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Paul Goodey, Markus Kiderlen and Wolfgang Weil

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Paul Goodey, Markus Kiderlen*and Wolfgang Weil

Abstract

We consider a variety of integral transforms arising in Geometric Tomography. It will be shown that these can be put into a common framework using spherical projection and lifting operators. These operators will be applied to support functions and surface area measures of convex bodies and to radial functions of star bodies. We then investigate averages of lifted projections and show that they correspond to self-adjoint intertwining operators. We obtain formulas for the eigenvalues of these operators and use them to ascertain circumstances under which tomographic measurements determine the original bodies. This approach via mean lifted projections leads us to some unexpected relationships between seemingly disparate geometric constructions.

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1 Introduction

Geometric Tomography deals with the retrieval of information about a compact set K in $\mathbb{R}^d$ from data arising from sections or projections of K. In particular, one can ask which information about sections and projections suffices to determine K uniquely (possibly up to a translation or another simple transformation). In case K is convex or star-shaped (with respect to the origin o), K can be conveniently (and uniquely) described by its support function $h(K,\cdot)$ (in the convex case) respectively its radial function $\rho(K,\cdot)$ (in the star-shaped case, which includes convex sets, of course). Consequently, these sets may be associated with functions defined on the unit sphere S^{d-1}. For convex bodies, further analytical descriptions exist, for example using the surface area measures $S_j(K,\cdot), j = 1,\dots,d-1$, which are finite Borel measures on S^{d-1}.

Since the projection $K|L$ of a convex body K onto a subspace L is convex and the intersection $K \cap L$ of a star-shaped body K with a subspace L (through o) is star-shaped, the analytical description of $K|L$, respectively $K \cap L$, yields a function $f(K,L,\cdot)$ (or measure $\mu(K,L,\cdot)$) on the subsphere $S^{d-1} \cap L$. Exploiting these data in various ways (for example by averaging over all L or by other geometrical procedures), one usually obtains a function (or measure) $F(K,\cdot)$ on S^{d-1} which represents the tomographic data, and the basic question is whether (or to what extent) the body K is determined by $F(K,\cdot)$. Analytically, this situation can be described

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by a spherical transform T which maps $h(K,\cdot)$ (or $\rho(K,\cdot)$ or $S_j(K,\cdot)$) to $F(K,\cdot)$ and is, in most cases, given in integral form. In this way, the tomographic uniqueness problem for convex or star-shaped bodies K corresponds to the injectivity properties of the integral transform T.

For a large class of tomographic uniqueness problems, the associated integral operator is linear. We briefly describe two classical examples of this kind. From now on, we concentrate on compact sets K with nonempty interior (either convex or star-shaped) and we assume, in the star-shaped case, that o is an interior point and that $\rho(K,\cdot)$ is continuous. As usual, we speak of these sets as *convex bodies*, respectively *star bodies*.

Example 1 (Radon transform). The first example concerns the determination of a star body K by the $(d-1)$-volume $V_{d-1}(K\cap u^\perp)$ of its hyperplane sections $K\cap u^\perp$, $u\in S^{d-1}$ (here $u^\perp$ denotes the subspace orthogonal to u). Since

$$V_{d-1}(K\cap u^\perp)=\frac{1}{d-1}\int_{S^{d-1}\cap u^\perp}\rho^{d-1}(K,v)\,dv,$$

where the integration is with respect to the spherical Lebesgue measure on $S^{d-1}\cap u^\perp$, the geometric transform $K\mapsto V_{d-1}(K\cap(\cdot)^\perp)$ is associated with the spherical *Radon transform* R,

$$(Rf)(u)=\int_{S^{d-1}\cap u^\perp}f(v)\,dv,\qquad u\in S^{d-1}.$$

Since R is known to be injective (only) on even functions f, the volumes of the central sections $K\cap(\cdot)^\perp$ determine K only if K is symmetric (with respect to o).

Example 2 (Cosine transform). The second example is, in some sense, a dual. For a convex body K, we consider the volumes $V_{d-1}(K\,|\,u^\perp)$ of the $(d-1)$-dimensional projections and ask whether they determine K. Here,

$$V_{d-1}(K\,|\,u^\perp)=\frac{1}{2}\int_{S^{d-1}}|\langle u,v\rangle|S_{d-1}(K,dv),\quad u\in S^{d-1},$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^d$. The associated integral transform C is the *cosine transform*, it maps measures μ on S^{d-1} to functions via

$$(C\mu)(u)=\int_{S^{d-1}}|\langle u,v\rangle|\mu(dv),\qquad u\in S^{d-1}.$$

C is known to be injective (only) on even measures. Therefore, the volumes of the projections $K\,|\,(\cdot)^\perp$ determine the surface area measure $S_{d-1}(K,\cdot)$ only, if $S_{d-1}(K,\cdot)$ is even. Since $S_{d-1}(K,\cdot)$ determines K up to a translation and is even, if and only if K is centrally symmetric, we obtain the statement that a centrally symmetric convex body K is uniquely determined (up to a translation) by the $(d-1)$-dimensional volumes of its projections (onto hyperplanes).

Various generalizations of the two examples above are possible. For instance, we can replace the $(d-1)$-volume of the sections or projections by the Hausdorff measure of their boundaries or by other geometric functionals. Another obvious generalization is to replace hyperplanes by j-dimensional subspaces, $j \in \{2, \dots, d-2\}$, yielding functions on the Grassmannian $\mathcal{L}_j^d$ of j-spaces instead of the sphere. For convex bodies K, we can consider intersections with affine hyperplanes (or more generally, affine j-flats), and again the resulting functions live on a more complicated (homogeneous) space. Finally, the sectional or projectional data can be averaged in many different ways. For instance, we can average the sets directly, building the Minkowski sum, the Blaschke sum or the radial sum. Here, we give two examples of that kind. In both cases, K is a convex body and $k \in \{1, \dots, d-1\}$ is a fixed dimension.

Example 3 (Projection mean body). For each k-dimensional subspace $L \subset \mathbb{R}^d$, we consider the projection $K|L$ as a (k-dimensional) body in $\mathbb{R}^d$ and average this set over all L. The resulting convex body $P_k(K)$ is called the k-th *projection mean body* of K. Analytically, it is defined by its support function,

$$h(P_k(K), \cdot) = \int_{\mathcal{L}_k^d} h(K|L, \cdot)\, dL,$$

where the integration is with respect to the invariant probability measure on $\mathcal{L}_k^d$. The projection mean body $P_{d-1}(K)$ was introduced by Schneider [27], who showed that K is a ball, if and only if $P_{d-1}(K)$ is homothetic to K. Spriestersbach [35] proved an injectivity result for the associated spherical operator $p_{d-1} : h(K, \cdot) \mapsto h(P_{d-1}(K), \cdot)$. Further injectivity results for the corresponding operators p_k are due to Goodey [7] and Goodey and Jiang [9]. In summary, p_k is injective for $k \geq d/2$ and for $k = 3$, whereas p_2 is injective only for $d \neq 14$. The cases $3 < k < d/2$ are still open.

Example 4 (Blaschke section body). In order to define the k-th *Blaschke section body* $B_k(K)$ of K, we first consider a fixed subspace $L \in \mathcal{L}_k^d$. Integration over all parallel sections of K, in the sense of Blaschke addition, defines a relative Blaschke section body $B_L(K) \subset L$. More precisely, $B_L(K)$ is given by its surface area measure in L,

$$S'_{k-1}(B_L(K), \cdot) = \int_{L^\perp} S'_{k-1}((K + x) \cap L, \cdot)\, dx,$$

where the integration is with respect to the Lebesgue measure in the orthogonal space $L^\perp$. Here, we used Minkowski's existence theorem to guarantee that the right hand side is indeed the $(k-1)$-st surface area measure of some convex body in L. By the same theorem, the invariant mean of the measures $S'_{k-1}(B_L(K), \cdot)$ (considered as measures on S^{d-1}) is the $(d-1)$-st surface area measure of a convex body $B_k(K)$:

$$S_{d-1}(B_k(K), \cdot) = \int_{\mathcal{L}_k^d} S'_{k-1}(B_L(K), \cdot)\, dL.$$

The Blaschke section body $B_k(K)$ was first studied in [10]. Surprisingly, the associated transform $b_k : S_{d-1}(K, \cdot) \mapsto S_{d-1}(B_k(K), \cdot)$ is (up to linearity) just the extension of p_k to measures. It therefore has similar injectivity properties.

A different averaging procedure arises if geometric functionals of sections or projections are integrated over all k-dimensional subspaces which contain a given direction u. Again, the resulting function or measure will live on S^{d-1} and gives rise to a spherical operator. We conclude this introduction with an example of that kind.

Example 5 (**Directed section mean**)**.** Let K be a star body, $u \in S^{d-1}$ and $L \in \mathcal{L}_k^d$ a subspace containing u. If u^+ denotes the half-space bounded by $u^\perp$ and containing u, $L \cap u^+$ is a k-dimensional half-space. We define the *averaged directed section function* $\overline{s}_k(K, \cdot)$ as the volume of $K \cap L \cap u^+$, integrated over the set $\mathcal{L}_k^{[u]}$ of all k-spaces L which contain u,

$$\overline{s}_k(K,u) = \frac{1}{k} \int_{\mathcal{L}_k^{[u]}} \int_{S^{d-1}\cap L \cap u^+} \rho^k(K,v)\, dv\, dL, \quad u \in S^{d-1}.$$

The associated spherical operator A_k^d satisfies

$$(A_k^d f)(u) = \int_{\mathcal{L}_k^{[u]}} \int_{S^{d-1}\cap L \cap u^+} f(v)\, dv\, dL, \quad u \in S^{d-1}.$$

As we shall see, it exhibits a much more diverse injectivity behavior.

In the following, we discuss tomographic transforms involving sections and projections and the associated spherical operators in greater generality. We introduce families of spherical projections and liftings and study their averages. In order to investigate the injectivity properties, we make use of the fact that the resulting integral operators intertwine the group SO_d of rotations. For such operators T, the methods of spherical harmonic analysis can be applied. In particular, the eigenfunctions of T are the spherical harmonics, and the injectivity properties of T require us to know which of the corresponding eigenvalues are nonzero, see [6] for a discussion of these ideas in the context of convexity.

2 Basic notations

In this section, we describe some standard notations, including those which were already mentioned above.

Throughout the following, we work in d-dimensional Euclidean space $\mathbb{R}^d$. We generally assume $d \geq 3$, although many operators are also defined for $d = 2$ or even $d = 1$. However, these cases often require additional interpretations and frequently their injectivity behavior is different. From the point of view of applications, the three-dimensional case is the most interesting one.

We use $\langle u, x\rangle$ for the standard scalar product of $u, x \in \mathbb{R}^d$ and $\|\cdot\|$ for the (Euclidean) norm. B^d is the unit ball and S^{d-1} the unit sphere. The Lebesgue measure in $\mathbb{R}^d$ is denoted by λ_d and the spherical Lebesgue measure on S^{d-1} is denoted by ω_{d-1}. The volume of B^d will be denoted by κ_d and so the surface area of S^{d-1} is $\varpi_d = \omega_{d-1}(S^{d-1}) = d\kappa_d$. Explicitly, we have

$$\varpi_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{2.1}$$

For $u \in S^{d-1}$, let $[u]$ be the line generated by u, $u^\perp$ the $(d-1)$-dimensional subspace orthogonal to u and $u^+ = \{x \in \mathbb{R}^d : \langle u, x\rangle \geq 0\}$ the closed half-space generated by u (and containing u). $\mathbf{1}_A$ is the indicator function of a set A and $A|L$ denotes the orthogonal projection of a set $A \subset \mathbb{R}^d$ onto a subspace L.

For $j \in \{0, \dots, d\}$, let $\mathcal{L}_j^d$ denote the Grassmannian of all j-dimensional subspaces in $\mathbb{R}^d$. If $j \neq k$ and $M \in \mathcal{L}_k^d$, we let $\mathcal{L}_j^M$ be the submanifold of all j-spaces containing or contained in the fixed k-space M (depending on whether $j > k$ or $j < k$). Also, for $M \in \mathcal{L}_k^d$ and $u \in S^{d-1}$, $M \vee u$ is the subspace generated by M and u. We denote the unique invariant probability measures on $\mathcal{L}_j^d$ and $\mathcal{L}_j^M$ by ν_j and ν_j^M, respectively. For $L \in \mathcal{L}_j^d$, we put $S^{j-1}(L) = L \cap S^{d-1}$ and denote the spherical Lebesgue measure on $S^{j-1}(L)$ by ω_{j-1}^L.

Within integrals, we simply write $dx, du, dL, \dots$ if integration is with respect to the corresponding invariant measure, as long as the latter is clear from the range given under the integral sign.

For a compact set $T \subset \mathbb{R}^d$, let $\mathcal{C}(T)$ be the Banach space of continuous functions f on T, supplied with the maximum norm $\|f\|_\infty$. The dual space $\mathcal{M}(T)$ is the space of finite signed Borel measures on T (and we supply the latter with the weak* topology). The Borel σ-algebra on T is denoted by $\mathcal{B}(T)$. Measurability of functions always refers to the corresponding Borel σ-algebra.

We shall frequently use the following decomposition of ω_{d-1} with respect to cylindrical coordinates (see e.g. [23]),

$$\int_{S^{d-1}} f(v)\,dv = \int_{S^{d-2}(u^\perp)} \int_{-1}^{1} f(tu + \sqrt{1-t^2}v)\,(1-t^2)^{(d-3)/2}\,dt\,dv. \tag{2.2}$$

Here, $u \in S^{d-1}$ is fixed and $f \geq 0$ is a measurable function. A more general result yields

$$\int_{S^{d-1}} f(v)\,dv = \int_{S^{k-1}(L)} \int_{H^{d-k}(L,v)} f(w)\langle v, w\rangle^{k-1}\,dw\,dv, \tag{2.3}$$

for $k \in \{1, \dots, d-1\}$ and $L \in \mathcal{L}_k^d$. Here,

$$H^{d-k}(L, v) = \{u \in S^{d-1} \setminus L^\perp : \mathrm{pr}_L(u) = v\}$$

is the relatively open $(d-k)$-dimensional half-sphere, generated by $L^\perp$ and $v \in S^{k-1}(L)$, and $\mathrm{pr}_L(u) = \frac{u|L}{\|u|L\|}$ is the *spherical projection* of $u \in S^{d-1} \setminus L^\perp$ onto $S^{k-1}(L)$. This decomposition (2.3) of the spherical Lebesgue measure was already used in [36] and [4]. For $k = d-1$ and $L = u^\perp$, and shifting to cylindrical coordinates, it yields formula (2.2).

We also recall an integral formula of Chern in the form quoted in [7]. For a measurable function $h \geq 0$ on $\mathcal{L}_k^d$ and $v \in S^{d-1}$, we have

$$\int_{\mathcal{L}_k^d} h(L)\,dL = \frac{\varpi_k}{2\varpi_d} \int_{\mathcal{L}_{k-1}^{v^\perp}} \int_{S^{d-k}(M^\perp)} h(M \vee u)|\langle u, v\rangle|^{k-1}\,du\,dM. \tag{2.4}$$

A similar formula, for $u \in S^{d-1}$ and a measurable function $h \geq 0$ on the sphere, can be obtained from a decomposition of the spherical Lebesgue measure in $M^\perp$ and an invariance argument,

$$\int_{S^{d-1}} h(v)(1 - \langle v, u\rangle^2)^{(1-k)/2}\,dv = \frac{\varpi_{d-1}}{\varpi_{d-k}} \int_{\mathcal{L}_{k-1}^{u^\perp}} \int_{S^{d-k}(M^\perp)} h(v)\,dv\,dM. \tag{2.5}$$

As we already mentioned, a *convex body* K in this work is a compact convex set in $\mathbb{R}^d$, which has interior points. The latter assumption is usually not made in convex geometry (see for example the book of Schneider [28] which we use as the standard reference, in the following). It is, however, helpful in our context since some injectivity results only hold with dimensional restrictions on the sets under consideration. The disadvantage is that some of the results are not stated in their utmost generality, but the reader should be able to make the necessary extensions. We denote by $\mathcal{K}^d$ the set of all convex bodies and supply it with the Hausdorff metric. A convex body K can be described by its *support function* $h(K,\cdot)$,

$$h(K,y) = \max\{\langle x,y\rangle : x \in K\},\ y \in \mathbb{R}^d.$$

$h(K,\cdot)$ is a continuous function on $\mathbb{R}^d$ which is homogeneous of degree 1. Therefore, we will mostly work with the restriction of $h(K,\cdot)$ to the unit sphere S^{d-1}. The mapping

$$\mathcal{K}^d \to \mathcal{C}(S^{d-1}), K \mapsto h(K,\cdot),$$

is then continuous and injective. It is, in addition, linear with respect to nonnegative linear combinations of convex bodies. An alternative description of convex bodies makes use of the *surface area measures*

$$S_j(K,\cdot), \qquad j = 1,\dots,d-1.$$

For a Borel set $A \in \mathcal{B}(S^{d-1})$, they are defined by the *local Steiner formula*

$$\lambda_d(B_\varepsilon(K,A)) = \frac{1}{d}\sum_{j=0}^{d-1} \varepsilon^{d-j}\binom{d}{j} S_j(K,A), \tag{2.6}$$

for $\varepsilon > 0$. Here, $B_\varepsilon(K,A)$ is the (Borel) set of all points $x \notin K$ whose nearest point $p(K,x) \in K$ satisfies $\|x - p(K,x)\| \le \varepsilon$ and $(x - p(K,x))/\|x - p(K,x)\| \in A$. Note, that the measure $S_0(K,\cdot)$, occurring in (2.6), is independent of K (it coincides with the spherical Lebesgue measure ω_{d-1}). For $j = 1,\dots,d-1$, the mapping

$$K \mapsto S_j(K,\cdot)$$

is again continuous but determines the body K only up to translation. $K \mapsto S_1(K,\cdot)$ is also linear.

For a star-shaped compact set K, containing o in its interior, we consider the *radial function*

$$\rho(K,u) = \max\{r \ge 0 : ru \in K\}, \quad u \in S^{d-1}.$$

As mentioned earlier, we assume that $\rho(K,\cdot) \in \mathcal{C}(S^{d-1})$ and refer to those sets as *star bodies*. Let $\mathcal{S}^d$ be the set of all star bodies, supplied with the radial metric. The standard reference here is the book of Gardner [5] (which uses a slightly more general definition of star bodies). Again,

$$K \mapsto \rho(K,\cdot)$$

is injective and linear (with respect to nonnegative radial combinations of star bodies), and continuous on $\mathcal{S}^d$. We note that a convex body K which contains o in its interior is a star body.

The geometric operations which we consider in the following are sections and projections. The projection $K|L$ of a convex body onto a subspace $L \in \mathcal{L}_j^d$ is a convex body (in L), the section by an affine subspace $K \cap (L + x)$, $x \in L^\perp$, is a convex body in L, as long as x is a relative interior point of $K|L^\perp$. The section $K \cap L$ of a star body K by $L \in \mathcal{L}_j^d$ is a star body in L. The support function of the projection is denoted by $h'(K|L, \cdot) \in \mathcal{C}(S^{j-1}(L))$ and the surface area measures by $S_i'(K|L, \cdot) \in \mathcal{M}(S^{j-1}(L)), i = 1, \ldots, j-1$. The radial function of the section is $\rho'(K \cap L, \cdot) \in \mathcal{C}(S^{j-1}(L))$.

3 Analytic tools

In this section, we describe in more detail the analytic tools which we use, including some basic facts from spherical harmonic analysis.

Differentiability properties of a function f on the sphere are understood as differentiability properties of its radial extension $\check{f}$ on $\mathbb{R}^d \setminus \{o\}$, where $\check{f}(x) = f(x/\|x\|)$; see [16, Section 1.2]. Let $k \in \mathbb{N}$ or $k = \infty$ be given and let $\mathcal{C}^k(\mathbb{R}^d \setminus \{o\})$ denote the space of k-times continuously differentiable real functions on $\mathbb{R}^d \setminus \{o\}$. We say that $f \in \mathcal{C}^k(S^{d-1})$ if $\check{f} \in \mathcal{C}^k(\mathbb{R}^d \setminus \{o\})$. If $f \in \mathcal{C}^k(S^{d-1})$ and $q = (q_1, \ldots, q_d)$ is a multi-index of nonnegative integers with $|q| = q_1 + \cdots + q_d \leq k$, we put

$$\partial^q f(u) = \frac{\partial^{|q|}}{\partial^{q_1} \cdots \partial^{q_d}} \check{f}(u), \qquad u \in S^{d-1}.$$

For finite k, the space $\mathcal{C}^k(S^{d-1})$ will be endowed with the norm topology generated by all derivatives ∂^q with $|q| \leq k$ and $\mathcal{C}^\infty(S^{d-1})$ will carry the projective topology. Thus, a sequence (f_n) of functions converges to f in $\mathcal{C}^k(S^{d-1}), k \in \mathbb{N} \cup \{\infty\}$, if and only if $(\partial^q f_n)$ converges uniformly to $\partial^q f$ for all multi-indices q, $|q| \leq k$. The dual space of $\mathcal{C}^\infty(S^{d-1})$ is called the space of *distributions* (or *generalized functions*) on S^{d-1}. A distribution F is called *regular*, if it can be represented by an integrable function f in the sense that

$$F(g) = \int_{S^{d-1}} f(u) g(u)\, du, \qquad g \in \mathcal{C}^\infty(S^{d-1}).$$

In the following, we will identify a regular distribution with its representation f and write $F = f$ *in the sense of distributions*. Of course, f is determined by F only ω_{d-1}-almost everywhere. Any finite signed measure μ on S^{d-1} can be interpreted as the distribution

$$g \mapsto \int_{S^{d-1}} g\, d\mu, \qquad g \in \mathcal{C}^\infty(S^{d-1}).$$

This distribution is regular if and only if μ is absolutely continuous with respect to ω_{d-1}.

The Laplace-Beltrami operator Δ on the sphere is a second order differential operator. If $f \in \mathcal{C}^2(S^{d-1})$, then Δf is obtained by applying the ordinary Laplace operator to $\check{f}$ and restricting the resulting function to the sphere. It can be extended to distributions. If the distribution F can be represented by $f \in \mathcal{C}^2(S^{d-1})$, then ΔF is, by definition, the distribution represented by Δf. For arbitrary F, we put

$$(\Delta F)(g) = F(\Delta g), \qquad g \in \mathcal{C}^\infty(S^{d-1}).$$

This is an extension of the definition for regular distributions, since

$$\int_{S^{d-1}} (\Delta f)(u)g(u)\,du = \int_{S^{d-1}} f(u)(\Delta g)(u)\,du;$$

holds for all $f, g \in \mathcal{C}^2(S^{d-1})$; see e.g. [16, (1.2.5)]. The Laplace-Beltrami operator plays an important role in convex geometry, since it connects the support function and the first surface area measure of a convex body K. We have

$$\left(\frac{\Delta}{d-1} + 1\right) h(K, \cdot) = S_1(K, \cdot) \tag{3.1}$$

in the sense of distributions.

In [2], it is shown that Δ commutes with rotations. The eigenspaces of the Laplace-Beltrami operator on $\mathcal{C}^2(S^{d-1})$ are the spaces $\mathcal{H}_n^d$ of spherical harmonics in dimension d of order n, $n = 0, 1, 2, \ldots$. We have

$$\Delta h_n = -n(n + d - 2)h_n, \qquad h_n \in \mathcal{H}_n^d.$$

For a square-integrable spherical function f, let f_n be the orthogonal projection of f on $\mathcal{H}_n^d$ in the Hilbert space $L^2(S^{d-1})$. The series $\sum_{n=0}^\infty f_n$ converges to f in the L^2-sense and is called the *spherical harmonic expansion* of f.

It is well known that the smoothness properties of f can be deduced from its spherical harmonic expansion. We state this result as a lemma.

Lemma 3.1. *Let $f \in L^2(S^{d-1})$ with spherical harmonic expansion $\sum_{n=0}^\infty f_n$ be given. Then, for any multi-index q, there is a constant $c = c(d, |q|)$ such that*

$$\|\partial^q f_n\|_\infty \le c n^{d/2+|q|-1} \|f_n\|_\infty. \tag{3.2}$$

Hence, if the sequence $(\|f_n\|_\infty)$ converges to 0 faster than any polynomial in n, then $f \in \mathcal{C}^\infty(S^{d-1})$ and $\sum_{n=0}^\infty f_n$ converges to f in $\mathcal{C}^\infty(S^{d-1})$.

Conversely, $f \in \mathcal{C}^\infty(S^{d-1})$ implies that $(\|f_n\|_\infty)$ converges to 0 faster than any polynomial in n.

Proof. (3.2) is taken from Lemma 3.6.5 in [16]. The remaining claims follow directly from this and Proposition 3.6.2 in [16], which guarantees the existence of a constant $c' = c'(d)$ with

$$\|f_n\|_\infty \le c' \|\Delta^k f\|_\infty n^{d/2-2k-1}$$

whenever $f \in \mathcal{C}^{2k}(S^{d-1})$, $k \in \{1, 2, \ldots\}$. □

In the following sections we will describe various geometric transformations, associated with certain linear operators on spaces of functions or measures on the sphere S^{d-1}. As we mentioned in the introduction, from the point of view of geometric tomography, it is important to know the injectivity properties of these transformations.

From the analytic point of view, this is equivalent to investigating the injectivity properties of the associated linear operators.

We will now briefly describe a general framework within which such questions can be answered. Our geometric transformations will typically intertwine the action of the rotation group SO_d. This means that these transformations are rotation covariant. If T is one of the associated linear operators acting on a space $\mathcal{F}$ of spherical functions we have

$$Tf_\rho = (Tf)_\rho \qquad \text{for } f \in \mathcal{F},\ \rho \in \mathrm{SO}_d, \tag{3.3}$$

here f_ρ is the rotation of f defined by $f_\rho(u) = f(\rho^{-1}u)$ for each $u \in S^{d-1}$. Of course, the Laplace-Beltrami operator Δ is an example of an intertwining operator.

A natural setting for the investigation of such operators is spherical harmonic analysis, which has group representation theory as its background. In order to apply harmonic analysis to our transformations, it is important to know that the spaces, $\mathcal{H}_n^d$ $(n = 0, 1, \dots)$, of spherical harmonics in dimension d are the irreducible invariant subspaces of $L^2(S^{d-1})$, see [3] for example. These are finite dimensional subspaces of $\mathcal{C}^\infty(S^{d-1})$; moreover, the union of their orthonormal bases forms a complete orthonormal system in the Hilbert space $L^2(S^{d-1})$. It is a consequence of Schur's Lemma that, for a linear operator $T : \mathcal{C}^\infty(S^{d-1}) \to \mathcal{C}(S^{d-1})$ which intertwines the group action of SO_d in the sense of (3.3), T acts as a multiple of the identity when restricted to any one of the spaces $\mathcal{H}_n^d$. It follows that, for any intertwining linear operator T, we have multipliers $a_1, a_2, \dots$ such that $Th_n^d = a_n h_n^d$, for all $h_n^d \in \mathcal{H}_n^d$.

If P_n^d denotes the Legendre polynomial of degree n in dimension d, then $P_n^d(\langle u, \cdot\rangle) \in \mathcal{H}_n^d$ for any fixed $u \in S^{d-1}$. Clearly this spherical harmonic is invariant under the action of SO_{d-1}, the subgroup of rotations leaving u fixed. In fact, it is the unique function having this property and mapping u to the number 1. This allows us to give a specific formulation for the multipliers a_n, namely

$$a_n = a_n P_n^d(\langle u, u\rangle) = (TP_n^d(\langle u, \cdot\rangle))(u). \tag{3.4}$$

The following proposition summarizes the facts mentioned so far and states, in addition, the relevant injectivity result. We denote by I the identity map.

Proposition 3.2. *Let $T : \mathcal{C}^\infty(S^{d-1}) \to \mathcal{C}(S^{d-1})$ be a linear operator which intertwines the group action of* SO_d.

(a) *For $n = 1, 2, \dots,$ we have*

$$T|_{\mathcal{H}_n^d} = a_n I,$$

where

$$a_n = T(P_n^d(\langle u, \cdot\rangle))(u),$$

for arbitrary $u \in S^{d-1}$.

(b) *If T is also continuous, then T is self-adjoint, in the sense that*

$$\int (Tf)(u) g(u)du = \int f(u)(Tg)(u)du, \tag{3.5}$$

for all $f, g \in \mathcal{C}^\infty(S^{d-1})$, and T is injective, if and only if $a_n \neq 0$ for all n.

The same statements hold true for a linear operator $T : \mathcal{C}(S^{d-1}) \to \mathcal{C}(S^{d-1})$ which intertwines the group action of SO_d.

Proof. It remains to prove (b) and we start by proving (3.5) for operators $T : \mathcal{C}^\infty(S^{d-1}) \to \mathcal{C}(S^{d-1})$. Fix $u \in S^{d-1}$. The mapping $A : f \mapsto (Tf)(u)$, $f \in \mathcal{C}^\infty(S^{d-1})$, is a spherical distribution. As S^{d-1} is compact, the order of A is finite: there is a $k \in \mathbb{N}$ and a constant $c > 0$ such that

$$|Af| \le c \max_{|q|\le k} \|\partial^q f\|_\infty,$$

cf. [34, pp. 31-32]. For $f = P_n^d(\langle u, \cdot\rangle)$, we have $Af = a_n$ and Lemma 3.1 implies

$$|a_n| \le c' n^{d/2+k-1} \tag{3.6}$$

for some $c' > 0$. This lemma also states that the spherical harmonic expansion $\sum_{n=0}^\infty f_n$ of $f \in \mathcal{C}^\infty(S^{d-1})$ converges in $\mathcal{C}^\infty(S^{d-1})$ to f and that $\sum_{n=0}^\infty a_n f_n$ converges uniformly due to (3.6). For $g \in \mathcal{C}^\infty(S^{d-1})$ this implies

$$\begin{aligned}\int_{S^{d-1}} Tf(u)g(u)\,du &= \int_{S^{d-1}} \sum_{n=0}^\infty (Tf_n)(u)g(u)\,du \\ &= \int_{S^{d-1}} \sum_{n=0}^\infty a_n f_n(u)g(u)\,du = \sum_{n=0}^\infty a_n \int_{S^{d-1}} f_n(u)g(u)\,du \\ &= \sum_{n=0}^\infty a_n \int_{S^{d-1}} f_n(u)g_n(u)\,du.\end{aligned}$$

Interchanging the roles of f and g yields (3.5).

For linear continuous operators $T : \mathcal{C}(S^{d-1}) \to \mathcal{C}(S^{d-1})$ which intertwine the group action of SO_d, (3.5) is now obtained as follows: the restriction of T to $\mathcal{C}^\infty(S^{d-1})$ is a linear continuous intertwining operator from $\mathcal{C}^\infty(S^{d-1})$ to $\mathcal{C}(S^{d-1})$, so (3.5) holds for all $f, g \in \mathcal{C}^\infty(S^{d-1})$ and by approximation for all continuous f and g.

The injectivity statement is now an easy consequence of (3.5). Clearly, T is not injective if we have $a_n = 0$ for some $n = 0, 1, \dots$. For the converse, assume that $a_n \ne 0$ for all $n = 0, 1, \dots$ and use the fact that the spherical harmonics in the expansion $\sum_{n=0}^\infty f_n$ of some function f can be written as

$$f_n = c_{n,d} \int_{S^{d-1}} f(u) P_n^d(\langle u, \cdot\rangle)\,du \tag{3.7}$$

where $c_{n,d} > 0$ are known constants; cf. [23]. If $Tf = 0$ for f in the appropriate class of functions on S^{d-1}, we have due to (3.7), (3.5), and (a),

$$\begin{aligned}0 = (Tf)_n &= c_{n,d} \int_{S^{d-1}} (Tf)(u) P_n^d(\langle u, \cdot\rangle)\,du \\ &= c_{n,d} \int_{S^{d-1}} f(u) \big(T P_n^d(\langle u, \cdot\rangle)\big)\,du \\ &= a_n c_{n,d} \int_{S^{d-1}} f(u) P_n^d(\langle u, \cdot\rangle)\,du \\ &= a_n f_n.\end{aligned}$$

It follows that $f_n = 0$ for each n and so $f = 0$. □

It is also clear, from similar considerations, that intertwining operators commute.

Corollary 3.3. *Fix* $u \in S^{d-1}$. *Let* $T : \mathcal{C}^\infty(S^{d-1}) \to \mathcal{C}(S^{d-1})$ *be a continuous and intertwining linear operator.*

Then $T = 0$, *if and only if*

$$T(\langle u, \cdot\rangle^k)(u) = 0, \qquad k = 0, 1, 2, \dots. \tag{3.8}$$

Proof. Using (3.4) and the fact that $P_n^d(x)$ is a polynomial, we see that (3.8) is equivalent to $a_n = 0$ for all $n = 0, 1, 2, \dots$. The convergence of spherical harmonic expansions for $\mathcal{C}^\infty$-functions implies that the latter is equivalent to $T = 0$. □

We now concentrate on transforms T given in integral form and seek an explicit representation of ΔT. Using Corollary 3.3, we need to consider the values

$$\big(T\Delta\varphi(\langle u, \cdot\rangle)\big)(u),$$

with $\varphi(x) = x^k$, $k = 0, 1, 2, \dots$. The calculation of these values can be reduced to the one-dimensional setting, since

$$\Delta\varphi(\langle u, \cdot\rangle) = (D_d\,\varphi)(\langle u, \cdot\rangle) \tag{3.9}$$

with the (one-dimensional) differential operator

$$D_d = (1 - x^2)\frac{d^2}{dx^2} - (d-1)x\frac{d}{dx}.$$

(3.9) can be seen by expressing the operator Δ in terms of cylindrical coordinates; see for example [23, equation (54)]. If F is a distribution on $\mathbb{R}$, then,

$$F(D_d\varphi) = (D_d^*F)(\varphi),$$

where

$$D_d^*F = \frac{d^2}{dx^2}(1 - x^2)F + (d-1)\frac{d}{dx}xF.$$

This follows directly from the definition of derivatives of distributions on $\mathbb{R}$. Let w be the weight function $x \mapsto (1 - x^2)^{(d-3)/2}$ and $L^1([-1, 1], w)$ be the space of w-integrable functions on the interval $[-1, 1]$.

Theorem 3.4. *For* $G \in L^1([-1, 1], w)$ *the transform* T, *given by*

$$Tf = \int_{S^{d-1}} G(\langle v, \cdot\rangle) f(v)\, dv, \qquad f \in \mathcal{C}(S^{d-1}),$$

has the following properties.

(a) T *is a continuous endomorphism of* $\mathcal{C}(S^{d-1})$ *which intertwines the action of the rotation group.*

(b) *If, in addition, $D_d^*(\mathbf{1}_{[-1,1]}Gw)$ is a regular distribution and $f \in \mathcal{C}(S^{d-1})$, then $\Delta(Tf)$ is also a regular distribution. In fact, if $D_d^*(\mathbf{1}_{[-1,1]}Gw)$ is represented by the integrable function g, then*

$$\Delta(Tf) = \int_{S^{d-1}} f(v)\frac{g(\langle \cdot, v\rangle)}{w(\langle \cdot, v\rangle)}\,dv \tag{3.10}$$

in the sense of distributions.

Proof. (a) It is easy to see from (2.2) that $G \in L^1([-1,1], w)$ if and only if $v \mapsto G(\langle u, v\rangle)$ is integrable with respect to ω_{d-1} (for any fixed unit vector u). Let $f \in \mathcal{C}(S^{d-1})$ and assume that the sequence $(u_n) \subset S^{d-1}$ converges to $u \in S^{d-1}$. There is a sequence $(\rho_n) \subset \mathrm{SO}_d$ converging to the identity on $\mathbb{R}^d$ with $\rho_n u = u_n$. Hence

$$\begin{aligned}
\lim_{n\to\infty}(Tf)(u_n) &= \lim_{n\to\infty}\int_{S^{d-1}} G(\langle \rho_n u, v\rangle)f(v)\,dv \\
&= \lim_{n\to\infty}\int_{S^{d-1}} G(\langle u, v\rangle)f(\rho_n v)\,dv \\
&= (Tf)(u)
\end{aligned}$$

by the uniform continuity of f on the compact set S^{d-1}. Thus, $T : \mathcal{C}(S^{d-1}) \to \mathcal{C}(S^{d-1})$. The continuity and the intertwining property of T follow easily from the properties of the kernel function $G(\langle u, \cdot\rangle)$.

To show (b) fix $f \in \mathcal{C}(S^{d-1})$. By definition, the operator ΔTf acts on $\mathcal{C}^\infty(S^{d-1})$ by

$$\begin{aligned}
(\Delta Tf)h &= \int_{S^{d-1}} (Tf)(u)(\Delta h)(u)\,du \\
&= \int_{S^{d-1}} f(u)T(\Delta h)(u)\,du, \quad h \in \mathcal{C}^\infty(S^{d-1}),
\end{aligned}$$

where we used (a) and Proposition 3.2(b) for the second equation. Due to (a), $S := T\Delta$ is a continuous linear operator from $\mathcal{C}^\infty(S^{d-1})$ to $\mathcal{C}(S^{d-1})$ intertwining the action of the rotation group. The right hand side of (3.10), considered as a regular distribution, maps $h \in \mathcal{C}^\infty(S^{d-1})$ to

$$\int_{S^{d-1}} h(u)\int_{S^{d-1}} f(v)\frac{g(\langle u, v\rangle)}{w(\langle u, v\rangle)}\,dvdu = \int_{S^{d-1}} f(v)S'h(v)\,dv,$$

with

$$S'h := \int_{S^{d-1}} h(u)\frac{g(\langle u, v\rangle)}{w(\langle u, v\rangle)}\,du.$$

Here, Fubini's theorem was applied using the fact that $g/w \in L^1([-1,1], w)$. The operator S' can be applied to continuous functions h and is then a continuous endomorphism of $\mathcal{C}(S^{d-1})$ that intertwines the action of the rotation group by (a). Hence its restriction to $\mathcal{C}^\infty(S^{d-1})$ is continuous, linear and intertwining with values in $\mathcal{C}(S^{d-1})$. To show (3.10), it is enough to prove $S = S'$. By Corollary 3.3, the latter is equivalent to the fact that

$$\big(S\varphi(\langle u, \cdot\rangle)\big)(u) = \big(S'\varphi(\langle u, \cdot\rangle)\big)(u)$$

with $\varphi(x) = x^k$, $k = 0, 1, 2, \ldots$. Using (3.9) and introducing cylindrical coordinates, we obtain

$$\begin{aligned}
\big(S\varphi(\langle u, \cdot\rangle)\big)(u) &= \int_{S^{d-1}} G(\langle u, v\rangle)(D_d\varphi)(\langle u, v\rangle)\, dv \\
&= \varpi_{d-1} \int_{-1}^{1} G(x)(D_d\varphi)(x)w(x)\, dx \\
&= \varpi_{d-1} \int_{-\infty}^{\infty} \mathbf{1}_{[-1,1]}(x)G(x)w(x)(D_d\varphi)(x)\, dx \\
&= \varpi_{d-1}\big(D_d^*(\mathbf{1}_{[-1,1]}Gw)\big)(\varphi). \\
&= \varpi_{d-1} \int_{-1}^{1} g(x)\varphi(x)\, dx \\
&= \int_{S^{d-1}} \varphi(\langle u, v\rangle) \frac{g(\langle u, v\rangle)}{w(\langle u, v\rangle)}\, dv \\
&= \big(S'\varphi(\langle u, \cdot\rangle)\big)(u),
\end{aligned}$$

and the proof is complete. □

4 Projection and lifting on the sphere

Operations like sections and projections of sets can frequently be represented analytically by projection and lifting operators on the sphere. Consider, for example, the projection $K|L$ of a convex body K onto a k-dimensional linear subspace L, $1 \leq k \leq d-1$. The support function $h'(K|L, \cdot)$ of $K|L$, as a convex body in L, is the restriction $h(K, \cdot)|_{S^{k-1}(L)}$ of $h(K, \cdot)$ to $S^{k-1}(L)$. The restriction operator

$$\pi_{L,\infty} : f \mapsto f|_{S^{k-1}(L)}, \qquad f \in \mathcal{C}(S^{d-1}),$$

will be represented in the sequel as a special spherical projection operator.

On the other hand, the support function $h(K|L, \cdot)$ of the same set $K|L$, embedded in $\mathbb{R}^d$, satisfies

$$h(K|L, u) = h(K, u|L) = \|u|L\|\, h(K, \mathrm{pr}_L(u)), \quad u \in S^{d-1} \setminus L^\perp. \tag{4.1}$$

Formula (4.1) defines an operator $\pi_{L,1}^* : \mathcal{C}(S^{k-1}(L)) \to \mathcal{C}(S^{d-1})$ by

$$(\pi_{L,1}^* f)(u) = \|u|L\|\, f(\mathrm{pr}_L(u)), \quad u \in S^{d-1} \setminus L^\perp,$$

and $(\pi_{L,1}^* f)(u) = 0$, for $u \in S^{d-1} \cap L^\perp$. $\pi_{L,1}^*$ is a particular case of a spherical lifting.

In the following, more general spherical projections and liftings will be introduced. As these operators are also applied to surface area measures, we define them as operators on (certain subspaces of) $\mathcal{M}(S^{d-1})$ and $\mathcal{M}(S^{k-1}(L))$, respectively.

In the subsequent definitions, we fix $k \in \{1, \ldots, d-1\}$ and $L \in \mathcal{L}_k^d$, and let $m > -k$ be an integer. We put

$$\mathcal{M}_{m,L}(S^{d-1}) = \Big\{\mu \in \mathcal{M}(S^{d-1}) : \int_{S^{d-1}} \|u|L\|^m\, |\mu|(du) < \infty\Big\}.$$

For $m \geq 0$, we have $\mathcal{M}_{m,L}(S^{d-1}) = \mathcal{M}(S^{d-1})$, whereas for $m < 0$, $\mathcal{M}_{m,L}(S^{d-1})$ is a proper subspace of $\mathcal{M}(S^{d-1})$.

Definition 4.1. *The* m-weighted spherical projection $\pi_{L,m}$ *is the mapping from* $\mathcal{M}_{m,L}(S^{d-1})$ *into* $\mathcal{M}\big(S^{k-1}(L)\big)$*, given by*

$$\pi_{L,m}\mu = \int_{S^{d-1}\setminus L^\perp} \mathbf{1}_{(\cdot)}(\mathrm{pr}_L(u))\|u|L\|^m\,\mu(du).$$

Definition 4.2. *The operator* $\pi^*_{L,m} : \mathcal{M}\big(S^{k-1}(L)\big) \to \mathcal{M}\big(S^{d-1}\big)$*, given by*

$$\pi^*_{L,m}\mu = \int_{S^{k-1}(L)} \int_{H^{d-k}(L,v)\cap(\cdot)} \langle v, w\rangle^{k+m-1}\,dw\,\mu(dv),$$

is called the m-weighted spherical lifting*.*

Note that the definitions would make sense for more general values of m, in particular for $m \le -k$ and for real values of m (under appropriate integrability assumptions on the measures μ). However, the current definitions cover all applications which will occur. For any $m > -k$, the family $\{\pi^*_{L,m} : L \in \mathcal{L}^d_k\}$ is a family of liftings in the sense of [20].

The lifting $\pi^*_{L,m}\mu$ of a measure $\mu \in \mathcal{M}\big(S^{k-1}(L)\big)$ can be interpreted as a dispersion of μ, in a weighted manner, along half-spheres orthogonal to L. The weight depends on m and is chosen in such a way that $\pi^*_{L,0}\,\omega^L_{k-1} = \omega_{d-1}$, which amounts to (2.3).

Both the spherical projection and the spherical lifting are weakly continuous linear operators that map positive measures to positive measures. By identifying a measurable function f (which is integrable with respect to ω_{d-1}, respectively ω^L_{k-1}) with the measure $\int_{(\cdot)} f(u)\,du$, we can apply the operators $\pi_{L,m}$ and $\pi^*_{L,m}$ also to functions, and the results will again be functions. Here, (2.3) implies that, for an integrable function f on S^{d-1}, we have

$$\int_{(\cdot)} f(u)\,du \in \mathcal{M}_{m,L}(S^{d-1}),$$

for all $m > -k$. As a further consequence of (2.3), we obtain

$$(\pi_{L,m}f)(u) = \int_{H^{d-k}(L,u)} f(v)\langle u, v\rangle^{k+m-1}\,dv, \quad u \in S^{k-1}(L).$$

In particular, for $f \equiv 1$, we obtain

$$\pi_{L,m}1 = \int_{H^{d-k}(L,v)} \langle v, \cdot\rangle^{k+m-1}\,dv = \frac{\varpi_{d+m}}{\varpi_{k+m}}. \tag{4.2}$$

Similarly, for an integrable function h on $S^{k-1}(L)$, we get

$$(\pi^*_{L,m}h)(u) = \|u|L\|^m h(\mathrm{pr}_L(u)), \quad u \in S^{d-1} \setminus L^\perp, \tag{4.3}$$

and we put $(\pi^*_{L,m}h)(u) = 0$, for $u \in S^{d-1} \cap L^\perp$. In particular, $\pi_{L,m}f$ is defined for all $m > -k$ and all $f \in \mathcal{C}\big(S^{d-1}\big)$, and is then a continuous function. On the other hand,

$\pi^*_{L,m}h$ is defined for all $m > -k$ and and all $h \in \mathcal{C}\big(S^{k-1}(L)\big)$, and gives a continuous function if $m > 0$. However, for $m \le 0$, the function $\pi^*_{L,m}h$ is measurable and

$$\int_{S^{d-1}} |\pi^*_{L,m}h|(u)\,\mu(du) < \infty,$$

for all $\mu \in \mathcal{M}_{m,L}(S^{d-1})$.

We now see that $\pi^*_{L,m}$ can be considered as the transpose of $\pi_{L,m}$ (and vice versa), since

$$\int_{S^{k-1}(L)} f\,d(\pi_{L,m}\mu) = \int_{S^{d-1}} (\pi^*_{L,m}f)\,d\mu, \tag{4.4}$$

for all integrable functions f on $S^{k-1}(L)$ and $\mu \in \mathcal{M}_{m,L}(S^{d-1})$, and

$$\int_{S^{k-1}(L)} \pi_{L,m}f\,d\mu = \int_{S^{d-1}} f\,d(\pi^*_{L,m}\mu), \tag{4.5}$$

for all $\mu \in \mathcal{M}\big(S^{k-1}(L)\big)$ and integrable functions f on S^{d-1}.

Lemma 4.3. *Let $m > -k$ be an integer and $\mu \in \mathcal{M}(S^{d-1})$. Then, we have $\mu \in \mathcal{M}_{m,L}(S^{d-1})$, for ν_k-almost all L.*

Proof. We may assume $\mu \ge 0$. Fubini's theorem implies

$$\int_{\mathcal{L}^d_k}\int_{S^{d-1}} \|u|L\|^m\,\mu(du)\,dL$$
$$= \int_{S^{d-1}}\int_{\mathcal{L}^d_k} \|u|L\|^m\,dL\,\mu(du) = c_{d,m,k}\mu(S^{d-1}) < \infty.$$

with the constant

$$c_{d,m,k} = \int_{\mathcal{L}^d_k} \|u|L\|^m\,dL = \frac{1}{\varpi_d}\int_{S^{d-1}} \|u|L_0\|\,du = \frac{\varpi_{d+m}\varpi_k}{\varpi_{k+m}\varpi_d} \tag{4.6}$$

(with $L_0 \in \mathcal{L}^d_k$ fixed). Hence,

$$\int_{S^{d-1}} \|u|L\|^m\,\mu(du) < \infty,$$

for ν_k-almost all L. □

Some of the above definitions and considerations have a natural extension to the case $m = \infty$. We have already introduced the restriction operator $\pi_{L,\infty}$ which can be applied to any measurable function f on S^{d-1}. If $\mathcal{M}\big(S^{k-1}(L)\big)$ and $\mathcal{M}\big(S^{d-1}\big)$ are endowed with the total variation norm, $\pi^*_{L,m}$ becomes a continuous operator with operator norm $\varpi_{d+m}/\varpi_{k+m}$, due to (4.2). If the operators are normalized, we have for $\mu \in \mathcal{M}\big(S^{k-1}(L)\big)$

$$\frac{\varpi_{k+m}}{\varpi_{d+m}}\pi^*_{L,m}\mu \to \pi^*_{L,\infty}\mu, \qquad m \to \infty, \tag{4.7}$$

in the weak sense, where $\pi^*_{L,\infty}\mu = \mu(\,\cdot\,\cap S^{k-1}(L))$ is the *trivial extension* of μ to the whole of S^{d-1}. The formula for $m = \infty$, corresponding to (4.5), then reads

$$\int_{S^{k-1}(L)} \pi_{L,\infty} f \, d\mu = \int_{S^{d-1}} f \, d(\pi^*_{L,\infty}\mu), \tag{4.8}$$

thus $\pi^*_{L,\infty}$ is the transpose of the restriction operator $\pi_{L,\infty}$.

We emphasize the fact, that $\pi_{L,\infty}$ cannot be extended, in a weakly continuous manner, to an operator on measures μ. Hence, the kind of symmetry between functions and measures, which we encountered in the case of the operators $\pi_{L,m}$ and $\pi^*_{L,m}$, does not extend to $m = \infty$. In fact, if we consider $\pi^*_{L,m} f$ for a continuous function f and let $m \to \infty$, (4.3) produces a different natural limit operator τ^*_L, namely

$$(\pi^*_{L,m} f)(u) \to (\tau^*_L f)(u) = \begin{cases} f(u), & \text{if } u \in L, \\ 0, & \text{otherwise,} \end{cases}$$

for $f \in \mathcal{C}\big(S^{k-1}(L)\big)$ and $u \in S^{d-1}$. The transpose τ_L of τ^*_L would be defined by

$$\int_{S^{k-1}(L)} f \, d(\tau_L \mu) = \int_{S^{d-1}} (\tau^*_L f) \, d\mu,$$

for $f \in \mathcal{C}\big(S^{k-1}(L)\big)$ and $\mu \in \mathcal{M}(S^{d-1})$. Hence $\tau_L\mu$ must be the restriction of the measure μ to the Borel sets in $S^{k-1}(L)$. If μ is a function f, we therefore have $\tau_L f \equiv 0$, which is, of course, very different from $\pi_{L,\infty} f$. We will not need the operators τ_L and τ^*_L, in the following.

Next we will prove a lemma, which allows us to reduce certain considerations involving projections or lifting to the case where L is a hyperplane. It generalizes [4, Lemma 5.2], where $m = 1$ and $m = -1$ are treated. Assume that L and M are two linear subspaces with $\{o\} \neq M \subset L \neq \mathbb{R}^d$ and denote by $\pi^L_{M,m}$ and $\pi^{*L}_{M,m}$ the projection and lifting operator in L.

Lemma 4.4. *Let L and M be linear subspaces with $\{o\} \neq M \subset L \neq \mathbb{R}^d$ and let $m > -\dim M$ be an integer. Then*

$$\pi^L_{M,m}\pi_{L,m}\mu = \pi_{M,m}\mu,$$

for measures $\mu \in \mathcal{M}_{m,M}(S^{d-1})$, and

$$\pi^*_{L,m}\pi^{*L}_{M,m}\mu = \pi^*_{M,m}\mu,$$

for $\mu \in \mathcal{M}(S^{k-1}(L))$.

Proof. For $u \in S^{d-1} \setminus M^\perp$, we have $(u|L)|M = u|M$ and thus

$$\|u|L\|(\mathrm{pr}_L(u)|M) = u|M.$$

This gives $\mathrm{pr}_M\big(\mathrm{pr}_L(u)\big) = \mathrm{pr}_M(u)$, for $u \notin M^\perp$ and therefore almost surely with respect to the measure $\int_{(\cdot)} \|u|M\|^m \, |\mu|(du)$. The definition of projections now implies

$$\begin{aligned} \pi^L_{M,m}\pi_{L,m}\mu &= \int_{S^{d-1}\setminus M^\perp} \mathbf{1}_{(\cdot)}(\mathrm{pr}_M(\mathrm{pr}_L(u))\|\mathrm{pr}_L(u)|M\|^m \|u|L\|^m \, \mu(du) \\ &= \int_{S^{d-1}\setminus M^\perp} \mathbf{1}_{(\cdot)}(\mathrm{pr}_M(u))\|u|M\|^m \, \mu(du) \\ &= \pi_{M,m}\mu. \end{aligned}$$

This shows the first claim. The second follows by transposition. □

5 Tomographic interpretations

In this section, we give some tomographic interpretations of spherical liftings and projections. The following results were originally obtained in [19].

Assume $L \in \mathcal{L}_k^d$ and that K is a convex body in L. If we consider the support function $h(K,\cdot)$ of K in $\mathbb{R}^d$, the definition of $\pi^*_{L,m}$ implies

$$(\pi^*_{L,m} h'(K,\cdot))(u) = \|u|L\|^{m-1} h(K,u), \tag{5.1}$$

for all $u \in S^{d-1}$, where $u \notin L^\perp$ is required in addition, whenever $m < 1$. For an arbitrary convex body $K \subset \mathbb{R}^d$, we have already seen that $h'(K|L,\cdot) = \pi_{L,\infty} h(K,\cdot)$ (on $S^{k-1}(L)$) and thus

$$h(K|L,\cdot) = \pi^*_{L,1}\pi_{L,\infty} h(K,\cdot) \tag{5.2}$$

on S^{d-1}. This shows that, for $m = \infty$, the lifted projection $\pi^*_{L,1}\pi_{L,m}$ maps the cone of support functions into itself. The following theorem yields an extension of this fact to all $m \in (-k,\infty]$.

Theorem 5.1. *Let $m \in (-k,\infty]$ be an integer and $K \in \mathcal{K}^d$. Then, $\pi_{L,m} h(K,\cdot)$ is the support function of some convex body in L. In particular, $\pi^*_{L,1}\pi_{L,m} h(K,\cdot)$ is the support function of a convex body in L, embedded in $\mathbb{R}^d$.*

Proof. Since $\pi_{L,\infty} h(K,\cdot) = h'(K|L,\cdot)$ we may assume $m < \infty$ and, by Lemma 4.4 we may restrict our attention to the case $L = x^\perp$ for some $x \in S^{d-1}$. For $u \in S^{d-2}(x^\perp)$ put

$$\begin{aligned} h(u) &= \big(\pi_{x^\perp,m} h(K,\cdot)\big)(u) \\ &= \int_{H^1(x^\perp,u)} h(K,z)\langle u,z\rangle^{d+m-2}\,dz \\ &= \int_{-1}^{1} h(K, sx + \sqrt{1-s^2}u)(1-s^2)^{(d+m-3)/2}\,ds. \end{aligned}$$

It follows from [28, Theorem 1.7.1] that h is a support function on $x^\perp$ if and only if

$$\frac{h(u_1)+h(u_2)}{\rho} \ge h\left(\frac{u_1+u_2}{\rho}\right) \tag{5.3}$$

holds for all linearly independent $u_1, u_2 \in S^{d-2}(x^\perp)$, where $\rho = \|u_1+u_2\| \in (0,2)$. We will establish (5.3). The support function $h(K,\cdot)$ satisfies

$$h(K, sx+\sqrt{1-s^2}u_1) + h(K, sx+\sqrt{1-s^2}u_2) \ge h(K, 2sx+\sqrt{1-s^2}(u_1+u_2))$$

and thus

$$\frac{h(u_1)+h(u_2)}{\rho} \ge \frac{1}{\rho}\int_{-1}^{1} h(K, 2sx+\sqrt{1-s^2}(u_1+u_2))(1-s^2)^{(d+m-3)/2}\,ds.$$

Substituting

$$t = \frac{2s}{\|2sx + \sqrt{1-s^2}(u_1+u_2)\|} = \frac{s}{\sqrt{(\rho/2)^2 + (1-(\rho/2)^2)s^2}}$$

and using the abbreviations $u(t) = tx + \sqrt{1-t^2}(u_1+u_2)/\rho$ and

$$g_\rho(t) = \frac{\rho/2}{\big(1-(1-(\rho/2)^2)t^2\big)^{(d+m+1)/2}}$$

yields

$$\frac{h(u_1)+h(u_2)}{\rho} \geq \int_{-1}^{1} g_\rho(t) h\big(K, u(t)\big)(1-t^2)^{(d+m-3)/2}\, dt$$

We conclude that (5.3) holds if

$$J(K) = \int_{-1}^{1} (g_\rho(t)-1) h\big(K, u(t)\big)(1-t^2)^{(d+m-3)/2}\, dt \geq 0. \tag{5.4}$$

The integral in (5.4) does not change if K is replaced by its reflection in $x^\perp$ and we may therefore assume that K is symmetric with respect to $x^\perp$, that is, $h(K,u(t)) = h(K,u(-t))$ for all $t \in [-1,1]$. The function $t \mapsto g_\rho(t) - 1$ is strictly increasing on $[0,1]$ and has a zero at

$$t_\rho = \sqrt{\frac{1-(\rho/2)^{2/(d+m+1)}}{1-(\rho/2)^2}}.$$

For $0 \leq t \leq t_\rho$, we have $g_\rho(t) - 1 \leq 0$ and $u(t) = \gamma u(t_\rho) + \delta u(-t_\rho)$ with

$$2\gamma = \frac{t}{t_\rho} + \frac{\sqrt{1-t^2}}{\sqrt{1-t_\rho^2}} \geq 0 \quad \text{and} \quad 2\delta = -\frac{t}{t_\rho} + \frac{\sqrt{1-t^2}}{\sqrt{1-t_\rho^2}} \geq 0.$$

Using the symmetry property of K, this implies

$$h\big(K, u(t)\big) \leq \gamma h\big(K, u(t_\rho)\big) + \delta h\big(K, u(-t_\rho)\big) = \frac{\sqrt{1-t^2}}{\sqrt{1-t_\rho^2}} h\big(K, u(t_\rho)\big).$$

If $t_\rho \leq t \leq 1$, we have $g_\rho(t) - 1 \geq 0$ and $u(t_\rho) = \tilde\gamma u(-t_\rho) + \tilde\delta u(t)$ with

$$\tilde\gamma = \frac{-t_\rho\sqrt{1-t^2} + t\sqrt{1-t_\rho^2}}{t_\rho\sqrt{1-t^2} + t\sqrt{1-t_\rho^2}} \geq 0 \quad \text{and} \quad \tilde\delta = \frac{2t_\rho\sqrt{1-t_\rho^2}}{t_\rho\sqrt{1-t^2} + t\sqrt{1-t_\rho^2}} > 0.$$

Hence

$$h\big(K, u(t)\big) \geq \frac{1}{\tilde\delta}\Big(h\big(K, u(t_\rho) - \tilde\gamma h\big(K, u(-t_\rho)\big)\big)\Big) = \frac{\sqrt{1-t^2}}{\sqrt{1-t_\rho^2}} h\big(K, u(t_\rho)\big).$$

Substituting this into the integral in (5.4), we obtain

$$\begin{aligned} J(K) &= 2\int_0^1 (g_\rho(t)-1) h\big(K, u(t)\big)(1-t^2)^{(d+m-3)/2}\, dt \\ &\geq \frac{2h\big(K, u(t_\rho)\big)}{\sqrt{1-t_\rho^2}} \int_0^1 (g_\rho(t)-1)(1-t^2)^{(d+m-2)/2}\, dt. \end{aligned}$$

The latter integral vanishes, since

$$\int_0^1 g_\rho(t)(1-t^2)^{(d+m-2)/2}\,dt = \int_0^1 (1-s^2)^{(d+m-2)/2}\,ds,$$

(re-substitute $s = \frac{(\rho/2)t}{\sqrt{1-(1-(\rho/2)^2)t^2}}$), so (5.4) holds.

In view of our definition of convex bodies, it remains to show that the compact convex set $M \subset x^\perp$ with support function $h'(M,\cdot) = \pi_{x^\perp,m} h(K,\cdot)$ contains a $(d-1)$-dimensional ball. Replacing K with $K+a$ for some translation vector $a \in \mathbb{R}^d$ leads to a translation of M by $(\gamma a)|x^\perp$, $\gamma = \varpi_{d+m+1}/\varpi_{d+m}$, since $h(K+a,\cdot) = h(K,\cdot) + \langle a,\cdot\rangle$ and

$$\begin{aligned}
(\pi_{x^\perp,m}\langle a,\cdot\rangle)(u) &= \Big\langle a, \int_{H^1(x^\perp,u)} z\langle z,u\rangle^{d+m-2}\,dz\Big\rangle \\
&= \Big\langle a, \int_{-1}^1 (tx+\sqrt{1-t^2}u)(1-t^2)^{(d+m-3)/2}\,dt\Big\rangle \\
&= \langle a,u\rangle \int_{-1}^1 (1-t^2)^{(d+m-2)/2}\,dt \\
&= \langle \gamma a,u\rangle, \qquad u \in S^{d-2}(x^\perp).
\end{aligned}$$

We may therefore assume that the origin is an interior point of K. If $\delta > 0$ is such that $\delta B^d \subset K$, then

$$h'(M,\cdot) = \pi_{x^\perp,m} h(K,\cdot) \geq \pi_{x^\perp,m} h(\delta B^d,\cdot) \equiv \text{const} > 0,$$

shows that M contains a $(d-1)$-dimensional ball centered at the origin. □

Theorem 5.1 can be seen as a generalized dual statement to Busemann's theorem. In terms of liftings, the latter theorem states that if K is a convex body with $o \in \operatorname{int} K$ and $L \in \mathcal{L}_k^d$, then $\pi_{L,1-k}\rho^{d-k+1}(K,\cdot)$ is the radial function of a convex body in L. Theorem 5.1 with $m = 1-k$ implies that $\pi_{L,1-k}h(K,\cdot)$ is the support function of a convex body in L.

There is no similar result for the lifting operator $\pi^*_{L,m}$. For $m < 0$ and a convex body M in L, the function $\pi^*_{L,m}h'(M,\cdot)$ is unbounded and therefore not a support function. For $m = 0$, it is only a support function, if it is constant, that is M is a ball (in L). For $m > 0$, the lifted function vanishes on $L^\perp$. Hence, if it is the support function of a body K, we must have $K \subset L$. This immediately implies $K = M$ and $m = 1$, the trivial case.

Next, let $1 \leq j < k \leq d-1$ and assume that $K \subset \mathbb{R}^d$ is a convex body in $L \in \mathcal{L}_k^d$. In the following result, we show the connection between $S'_j(K,\cdot)$, the j-th surface area measure of K as a subset of L, and $S_j(K,\cdot)$, the j-th surface area measure of K in $\mathbb{R}^d$.

Theorem 5.2. *Let $1 \leq j < k \leq d-1$, $L \in \mathcal{L}_k^d$, and $K \subset \mathbb{R}^d$ be a convex body in L. Then,*

$$\binom{k-1}{j}\pi^*_{L,-j}S'_j(K,\cdot) = \binom{d-1}{j}S_j(K,\cdot). \tag{5.5}$$

For centrally symmetric K and $j = 1$, (5.5) was shown earlier in [4] (up to missing constants).

Proof. As surface area measures depend weakly continuously on K and the lifting is a weakly continuous operator, it is enough to prove (5.5) for polytopes K and then use approximation. Let $K \subset L$ be a polytope and let $\mathcal{F}_j(K)$ be the family of its j-faces. For $F \in \mathcal{F}_j(K)$, let $N(K,F)$ and $N'(K,F)$ be the normal cones of K at F in $\mathbb{R}^d$, respectively in L. Then, we have

$$N(K,F) = N'(K,F) + L^{\perp}. \tag{5.6}$$

The j-th surface area measure of K relative to L can be written as

$$\binom{k-1}{j} S'_j(K,\cdot) = \sum_{F\in\mathcal{F}_j(K)} \omega^{L'}_{k-j-1}(N'(K,F)\cap(\cdot))V_j(F),$$

see [28, (4.2.11) and (4.2.18)]. Here $L' \subset L$ is the linear hull of $N'(K,F)$. We apply the definition of the lifting $\pi^*_{L,-j}$, equation (5.6) and the decomposition (2.3) in the space $L' + L^{\perp}$:

$$\begin{aligned}
&\pi^*_{L,-j}\big(\omega^{L'}_{k-j-1}(N'(K,F)\cap(\cdot))\big)\\
&\quad= \int_{S^{k-j-1}(L')\cap N'(K,F)} \int_{H^{d-k}(L,v)\cap(\cdot)} \langle v,w\rangle^{k-j-1}\,dw\,dv\\
&\quad= \int_{S^{k-j-1}(L')} \int_{H^{d-k}(L,v)\cap(\cdot)} \mathbf{1}_{N(K,F)}(v)\langle v,w\rangle^{k-j-1}\,dw\,dv\\
&\quad= \big(\pi^{*(L'+L^{\perp})}_{L',0}\,\omega^{L'}_{k-j-1}\big)(N(K,F)\cap(\cdot))\\
&\quad= \omega^{L'+L^{\perp}}_{d-j-1}(N(K,F)\cap(\cdot)).
\end{aligned}$$

This implies

$$\begin{aligned}
\binom{k-1}{j}\pi^*_{L,-j}S'_j(K,\cdot) &= \sum_{F\in\mathcal{F}_j(K)} \omega^{L'+L^{\perp}}_{k-j-1}(N(K,F)\cap(\cdot))V_j(F)\\
&= \binom{d-1}{j}S_j(K,\cdot).
\end{aligned}$$

□

A translative Crofton-type formula for surface area measures involving $\pi_{L,1}$ is given in [10]; see also [25]. It contains the special case

$$\int_{L^{\perp}} S'_{k-1}((K+x)\cap L,\cdot)\,dx = \pi_{L,1}S_{d-1}(K,\cdot), \tag{5.7}$$

where K is an arbitrary convex body in $\mathbb{R}^d$. Lifting with $\pi^*_{L,1-k}$ implies

$$\binom{d-1}{k-1}\int_{L^{\perp}} S_{k-1}((K+x)\cap L,\cdot)\,dx = \pi^*_{L,1-k}\pi_{L,1}S_{d-1}(K,\cdot), \tag{5.8}$$

a result which will be used later.

If K is a star body in $\mathbb{R}^d$, we have

$$\rho(K\cap L,\cdot) = \pi_{L,\infty}\rho(K,\cdot)$$

on $S^{k-1}(L)$.

In addition to the previously mentioned connections between geometric operations on the one hand and spherical liftings and projections on the other hand, the latter operators also play an important role in the theory of zonoids and their generalizations, see [4] (where $\pi_{L,1}$ is denoted π_L and $\pi_{L,-1}$ is denoted τ_L) and the references therein.

6 Averaging of sets

In the introduction, we described two general principles for averaging lower dimensional data. The first one was to average the (lower dimensional) sets directly. In the case of (central) sections of star bodies, this will usually be based on radial addition. For projections (or affine sections) of convex bodies we have two different additions, Minkowski and Blaschke addition. Moreover, different averages arise if we interpret lower dimensional sets as sets in the pertinent subspace or in $\mathbb{R}^d$. As we shall see, all these averages can be expressed as integrals of combinations of liftings and projections.

The first results in this section are mostly taken from [19], the case $j = 1$ is also treated in [20].

Definition 6.1. *For $1 \leq k \leq d-1$ and $-k < m, j < \infty$, we define the* mean lifted projection $\pi_{m,j}^{(k)}$ *by*

$$(\pi_{m,j}^{(k)}\mu)(A) = \int_{\mathcal{L}_k^d} (\pi_{L,m}^* \pi_{L,j}\mu)(A)\, dL, \tag{6.1}$$

for $\mu \in \mathcal{M}(S^{d-1})$ and $A \in \mathcal{B}(S^{d-1})$.

This definition makes sense, since the integrand is defined for ν_k-almost all L by Lemma 4.3. Moreover, $\pi_{m,j}^{(k)}$ is a weakly continuous linear operator, mapping the space $\mathcal{M}(S^{d-1})$ into itself, and commuting with rotations in SO_d. By definition, $\pi_{m,j}^{(k)}$ can also be applied to integrable functions f on S^{d-1}, and then yields a function $\pi_{m,j}^{(k)} f$. Due to (4.5), (4.4) and Fubini's theorem, we obtain

$$\int_{S^{d-1}} f\, d(\pi_{m,j}^{(k)}\mu) = \int_{S^{d-1}} (\pi_{j,m}^{(k)} f)\, d\mu, \tag{6.2}$$

for any such f and $\mu \in \mathcal{M}(S^{d-1})$. In order to obtain some continuity properties of $\pi_{m,j}^{(k)}$ on functions, we first derive a suitable representation of $\pi_{m,j}^{(k)} f$. Using formula (2.4) in the definition of $\pi_{m,j}^{(k)}$, we get

$$\begin{aligned}
&(\pi_{m,j}^{(k)} f)(v) \\
&\quad = \int_{\mathcal{L}_k^d} (\pi_{L,m}^* \pi_{L,j} f)(v)\, dL \\
&\quad = \frac{\varpi_k}{\varpi_d} \int_{\mathcal{L}_{k-1}^{v^\perp}} \int_{S^{d-k}(M^\perp)} \mathbf{1}_{v^+}(u) (\pi_{M\vee u,m}^* \pi_{M\vee u,j} f)(v) \langle u, v\rangle^{k-1}\, du\, dM \\
&\quad = \frac{\varpi_k}{\varpi_d} \int_{\mathcal{L}_{k-1}^{v^\perp}} \int_{S^{d-k}(M^\perp)} \int_{S^{d-k}(M^\perp)\cap u^+} \mathbf{1}_{v^+}(u) \|v|(M\vee u)\|^m \langle u, v\rangle^{k-1} \\
&\qquad\qquad \times f(w) \langle u, w\rangle^{k+j-1}\, dw\, du\, dM \\
&\quad = \frac{\varpi_k}{\varpi_d} \int_{\mathcal{L}_{k-1}^{v^\perp}} \int_{S^{d-k}(M^\perp)} f(w) I(v,w)\, dw\, dM
\end{aligned} \tag{6.3}$$

where

$$I(v,w) = \int_{S^{d-k}(M^\perp)} \mathbf{1}_{v^+ \cap w^+}(u) \langle u, v\rangle^{k+m-1} \langle u, w\rangle^{k+j-1}\, du.$$

Obviously, $0 \leq I(v,w) \leq \varpi_{d-k+1}$ and $I(v,w)$ is continuous in v and w. Consequently, $\pi^{(k)}_{m,j}$ is a continuous endomorphism on the Banach space $\mathcal{C}(S^{d-1})$. Proposition 3.2 shows that it is self-adjoint, that is,

$$\int_{S^{d-1}} (\pi^{(k)}_{m,j} f)(u)\, g(u)\, du = \int_{S^{d-1}} f(u)\, (\pi^{(k)}_{m,j} g)(u)\, du,$$

for $f, g \in \mathcal{C}(S^{d-1})$. A comparison with (6.2) now shows that

$$\pi^{(k)}_{m,j} = \pi^{(k)}_{j,m},$$

a relation that can also derived from the next theorem, where we explicitly describe the integral kernel of the spherical integral operator $\pi^{(k)}_{m,j}$. This result follows ideas in [7], the special case $\pi^{(k)}_{1,1-k}$ has been treated in [20, Theorem 3.2].

Theorem 6.2. *Assume* $1 \leq k \leq d-1$, $-k < m, j < \infty$ *and* $\mu \in \mathcal{M}(S^{d-1})$. *Then*

$$\pi^{(k)}_{m,j}\mu = \gamma \int_{S^{d-1}\cap(\cdot)} \int_{S^{d-1}} K^{(k)}_{m,j}(\langle u,v\rangle)\, \mu(du)\, dv, \tag{6.4}$$

where

$$\gamma = \frac{\varpi_{d+k+m+j-1}\varpi_{d-k}\varpi_k}{\varpi_{2k+m+j}\varpi_d\varpi_{d-1}} \tag{6.5}$$

and

$$K^{(k)}_{m,j}(t) = (1-t^2)^{(1-k)/2} \int_0^{\pi-\arccos(t)} \sin^{k+m-1}(s) \sin^{k+j-1}(s+\arccos(t))\, ds,$$

for $-1 \leq t \leq 1$.

Proof. Let γ be the constant given by (6.5). By (6.2) and Fubini's theorem, it is enough to show

$$(\pi^{(k)}_{m,j} f)(v) = \gamma \int_{S^{d-1}} f(w) K^{(k)}_{m,j}(\langle v,w\rangle)\, dw,$$

for all continuous functions f and all unit vectors v. Combining (6.3) with (2.5) gives

$$\begin{aligned}(\pi^{(k)}_{m,j} f)(v) &= \frac{\varpi_k}{\varpi_d} \int_{\mathcal{L}^{v\perp}_{k-1}} \int_{S^{d-k}(M^\perp)} f(w) I(v,w)\, dw\, dM \\ &= \frac{\varpi_k \varpi_{d-k}}{\varpi_d \varpi_{d-1}} \int_{S^{d-1}} f(w) I(v,w)(1-\langle v,w\rangle^2)^{(1-k)/2}\, dw.\end{aligned}$$

We can identify $M^\perp$ with $\mathbb{R}^{d-k+1}$ and decompose $\omega^{M^\perp}_{d-k}$ using (2.3) in this space to obtain

$$I(v,w) = \int_{S^1(v\vee w)} \int_{H^{d-k-1}(v\vee w,z)} \mathbf{1}_{v^+\cap w^+}(u) \langle u,v\rangle^{k+m-1} \times \langle u,w\rangle^{k+j-1} \langle u,z\rangle\, du\, dz.$$

For any pair (u,z) of integration variables, we have

$$\langle u,v\rangle = \langle u,z\rangle\langle v,z\rangle \quad \text{and} \quad \langle u,w\rangle = \langle u,z\rangle\langle w,z\rangle.$$

In particular, $u \in v^+ \cap w^+$ if and only if $z \in v^+ \cap w^+$ and we obtain

$$\begin{aligned} I(v,w) &= \int_{S^1(v\vee w)\cap v^+\cap w^+} \langle v, z\rangle^{k+m-1} \langle w, z\rangle^{k+j-1} \\ &\quad \times \int_{H^{d-k-1}(v\vee w,z)} \langle u, z\rangle^{2k+m+j-1}\, du\, dz \\ &= \frac{\varpi_{d+k+j+m-1}}{\varpi_{2k+m+j}} \int_{S^1(v\vee w)\cap v^+\cap w^+} \langle v, z\rangle^{k+m-1} \langle w, z\rangle^{k+j-1}\, dz, \end{aligned}$$

where in the last step we used the fact that the inner integral is independent of z and can be determined using (4.2).

Finally, we identify $v \vee w$ with $\mathbb{R}^2$. Due to the invariance properties of ω_1 with respect to rotations and reflections (at arbitrary lines through the origin), we may assume $v = (0, 1)$ and w to be in the right closed half-plane. Let $\alpha = \alpha(v, w) = \arccos(\langle v, w\rangle)$ be the (smaller) angle between v and w. The parametrization $z = (\cos s, \sin s)$ with $s \in [0, 2\pi)$ implies

$$I(v,w) = \frac{\varpi_{d+k+j+m-1}}{\varpi_{2k+m+j}} \int_0^{\pi-\alpha} \sin^{k+m-1}(s)\, \sin^{k+j-1}(s+\alpha)\, ds.$$

This representation of $I(v, w)$ can be substituted into (5.1), to yield the required result. □

Definition 6.1 can be extended to the case $m = \infty$.

Definition 6.3. *For $1 \le k \le d-1$ and $-k < j < \infty$, we define the* mean lifted projection $\pi^{(k)}_{\infty,j}$ *by*

$$(\pi^{(k)}_{\infty,j}\mu)(A) = \int_{\mathcal{L}^d_k} (\pi^*_{L,\infty}\pi_{L,j}\mu)(A)\, dL,$$

for $\mu \in \mathcal{M}(S^{d-1})$ and $A \in \mathcal{B}(S^{d-1})$.

Again, we observe that $\pi^{(k)}_{\infty,j}$ is a weakly continuous linear operator, mapping $\mathcal{M}(S^{d-1})$ to itself, and commuting with rotations ϑ. The kernel representation of $\pi^{(k)}_{\infty,j}$ is given in the following theorem; cf. [20, Theorem 3.2] for $j = 1$.

Theorem 6.4. *Let $-k < j < \infty$, and $\mu \in \mathcal{M}(S^{d-1})$. Then*

$$\pi^{(k)}_{\infty,j}\mu = \frac{\varpi_{d-k}\varpi_k}{\varpi_d\varpi_{d-1}} \int_{S^{d-1}\cap(\cdot)} \int_{S^{d-1}} K^{(k)}_{\infty,j}(\langle u,v\rangle)\, \mu(du)\, dv \tag{6.6}$$

with

$$K^{(k)}_{\infty,j}(t) = 1_{t\ge 0} t^{k+j-1}(1-t^2)^{(1-k)/2}, \quad -1 \le t \le 1.$$

Proof. It would be possible to repeat the proof of Theorem 6.2 with appropriate modifications. We prefer however an approximation argument, which will make use of the statement of Theorem 6.2. For fixed $\mu \in \mathcal{M}(S^{d-1})$ and $L \in \mathcal{L}^d_k$, the weak convergence

$$\lim_{m\to\infty} \frac{\varpi_{k+m}}{\varpi_{d+m}} \pi^*_{L,m}\pi_{L,j}\mu = \pi^*_{L,\infty}\pi_{L,j}\mu$$

follows from (4.7). The total variations of all the members of this sequence are bounded from above by the total variation of μ, so Lebesgue's dominated theorem yields

$$\lim_{m\to\infty} \frac{\varpi_{k+m}}{\varpi_{d+m}} \pi^{(k)}_{m,j}\mu = \pi^{(k)}_{\infty,j}\mu.$$

Theorem 6.2 states that $\frac{\varpi_{k+m}}{\varpi_{d+m}} \pi^{(k)}_{m,j}\mu$ has Radon-Nikodym derivative

$$\gamma_m \int_{S^{d-1}} \frac{\varpi_{k+m}}{\varpi_{k+m+1}} K^{(k)}_{m,j}(\langle u, \cdot\rangle)\, \mu(du)$$

with

$$\gamma_m = \frac{\varpi_{k+m+1}\varpi_{d+k+m+j-1}\varpi_{d-k}\varpi_k}{\varpi_{d+m}\varpi_{2k+m+j}\varpi_d\varpi_{d-1}} \to \frac{\varpi_{d-k}\varpi_k}{\varpi_d\varpi_{d-1}}$$

as $m \to \infty$, by Stirling's formula. The functions

$$\begin{aligned}(u, v) \mapsto & \frac{\varpi_{k+m}}{\varpi_{k+m+1}} K^{(k)}_{m,j}(\langle u, v\rangle) \\ & \le (1 - \langle u, v\rangle^2)^{(1-k)/2} \frac{\varpi_{k+m}}{\varpi_{k+m+1}} \int_0^{\pi} \sin^{k+m-1}(s)\, ds \\ & = (1 - \langle u, v\rangle^2)^{(1-k)/2}\end{aligned}$$

are bounded uniformly in m by a $\mu \times \omega_{d-1}$-integrable function. For $t \neq 0$, we have

$$\lim_{m\to\infty} \frac{\varpi_{k+m}}{\varpi_{k+m+1}} K^{(k)}_{m,j}(t) = K^{(k)}_{\infty,j}(t),$$

as the probability measures

$$\frac{\varpi_{k+m}}{\varpi_{k+m+1}} \int_{(\cdot)} \sin^{k+m-1}(s) 1_{0\le s\le \pi}\, ds$$

converge weakly to the Dirac measure $\delta_{\pi/2}$, as $m \to \infty$. The claim therefore follows once more by an application of the Lebesgue dominated theorem. □

The operator $\pi^{(k)}_{\infty,j}$ is self-adjoint, if applied to continuous functions, in the sense that

$$\int_{S^{d-1}} (\pi^{(k)}_{\infty,j} f)(u)\, g(u)\, du = \int_{S^{d-1}} f(u)\, (\pi^{(k)}_{\infty,j} g)(u)\, du, \tag{6.7}$$

for $f, g \in \mathcal{C}(S^{d-1})$. Note that, by (4.8) and (4.4), the transpose of $\pi^{(k)}_{\infty,j}$ is the operator $\pi^{(k)}_{j,\infty}$ on $\mathcal{C}(S^{d-1})$, given by

$$(\pi^{(k)}_{j,\infty} f)(u) = \int_{\mathcal{L}^d_k} (\pi^*_{L,j}\pi_{L,\infty} f)(u)\, dL, \qquad u \in S^{d-1}.$$

(6.7) shows that $\pi^{(k)}_{\infty,j}$, restricted to $\mathcal{C}(S^{d-1})$, coincides with $\pi^{(k)}_{j,\infty}$. However, $\pi^{(k)}_{j,\infty}\mu$ is not defined for measures μ.

We mention some special cases. For $k = j = 1$, $f \in \mathcal{C}(S^{d-1})$ and $u \in S^{d-1}$, we obtain

$$(\pi^{(1)}_{\infty,1} f)(u) = 2\varpi_d^{-1} \int_{S^{d-1}\cap u^+} \langle u, v\rangle f(v)\, dv.$$

For even f, this yields

$$\varpi_d(\pi^{(1)}_{\infty,1}f)(u) = (Cf)(u), \tag{6.8}$$

hence $\pi^{(1)}_{\infty,1}f$ coincides up to a factor with the cosine transform Cf of f, which we discussed in Example 1.

The operator $\pi^{(1)}_{\infty,0}$ is a multiple of the *spherical cap transform* (also called the *hemispherical transform*, see [16]), since

$$\frac{\varpi_d}{2}(\pi^{(1)}_{\infty,0}f)(u) = \int_{S^{d-1}\cap u^+} f(v)\,dv, \quad u \in S^{d-1},$$

for all continuous f.

Concerning Example 3 (the projection mean body), we now see from (5.2) and (6.1), that

$$h(P_k(K),\cdot) = \pi^{(k)}_{1,\infty}h(K,\cdot).$$

Thus, (6.6) implies

$$h(P_k(K),u) = \frac{\varpi_{d-k}\varpi_k}{\varpi_d\varpi_{d-1}}\int_{S^{d-1}\cap u^+}\frac{\cos^k\alpha(u,v)}{\sin^{k-1}\alpha(u,v)}h(K,v)\,dv, \qquad u \in S^{d-1},$$

where $\alpha(u,v)$ is again the (smaller) angle between u and v.

For the Blaschke section body in Example 4, (5.7) implies that

$$S_{d-1}(B_k(K),\cdot) = \pi^{(k)}_{\infty,1}S_{d-1}(K,\cdot).$$

Since $\pi^{(k)}_{\infty,1} = \pi^{(k)}_{1,\infty}$, this shows the result mentioned in the introduction, namely that $B_k(K)$ and $P_k(K)$ are associated with the same operator. In particular, $S_{d-1}(B_k(K),\cdot)$ has ω_{d-1}-density

$$\frac{\varpi_{d-k}\varpi_k}{\varpi_d\varpi_{d-1}}\int_{S^{d-1}\cap(\cdot)^+}\frac{\cos^k\alpha(\cdot,v)}{\sin^{k-1}\alpha(\cdot,v)}S_{d-1}(K,dv).$$

This result was already stated in [25, Theorem 2], in even more general form (for support measures of sets of positive reach).

The following two examples concern variations of section and projection means.

Example 6 (**Blaschke section body of the second kind**). As a variant, we consider the k-th *Blaschke section body* $\tilde{B}_k(K)$ *of the second kind*. In contrast to Example 4, the integrand is the $(k-1)$-st surface area measure of $(K+x)\cap L$, considered as a subset of $\mathbb{R}^d$. The convex body $\tilde{B}_k(K)$ is defined by

$$S_{d-1}(\tilde{B}_k(K),\cdot) = \int_{\mathcal{L}^d_k}\int_{L^\perp} S_{k-1}((K+x)\cap L,\cdot)\,dx\,dL.$$

Relation (5.8) implies

$$S_{d-1}(\tilde{B}_k(K),\cdot) = \binom{d-1}{k-1}^{-1}\pi^{(k)}_{1,1-k}S_{d-1}(K,\cdot).$$

Now (6.4) shows that this measure has the ω_{d-1}-density

$$\int_{S^{d-1}} \sin^{1-k} \alpha(\cdot, v) f(\alpha(\cdot, v))\, S_{d-1}(K, dv)$$

with

$$f(t) = \binom{d-1}{k-1}^{-1} \frac{\varpi_{d+1}\varpi_{d-k}\varpi_k}{\varpi_d\varpi_{d-1}\varpi_{k+2}} \int_{-\cos t}^{1} (1-s^2)^{(k-1)/2} ds.$$

This result is taken from [20, Theorem 3.2].

In the following we introduce variations of the projection mean bodies $P_k(K)$ These variations were already mentioned in [19, p. 110] and are derived from K by an m-weighted projection which produces a convex body in $L \in \mathcal{L}_k^d$. These bodies are then thought of as bodies in $\mathbb{R}^d$ and averaged over all L.

Example 7 (**Support body**). In order to define the *k-th m-weighted support body $W_{k,m}(K)$ of K*, let L be some k-dimensional subspace and $m > -k$. Theorem 5.1 implies that the function

$$u \mapsto \int_{H^{d-k}(L,u)} h(K, v)\langle u, v\rangle^{k+m-1}\, dv = (\pi_{L,m} h(K, \cdot))(u)$$

on $S^{k-1}(L)$ is the support function of a convex body $W_{L,m}(K)$ in L. Interpreting this set as a (lower dimensional) convex body in $\mathbb{R}^d$, we can average in the sense of Minkowski addition with respect to all k-dimensional planes L and obtain a support function of a convex body $W_{k,m}(K)$ given by

$$h(W_{k,m}(K), u) = \int_{\mathcal{L}_k^d} h(W_{L,m}(K), u)\, dL = (\pi_{1,m}^{(k)} h(K, \cdot))(u), \qquad u \in S^{d-1}.$$

Due to Corollary 8.2 below, the 1-weighted support body $W_{k,1}(K)$ determines K for all k. The same holds true for $W_{k,2}(K)$.

We see again that the Examples 6 and 7 with $m = 1 - k$ are both related to the same spherical operator $\pi_{1,1-k}^{(k)}$.

With respect to star bodies K, the situation is slightly different. It is not meaningful to average the sections $K \cap L$ over all $L \in \mathcal{L}_k^d$ as an integral of radial functions, since $\rho(K \cap L, \cdot) = 0$ outside $S^{k-1}(L)$. Instead, we can consider $\rho(K \cap L, \cdot) = \pi_{L,\infty}\rho(K, \cdot)$ as a measure on $S^{k-1}(L)$, extend it to S^{d-1} by $\pi_{L,\infty}^*\pi_{L,\infty}\rho(K, \cdot)$ and average over L. The resulting measure is however proportional to $\int_{(\cdot)} \rho(K, u)\, du$, as a simple invariance argument shows, hence the corresponding average operator is trivial.

We now use the Laplace-Beltrami operator Δ to obtain connections between different operators $\pi_{\infty,j}^{(k)}$. Theorem 3.4 can be applied to the operators $T = \pi_{\infty,j}^{(k)}$. It turns out that $\pi_{\infty,j}^{(k-2)}$, $\pi_{\infty,j}^{(k)}$ and $\pi_{\infty,j}^{(k+2)}$ are closely related by the application of Δ. This is made precise in the next proposition, see also [21, Proposition 2] for the case $j = 1$. It is convenient to extend the definition of $\pi_{\infty,j}^{(k)}$ to $k = -1$ and $k = 0$ using Theorem 6.4. By definition, $\pi_{\infty,j}^{(-1)}$ and $\pi_{\infty,j}^{(0)}$ map the measure μ to the function given by (6.6) with $k = -1$ and $k = 0$, respectively, where ϖ_{-1} is given by (2.1) and $\varpi_0 = 1$.

Proposition 6.5. *For $d \geq 3$ and $f \in \mathcal{C}(S^{d-1})$ the following equalities hold in the sense of distributions.*

(a) *For $2 \leq k \leq d-3$ and $3-k \leq j < \infty$ we have*

$$\big(\Delta - a_k\big)\pi^{(k)}_{\infty,j} f = b_k \pi^{(k-2)}_{\infty,j} f - c_k \pi^{(k+2)}_{\infty,j} f. \tag{6.9}$$

Here $b_2 = (d-2)(j+1)j$ and the other coefficients are given by

$$a_k = (2k-d)(k+j-1) - j,\; b_k = \frac{d-k}{k-2}(k+j-1)(k+j-2),\; c_k = k(k-1).$$

(b) *For $k = d-2 \geq 3$ and $3-k \leq j < \infty$ we have*

$$\big(\Delta - a_{d-2}\big)\pi^{(d-2)}_{\infty,j} f = b_{d-2}\pi^{(d-4)}_{\infty,j} f - p\, f$$

with $p = \frac{(d-2)(d-3)}{\varpi_{d-1}}$.

(c) *For $k = 1$ and $2 \leq j < \infty$ we have*

$$\big(\Delta - a_1\big)\pi^{(1)}_{\infty,j} f = b_1 \pi^{(-1)}_{\infty,j} f$$

For $k = j = 1$, we have

$$\big(\Delta + (d-1)\big)\pi^{(1)}_{\infty,1} f = \frac{2}{\varpi_d} Rf, \tag{6.10}$$

where R is the spherical Radon transform.

Proof. By Theorem 6.4, $T = \pi^{(k)}_{\infty,j}$ satisfies the conditions of Theorem 3.4 with the function

$$G_k(x) = \frac{\varpi_{d-k}\varpi_k}{\varpi_d \varpi_{d-1}} \mathbf{1}_{[0,1]}(x) x^{k+j-1}(1-x^2)^{(1-k)/2}.$$

By Theorem 3.4(b), the main step is to calculate $D^*_d F_k$ for

$$F_k(x) = \mathbf{1}_{[-1,1]}(x) G_k(x) w(x) = \frac{\varpi_{d-k}\varpi_k}{\varpi_d \varpi_{d-1}} \mathbf{1}_{[0,1]}(x)\, x^{k+j-1}(1-x^2)^{(d-k-2)/2},$$

$x \in \mathbb{R}$. This can be done introducing the functions

$$f_{r,s}(x) = \mathbf{1}_{[0,1]}(x)\, x^r (1-x^2)^{s/2}, \quad x \in \mathbb{R}, \tag{6.11}$$

and using the fact that

$$f'_{r,s} = \begin{cases} r f_{r-1,s} - s f_{r+1,s-2}, & \text{if } r > 0,\ s > 0 \\ r f_{r-1,s} - \delta_1, & \text{if } r > 0,\ s = 0 \\ \delta_0 - s f_{1,s-2}, & \text{if } r = 0,\ s > 0 \\ \delta_0 - \delta_1, & \text{if } r = 0,\ s = 0 \end{cases}$$

holds in the sense of distributions. For example, in case (a), we obtain

$$D^*_d F_k = a_k F_k + b_k F_{k-2} + c_k F_{k+2}.$$

The right hand side is an integrable function, so (3.10) implies (6.9).

The case (b) and the first case of (c) are treated the same way. Note however, that the case $k = j = 1$ in (c) leads us out of the class of mean lifted projections, as we obtain

$$D_d^* F_1 = (1-d)F_1 + \frac{2}{\varpi_d}\delta_0.$$

The right hand side is *not* a regular distribution, so (3.10) cannot be used. One could extend Theorem 3.4(b) to measures (replacing regular distributions) on $\mathbb{R}$, but (6.10) also follows more directly: It is well known ([12, Proposition 2.1]) that $\big(\Delta + (d-1)\big)C = 2R$ holds in the sense of distributions. Together with (6.8), this implies (6.10). □

Theorem 3.4 can also be applied to determine $\Delta\pi_{m,j}^{(k)}$, for $m \neq \infty$. Typically, the result is again an integral transform with an integrable kernel function. However, it cannot be written as a linear combination of mean lifted projections, and we therefore omit the explicit formulas here.

7 Directed averages

A second possible average process which produces functions on S^{d-1} arises if, for a given $u \in S^{d-1}$, a scalar functional φ of $K|L$ (convex case) or $K \cap L$ (star body case) is averaged over all $L \in \mathcal{L}_k^{[u]}$, that is, over all k-spaces containing the given direction u.

For example, if $K \in \mathcal{K}^d$, $\varphi(K|L)$ could be the volume $V_k(K|L)$, the averaged function is then

$$\overline{V}_k(K,u) = \int_{\mathcal{L}_k^{[u]}} V_k(K|L)\, dL, \quad u \in S^{d-1}.$$

Since $V_k(K|L) = V_k(-K|L)$, the corresponding operator can only be injective on even functions. A similar approach is possible for central sections of star bodies K, the averaged section function is then

$$\overline{R}_k(K,u) = \int_{\mathcal{L}_k^{[u]}} V_k(K \cap L)\, dL, \quad u \in S^{d-1}. \tag{7.1}$$

Here, we recall that

$$V_k(K \cap L) = \frac{1}{k}\int_{S^{k-1}(L)} \rho^k(K,v)\, dv. \tag{7.2}$$

Again, the corresponding operator can only be injective on even functions.

In order to obtain better injectivity results, we may allow the functional φ not only to depend on K and L, but also on u, for example by considering only the "half" of the projection $K|L$ or section $K \cap L$ which is in direction u. This slightly vague formulation can be made precise analytically by restricting functions and measures to the half-sphere $u^+ \cap S^{d-1}$, respectively $u^+ \cap S^{k-1}(L), L \in \mathcal{L}_k^d$.

A first example of this kind is the directed section mean presented in the introduction (Example 5). As a generalization of (7.1) and (7.2), this function was defined by

$$\overline{s}_k(K,u) = \frac{1}{k}\int_{\mathcal{L}_k^{[u]}}\int_{S^{k-1}(L)\cap u^+} \rho^k(K,v)\,dv\,dL, \quad u \in S^{d-1}.$$

For the question as to whether K is determined by the function $\overline{s}_k(K,\cdot)$, the power k in the integrand does not play a role. We therefore may and will replace it by 1, that is, we consider

$$\overline{s}_{1k}(K,u) = \frac{1}{k}\int_{\mathcal{L}_k^{[u]}}\int_{S^{k-1}(L)\cap u^+} \rho(K,v)\,dv\,dL, \quad u \in S^{d-1}.$$

Example 8 (**Directed projection mean**)**.** The analogue for projections of a convex body K is the *directed projection mean*

$$\overline{p}_{1k}(K,u) = \frac{1}{k}\int_{\mathcal{L}_k^{[u]}}\int_{S^{k-1}(L)\cap u^+} h(K,v)\,dv\,dL, \quad u \in S^{d-1}.$$

Up to a constant, $\overline{p}_{1k}(K,\cdot)$ is the directed version of the averaged mean width of the projections, since

$$\int_{S^{k-1}(L)\cap u^+} h(K,v)\,dv + \int_{S^{k-1}(L)\cap(-u)^+} h(K,v)\,dv = c_d V_1(K|L).$$

Directed versions of other intrinsic volumes of $K|L$ (in particular, a directed version of the k-dimensional volume $V_k(K|L)$, based on the tensor formulas of [29] and [31]), have also been studied (see [14] and [15]). However, their averages depend in a nonlinear way on K (as long as Minkowski addition is considered), therefore we concentrate on $\overline{p}_{1k}(K,\cdot)$, in the following. Later, we also discuss a variant of $\overline{p}_{1k}(K,\cdot)$, where the support function is replaced by the first surface area measure.

It is obvious that $K \mapsto \overline{s}_{1k}(K,\cdot)$ (for star bodies K) and $K \mapsto \overline{p}_{1k}(K,\cdot)$ (for convex bodies K) lead to the same linear operator on $\mathcal{C}(S^{d-1})$ and we now aim to describe it in terms of spherical projections. Due to the different nature of the averaging process, there are no liftings involved.

Definition 7.1. *For $1 \le k \le d-1$ and $0 \le m, j < \infty$, we define the* mean directed projection $\tau_{m,j}^{(k)}$ *by*

$$(\tau_{m,j}^{(k)}\mu)(u) = \int_{\mathcal{L}_k^{[u]}} (\pi_{[u],m}^{L}\pi_{L,j}\mu)(\{u\})\,dL, \tag{7.3}$$

for $u \in S^{d-1}$ and $\mu \in \mathcal{M}(S^{d-1})$.

For $0 \le m, j < \infty$, we have

$$\begin{aligned}(\pi^L_{[u],m}\pi_{L,j}\mu)(u) &= \int_{S^{k-1}(L)\setminus L^\perp} \mathbf{1}_{\{\mathrm{pr}_{[u]}(w)\in\{u\}\}}\|w|[u]\|^m\,(\pi_{L,j}\mu)(dw)\\ &= \int_{S^{k-1}(L)\cap(u^+\setminus u^\perp)} \|w|[u]\|^m\,(\pi_{L,j}\mu)(dw)\\ &= \int_{S^{d-1}\cap(u^+\setminus u^\perp)} (\pi^*_{L,j}\|(\cdot)|[u]\|^m)(x)\,\mu(dx)\\ &= \int_{S^{d-1}\cap(u^+\setminus u^\perp)} \|x|L\|^j\,\|\mathrm{pr}_L(x)|[u]\|^m\,\mu(dx).\end{aligned}$$

For $\mu = \int_{(\cdot)} f d\omega_{d-1}$, we consider $\tau^{(k)}_{m,j}$ also as an operator on functions, hence

$$(\tau^{(k)}_{m,j}f)(u) = \int_{\mathcal{L}^{[u]}_k}\int_{S^{d-1}\cap u^+} \|x|L\|^j\,\|\mathrm{pr}_L(x)|[u]\|^m\,f(x)\,dx\,dL. \tag{7.4}$$

In Definition 7.1 we excluded the cases where one of the numbers m or j equals ∞, as the corresponding spherical projections cannot naturally be defined on the space of measures. On the space of continuous functions we may use (7.3) with μ replaced by f and the singleton $\{u\}$ replaced by u. This yields

$$(\tau^{(k)}_{m,\infty}f)(u) = \int_{\mathcal{L}^{[u]}_k}\int_{S^{k-1}(L)\cap u^+} f(v)\langle u,v\rangle^m\,dv\,dL,$$

and

$$(\tau^{(k)}_{\infty,j}f)(u) = \int_{\mathcal{L}^{[u]}_k}\int_{H^{d-k}(L,u)} f(v)\langle u,v\rangle^j\,dv\,dL,$$

$0 \le m, j < \infty$. Both these relations can alternatively be obtained by considering the limits $j \to \infty$ and $m \to \infty$ in the appropriately normalized right hand side of (7.4). We see that

$$\overline{s}_{1k}(K,\cdot) = \frac{1}{k}\tau^{(k)}_{0,\infty}\rho(K,\cdot), \quad \overline{p}_{1k}(K,\cdot) = \frac{1}{k}\tau^{(k)}_{0,\infty}h(K,\cdot).$$

As we shall now show, the operator $\tau^{(j)}_{s,\infty}$ fits into our series of mean lifted projections $\pi^{(k)}_{m,\infty}$.

Theorem 7.2. *For $k \in \{1,\dots,d-1\}$ and $m > -k$, we have*

$$\tau^{(d-k+1)}_{k+m-1,\infty} = \pi^{(k)}_{m,\infty}.$$

In particular $\tau^{(j)}_{0,\infty} = \pi^{(d-j+1)}_{j-d,\infty}$, for $j = 1,\dots,d-1$.

Proof. For $f, g \in \mathcal{C}(S^{d-1})$, the self-adjointness of $\pi^{(k)}_{m,\infty}$ and Fubini's theorem imply

$$\begin{aligned}&\int_{S^{d-1}} (\pi^{(k)}_{m,\infty}f)(u)g(u)\,du\\ &\quad= \int_{S^{d-1}} f(u)(\pi^{(k)}_{m,\infty}g)(u)\,du\\ &\quad= \int_{\mathcal{L}^d_k}\int_{S^{d-1}} \|u|L\|^m f(u)g\left(\mathrm{pr}_L(u)\right)\,du\,dL.\end{aligned}$$

Applying (2.3) yields

$$\begin{aligned}
&\int_{S^{d-1}} (\pi^{(k)}_{m,\infty} f)(u) g(u)\, du \\
&\qquad = \int_{\mathcal{L}^d_k} \int_{S^{k-1}(L)} g(v) \int_{H^{d-k}(L,v)} \langle v, w\rangle^{k+m-1} f(w)\, dw\, dv\, dL.
\end{aligned}$$

Finally, we use [33, Satz 6.1.1] to obtain

$$\begin{aligned}
&\int_{S^{d-1}} (\pi^{(k)}_{m,\infty} f)(u) g(u)\, du \\
&\quad = \int_{S^{d-1}} g(u) \int_{\mathcal{L}^{[u]}_k} \int_{S^{d-k}(L^\perp \vee u)\cap u^+} \langle u, w\rangle^{k+m-1} f(w)\, dw\, dL\, du \\
&\quad = \int_{S^{d-1}} g(u) \int_{\mathcal{L}^{[u^\perp]}_{d-k}} \int_{S^{d-k}(M \vee u)\cap u^+} \langle u, w\rangle^{k+m-1} f(w)\, dw\, dM\, du \\
&\quad = \int_{S^{d-1}} g(u) \int_{\mathcal{L}^{[u]}_{d-k+1}} \int_{S^{d-k}(L)\cap u^+} \langle u, w\rangle^{k+m-1} f(w)\, dw\, dL\, du \\
&\quad = \int_{S^{d-1}} g(u) (\tau^{(d-k+1)}_{k+m-1,\infty} f)(u)\, du.
\end{aligned}$$

Since g is arbitrary, this proves the result. □

If we concentrate on even functions, we obtain a further connection between mean lifted projections and mean directed projections. Namely, for $f \in \mathcal{C}(S^{d-1})$ even and $u \in S^{d-1}$, we have

$$(\pi^{(k)}_{m,1-k} f)(u) = \frac{1}{2} \int_{\mathcal{L}^d_k} \|u|L\|^m \int_{S^{d-k}(L^\perp \vee u)} f(v)\, dv\, dL.$$

An invariance argument shows that the image measure of $\int_{(\cdot)} \|u|L\|^m\, dL$ under the mapping $L \mapsto L^\perp \vee u$ is a multiple of $\nu^{[u]}_{d-k+1}$. Hence,

$$(\pi^{(k)}_{m,1-k} f)(u) = \frac{c_{d,m,k}}{2} \int_{\mathcal{L}^{[u]}_{d-k+1}} \int_{S^{d-k}(L)} f(v)\, dv\, dL, \tag{7.5}$$

where $c_{d,m,k}$ is given by (4.6). In view of the definition of $\tau^{(d-k+1)}_{0,\infty}$ and Theorem 7.2 we obtain

$$\pi^{(k)}_{m,1-k} f = c_{d,m,k} \tau^{(d-k+1)}_{0,\infty} f = c_{d,m,k} \pi^{(k)}_{\infty,1-k} f$$

for all even continuous functions f. Consequently, on even functions, the operators $\pi^{(k)}_{m,1-k}$, $m = 1-k, 2-k, \ldots$ are proportional to $\tau^{(d-k+1)}_{0,\infty}$ and $\pi^{(k)}_{\infty,1-k}$.

In particular, for centrally symmetric K, (7.5) implies

$$\begin{aligned}
(k c_{d,1,d-k+1}) \overline{p}_{1k}(K,\cdot) &= c_{d,1,d-k+1} \tau^{(k)}_{0,\infty} h(K,\cdot) \\
&= \pi^{(d-k+1)}_{1,k-d} h(K,\cdot) = h(W_{d-k+1,k-d}(K),\cdot).
\end{aligned}$$

Hence $\overline{p}_{1k}(K,\cdot)$ and $h(W_{d-k+1,k-d}(K),\cdot)$ coincide (up to constants) with the invariant average of the mean width of all projections of the symmetric body K onto k-planes containing u.

Another instance, where we get a connection between directed means and mean lifted projections, is the class of transforms $\tilde{\tau}_m^{(k)}$ defined on $\mathcal{C}^\infty(S^{d-1})$ by

$$(\tilde{\tau}_m^{(k)} f)(u) = \int_{\mathcal{L}_k^{[u]}} \int_{S^{k-1}(L)\cap u^+} \big(\Delta_L \pi_{L,\infty} f\big)(v)\langle u,v\rangle^m \, dv\, dL,$$

where, $2 \le k \le d$ and $m \ge 0$. This can be written more concisely as

$$(\tilde{\tau}_m^{(k)} f)(u) = \int_{\mathcal{L}_k^{[u]}} \big(\pi^L_{[u],m} \Delta_L \pi_{L,\infty} f\big)(u)\, dL.$$

We include here the case $k = d$, which reads

$$(\tilde{\tau}_m^{(d)} f)(u) = \int_{S^{d-1}\cap u^+} \big(\Delta f\big)(v)\langle u,v\rangle^m \, dv = \varpi_d/2\big((\pi^{(1)}_{\infty,m}\Delta)f\big)(u),$$

by Theorem 6.4.

Theorem 7.3. *Fix $2 \le k \le d$. Then we have*

(a) $\frac{1}{\varpi_{d-1}}\tilde{\tau}_0^{(k)} = \frac{1}{\varpi_{k-1}}\tilde{\tau}_0^{(d)} = \frac{\varpi_d}{2\varpi_{k-1}}\pi^{(1)}_{\infty,0}\Delta,$

(b) $\tilde{\tau}_1^{(k)} = \frac{\varpi_{k-1}}{\varpi_{d-1}} R - p_k\, \pi^{(d-k+1)}_{\infty,k-d+1},$

(c) $\tilde{\tau}_m^{(k)} = q_k \pi^{(d-k-1)}_{\infty,k-d+m} - p_k\, \pi^{(d-k+1)}_{\infty,k-d+m}$ *for* $m \ge 2$.

Here the constants are given by

$$q_k = \frac{m(m-1)\varpi_d\varpi_{d-1}\varpi_{k-1}}{\varpi_{k+1}\varpi_{d-k-1}} \qquad \text{and} \qquad p_k = \frac{(k-1)\varpi_d\varpi_{d-1}}{\varpi_{d-k+1}}.$$

Proof. For an orthonormal basis $x_1, \dots, x_k$ of L, we have

$$\|\frac{\partial^2}{\partial x_i^2}\big(\pi_{L,\infty} f\big)\check{}\ \|_\infty = \|\frac{\partial^2}{\partial x_i^2}\check{f}\|_\infty \le \max_{q\in\mathbb{N}_0^d, |q|\le 2} \|\partial^q f\|_\infty =: \|f\|_{\mathcal{C}^2},$$

so we get $\|\Delta_L \pi_{L,\infty} f\|_\infty \le k\|f\|_{\mathcal{C}^2}$ and

$$\|\tilde{\tau}_m^{(k)} f\|_\infty \le k\varpi_k/2\|f\|_{\mathcal{C}^2}.$$

Thus, $\tilde{\tau}_m^{(k)}$ is a linear continuous operator from $\mathcal{C}^\infty(S^{d-1})$ to $\mathcal{C}(S^{d-1})$ and it intertwines the action of the rotation group. As the right hand sides of (a), (b) and (c) have the same properties, we may apply Corollary 3.3. It is therefore enough to prove the identities when applied to functions of the form $\varphi(\langle u,\cdot\rangle)$ with $\varphi(x) = x^n$, $n = 0, 1, 2, \dots$ and evaluated at u. As $u \in L$, we have

$$\Delta_L \pi_{L,\infty}\varphi(\langle u,\cdot\rangle) = \Delta_L\varphi(\langle u,\cdot\rangle) = (D_k\varphi)(\langle u,\cdot\rangle),$$

where (3.9) was used in the k-dimensional space L. Introducing cylindrical coordinates, we arrive at

$$\begin{aligned}\big(\tilde{\tau}_m^{(k)}\varphi(\langle u,\cdot\rangle)\big)(u) &= \int_{\mathcal{L}_k^{[u]}}\int_{S^{k-1}(L)\cap u^+}(D_k\varphi)(\langle u,v\rangle)\langle u,v\rangle^m\,dv\,dL\\ &= \varpi_{k-1}\int_0^1 (D_k\varphi)(x)x^m(1-x^2)^{(k-3)/2}dx\\ &= \varpi_{k-1}\big(D_k^* f_{m,k-3}\big)(\varphi),\end{aligned}$$

where the functions (6.11) are used. Explicit calculation for $m=0$ shows that

$$\big(D_k^* f_{0,k-3}\big) = \delta_0' \tag{7.6}$$

is independent of k and thus

$$\varpi_{d-1}\big(\tilde{\tau}_0^{(k)}\varphi(\langle u,\cdot\rangle)\big)(u) = \varpi_{k-1}\big(\tilde{\tau}_0^{(d)}\varphi(\langle u,\cdot\rangle)\big)(u),$$

which implies (a). (Actually, (7.6) shows that $\tilde{\tau}_m^{(k)}$ is, up to a constant, equal to the derivative of the generalized spherical Radon transform at height $x=0$, see, for example, [22] where this transform is defined.)

For $m=1$ we get

$$\big(D_k^* f_{1,k-3}\big) = \delta_0 - (k-1)f_{1,k-3}.$$

Using

$$\big(R(\varphi(\langle u,\cdot\rangle)\big)(u) = \varpi_{d-1}\varphi(0)$$

and Theorem 6.4, (b) is obtained. Finally, $m\ge 2$ implies

$$\big(D_k^* f_{m,k-3}\big) = m(m-1)f_{m-2,k-1} - (k-1)f_{m,k-3},$$

which, in view of Theorem 6.4, implies (c). □

Explicitly, Theorem 7.3(a) reads

$$\int_{\mathcal{L}_k^{[u]}}\int_{S^{k-1}(L)\cap u^+}\big(\Delta_L\pi_{L,\infty}f\big)(v)\,dv\,dL = \int_{S^{d-1}\cap u^+}\big(\Delta f\big)(v)\,dv,$$

which was shown in [15, Theorem 7 and 8] by calculating and comparing the multipliers of the operators on both sides.

8 Harmonic analysis of the operators $\pi_{m,j}^{(k)}$

As we have seen, for finite values of m and j, the operator $\pi_{m,j}^{(k)}$ can be considered as a self-adjoint and intertwining continuous linear operator on $\mathcal{C}(S^{d-1})$. According to (3.4) the corresponding multipliers are given by

$$a_{d,k,m,j,n} = \big(\pi_{m,j}^{(k)}P_n^d(\langle u,\cdot\rangle)\big)(u). \tag{8.1}$$

If $\mu \in \mathcal{M}(S^{d-1})$ with $\pi_{m,j}^{(k)}\mu = 0$, we have

$$a_{d,k,m,j,n} \int_{S^{d-1}} h_n(u)\,\mu(du) = \int_{S^{d-1}} \left(\pi_{j,m}^{(k)} h_n\right)(u)\,\mu(du) = 0,$$

for all $h \in \mathcal{H}_n^d$ and for all $n = 0, 1, \ldots$. If the multipliers $a_{d,k,m,j,n}$ are all nonzero, the measure μ would annihilate all spherical harmonics and, therefore, all continuous functions. Consequently, $\mu = 0$ which implies that $\pi_{m,j}^{(k)} : \mathcal{M}(S^{d-1}) \to \mathcal{M}(S^{d-1})$ would be injective.

In order to find an explicit representation for the multipliers $a_{d,k,m,j,n}$, we will make use of the connection coefficients $c_{n,t}^{d,d'}$. These are positive numbers which allow us to express d-dimensional Legendre polynomials in terms of Legendre polynomials of dimension $d' \in \{2, \ldots, d-1\}$, see [1]. We have

$$P_n^d(x) = \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d'} P_{n-2t}^{d'}(x), \tag{8.2}$$

where

$$c_{n,t}^{d,d'} = \frac{(2(n-2t)+d'-2)n!}{(d'-2)t!(n-2t)!} \frac{\left(\frac{d-2}{2}\right)_{n-t} \left(\frac{d-d'}{2}\right)_t (d'-2)_{n-2t}}{\left(\frac{d'}{2}\right)_{n-t} (d-2)_n}.$$

Here, we have used the Pochhammer symbol

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad \text{for } a \in \mathbb{R};$$

we also put $(a)_0 = 1$, for all $a \in \mathbb{R}$. In the sequel, it will be convenient to extend (8.2) to include the case $d' = d \geq 2$. For this extension, we put $c_{n,t}^{d,d} = 0$ for $t > 0$ and $c_{n,0}^{d,d} = 1$. The connection coefficients will allow us to relate the operators $\pi_{m,j}^{(k)}$ to certain intertwining operators in lower dimensions.

We define the operator $I_{d,p}$ on $\mathcal{C}(S^{d-1})$ by

$$\left(I_{d,p} f\right)(v) = \int_{S^{d-1} \cap v^+} f(u) \langle u, v\rangle^p \, du.$$

By Theorem 3.4(a) this is a continuous endomorphism of $\mathcal{C}(S^{d-1})$ which intertwines the action of SO_d. If f is continuous on $[-1,1]$, we can use cylindrical coordinates to deduce

$$\left(I_{d,p} f(\langle v, \cdot\rangle)\right)(v) = \varpi_{d-1} \int_0^1 x^p f(x) \left(1-x^2\right)^{(d-3)/2} dx. \tag{8.3}$$

We use Proposition 3.2 and (3.4) to see that the multipliers of $I_{d,p}$ are given by

$$\beta_{d,p,n} = \varpi_{d-1} \int_0^1 P_n^d(x) x^p (1-x^2)^{(d-3)/2} \, dx \qquad (n = 0, 1, \ldots).$$

If $d \geq 2$, we have

$$\beta_{d,p,n} = \begin{cases} \frac{p!}{2^n (p-n)! \left(\frac{d-1}{2}\right)_n} \frac{\varpi_{d-1}\varpi_{d+p+n}}{\varpi_{d+2n-1}\varpi_{p-n+1}}, & \text{if } 0 \leq n \leq p, \\ \frac{p!}{2^{p+1}\left(\frac{d-1}{2}\right)_{p+1}} \varpi_{d-1} P_{n-p-1}^{d+2p+2}(0), & \text{if } p < n \end{cases} \tag{8.4}$$

for $p > 0$ and

$$\beta_{d,0,n} = \frac{\varpi_{d-1}}{d-1} P_{n-1}^{d+2}(0). \tag{8.5}$$

Both results can be found in [21, proof of Proposition 3], where $2 \leq p \leq d-1$ was assumed, but not needed. Since $P_{2n+1}^d(0) = 0$ and

$$P_{2n}^d(0) = (-1)^n \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{d-1}{2}\right)_n}, \qquad n = 0, 1, 2, \ldots$$

(see [16, Lemma 3.3.8]), the multipliers $\beta_{d,p,n}$ are explicitly known. In particular, $\varpi_{d-1}^{-1}\beta_{d,p,n}$ is a rational multiple of π for even d and even $p+n$ with $n \leq p$. In all other cases $\varpi_{d-1}^{-1}\beta_{d,p,n} \in \mathbb{Q}$.

Theorem 8.1. *Fix $1 \leq k \leq d-1$ and $-k < m, j < \infty$.*

(a) *The multipliers of $\pi_{m,j}^{(k)}$ satisfy*

$$a_{d,k,m,j,n} = \frac{\varpi_k}{\varpi_d} \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d-k+1} \beta_{d-k+1,k+j-1,n-2t} \beta_{d-k+1,k+m-1,n-2t},$$

(b) *The multipliers of $\pi_{\infty,j}^{(k)}$ satisfy*

$$a_{d,k,\infty,j,n} = \frac{\varpi_k}{\varpi_d} \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d-k+1} \beta_{d-k+1,k+j-1,n-2t}. \tag{8.6}$$

Proof. To prove (a) we use (8.1), definitions of $\pi_{m,j}^{(k)}$, $I_{s,p}$ and $\beta_{s,p,n}$ and obtain

$$\begin{aligned} a_{d,k,m,j,n} &= \left(\pi_{m,j}^{(k)} P_n^d(\langle u, \cdot \rangle)\right)(u) \\ &= \int_{\mathcal{L}_k^d} \|u|L\|^m \int_{H^{d-k}(L,\mathrm{pr}_L(u))} P_n^d\left(\langle u, v\rangle\right) \langle v, \mathrm{pr}_L(u)\rangle^{k+j-1}\, dv\, dL \\ &= \int_{\mathcal{L}_k^d} \|u|L\|^m \left(I_{d-k+1,k+j-1} P_n^d\left(\langle u, \cdot\rangle\right)\right)(\mathrm{pr}_L(u))\, dL \\ &= \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d-k+1} \beta_{d-k+1,k+j-1,n-2t} \int_{\mathcal{L}_k^d} \|u|L\|^m P_{n-2t}^{d-k+1}\left(\langle u, \mathrm{pr}_L(u)\rangle\right) dL \\ &= \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d-k+1} \beta_{d-k+1,k+j-1,n-2t} \int_{\mathcal{L}_k^d} \|u|L\|^m P_{n-2t}^{d-k+1}\left(\|u|L\|\right) dL. \end{aligned}$$

In order to continue, we consider integrals of the form

$$\int_{\mathcal{L}_k^d} f\left(\|u|L\|\right)\, dL,$$

where $f \in \mathcal{C}(S^{d-1})$ and $u \in S^{d-1}$. The rotation invariance of ν_k implies that this integral is independent of the choice of $u \in S^{d-1}$. A further application of rotation invariance gives

$$\begin{aligned}\varpi_d \int_{\mathcal{L}_k^d} f\left(\|u|L\|\right)\, dL &= \int_{S^{d-1}} \int_{\mathcal{L}_k^d} f\left(\|u|L\|\right)\, dL\, du \\ &= \int_{S^{d-1}} f\left(\|u|L\|\right)\, du.\end{aligned}$$

We deduce that, for any $L \in \mathcal{L}_k^d$,

$$\begin{aligned}\varpi_d \int_{\mathcal{L}_k^d} f\left(\|u|L\|\right)\, dL &= \int_{S^{k-1}(L)} \int_{H^{d-k}(L,v)} |\langle w, v\rangle|^{k-1} f\left(\|w|L\|\right)\, dw\, dv \\ &= \int_{S^{k-1}(L)} \int_{H^{d-k}(L,v)} |\langle w, v\rangle|^{k-1} f\left(|\langle w, v\rangle|\right)\, dw\, dv \\ &= \int_{S^{k-1}(L)} \left(I_{d-k+1,k-1} f\left(|\langle v, \cdot\rangle|\right)\right)(v)\, dv.\end{aligned}$$

Combining this with our earlier calculation of $a_{d,k,m,j,n}$ and (8.3) gives (a).

To show (b), the same arguments as above can be used:

$$\begin{aligned}a_{d,k,\infty,j,n} &= \left(\pi_{\infty,j}^{(k)} P_n^d\left(\langle u, \cdot\rangle\right)\right)(u) = \int_{\mathcal{L}_k^d} \|u|L\|^j P_n^d\left(\|u|L\|\right)\, dL \\ &= \frac{1}{\varpi_d} \int_{S^{d-1}} \|u|L\|^j P_n^d\left(\|u|L\|\right)\, du \\ &= \frac{\varpi_{d-k}\varpi_k}{\varpi_d} \int_0^1 x^{k+j-1} P_n^d(x)\left(1 - x^2\right)^{(d-k-2)/2}\, dx \\ &= \frac{\varpi_k}{\varpi_d} \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d-k+1} \beta_{d-k+1,k+j-1,n-2t}.\end{aligned}$$

This completes the proof. □

Corollary 8.2. *Let $2 \le k \le d$ be given.*
Then $\pi_{j,j}^{(k)}$ and $\pi_{j,j+1}^{(k)}$ are injective for all $-k < j < \infty$.

Proof. The injectivity of $\pi_{j,j}^{(k)}$ follows directly from Theorem 8.1(a), which shows that all the multipliers are positive.

To show that the multipliers of $\pi_{j,j+1}^{(k)}$ are all positive, we use the abbreviations $d' = d - k + 1$ and $p = k + j$. According to Theorem 8.1(a) we have

$$a_{d,k,j,j+1,n} = \frac{\varpi_k}{\varpi_d} \sum_{t=0}^{\left[\frac{n}{2}\right]} c_{n,t}^{d,d'} \beta_{d',p-1,n-2t} \beta_{d',p,n-2t}.$$

All summands with $n-2t>p$ (if any) vanish, as evaluation with (8.4) involves a Legendre polynomial of odd degree. Here we used the fact that $p-1$ and p obviously have different parity. All summands with $n-2t\le p-1$ are positive due to (8.4). If $n-p$ is odd, all summands have been treated and we have $a_{d,k,j,j+1,n}>0$, since there is at least one natural $t\le[n/2]$ with $n-2t\le p-1$. Otherwise,

$$a_{d,k,j,j+1,n}\ge\frac{\varpi_k}{\varpi_d}c^{d,d'}_{n,(n-p)/2}\beta_{d',p-1,p}\beta_{d',p,p}>0$$

again by (8.4) and (8.5). □

Although $\pi^{(k)}_{\infty,j}$ need not be injective for a fixed k, we can show that certain linear combinations of these operators for two different dimensions k are injective. Let the field extension $\mathbb{Q}(\pi)$ be the smallest subfield of $\mathbb{R}$ which contains π and all rationals.

Proposition 8.3. *Let integers $2\le k<k'\le d-1$ and $j\ge 1-k$ be given. If $a,b\in\mathbb{R}$, $b\neq 0$ are such that $a/b\notin\mathbb{Q}(\pi)$, then*

$$a\varpi_k^{-1}\pi^{(k)}_{\infty,j}+b\varpi_{k'}^{-1}\pi^{(k')}_{\infty,j}$$

is injective on $\mathcal{M}(S^{d-1})$.

Proof. It can be seen from (8.6) and (8.4) that the multipliers of

$$\frac{\varpi_d}{\varpi_{d-1}\varpi_k}\pi^{(k)}_{\infty,j}\qquad\text{and}\qquad\frac{\varpi_d}{\varpi_{d-1}\varpi_{k'}}\pi^{(k')}_{\infty,j}$$

are all in $\mathbb{Q}(\pi)$. It is therefore enough to show that for all $n=0,1,2,\dots,$ at least one of the multipliers $a_{d,k,\infty,j,n}$ and $a_{d,k',\infty,j,n}$ is nonzero. We distinguish two cases depending on the parities of k and k'.

If k and k' have different parity, the claim follows from the fact that

$$a_{d,k,\infty,j,n}>0,\ \text{ for all } n \text{ such that } k+j+n \text{ is odd.} \tag{8.7}$$

We show (8.7) for the case, where $k+j$ is odd, the even case follows analogously. Let n be even. Again from (8.6) and (8.4), and the fact that $P_q^d(0)=0$ for odd q it follows that

$$a_{d,k,\infty,j,n}=\sum_{t=0}^{\frac{k+j-1}{2}}c^{d,d-k+1}_{n,n/2-t}\beta_{d-k+1,k+j-1,2t}$$

is a sum of positive numbers and hence positive.

The remaining case, where k and k' have the same parity, can be treated as in [21, Lemma 1]. Therefore, we keep the arguments short. Assume that there is an $n\in\mathbb{N}$ such that $a_{d,k,\infty,j,n}=a_{d,k',\infty,j,n}=0$. (8.6) and (8.4) imply $n\ge k'+j+1$. From $a_{d,k,\infty,j,n}=0$ and repeated application of Proposition 6.5(a) we conclude that for all $p>0$ such that $k+2p\le d-1$,

$$a_{d,k+2p,\infty,j,n}=c(d,p,n)a_{d,k+2,\infty,j,n} \tag{8.8}$$

with some constants $c(d,p,n)$. The condition $n\ge k'+j+1$ and the fact that the n-th multiplier of Δ is $-n(n+d-2)$ implies that $c(d,p,n)>0$ for $k+2p\le k'$.

But $a_{d,k',\infty,j,n} = 0$, so (8.8) implies first $a_{d,k+2,\infty,j,n} = 0$ and then $a_{d,k+2p,\infty,j,n} = 0$ for all $k+2p \leq k'$. Repeated application of Proposition 6.5(a), starting with k' and increasing the dimension k implies $a_{d,d-1,\infty,j,n} = 0$ or $a_{d,d-2,\infty,j,n} = 0$. As $d \geq 4$ we have found $p \geq d/2$ with $a_{d,p,\infty,j,n} = 0$. But this is impossible:

As the multipliers of $\pi^{(p)}_{\infty,j}$ are, up to non-vanishing multiplicative constants, equal to

$$\int_0^1 x^{p+j-1}(1-x^2)^{(d-p-2)/2} P^d_n(x)\, dx,$$

the arguments in [7] imply that the multipliers of $\pi^{(p)}_{\infty,j}$ are all positive as long as $d/2 \leq p \leq d-1$ and $-p < j < \infty$. □

As mentioned previously in Section 6, the mean projection operator (Example 3) and the Blaschke section operator (Example 4) both correspond to $\pi^{(k)}_{1,\infty}$. Here, we have incomplete injectivity results. Roughly speaking, the operators are known to be injective for $k \geq d/2$ and for $k = 3$, however, they are known to be non-injective for $k = 2$ in the dimension $d = 14$ (see [7], [9] and [10]).

The Blaschke section bodies of the second kind (Example 6) and the k-th support bodies with weight $1-k$ (Example 7) both correspond to the operator $\pi^{(k)}_{1,1-k}$; see Section 6. In [19, proof of Satz 3.20] (see also [20, p. 517]) the multipliers of $\pi^{(k)}_{1,1-k}$ are calculated explicitly for $1 < k < d$:

$$a_{d,k,1-k,1,n} = \frac{(k-1)(d-2)}{16\binom{d+n-3}{n}} \frac{\varpi_{d+1}\varpi_{d-k+n+1}\varpi_k\varpi_{n+2}}{\varpi_{d+n-2}\varpi_{k+n-1}\varpi_6}$$

for $n \neq 1$ and $a_{d,k,1-k,1,1} = \frac{k}{d(d-k)}\varpi_{d-k}$. This shows that $\pi^{(k)}_{1,1-k}$ is injective on $\mathcal{M}(S^{d-1})$ for all $1 < k < d$, and so here we always have injectivity for these operators.

The directed section mean (Example 5) and the directed projection mean (Example 8) correspond to the operator $\pi^{(d-k+1)}_{k-d,\infty}$ (see Section 7). Building on the ideas in [17] and [18], these were studied in [13] and [14] where equation 8.6 was used to show that $\pi^{(d-k+1)}_{k-d,\infty}$ is injective for $2 \leq k < (2d-3)/5$ and for $(d-2)/2 \leq k \leq d-1$. These injectivity properties are based on recursion formulas which were derived using an algorithm of Zeilberger, see [24]. This algorithm finds formulas for the n-th multiplier of the form $a_n = p_n(d,k)/q_n(d,k)$, where p_n, q_n are polynomials in the dimensions d of the ambient space and k of the sections (projections). Hence, we have $a_n = 0$ (and therefore non-injectivity), if (d,k) is an integer point on the algebraic curve $p_n(d,k) = 0$. If the curve is singular, there may be an infinite family of such points and consequently a whole family of pairs (d,k), where injectivity fails. If the curve is non-singular, Siegel's theorem guarantees that there are at most finitely many integer solutions of $p_n(d,k) = 0$. The integer solutions (d,k) which satisfy $2 \leq k \leq d-1$ correspond to non-injectivity cases of the integral operator, and these are usually of an isolated nature. To give an example of these phenomena, we note that, for $\pi^{(d-k+1)}_{k-d,\infty}$, we have $p_5(d,k) = 4(d+4)(2d-5k-3)$. It follows that $\pi^{(3i+1)}_{-3i-3,\infty}$ is not injective for any $i = 1, 2, \ldots$. On the other hand $p_9(d,k) = 24(d+8)f(d,k)$ where

$$f(d,k) = 16d^3 - 72d^2k + 126dk^2 - 105k^3 + 72d^2 - 72dk - 315k^2 + 170d - 975k - 525.$$

The corresponding algebraic curve can have only finitely many integer points. However, these do occur. In fact $f(60, 25) = 0$ and so $\pi^{(36)}_{-35,\infty}$ is not injective.

9 Related tomographic transforms

In this section we collect some examples of tomographic data that can be written in terms of mean lifted projections, although they are not defined as averages with respect to k-dimensional planes. The first two examples are tomographic data which can be expressed using $\pi^{(k)}_{m,j}$ directly, where $m < \infty$, but $k = 0$. We therefore extend the definition of mean lifted projections to the case $k = 0$ using Theorem 6.4. For this purpose we let $\pi^{(0)}_{m,j}$ be the operator defined by (6.6) with $k = 0$ (where $\varpi_0 = 1$). The third example consists of tomographic data that can be expressed as combinations of two mean lifted projections and the Laplace-Beltrami operator.

Example 9 (Mean section body). For $1 \le k \le d-1$, the *mean section body* $M_k(K)$ of $K \in \mathcal{K}^d$ is defined by

$$h(M_k(K), \cdot) = \int_{\mathcal{L}^d_k} \int_{L^\perp} h(K \cap (L + x), \cdot)\, dx\, dL.$$

Up to a multiplicative constant, $M_k(K)$ can be interpreted as the invariant Minkowski-average of all intersections $K \cap E$, where E runs through all k-dimensional affine subspaces hitting K.

For $k = 2$, Goodey and Weil [11] showed that $h(M_k(K), \cdot)$ coincides with an integral transform of the surface area measure of K up to a linear term (which corresponds to a translation of $M_k(K)$). If $M^*_k(K)$ denotes the centered version of $M_k(K)$ (the translate of $M_k(K)$ with Steiner point at the origin), then

$$h(M^*_2(K), u) = \frac{1}{(d-1)\pi} \int_{S^{d-1}} \alpha(u, -v) \sin \alpha(u, -v)\, S_{d-1}(K, dv).$$

(The constant here is corrected for a missing multiplicative term $1/(2\pi)$ in [11].) As $\alpha(u, -v) + \alpha(u, v) = \pi$, this gives

$$h(M^*_2(K), \cdot) = \frac{1}{\pi} \pi^{(0)}_{1,1} S_{d-1}(K, \cdot).$$

In [11], the multipliers of this transform are calculated explicitly and shown to be all nonzero.

Explicit integral representations of $h(M_k(K), \cdot)$ for arbitrary convex bodies K in the case $k \ge 3$ are not known. However, using Radon transforms of projection functions, Goodey [8] showed that in this case $M_k(K)$ determines K in the class of all centrally symmetric convex bodies.

We also extend the definition of mean lifted projections with $m = \infty$ to the case $k = 0$. Let $\pi^{(0)}_{\infty,j}$ be the integral transform given by (6.6), where we put $\varpi_0 = 1$. In particular, $\pi^{(0)}_{\infty,1}$ satisfies

$$(\pi^{(0)}_{\infty,1} \mu)(u) = \frac{1}{\varpi_{d-1}} \int_{S^{d-1} \cap u^+} \sin \alpha(u, v)\, \mu(dv)$$

$u \in S^{d-1}$, $\mu \in \mathcal{M}(S^{d-1})$.

Example 10 (Upper surface integral). For a convex body K and $u \in S^{d-1}$ define the *upper boundary* $\partial^u K$ of K as the set of all boundary points of K which have an outer unit normal in u^+. Then $\partial^u K$ is a compact $(d-1)$-surface. For $1 \le k \le d-1$ and $u \in S^{d-1}$ put

$$H_k(K,u) = \int_{\mathcal{L}_k^{u^\perp}} \int_{L^\perp} \mathcal{H}^{k-1}\big(\partial^u K \cap (L+x)\big)\, dx\, dL \tag{9.1}$$

($\mathcal{H}^j$ denotes j-dimensional Hausdorff measure). These are the tomographic measurements obtained from contents of sections of the upper boundary with affine k-planes orthogonal to u.

Note that the subset $\partial^u_+ K$ of $\partial^u K$ consisting of all points with an outer normal in the *open* half-space $u^+ \setminus u^\perp$ can be interpreted as the set of illuminated boundary points of K from direction u. For ω_{d-1}-almost all u we have $\mathcal{H}^{d-1}(\partial^u K \setminus \partial^u_+ K) = 0$ and for these u we could just as well work with the illuminated part instead of $\partial^u K$.)

Proposition 9.1. *Let $1 \le k \le d-1$, K a convex body and let $H_k(K,\cdot)$ be given by* (9.1)*. Then we have*

$$H_k(K,\cdot) = \frac{\varpi_d \varpi_k}{\varpi_{k+1}} \pi^{(0)}_{\infty,1} S_{d-1}(K,\cdot).$$

In particular, the transforms $H_k(K,\cdot)$, $1 \le k \le d-1$, differ only by a multiplicative constant.

Proof. The inner integral in (9.1) can be expressed in terms of the surface area measure of K,

$$\int_{L^\perp} \mathcal{H}^{k-1}\big(\partial^u K \cap (L+x)\big)\, dx = \int_{S^{d-1}\cap u^+} \|v|L\|\, S_{d-1}(K,dv). \tag{9.2}$$

This is a special case of a translative integral formula of Crofton type [37, Corollary 1.3.1]. In view of [28, Theorem 4.2.5, (4.2.10)], we have

$$\mathcal{H}^{k-1}\big(\partial^u K \cap (L+x)\big) = 2\Phi_{k-1}(K \cap (L+x), \partial^u K)$$

for almost all $x \in L^\perp$ (here, $\Phi_{k-1}(M,\cdot)$ is the $(k-1)$-st curvature measure of $M \in \mathcal{K}$), so (9.2) can also be derived from a translative integral formula for curvature measures, see e.g. [32, Corollary B].

Using (9.2) and Fubini's theorem, (9.1) becomes

$$H_k(K,u) = \int_{S^{d-1}\cap u^+} \int_{\mathcal{L}_k^{u^\perp}} \|v|L\|\, dL\, S_{d-1}(K,dv).$$

As $L \subset u^\perp$, we have, for $v \ne \pm u$,

$$\|v|L\| = \|(v|u^\perp)|L\| = \|\mathrm{pr}_{u^\perp}(v)|L\|\|v|u^\perp\| = \|\mathrm{pr}_{u^\perp}(v)|L\| \sin\alpha(u,v),$$

which implies

$$H_k(K,u) = \int_{S^{d-1}\cap u^+} \sin\alpha(u,v) \int_{\mathcal{L}_k^{u^\perp}} \|\mathrm{pr}_{u^\perp}(v)|L\|\, dL\, S_{d-1}(K,dv).$$

Relation (4.6) applied in $u^\perp$ with $m = 1$ gives

$$H_k(K,u) = \frac{\varpi_d \varpi_k}{\varpi_{d-1}\varpi_{k+1}} \int_{S^{d-1}\cap u^+} \sin\alpha(u,v)\, S_{d-1}(K,dv). \tag{9.3}$$

The definition of $\pi^{(0)}_{\infty,1}$ implies the assertion. □

Motivated by stereological questions, the transform $H_{d-1}(K,\cdot)$ was introduced by Schneider in [30], where (9.3) is derived for $k = d-1$ and all the multipliers of this transform are shown to be nonzero. Together with the above result, this implies that $H_k(K,\cdot)$ determines K if known for one $k \in \{1,\dots,d-1\}$.

It should be noted that the above considerations can also be applied in a non-directed setting. If we replace $\partial^u K$ in (9.1) with ∂K, then the resulting function is a multiple of the sine transform of $S_{d-1}(K,\cdot)$. The special case $k = d-1$ then reads

$$\int_{-\infty}^{\infty} \mathcal{H}^{d-2}(\partial K \cap (u^\perp + tu))\, dt = \int_{S^{d-1}} \sin\alpha(u,v)\, S_{d-1}(K,dv), \tag{9.4}$$

and was already observed by Schneider [26]. As $\sin\alpha(v,\cdot)$ is the support function of the unit ball in $v^\perp$, (9.4) implies that $H_k(K,\cdot)$ is the support function of a convex body, whenever K is origin symmetric. This is not true in general; note that $H_k(K,\cdot)$ need not even be continuous on S^{d-1}.

We turn to the second class of examples, where combinations of mean lifted projections and Δ can be used to express the tomographic data under consideration.

Example 11 (Directed projection mean (first kind)). For $1 \le i < k \le d-1$, the *average directed projection function (of the first kind)* $\overline{v}_{ik}(K,\cdot)$ of $K \in \mathcal{K}^d$ is defined as

$$\overline{v}_{ik}(K,u) = \int_{\mathcal{L}_k^{[u]}} S_i'(K|L, u^+ \cap S^{k-1}(L))\, dL, \quad u \in S^{d-1}.$$

For $i = 1$, this function can be expressed more explicitly using

$$\left(\frac{\Delta_L}{k-1} + 1\right) h'(M,\cdot) = S_1'(M,\cdot),$$

which is (3.1) applied to a convex body $M \subset L$. Recalling the facts that $h'(K|L,\cdot) = \pi_{L,\infty} h(K,\cdot)$ and that the hemispherical transform in L can be expressed using $\pi^L_{[u],0}$, we get

$$\begin{aligned}\overline{v}_{1k}(K,u) &= \int_{\mathcal{L}_k^{[u]}} \big(\pi^L_{[u],0}(\frac{\Delta_L}{k-1} + 1)\pi_{L,\infty} h(K,\cdot)\big)(u)\, dL \\ &= \frac{1}{k-1}\big(\tilde{\tau}_0^{(k)} h(K,\cdot)\big)(u) + \big(\tau^{(k)}_{0,\infty} h(K,\cdot)\big)(u).\end{aligned}$$

Using Theorems 7.3(a) and 7.2 we conclude

$$\overline{v}_{1k}(K,\cdot) = \left(\frac{\varpi_d \varpi_{d-1}}{2(k-1)\varpi_{k-1}} \pi^{(1)}_{\infty,0}\Delta + \pi^{(d-k+1)}_{\infty,k-d}\right) h(K,\cdot).$$

This shows that $\overline{v}_{1k}(K, \cdot)$ can be written as combination of mean lifted projections and the Laplace-Beltrami operator, applied to the support function of K. Estimating the multipliers of this transform, it was shown in [14] that K is determined up to translation by $\overline{v}_{1k}(K, \cdot)$ whenever $2 \leq k \leq (2d-3)/5$ or $(d-2)/2 \leq k \leq d-1$. A corresponding stability result for these cases is derived there, as well.

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Paul Goodey
University of Oklahoma
Norman, OK, USA
pgoodey@ou.edu

Markus Kiderlen
University of Aarhus
Aarhus, Denmark
kiderlen@imf.au.dk

Wolfgang Weil
University of Karlsruhe
Karlsruhe, Germany
weil@math.uka.de