.2] there exists an $\{L_1^\perp, \dots, L_m^\perp\}$-switching component. This is a union $A \cup A'$ of finite, disjoint and non-empty sets A and A' with

$$\operatorname{card}\big(A \cap (x + L_i^\perp)\big) = \operatorname{card}\big(A' \cap (x + L_i^\perp)\big) \text{ for all } x \in \mathbb{R}^d, i = 1, \dots, m. \tag{18}$$

Hence, A and A' have the same discrete X-ray functions in the directions $L_1^\perp, \dots, L_m^\perp$ although they are disjoint. (18) is invariant with respect to arbitrary translations of $A \cup A'$. It is also unchanged when $A \cup A'$ is reflected at the origin. We can therefore assume that

$$A \cap \bigcup_{i=1}^m L_i^\perp = A' \cap \bigcup_{i=1}^m L_i^\perp = \emptyset,$$

that every ray starting at 0 hits $A \cup A'$ in at most one point and that A and A' are both origin symmetric. These conditions imply that there are finite even measures μ and μ' with $\mathcal{P}(\mu) = A$ and $\mathcal{P}(\mu') = A'$ with disjoint finite supports. Due to (16) the spherical projections of these measures can be expressed using the X-ray functions of A and A', which coincide by (18). We thus obtain $\pi_{L_i}\mu = \pi_{L_i}\mu'$ for all $i = 1, \dots, m$.

Gardner [4, p. 64] introduces the notion of *verification* of sets by X-rays. We transfer this notion to the present context for measures in a sub-cone $\mathcal{M}'$ of $\mathcal{M}$: A measure $\mu \in \mathcal{M}'$ can *verified by spherical projections on m planes in $\mathcal{L}_k^d$*, if $L_1, \dots, L_m \in \mathcal{L}_k^d$ can be chosen (depending on μ), such that if $\mu' \in \mathcal{M}'$ and $\pi_{L_i}\mu = \pi_{L_i}\mu'$ for $1 \le i \le m$, then $\mu = \mu'$. We say that a measure $\mu \in \mathcal{M}'$ can be *verified by spherical projections on almost all m-tuples of planes in $\mathcal{L}_k^d$*, if for all $(L_1, \dots, L_m) \in \mathcal{L}_k^d \times \dots \times \mathcal{L}_k^d$ with the possible exception of a set of $\nu_k^{\otimes m}$-measure zero, again, $\mu' \in \mathcal{M}'$ and $\pi_{L_i}\mu = \pi_{L_i}\mu'$ for $1 \le i \le m$ implies $\mu = \mu'$. Note that the set of measure zero may depend on μ. This subtle dependence on μ is the reason for the use of the term *verification* instead of *determination*, cf. [4, p. 64]. Verification by spherical projections on almost all m-tuples of planes is stronger than verification by projections on m planes.

Proposition 3.2. *Let $2 \le k \le d-1$ and $m = \lfloor d/k \rfloor + 1$.*

Then any $\mu \in \mathcal{M}_d^+$ can be verified by spherical projections on almost all m-tuples of planes in $\mathcal{L}_k^d$. The value of m is best possible here.

Proof. Let $\mu \in \mathcal{M}_d^+$, $F := \mathcal{P}(\mu)$ and $m = \lfloor d/k \rfloor + 1$. We modify the proof of [5, Theorem 7.4] appropriately. We first choose $m-1$ subspaces $L_1, \dots, L_{m-1} \in \mathcal{N}_k^d(\mu)$ such that $L_1^\perp, \dots, L_{m-1}^\perp$ are in general position. Note that $\nu_k^{\otimes(m-1)}$-almost all $(L_1, \dots, L_{m-1})$ satisfy these conditions. The subspace $T := \bigcap_{i=1}^{m-1} L_i^\perp$ has dimension

$$\dim T = (m-1)(d-k) - (m-2)d = d - \lfloor d/k \rfloor k < k.$$

The set

$$G := \bigcap_{i=1}^{m-1} \{x + L_i^\perp : x \in F|L_i\}$$

is a finite union of translates of T, $G = \{y_1 + T, \dots, y_s + T\}$, say. Let L_{ij} be the maximal linear subspace parallel to the affine hull of $(y_i+T)\cup(y_j+T)$, $1 \le i < j \le s$. As $\dim T < k$, we have $\dim L_{ij} \le k$ and almost all $L_m \in \mathcal{L}_k^d$ have the property that $L_m^\perp$ intersects L_{ij} in one point for all $1 \le i < j \le s$. Hence all $(d-k)$-dimensional planes $x + L_m^\perp$, $x \in \mathbb{R}^d$, intersect at most one of the translates of T in G, and each of these intersections is a single point. Fix one of these subspaces $L_m \in \mathcal{N}_k^d(\mu)$. It is easy to see that

$$F = \bigcap_{i=1}^{m} \{x + L_i^\perp : x \in F|L_i\},$$

where the right hand side is determined by the spherical projections of μ due to (17). But $F = \mathcal{P}(\mu)$ determines μ. Note that $(L_1, \dots, L_m)$ was chosen arbitrarily in $\mathcal{L}_k^d \times \cdots \times \mathcal{L}_k^d$ excluding only a set of measure zero (which may depend on μ).

It remains to show that the number m is best possible. In the proof of [5, Theorem 7.4], the existence of a finite set $F \subset \mathbb{R}^d$ (being actually the vertex set of a zonotope with interior points) is shown such that for any set $\mathcal{L} \subset \mathcal{L}_k^d$ of $\lfloor d/k \rfloor$ subspaces, there is a different finite set $F' \subset \mathbb{R}^d$ with the same X-rays as F in directions $L^\perp$ for all $L \in \mathcal{L}$. The two measures μ and μ' with $\mathcal{P}(\mu) = F$ and $\mathcal{P}(\mu') = F'$ show that m is optimal.

In view of (11), the above results can be reformulated in terms of random sets. We call a random set Z *polyhedral*, if it is almost surely a locally finite union of convex polytopes, and *simple polyhedral*, if there is an at most countable set of unit vectors containing almost surely all facet normals of Z. A stationary random set Z is simple polyhedral if and only if $\overline{S}(Z, \cdot)$ is discrete. Typical examples of polyhedral sets are crystalline media. The random sets in part (a) of the following theorem can actually be chosen to be Boolean models (with deterministic grains); see [26] for a definition.

Theorem 3.2.

(a) For any $1 \le k \le d-1$ and any finite $\mathcal{L} \subset \mathcal{L}_k^d$ there are two stationary simple polyhedral random sets Z and Z' with different mean normal measures, but such that

$$\overline{S}'(Z \cap L, \cdot) = \overline{S}'(Z' \cap L, \cdot)$$

holds for all $L \in \mathcal{L}$.

(b) Let $2 \le k \le d-1$, $m = \lfloor d/k \rfloor + 1$, and Z be a stationary simple polyhedral random set obeying (6). Then, for $\nu_k^{\otimes m}$-almost all $(L_1, \dots, L_m) \in (\mathcal{L}_k^d)^m$ and all stationary polyhedral random sets Z'

$$\overline{S}'(Z \cap L_i, \cdot) = \overline{S}'(Z' \cap L_i, \cdot), \qquad i = 1, \dots, m,$$

implies $\overline{S}(Z, \cdot) = \overline{S}(Z', \cdot)$.

Proof. As (b) is a direct consequence of Proposition 3.2 and (11), it remains to show (a). According to Proposition 3.1, there are two different finite even measures μ and μ' with finite supports such that $\pi_L \mu = \pi_L \mu'$ holds for all $L \in \mathcal{L}$. We may assume that the supports of μ and μ' are not concentrated on great circles of S^{d-1} (otherwise add a suitable even measure in $\mathcal{M}_d^+$ to both). Due to Minkowski's existence theorem (see e.g. [23, Section 7.1]) there are two convex polytopes K and K' (origin symmetric and with interior points) with $S(K,\cdot) = \mu$ and $S(K',\cdot) = \mu'$. Now let Z and Z' be stationary Boolean models with deterministic typical grains K and K', respectively. Their positive intensities γ and γ' are depending on the volumes $V(K)$ and $V(K')$ of K and K' and chosen such that

$$\gamma e^{-\gamma V(K)} = \gamma' e^{-\gamma' V(K')} =: \alpha.$$

Using [33, Section 3.2], we see that

$$\overline{S}(Z,\cdot) = \gamma e^{-\gamma V(K)} S(K,\cdot) = \alpha\mu$$

and $\overline{S}(Z',\cdot) = \alpha\mu'$. Hence $\overline{S}(Z,\cdot) \neq \overline{S}(Z',\cdot)$ but

$$\overline{S}'(Z\cap L,\cdot) = \alpha\pi_L\mu = \alpha\pi_L\mu' = \overline{S}'(Z'\cap L,\cdot)$$

holds for all $L \in \mathcal{L}$ according to (11).

4. Estimation of the mean normal measure from vertical sections

From a stereological point of view, an estimation procedure for $\overline{S}(Z,\cdot)$ should only require simple data acquisition like measurements of lengths or counting. Schneider [24] suggests to use surface area measurements in intersections with hyperplanes. He considers the mean normal measure $\overline{S}(X,\cdot)$ of a stationary particle process X of convex particles, and shows that the family of densities

$$\overline{\mathcal{H}}^{d-2}(\partial^u X \cap u^\perp), \quad u \in S^{d-1},$$

determines $\overline{S}(X,\cdot)$ uniquely. Here $\overline{\mathcal{H}}^{d-2}$ denotes the density of the $(k-2)$-dimensional Hausdorff measure and $\partial^u X$ is the surface process of all boundary points of X-particles having an outer unit normal in $S^{d-1} \cap u^\oplus$ (Schneider actually works with normals in the closed half sphere $S^{d-1} \setminus u^\ominus$, which leads to the same densities for almost all u). He also suggests to estimate $\overline{\mathcal{H}}^{d-2}(\partial^u X \cap u^\perp)$ using a pair of parallel hyperplanes at small distance to avoid an explicit determination of the outer normal in $\mathbb{R}^d$. His uniqueness proof is based on injectivity properties of the hemispherical sine transform. As there is no discrete algorithm available which allows to invert this integral transform, the suggested method is of limited value for applications, at least at the time being.

We suggest here a method that is based on vertical sections in the spirit of the proof of Theorem 3.1. It allows to replace inversion of the hemispherical sine transform by inversion of the cosine transform, for which non-parametric algorithms exist. The common direction u of the sheaf of vertical intersection planes will be considered fixed and called *reference direction* in the following. For $L \in \mathcal{L}_k^d(u)$, let $\partial_L^u(Z\cap L)$ be the set of all boundary points of $Z\cap L$ (relative to L) that have an outer unit normal in $S^{k-1}(L) \cap u^\oplus$. For a test line $g = \operatorname{lin} v \subset L$, $v \in S^{k-1}(L)$, and a measurable set $W \subset g$ with $0 < \mathcal{H}^1(W) < \infty$ set

$$\gamma^+(v) := \frac{1}{\mathcal{H}^1(W)} \operatorname{card}\left(W \cap \partial_L^u(Z\cap L)\right). \tag{19}$$

Obviously, $\gamma^+(v)$ only requires the knowledge of $Z \cap L$, and $\mathbb{E}\gamma^+(v)$ is independent on the choice of W. Moreover, γ^+ is an unbiased estimator of the cosine transform of $\overline{S}(Z, \cdot \cap u^\oplus)$, which can be shown using the concept of weighted surface processes in [19]. We present an alternative proof here.

Proposition 4.1. *Let Z be topologically regular ($Z = \operatorname{cl}\operatorname{int} Z$) and assume that* (6) *holds. With the notations as above, we have*

$$\mathbb{E}\gamma^+(v) = \int_{u^\oplus} |\langle w, v\rangle|\, \overline{S}(Z, dw) = C\left[\overline{S}(Z, \cdot \cap u^\oplus)\right](v) \tag{20}$$

for all $v \in S^{d-1}$.

Proof. Let D^1 and D^{d-1} be relative open unit cubes in g and $v^\perp$, respectively, and put $C^d := \operatorname{cl}(D^1 + D^{d-1})$. We may assume $W = D^1$. As $\partial^u_L(Z \cap L) = (\partial^u Z) \cap L$, we have

$$\mathbb{E}\gamma^+(v) = \mathbb{E}\Theta'\big(Z \cap g, (\partial^u Z \cap D^1) \times S^1(g)\big)\,.$$

As support measures are defined locally, the stationarity of Z, Fubini's theorem, and (1) imply

$$\begin{aligned}
\mathbb{E}\gamma^+(v) &= \int_{v^\perp} \mathbb{E}\Theta'\big([Z \cap g] \cap [C^d + z], (\partial^u Z \cap D^1) \times S^1(g)\big)\mathcal{H}^{d-1}(dz)\\
&= \mathbb{E}\int_{v^\perp} \Theta'\big([(Z \cap C^d) + z] \cap g, ([\partial^u Z + z] \cap D^1) \times S^1(g)\big)\mathcal{H}^{d-1}(dz)\\
&= \mathbb{E}\int_{\mathbb{R}^d \times S^{d-1}} \mathbf{1}_{x \in \partial^u Z}\mathbf{1}_{x|g \in D^1} \|w|g\|\, \Theta\big(Z \cap C^d, d(x,w)\big)\\
&= \mathbb{E}\int_{[\operatorname{int} C^d \cap \partial^u Z] \times S^{d-1}} |\langle w, v\rangle|\, \Theta\big(Z \cap C^d, d(x,w)).
\end{aligned}$$

In the last step we used that $\Theta(Z \cap C^d, \cdot)$ vanishes outside $C^d \times S^{d-1}$, and that $\Theta(Z \cap C^d, \cdot)$-almost all (x, w) with $x \in D^1 \times \partial D^{d-1}$ satisfy $w \in v^\perp$ and do therefore not contribute to the integral. Using again that support measures are defined locally, we may replace $Z \cap C^d$ by Z. As $Z = \operatorname{cl}\operatorname{int} Z$, $\mathcal{H}^{d-1}$-almost all $x \in \partial Z$ have a unique outer unit normal w and $\mathbf{1}_{x \in \partial^u Z} = \mathbf{1}_{w \in u^\oplus}$ holds. Another application of Fubini's Theorem and (10) therefore yield

$$\begin{aligned}
\mathbb{E}\gamma^+(v) &= \int_{u^\oplus} |\langle w, v\rangle|\, \mathbb{E}\left[\Theta\big(Z, \operatorname{int} C^d \times \cdot\big)\right](dw)\\
&= \int_{u^\oplus} |\langle w, v\rangle|\, \overline{S}(Z, dw).
\end{aligned}$$

Measurability issues in the preceding line of arguments follow from the weak continuity of support measures and the general considerations in [25, Anhang II].

We remark that Proposition 4.1 is still valid when Z is not topologically regular, but the counts in the definition of γ^+ must be modified: any isolated point of $g \cap \partial^u_L(Z \cap L)$ in W must be counted twice, as it (almost surely) comes from a boundary point of Z with exactly two antipodal outer normals. Proposition 4.1 implies that $\mathbb{E}\gamma^+$ determines $\overline{S}(Z, \cdot \cap u^\oplus)$ uniquely, as this measure is concentrated on a set without antipodal pairs. Replacing u by $-u$, we can define a variable $\gamma^-(v)$ like in (19). It satisfies

$$\mathbb{E}\gamma^- = C\left[\overline{S}(Z, \cdot \cap u^\ominus)\right]. \tag{21}$$

An estimation procedure for $\overline{S}(Z, \cdot)$ is now straightforward, if we use known inversion algorithms from [12], [6] to find a discrete approximation of a measure from finitely many (approximate) values of its cosine transform. We choose here the least squares approach, as it is more robust against measurement errors than the linear program approach. To ease presentation, we restrict considerations to two-dimensional vertical sections in $\mathbb{R}^3$ and assume $Z = \operatorname{cl}\operatorname{int} Z$. An extension to k-dimensional vertical sections for not necessarily topologically regular sets $Z \subset \mathbb{R}^d$ is straightforward, as long as $2 \le k \le d-1$.

(a) Choose a reference direction $u \in S^2$ such that $\overline{S}(Z, \cdot) \llcorner u^\circ = \mathbf{o}$; see the comments after Corollary 3.1.

(b) Choose $m \in \mathbb{N}$ vertical planes $L_1, \dots, L_m \in \mathcal{L}_2^3(u)$ and consider *independent* realizations of the intersections $Z \cap L_i$, $i = 1, \dots, m$.

(c) In each L_i, choose $n \in \mathbb{N}$ test directions $v_{i1}, \dots, v_{in} \in S^1(L_i)$ and determine the counts $\gamma^-(v_{ij})$ and $\gamma^+(v_{ij})$ of directed boundary points of $Z \cap L_i$ in unit intervals, $j = 1, \dots, n$, $i = 1, \dots, m$. Define the mn-dimensional vectors of observations

$$\Gamma^- := (\gamma^-(v_{ij}))_{ij}, \text{ and } \Gamma^+ := (\gamma^+(v_{ij}))_{ij}.$$

According to (20), and (21) these vectors are unbiased estimators for

$$([C\overline{S}(Z, \cdot \cap u^\ominus)](v_{ij}))_{ij}, \text{ and } ([C\overline{S}(Z, \cdot \cap u^\oplus)](v_{ij}))_{ij},$$

respectively.

(d) For $\mu \in \mathcal{M}_+$ set $\mu^\ominus := \mu\big(\cdot \cap u^\ominus\big)$ and

$$\mathbf{C}^-(\mu) := (C\mu^\ominus(v_{ij}))_{ij} \in \mathbb{R}^{mn},$$

and define $\mathbf{C}^+(\mu)$ analogously. Find solutions $\hat{\mu}^-_{m,n}$ and $\hat{\mu}^+_{m,n}$ of the optimization problems

$$\begin{array}{ll} \text{minimize} & \|\mathbf{C}^-(\mu) - \Gamma^-\| \\ \text{subject to} & \mu \in \mathcal{M}_+ \text{ is even,} \end{array} \tag{22}$$

and

$$\begin{array}{ll} \text{minimize} & \|\mathbf{C}^+(\mu) - \Gamma^+\| \\ \text{subject to} & \mu \in \mathcal{M}_+ \text{ is even,} \end{array} \tag{23}$$

respectively. As these solutions are even measures whose cosine transforms best fit the measurements (in the least squares sense), they can be considered as estimators for the even parts of $\overline{S}(Z, \cdot \cap u^\ominus)$ and $\overline{S}(Z, \cdot \cap u^\oplus)$, respectively.

(e) The measure

$$\hat{\mu}_{m,n} = 2\left(\hat{\mu}^-_{m,n}(\cdot \cap u^\ominus) + \hat{\mu}^+_{m,n}(\cdot \cap u^\oplus)\right) \tag{24}$$

is an estimator for $\overline{S}(Z, \cdot)$.

The infinite dimensional least squares problems (22) and (23) can be discretized in a loss-free way; see e.g. [6]. This means that among all solutions of (22) (and similar among those of (23)) there is one with support in a *finite* set T of prescribed directions, where T is only depending on the measurement directions v_{ij}. This allows to replace (22) and (23) by quadratic programs with non-negativity constraints. The latter can then be solved by standard software. Note that the resulting estimator $\hat{\mu}_{m,n}$ in (24) is not necessarily a mean normal measure of some stationary random set, as its centroid need not coincide with the origin. Asymptotically, however, $\hat{\mu}_{m,n}$ converges to the mean normal measure of Z if the number of test directions v_{ij} increases and these directions are chosen properly. To make this strong consistency result precise, we impose a condition on v_{ij}, which is slightly stronger than the denseness of the symmetrized sequence in S^2. Following [12, p. 14], the sequence

$$(v_{ij})_{ij} = (v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}, \ldots)$$

is called *asymptotically dense* in S^2 if

$$\liminf_{k\to\infty} \frac{1}{k} \operatorname{card}(F_k \cap G) > 0,$$

for all origin-symmetric open sets $G \neq \emptyset$ in S^2. Here F_k is the set of the first k members of (v_{ij}). Gardner et al. [6] discuss related notions.

Theorem 4.1. *Let Z be a stationary random set in $\mathcal{S}$ with $Z = \operatorname{cl}\operatorname{int} Z$ and*

$$\mathbb{E}[N(Z \cap K)]^2 < \infty \tag{25}$$

for one (and hence all) convex bodies K with interior points. Assume that $(v_{ij})_{ij}$ is an asymptotically dense sequence in S^2, and n is fixed. Then, almost surely, the weak convergence

$$\lim_{m\to\infty} \hat{\mu}_{m,n} = \overline{S}(Z, \cdot)$$

holds.

Proof. Due to (24), it is enough to show that $\mu^+_{m,n}$ converges weakly and almost surely to the even part ν^+ of $\overline{S}(Z, . \cap u^\oplus)$ (and the corresponding result for $\mu^-_{m,n}$, which follows by the replacement of u by $-u$). Note that (20) reads $\mathbb{E}\gamma^+ = C\nu^+$ with this notation. The convergence claim is shown in several steps:

(α) The total masses $\mu^+_{m,n}(S^2)$ of $\mu^+_{m,n}$ are almost surely uniformly bounded.

(β) Let (τ_m) be the sequence of even probability measures

$$\tau_m := \frac{1}{2mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (\delta_{v_{ij}} + \delta_{-v_{ij}})$$

and (m') a subsequence of (m) such that $(\tau_{m'})$ converges weakly to a measure τ, say. Then, almost surely,

$$\frac{1}{n} \sum_{i=1}^{n} \lim_{m\to\infty} \frac{1}{m'} \sum_{i=1}^{m'} f(v_{ij})\gamma^+_{ij} = \int_{S^2} (C\nu^+)\, f d\tau$$

for any even continuous function f on the sphere.

(γ) If (m') is a subsequence of (m) such that a realization of $\mu^+_{m',n}$ converges weakly to μ^+ and $\tau_{m'}$ to τ, then

$$\frac{1}{n}\sum_{i=1}^{n}\lim_{m\to\infty}\sum_{i=1}^{m'}[C\mu^+_{m',n}](v_{ij})\gamma^+_{ij} = \int_{S^2}(C\mu^+)(C\nu^+)d\tau.$$

(δ) Excluding a set of measure zero, we have that any accumulation point of $\mu^+_{m,n}$ is ν^+.

(ε) $\mu^+_{m,n}$ converges almost surely weakly to ν^+.

We start by showing (β) and write $\|f\|_\infty$ for the maximum norm of f. Consider the independent random variables

$$X_j := \frac{1}{n}\sum_{i=1}^{n} f(v_{ij})\gamma^+_{ij}.$$

Their variance can be bounded by

$$\mathrm{Var}(X_j) \le \frac{\|f\|^2_\infty}{n^2}\mathbb{E}\left(\sum_{i=1}^{n}\gamma^+_{ij}\right)^2 \le \|f\|^2_\infty \max_{i=1}^{n}\mathbb{E}(\gamma^+_{ij})^2.$$

γ^+_{ij} counts (certain) boundary points of Z in a unit line segment s, say. Assuming general position, a convex set can have at most two boundary points in s, so $\gamma^+_{ij} \le 2N(Z\cap s)$ and stationarity implies

$$\mathbb{E}(\gamma^+_{ij})^2 \le 4\mathbb{E}[N(Z\cap 1/2B^3]^2.$$

By (25), the Kolmogorov criterion $\sum_{j=1}^\infty \mathrm{Var}(X_j)/j^2 < \infty$ is satisfied and (X_j) obeys the strong law of large numbers. The set of measure zero to be excluded in the strong law of large numbers can be chosen independent of f, as the Banach space of continuous functions on S^2 is separable.

The claim (γ) follows from (β), as the weak convergence of $\mu^+_{m'',n}$ to μ^+ implies uniform convergence of the corresponding cosine transforms; see e.g. [23, Theorem 1.8.12].

To show (α), we use that $\mu^+_{m,n}$ is a solution of (23). It therefore yields a better objective function value than the zero measure, so

$$\|\mathbf{C}(\mu^+_{m,n}) - \Gamma^+\|^2 \le \|\Gamma^+\|^2.$$

Let $c_{ij} = C(\mu^+_{m,n})(v_{ij})$ be the components of $\mathbf{C}^+(\mu^+_{m,n})$. Cauchy-Schwarz inequality yields

$$\frac{1}{mn}\sum_{ij} c_{ij} \le (mn)^{-1/2}\|\mathbf{C}^+(\mu^+_{m,n})\| \le \left(\frac{2}{mn}\sum_{ij} c_{ij}\gamma^+_{ij}\right)^{1/2}. \qquad (26)$$

As (v_{ij}) is asymptotically dense, there is a constant $c > 0$ such that

$$\frac{1}{c} \le \frac{1}{mn}\sum_{ij}|\langle v_{ij},\cdot\rangle|$$

holds on S^2 for all sufficiently large m. The left hand side of (26) is therefore bounded from below by $\mu^+_{m,n}(S^2)/c$. As $|\langle v_{ij},\cdot\rangle| \leq 1$, its right hand side can be estimated from above, and we obtain

$$\mu^+_{m,n}(S^2) \leq \frac{2c^2}{mn}\sum_{ij}\gamma^+_{ij}.$$

Again, the strong law of large numbers implies that the right hand side of the last inequality is almost surely bounded.

To show (δ) we fix a realization such that $(\mu^+_{m,n}(S^2))$ is bounded. Due to (α) this holds for almost all realizations. Let μ^+ be an arbitrary accumulation point of this sequence and assume that $(\mu^+_{m',n})$ converges to it. Extracting a subsequence (m'') of (m'), we may assume $\tau_{m''} \to \tau$.

By definition of $\mu^+_{m',n}$, no even measure in $\mathcal{M}_+$ can yield a better objective function value than $\mu^+_{m',n}$ in (23). In particular, comparison with ν^+ yields

$$\|\mathbf{C}(\mu^+_{m,n}) - \Gamma^+\|^2 \leq \|\mathbf{C}(\nu^+) - \Gamma^+\|^2.$$

This gives

$$\frac{1}{m''n}\left(\|\mathbf{C}(\mu^+_{m'',n})\|^2 - \|\mathbf{C}(\nu^+)\|^2\right) \leq \frac{2}{m''n}\sum_{ij}\left([C\mu^+_{m'',n}](v_{ij}) - C\nu^+(v_{ij})\right)\gamma^+_{ij}.$$

In view of (β) and (γ), taking $m \to \infty$ implies

$$\int_{S^2}[C\mu^+ - C\nu^+]^2 d\tau \leq 0.$$

As (v_{ij}) is asymptotically dense in S^2, the support of τ is S^2 and thus $C\mu^+ - C\nu^+ = 0$ on S^2. The two measures involved are even and therefore uniquely determined by their cosine transforms, which gives $\mu^+ = \nu^+$, as required.

Finally (ε) follows from the fact that a set of uniformly bounded measures is relatively compact in the weak topology. Hence, convergence of $(\mu^+_{m,n})$ to ν^+ is equivalent to the fact that ν^+ is the only accumulation point of $(\mu^+_{m,n})$. The latter was shown in (δ).

The proof of Theorem 4.1 is following Männle [16], who showed the consistency of a least squares estimator of a measure from cosine transforms under stochastic independence of the measurements. In general, independence of the sequence γ^+_{ij} can only be assured if each of the counts is obtained from an independent realization of Z – a sampling scheme, which is not realistic in practice. If long-range dependence in Z is not present or negligible (like in the example of a Boolean model of convex grains with uniformly bounded diameter), this independence can (approximately) be assured by placing the test line segments W in the definition of γ^+ far enough from each other. We mention that the above convergence result can also be derived under stronger (but somewhat unrealistic) assumptions from [6, Section 9]. The speed of convergence result, also shown there, cannot be transferred directly to the present situation, as the restriction of measures in (24) is not a Lipschitz mapping in the Prohorov metric. Our last result shows that the requirements in Theorem 4.1 are in particular satisfied, when $u, L_1, L_2, \ldots$ and the test directions are randomized. To do so, we denote the unique normalized Haar measure on $\mathcal{L}^3_2(u)$ by ν^u_2.

Corollary 4.1. *Let Z be a stationary random set in $\mathcal{S}$ with $Z = \operatorname{cl}\operatorname{int} Z$ such that* (25) *holds. Assume that $n \in \mathbb{N}$ is fixed and*

(a) *$u \in S^2$ is a random isotropic reference direction in S^2 independent of Z,*

(b) *given u, the planes $L_1, L_2, \ldots$ are independent identically distributed (i.i.d.) in $\mathcal{L}_2^3(u)$ with distribution ν_2^u independent of Z,*

(c) *given u and $L_1, L_2, \ldots$, for each $i = 1, 2, \ldots$, the directions $v_{i1}, \ldots, v_{in}$ are i.i.d. isotropic random vectors in $S^1(L_i)$, independent of Z.*

Then $\hat{\mu}_{m,n}$ converges almost surely weakly to $\overline{S}(Z, \cdot)$, as $m \to \infty$.

Proof. In view of Theorem 4.1 it remains to show that the (random) sequence (v_{ij}) is almost surely asymptotically dense in S^2. Given u, the distribution of v_{11} is up to a normalizing constant the measure

$$\int_{S^2} 1_{w \in \cdot} \sqrt{1 - \langle u, w \rangle^2}^{-1} dw,$$

which has S^2 as its support. The strong law of large numbers now shows that (v_{ij}) is almost surely asymptotically dense.

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