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On the Esscher transforms and other equivalent martingale measures for Barndorff-Nielsen and Shephard stochastic volatility models with jumps*

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Abstract

We compute resp. discuss the Esscher martingale transform for exponential processes, the Esscher martingale transform for linear processes, the minimal martingale measure, the class of structure preserving martingale measures, and the minimum entropy martingale measure for stochastic volatility models of Ornstein-Uhlenbeck type as introduced by Barndorff-Nielsen and Shephard. We show, that in the model with leverage, with jumps both in the volatility and in the returns, all those measures are different, whereas in the model without leverage, with jumps in the volatility only and a continuous return process, several measures coincide, some simplifications can be made and the results are more explicit. We illustrate our results with parametric examples used in the literature.

Keywords: Esscher martingale transform for stochastic processes, stochastic volatility models with jumps, optimal martingale measures, option pricing.

1 Introduction

Lévy processes provide a lot of flexibility in financial modelling. Although financial returns increments exhibit some kind of serial dependence, many of their essential features are captured by this class of models: heavy tails, aggregational Gaussianity, volatility clustering are some of their features easily described by means of models based on Lévy processes. Introduction of jumps anyway rises the problem of dealing

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with incomplete market models; that means that there exist infinitely many martingale measures, compatible with the no arbitrage requirement and equivalent to the physical measure describing the underlying evolution, one can use to price derivative securities.

One reasonable way to solve this problem is based on the observation that in incomplete markets the "correct" equivalent martingale measure (EMM from now on) could not be independent on the preferences of investors any more, so by guessing a suitable utility function describing these preferences, the "optimal" EMM should maximize the expected value of this utility. It has been proved that for many interesting cases of utility functions this problem admits a dual formulation: finding an EMM maximizing some classes of utility functions is equivalent to find EMM minimizing some kind of distances [BF02]. Of particular relevance in the framework of utility maximization are the equivalent martingale measures maximizing exponential utility (and minimizing, by duality, the relative entropy) and those maximizing quadratic utility (and minimizing, always by duality, an appropriate L^2-distance).

Another popular approach to option pricing for incomplete models had been related to the construction of the Esscher martingale transform. As it has been already pointed out in [KS02] two different Esscher martingale transforms exist for Lévy processes according to the choice of the parameter which defines the measure: one turning the ordinary exponential process into a martingale and another one turning into a martingale the stochastic exponential. They have been called the Esscher martingale transform for the exponential process and the Esscher martingale transform for the linear process respectively. It has been shown in [ES05] that for exponential Lévy models the Esscher martingale transform for the linear process is also the minimal entropy martingale measure, i.e., the equivalent martingale measure which minimizes the relative entropy, and that this measure has also the property of preserving the Lévy structure of the model, see also [HS06].

The definition and the abstract theory of the Esscher martingale transforms for general semimartingales has been given in [KS02], following previous results in discrete time in [BDES96] and [BDES98]. Some recent results related to generalization of the Esscher transform to a non-Lévy setting are in [BM07, STY04, ECS05].

Since a few years interest is grown also in a "second generation" of models based on Lévy processes, i.e., the stochastic volatility models driven by Lévy processes; the model introduced by O.E. Barndorff-Nielsen and N. Shephard belongs to this class [BNS01b], [BNS01a]. In this model, or we could better say, class of models, the volatility is described by an Ornstein-Uhlenbeck process driven by a Lévy process with positive increments, i.e., a subordinator.

For these models (from now on BNS) some results are already available in the context of option pricing: E. Nicolato and E. Venardos [NV03] introduced a class of *structure preserving* equivalent martingale measures, under which the stochastic process describing the evolution of the underlying asset follows a stochastic differential equation with the same structure although with possibly different parameters. Under such measures the problem of pricing options could be solved by using a transform-based technique.

The purpose of this paper is to give the explicit construction of the Esscher transform both for the linear and for the exponential processes and the minimal martingale measure for BNS models and to present a systematic comparison of all the measures available in the literature (including those we will obtain in the present work) and their relations.

In Section 2 we will recall the essential features of BNS models and we will give the explicit calculations of the characteristic triplet characterizing them as a semimartingale process, in order to apply the general theory of the Esscher transform introduced in [KS02] in Section 3. There we will present the explicit construction of the Esscher martingale transforms both for the linear and the exponential processes.

In Section 4 we will discuss the existence of the Esscher martingale transforms for some relevant examples of BNS models, and present their construction, if they exist.

In Section 5 we will recall the main results obtained in [NV03] about the equivalent martingale measures which preserves the model structure and we will show that they do not coincide with neither of the Esscher martingale transforms. Next we will give the expression of the minimal martingale measure for BNS. At the end of this section we briefly recall the main result obtained for the minimal entropy martingale measure in the no-leverage case, and compare it with the previously discussed measures.

Finally in Section 6 we show that both Esscher transforms and the minimal martingale measure coincide in the non-leverage case. We then recall the results of [BMB05] for the minimal entropy martingale measure in the non-leverage case.

In appendices A–C we give sufficient conditions to assure that the candidates for the density processes for the exponential Esscher, the linear Esscher and the minimal martingale measure exist and are proper martingales.

We use the BNS model, because it is a model that exhibits a connection of jumps and stochastic volatility, but yet allows very explicit calculations. In [CT04] it is argued, that the Bates model is simpler and sufficient for most purposes (and thus perhaps preferable), but as jumps and stochastic volatility are independent in that model, it is less interesting from a mathematical perspective.

Throughout the paper we use the notation of [JS02] for semimartingale theory, stochastic calculus, and stochastic integration. In particular, if X is a semimartingale, then $L(X)$ denotes the set of predictable X-integrable processes, and for $H \in L(X)$ the stochastic integral is sometimes written as $H \cdot X$.

2 The BNS model

2.1 Specification of the model

We will now focus our attention on the class of stochastic volatility models with jumps, that has been introduced by Barndorff-Nielsen and Shephard in [BNS01b, BNS01a].

Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$ carrying a standard Brownian motion W and an independent increasing pure jump Lévy process Z. The process Z is called the *background driving Lévy process*, or BDLP for short. We assume that the discounted stock price is given by

$$S_t = S_0 e^{X_t}, \tag{2.1}$$

where $S_0 > 0$ is a constant, logarithmic returns satisfy

$$dX_t = (\mu + \beta V_{t-})dt + \sqrt{V_{t-}}dW_t + \rho dZ_{\lambda t}, \tag{2.2}$$

starting from $X_0 = 0$, and the instantaneous variance satisfies

$$dV_t = -\lambda V_{t-}dt + dZ_{\lambda t} \tag{2.3}$$

with constant initial value $V_0 > 0$. The parameter range is

$$\mu \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad \rho \leq 0, \quad \lambda > 0. \tag{2.4}$$

We denote the cumulant function and the Lévy measure of Z by $k(z)$ resp. $U(dx)$. Since Z is increasing we have

$$k(z) = \int_0^\infty (e^{zx} - 1)U(dx). \tag{2.5}$$

We will work with the *usual natural* filtration $(\mathcal{F}_t)$ generated by the pair $(W_t, Z_{\lambda t})$, as it is defined in [HWY92, 2.63, p.63]. The solution to (2.3) is

$$V_t = V_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}. \tag{2.6}$$

Therefore we have the inequalities

$$V_0 e^{-\lambda t} \leq V_t \leq V_0 e^{-\lambda t} + Z_{\lambda t}. \tag{2.7}$$

The evolution of the stock price is governed by

$$dS_t = S_{t-} d\tilde{X}_t, \tag{2.8}$$

with

$$\tilde{X}_t = \int_0^t S_{u-}^{-1} dS_u. \tag{2.9}$$

We will discuss the process $\tilde{X}$, which is called the exponential transform of X, in more detail below.

Remark 2.1. *In principle the leverage parameter ρ could be an arbitrary real number. If $\rho = 0$ we call the model a BNS model without leverage. In that case the trajectories of logarithmic returns, and thus of the asset price are continuous. If $\rho \neq 0$ we call the model a BNS model with leverage, and returns and the asset price exhibit jumps. If $\rho > 0$ and the jumps of Z are unbounded, the asset price process is not locally bounded. If $\rho \leq 0$ the asset price process will be locally bounded. Typically $\rho \leq 0$ and we restrict our analysis to that case.*

Remark 2.2. *As V is of finite variation and W is continuous we could define the model pathwise by Riemann-Stieltjes integrals without reference to stochastic integration.*

2.2 Semimartingale characteristics and cumulants for logarithmic returns

The background driving Lévy process Z, or BDLP for short, is increasing, and as a consequence we can always use the *zero truncation function* $h(x) = 0$, and we shall do so for most of the paper. If we assume also that $E[Z_1] < \infty$, we can use the *identity truncation function* $h(x) = x$, which corresponds to the Doob-Meyer decomposition of a special semimartingale.

For later usage it is convenient to introduce

$$U_\rho(dx) = U\left(\frac{dx}{\rho}\right), \tag{2.10}$$

and

$$k_\rho(z) = k(\rho z), \tag{2.11}$$

which are simply the Lévy measure resp. the cumulant function corresponding to the process ρZ. Let us recall that the jumps of the processes X and Z are related by $\Delta X_t = \rho \Delta Z_{\lambda t}$ and therefore the jump measure $\mu_X(dx, dt)$ of the process X is related to the jump measure $\mu_Z(dx, dt)$ of the process Z by

$$\mu_X(dx, dt) = \mu_Z\left(\frac{dx}{\rho}, \lambda dt\right). \tag{2.12}$$

We will denote by $\nu(dx, dt)$ the predictable compensator of $\mu_X(dx, dt)$. Let us provide now the semimartingale characteristics of X.

Lemma 2.3. *The semimartingale characteristics of X with respect to the zero truncation function are given by (B, C, ν), which satisfy*

$$dB_t = b_t dt, \quad dC_t = c_t dt, \quad \nu(dt, dx) = F(t, dx)dt, \tag{2.13}$$

where

$$b_t = \mu + \beta V_{t-}, \quad c_t = V_{t-}, \quad F(t, dx) = \lambda U_\rho(dx). \tag{2.14}$$

Proof. The jump measure of (X_t) coincides trivially with the jump measure of the Lévy process $\rho Z_{\lambda t}$. Hence its predictable compensator is the measure $\lambda U_\rho(dx)dt$. Using the definition of semimartingale characteristics as given in [JS02, II.2a, p.75f] for $h(x) = 0$ we have

$$X_t - \sum_{s \le t} \Delta X_s = \int_0^t (\mu + \beta V_{s-})ds + \int_0^t \sqrt{V_{s-}} dW_s, \tag{2.15}$$

and we can identify the corresponding drift and the quadratic variation of the continuous martingale part. □

Remark 2.4. *If we used the identity truncation function, we had $b_t = \mu + \rho\lambda\zeta + \beta V_{t-}$ with $\zeta = E[Z_1]$ for the first differential characteristic.*

For the next lemma we need the notions of an exponentially special semimartingale and its (modified) Laplace cumulant process from [KS02, Def.2.12, p.402 and Def.2.16, p.403].

Lemma 2.5. *Let $\theta \in L(X)$ be such that $\theta \cdot X$ is exponentially special. The modified Laplace cumulant process of X in θ is then given by*

$$K^X(\theta)_t = \int_0^t \tilde{\kappa}^X(\theta)_s ds, \tag{2.16}$$

where

$$\tilde{\kappa}^X(\theta)_t = b_t\theta_t + \frac{1}{2}c_t\theta_t^2 + \lambda k(\rho\theta_t). \tag{2.17}$$

Proof. This follows immediately from the characteristics computed above and [KS02, Theorem 2.18.1, p.404]. □

2.3 Semimartingale characteristics and cumulants for the exponential transform

In the following it is useful to rewrite the ordinary exponential in (2.1) as stochastic exponential. This can be done by using $\tilde{X}$, the exponential transform of X. and we obtain

$$S_t = S_0\mathcal{E}(\tilde{X})_t. \tag{2.18}$$

According to [KS02, Lemma 2.6.1, p.399] we have $\tilde{X} = X + \frac{1}{2}\langle X^c, X^c\rangle + (e^x - 1 - x) * \mu_X$. This means

$$\begin{aligned}\tilde{X}_t = &\int_0^t (\mu + \beta V_{s-})ds + \int_0^t \sqrt{V_{s-}}dW_s + \rho Z_{\lambda t} \\ &+ \frac{1}{2}\int_0^t V_{s-}ds + \sum_{s\le t}(e^{\rho\Delta Z_{\lambda s}} - 1 - \rho\Delta Z_{\lambda s})\end{aligned} \tag{2.19}$$

This can be rewritten as

$$\tilde{X}_t = \int_0^t (\mu + \tilde{\beta}V_{s-})ds + \int_0^t \sqrt{V_{s-}}dW_s + \sum_{s\le t}(e^{\rho\Delta Z_{\lambda s}} - 1), \tag{2.20}$$

where

$$\tilde{\beta} = \beta + \tfrac{1}{2}. \tag{2.21}$$

We have clearly

$$\tilde{X}^c = X^c \tag{2.22}$$

and the jumps are

$$\Delta\tilde{X}_t = e^{\rho\Delta Z_{\lambda t}} - 1. \tag{2.23}$$

We want to introduce the process

$$M_t = \rho Z_{\lambda t}, \tag{2.24}$$

and its exponential transform given by

$$\tilde{M}_t = \sum_{s\le t}(e^{\rho\Delta Z_{\lambda s}} - 1). \tag{2.25}$$

Then we can write

$$dX_t = (\mu + \beta V_{t-})dt + \sqrt{V_{t-}}dW_t + dM_t, \tag{2.26}$$

and

$$d\tilde{X}_t = (\mu + \tilde{\beta} V_{t-})dt + \sqrt{V_{t-}}dW_t + d\tilde{M}_t. \tag{2.27}$$

We note that $\tilde{M}$ is a Lévy process. It is helpful to introduce the function

$$g_\rho(x) = e^{\rho x} - 1, \tag{2.28}$$

its inverse function

$$g_\rho^{-1}(x) = \frac{1}{\rho}\ln(1+x), \tag{2.29}$$

and the induced measure

$$\tilde{U}_\rho = U \circ g_\rho^{-1}. \tag{2.30}$$

If U admits a density u then $\tilde{U}_\rho$ admits a density $\tilde{u}_\rho$ given by

$$\tilde{u}_\rho(x) = \frac{1}{\rho(1+x)} u\left(\frac{1}{\rho}\ln(1+x)\right). \tag{2.31}$$

The cumulant function of $\tilde{M}$ is

$$\tilde{k}_\rho(z) = \int_0^\infty (e^{z(e^{\rho x}-1)} - 1)U(dx). \tag{2.32}$$

Lemma 2.6. *The semimartingale characteristics of $\tilde{X}$ with respect to the zero truncation function are given by $(\tilde{B}, \tilde{C}, \tilde{\nu})$, which satisfy*

$$d\tilde{B}_t = \tilde{b}_t dt, \quad d\tilde{C}_t = \tilde{c}_t dt, \quad \tilde{\nu}(dt, dx) = \tilde{F}(t, dx)dt, \tag{2.33}$$

where

$$\tilde{b}_t = \mu + \tilde{\beta} V_{t-} \quad \tilde{c}_t = V_{t-} \quad \tilde{F}(t, dx) = \lambda \tilde{U}_\rho(dx). \tag{2.34}$$

Proof. This follows from the characteristics of X given above and [JS02, Theorem II.8.10, p.136]. □

Remark 2.7. *If we used the identity truncation function, we had $\tilde{b}_t = \mu + \lambda k(\rho) + \beta V_{t-}$ for the first differential characteristic.*

In the following lemma we need the notion of the derivative of a cumulant process from [KS02, Def.2.22, p.407].

Lemma 2.8. *Let $\theta \in L(\tilde{X})$ be such that $\theta \cdot \tilde{X}$ is exponentially special. The modified Laplace cumulant process of $\tilde{X}$ in θ is then given by*

$$K^{\tilde{X}}(\theta)_t = \int_0^t \tilde{\kappa}^{\tilde{X}}(\theta)_s ds, \tag{2.35}$$

where

$$\tilde{\kappa}^{\tilde{X}}(\theta)_t = \tilde{b}_t \theta_t + \frac{1}{2}\tilde{c}_t \theta_t^2 + \lambda \tilde{k}_\rho(\theta_t). \tag{2.36}$$

The derivative of the cumulant process $K^{\tilde{X}}(\theta)$ is given by

$$DK^{\tilde{X}}(\theta) = \int_0^t \tilde{\kappa}^{\tilde{X}}(\theta)_s ds, \tag{2.37}$$

where

$$D\tilde{\kappa}^{\tilde{X}}(\theta)_t = \tilde{b}_t + \tilde{c}_t\theta_t + \lambda\tilde{k}'_\rho(\theta_t). \tag{2.38}$$

Proof. The expression for $K^{\tilde{X}}(\theta)$ follows immediately from the characteristics of $\tilde{X}$ computed above and [KS02, Theorem 2.18.1–2, p.404]. The expression for $DK^{\tilde{X}}(\theta)$ follows from [KS02, Definition 2.22, p.407]. □

3 Esscher martingale transforms for BNS models

3.1 The Esscher martingale transform for exponential processes

Let us look at the Esscher martingale transform for exponential processes as described in [KS02, Theorem 4.1, p.421]. The discounted asset price S satisfies $S = S_0 e^X$. We have to find the solution to

$$K^X(\theta + 1) - K^X(\theta) = 0. \tag{3.1}$$

Suppose now we can establish a solution $\theta^\sharp_t$ to that equation, almost surely for all $t \in [0,T]$, and $G^\sharp_t = e^{\theta^\sharp \cdot X_t - K^X(\theta^\sharp)_t}$ defines a martingale $(G^\sharp_t)_{0\le t\le T}$, then we can define a probability measure $P^\sharp$ by

$$\frac{dP^\sharp}{dP} = e^{\theta^\sharp \cdot X_T - K^X(\theta^\sharp)_T}. \tag{3.2}$$

This measure is then the Esscher martingale transform for the exponential process e^X. If there is no solution with the required properties, we say the Esscher martingale transform for the exponential process does not exist.

Theorem 3.1. *Suppose there is $\theta^\sharp \in L(X)$, such that $\theta^\sharp \cdot X$ is exponentially special,*

$$K^X(\theta^\sharp + 1) - K^X(\theta^\sharp) = 0, \tag{3.3}$$

and

$$G^\sharp_t = \mathcal{E}(\tilde{N}^\sharp)_t, \tag{3.4}$$

with

$$\tilde{N}^\sharp_t = \int_0^t \psi^\sharp_s dW_s + \int_0^t \int (Y^\sharp(s,x) - 1)(\mu_X - \nu)(dx, ds), \tag{3.5}$$

$$\psi^\sharp_t = \theta^\sharp_t \sqrt{V_{t-}} \tag{3.6}$$

and

$$Y^\sharp(t,x) = e^{\theta^\sharp_t \rho x} \tag{3.7}$$

defines a martingale $(G^\sharp_t)_{0\le t\le T}$*. Then*

$$\frac{dP^\sharp}{dP} = \mathcal{E}(\tilde{N}^\sharp)_T \tag{3.8}$$

defines a probability measure $P^\sharp \sim P$ *on* $\mathcal{F}_T$*. The process* $(X_t)_{0\le t\le T}$ *is a semimartingale under* $P^\sharp$*; its semimartingale characteristics with respect to the zero truncation function are* $(B^\sharp, C^\sharp, \nu^\sharp)$ *which are given by*

$$dB^\sharp_t = b^\sharp_t dt, \quad dC^\sharp_t = c^\sharp_t dt, \quad \nu^\sharp(dt, dx) = F^\sharp(t, dx)dt, \tag{3.9}$$

where

$$b^\sharp_t = \mu + (\beta + \theta^\sharp_t)V_{t-}, \quad c^\sharp_t = V_{t-}, \quad F^\sharp(t, dx) = Y^\sharp(t, x)\lambda U_\rho(dx). \tag{3.10}$$

Proof. We can apply [KS02, Theorem 4.1, p.421] and conclude that the density in (3.8) defines an equivalent local martingale measure for e^X. By the Girsanov Theorem for general semimartingales, [JS02, 3.24, p.172f], the characteristics of X under $P^\sharp$ follow. □

In Appendix A we give sufficient conditions, that the solution $\theta^\sharp$ exists, which is then of the form $\theta^\sharp_t = \phi^\sharp(V_{t-})$ for some Borel function $\phi^\sharp : \mathbb{R}_+ \to \mathbb{R}$, and $G^\sharp$ is a proper martingale and thus a density process.

Remark 3.2. *If we used the identity truncation function we had* $b^\sharp_t = \mu + \lambda k'_\rho(\theta^\sharp_t) + (\beta + \theta^\sharp_t)V_{t-}$*.*

Remark 3.3. *From* (3.9) *and* (3.10)*, we see that in general the third characteristic of* X*, and thus of* Z*, under* $P^\sharp$ *will be non-deterministic and depend on time, hence* Z *is not a Lévy process under* $P^\sharp$ *any more.*

Remark 3.4. *Analyzing the above calculations we see that the concrete dynamics of the volatility process does not come into play, so analogous results hold for a quite general class of stochastic volatility models with jumps, including, for example, the Bates model [Bat96].*

3.2 The Esscher martingale transform for linear processes

Let us look at the Esscher martingale transform for linear processes as described in [KS02, Theorem 4.4, p.423]. Note, that in our notation the discounted asset price S satisfies $S = S_0\mathcal{E}(\tilde{X})$ and we must be careful not to confuse $\tilde{K}^X$ and $K^{\tilde{X}}$. We have to find the solution to

$$DK^{\tilde{X}}(\theta)_t = 0. \tag{3.11}$$

Suppose now we can establish a solution θ^*_t to that equation, almost surely for all $t \in [0, T]$, and $G^*_t = e^{\theta^*\cdot\tilde{X}_t - K^{\tilde{X}}(\theta^*)_t}$ defines a martingale $(G^*_t)_{0\le t\le T}$, then we can define a probability measure P^* by

$$\frac{dP^*}{dP} = e^{\theta^*\cdot\tilde{X}_T - K^{\tilde{X}}(\theta^*)_T}. \tag{3.12}$$

This measure is then the Esscher martingale transform for the linear process $\tilde{X}$. If there is no solution with the required properties, we say the Esscher martingale transform for the linear process does not exist.

Theorem 3.5. *Suppose there is $\theta^* \in L(\tilde{X})$, such that $\theta^* \cdot \tilde{X}$ is exponentially special,*

$$DK^{\tilde{X}}(\tilde{\theta}^*)_t = 0, \tag{3.13}$$

and

$$G^*_t = \mathcal{E}(\tilde{N}^*)_t \tag{3.14}$$

with

$$\tilde{N}^*_t = \int_0^t \psi^*_s dW_s + \int_0^t \int (Y^*(s,x) - 1)(\mu_X - \nu)(dx, ds), \tag{3.15}$$

$$\psi^*_t = \theta^*_t \sqrt{V_{t-}} \tag{3.16}$$

and

$$Y^*(t,x) = e^{\theta^*_t (e^x - 1)} \tag{3.17}$$

defines a martingale $(G^*_t)_{0 \le t \le T}$. Then

$$\frac{dP^*}{dP} = \mathcal{E}(\tilde{N}^*)_T \tag{3.18}$$

defines a probability measure $P^* \sim P$ on $\mathcal{F}_T$. The process $(X_t)_{0 \le t \le T}$ is a semimartingale under P^* with semimartingale characteristics (B^*, C^*, ν^*) given by

$$dB^*_t = b^*_t dt, \quad dC^*_t = c^*_t dt, \quad \nu^*(dt, dx) = F^*(t, dx)dt, \tag{3.19}$$

where

$$b^*_t = \mu + (\beta + \theta^*_t)V_{t-}, \quad c^*_t = V_{t-}, \quad F^*(t, dx) = Y^*(t,x)\lambda U_\rho(dx). \tag{3.20}$$

We can apply [KS02, Theorem 4.4, p.423] and conclude that the density in (3.14) defines an equivalent local martingale measure for $\mathcal{E}(\tilde{X})$. So we can apply again the Girsanov Theorem for general semimartingales, [JS02, 3.24, p.172f], to derive the characteristics of X under P^*. □

In Appendix B it is shown that there exists always a measurable function $\phi^* : \mathbb{R}_+ \to \mathbb{R}$, such that $\vartheta^*_t = \phi^*(V_{t-})$ is a solution to (3.11), and sufficient conditions are given that G^* is a proper martingale and thus a density process.

Remark 3.6. *If we used the identity truncation function we had $b^*_t = \mu + \lambda \tilde{k}'_\rho(\theta^*_t) + (\beta + \theta^*_t)V_{t-}$ for the first differential characteristic.*

Remark 3.7. *From* (3.19) *and* (3.20)*, we see that in general the third characteristic of X, and thus of Z, under P^* will be non-deterministic and depend on time, hence Z is not a Lévy process under P^* any more.*

Remark 3.8. *A comparison of [CS05b, Theorem 4.3, p.477] resp. [CS06, Theorem 3.3., p.8] and [KS02, Theorem 4.4 p.423] indicates that, for general semimartingales, the linear Esscher martingale transform and the minimum entropy-Hellinger martingale measure coincide, at least under the assumption [CS06,* (3.5) *p.8], which is equivalent to the existence of all exponential moments of $F(t, dx)$. This assumption holds for the Poisson toy example studied in the next section, but not for BNS models with a BDLP with semi-heavy tails, such as the Γ-OU and the IG-OU model.*

We conjecture that under the weaker conditions granting the existence of the linear Esscher transform given in the appendix, the linear Esscher measure and the minimum entropy-Hellinger martingale measure coincide also for the last two models mentioned. As systematic and and rigorous investigation of this relationship involves some technicalities and is left open for future research.

4 Examples

4.1 The Poisson toy example

4.1.1 Exponential Esscher martingale transform

This model is used for illustrative purposes, since all calculations are explicitly possible. Suppose

$$Z_t = \delta N_t \tag{4.1}$$

where $\delta > 0$ is the jump size and N is a standard Poisson process with intensity parameter $\gamma > 0$. Then

$$k(\theta) = \gamma(e^{\delta\theta} - 1) \tag{4.2}$$

and the solution of equation (3.3) is

$$\theta_t^\sharp = -\frac{\mu + \tilde{\beta} V_{t-}}{V_{t-}} - \frac{1}{\rho\delta} w\left(\frac{\delta\rho\lambda\gamma(e^{\delta\rho} - 1)}{V_{t-}} \exp\left(-\delta\rho\frac{\mu + \beta V_{t-}}{V_{t-}}\right)\right), \tag{4.3}$$

where w is known as (the principal branch of) the Lambert W (or polylogarithm) function. The function w is available in Mathematica, Maple, and many other computer packages and libraries. Basically it is the inverse function of xe^x. For further references and code for numerical evaluation see [CGH$^+$96] and [FSC73]. We need to know here only, that w is strictly increasing from 0 to ∞ as x goes from 0 to ∞.

For this model we have $E[e^{\xi Z_1}] < \infty$ for all $\xi \in \mathbb{R}$, so the condition (A.13) in Lemma A.2 is satisfied, and the exponential Esscher martingale transform exists.

4.1.2 Linear Esscher martingale transform

The jumps of $\tilde{X}$ are

$$\Delta\tilde{X}_t = e^{\rho\Delta X_t} - 1 \tag{4.4}$$

and since we have in the Poisson toy model only one jump size, this implies, that we can write

$$\tilde{X}_t = \tilde{\delta}_\rho N_t \tag{4.5}$$

where

$$\tilde{\delta}_\rho = e^{\rho\delta} - 1. \tag{4.6}$$

So the cumulant function is of the same form we have seen in the previous section, namely

$$\tilde{k}_\rho(z) = \gamma(e^{\tilde{\delta}_\rho z} - 1), \tag{4.7}$$

and its derivative is

$$\tilde{k}'_\rho(z) = \gamma\tilde{\delta}_\rho e^{\tilde{\delta}_\rho z}. \tag{4.8}$$

For the linear Esscher transform we have to solve (3.13), which becomes

$$\tilde{b}_t + \tilde{c}_t\theta + \lambda\gamma\tilde{\delta}_\rho e^{\tilde{\delta}_\rho\theta} = 0. \tag{4.9}$$

The solution will be given, again using the Lambert w function, as

$$\theta_t^* = -\frac{\mu + \tilde{\beta} V_{t-}}{V_{t-}} - \frac{1}{\tilde{\delta}_\rho} w \left(\frac{\lambda \gamma \tilde{\delta}_\rho^2}{V_{t-}} \exp \left(-\tilde{\delta}_\rho \frac{\mu + \tilde{\beta} V_{t-}}{V_{t-}} \right) \right). \tag{4.10}$$

As we have $E[e^{\xi Z_1}] < \infty$ for all $\xi \in \mathbb{R}$, the condition (B.8) in Lemma B.2 is satisfied, and the linear Esscher martingale transform exists.

4.2 The Γ-OU example

4.2.1 Exponential Esscher martingale transform

Suppose we have a stationary variance with $\Gamma(\delta, \gamma)$ distribution. Then the BDLP is a compound Poisson process with exponential jumps and has cumulant function

$$k(\theta) = \frac{\delta \theta}{\gamma - \theta} \tag{4.11}$$

for $\Re\theta < \gamma$. For the exponential Esscher transform we must have $\theta_t^\sharp = \phi^\sharp(V_{t-})$ where the function $\phi^\sharp$ is obtained by solving the equation

$$\mu + \tilde{\beta} v + v\phi + \lambda \frac{\delta \rho(\phi + 1)}{\gamma - \rho(\phi + 1)} - \lambda \frac{\delta \rho \phi}{\gamma - \rho \phi} = 0. \tag{4.12}$$

This equation can be transformed into a cubic polynomial equation in ϕ and thus, a real solution always exists.

Lemma A.2 provides sufficient conditions for $G^\sharp$ to be a true martingale, namely conditions (A.13), that can be written in this case as

$$\rho \left[\frac{(\mu + \lambda \delta \rho / (\gamma - \rho))_+}{V_0 e^{-\lambda T}} + \tilde{\beta} \right]_+ < \gamma \tag{4.13}$$

and

$$\frac{1}{2} \max \left\{ \left[\frac{(\mu + \lambda \delta \rho / (\gamma - \rho))_+}{V_0 e^{-\lambda T}} + \tilde{\beta} \right]_+, \left[\frac{-(\mu + \lambda \delta \rho / (\gamma - \rho))_-}{V_0 e^{-\lambda T}} + \tilde{\beta} \right]_- \right\}^2 < \gamma \tag{4.14}$$

4.2.2 Linear Esscher martingale transform

We have to solve the equation

$$\mu + \tilde{\beta}\theta + v\theta + \lambda \int_0^\infty e^{\theta(e^{\rho x} - 1)} (e^{\rho x} - 1) \delta x^{-1} e^{-\gamma x} dx = 0. \tag{4.15}$$

We do not have a closed form expression for the integral in the last equation, but we know from Lemma B.1, that there is always a real solution, that could be obtained numerically.

To apply Lemma B.2 we must have (4.14) and in that case we can conclude that the linear Esscher martingale transform exists.

4.3 The IG-OU example

4.3.1 Exponential Esscher martingale transform

The cumulant function of the BDLP in the IG-OU model is

$$k(\theta) = \frac{\delta\theta}{\sqrt{\gamma^2 - 2\theta}} \tag{4.16}$$

for $\Re(\theta) < \gamma^2/2$. To determine the exponential Esscher martingale transform we have to find a solution to (3.1), which becomes equivalent to solving $f(\theta; V_{t-}) = 0$ with $\theta > \gamma^2/(2\rho)$, where

$$f(\theta; v) = (\mu + \tilde{\beta}v) + v\theta + \lambda\delta\rho\left[\frac{\theta+1}{\sqrt{\gamma^2 - 2\rho(\theta+1)}} - \frac{\theta}{\sqrt{\gamma^2 - 2\rho\theta}}\right]. \tag{4.17}$$

In the notation of Lemma A.1 we have

$$\xi_1 = \gamma^2/2, \qquad \ell_0 = \infty \tag{4.18}$$

and so we know there is always a solution. The equation for $\theta^\sharp$ can be transformed into a polynomial equation of eighth order.

The conditions (A.13) in Lemma A.2 for this model can be written as

$$\rho\left[\frac{(\mu + \lambda\delta\rho/\sqrt{\gamma^2 - 2\rho})_+}{V_0 e^{-\lambda T}} + \tilde{\beta}\right]_+, < \frac{\gamma^2}{2} \tag{4.19}$$

and

$$\frac{1}{2}\max\left\{\left[\frac{(\mu + \lambda\delta\rho/\sqrt{\gamma^2 - 2\rho})_+}{V_0 e^{-\lambda T}} + \tilde{\beta}\right]_+, \left[\frac{-(\mu + \lambda\delta\rho/\sqrt{\gamma^2 - 2\rho})_-}{V_0 e^{-\lambda T}} + \tilde{\beta}\right]_-\right\}^2 \leq \frac{\gamma^2}{2} \tag{4.20}$$

and if so, the exponential Esscher martingale transform exists.

4.3.2 Linear Esscher martingale transform

We have to solve the equation

$$\mu + \tilde{\beta}\theta + v\theta + \lambda\int_0^\infty e^{\theta(e^{\rho x}-1)}(e^{\rho x} - 1)\frac{\delta}{\sqrt{2\pi}}x^{-3/2}e^{-\gamma^2 x/2}dx = 0. \tag{4.21}$$

We do not have a closed form expression for the integral in the last equation, but we know from Lemma B.1, that there is always a real solution, that could be obtained numerically.

To apply Lemma B.2 we must have (4.20) and in that case we can conclude that the linear Esscher martingale transform exists.

5 Other equivalent martingale measures for BNS models

E. Nicolato and E. Venardos [NV03] have given a complete characterization of all equivalent martingale measures for BNS models through the following theorem (slightly reformulated, see Remark 5.3 below).

Theorem 5.1. *Let Q be an EMM for the BNS model. Then the corresponding density process is given by the stochastic exponential*

$$G_t^Q = \mathcal{E}(\tilde{N}^Q)_t, \tag{5.1}$$

where

$$\tilde{N}_t^Q = \int_0^t \psi_s^Q dW_s + \int_0^t \int (Y^Q(s,x) - 1)(\mu_X - \nu)(dx, ds), \tag{5.2}$$

and where ψ^Q is a predictable process and Y^Q is a strictly positive predictable function such that

$$\int_0^T (\psi_s^Q)^2 ds < \infty \qquad P\text{-a.s.} \tag{5.3}$$

and

$$\int_0^T \int_0^\infty \left(\sqrt{Y^Q(s,x)} - 1\right)^2 U_\rho(dx) < \infty \qquad P\text{-a.s.} \tag{5.4}$$

The function Y^Q and the process ψ^Q are related by

$$\mu + \left(\beta + \frac{1}{2}\right) V_{t-} + \sqrt{V_{t-}}\psi_t^Q + \int_0^\infty Y^Q(x,t)(e^x - 1)\lambda U_\rho(dx) = 0 \qquad dP \otimes dt\text{-a.e.} \tag{5.5}$$

The process $(X_t)_{0\le t\le T}$ is a semimartingale under Q with semimartingale characteristics (B^Q, C^Q, ν^Q) with respect to the zero truncation function are given by

$$dB_t^Q = b_t^Q dt, \quad dC_t^Q = c_t^Q dt, \quad \nu^Q(dt, dx) = F^Q(t, dx)dt, \tag{5.6}$$

where

$$\begin{aligned} b_t^Q &= -\frac{1}{2}V_{t-} - \int (e^x - 1)Y^Q(t,x)\lambda U_\rho(dx), \\ c^Q &= V_{t-}, \qquad F^Q(t,dx) = Y^Q(t,x)\lambda U_\rho(dx). \end{aligned} \tag{5.7}$$

Remark 5.2. *If we used the identity truncation function we had $b_t^Q = -\frac{1}{2}V_{t-} - \int (e^x - 1 - x)Y^Q(t,x)\lambda U_\rho(dx)$ for the first differential characteristic.*

Remark 5.3. *The careful reader will note, that our b^Q and Y^Q have a meaning slightly different from that appearing in [NV03], and consequently equation* (5.5) *is modified. The reasons are that, while Nicolato and Venardos work with the jump measure of the process $(Z_{\lambda t})$, we use the jump measure of the process X in order to be notationally consistent with [KS02] and the rest of our paper, see* (2.12)*. Moreover in [NV03] b^Q and Y^Q correspond to the stochastic exponential, and the identity truncation function is used. Finally [NV03] allow a riskless interest rate $r \geq 0$, whereas we use discounted quantities throughout the paper.*

Remark 5.4. *We recall that the predictable process ψ^Q and the predictable function Y^Q can be interpreted as the market price of risk associated respectively to the diffusion and the jump part of the price process.*

5.1 The minimal martingale measure for BNS models

We will see in Section 6 that in the BNS model without leverage the minimal martingale measure coincides with both Esscher martingale transforms. In this section we compute the minimal martingale measure for the BNS model with leverage, and show in the general case the measures do not coincide.

Theorem 5.5. *Let*

$$\tilde{N}_t^\flat = \int_0^t \psi_s^\flat dW_s + \int_0^t \int (Y^\flat(s,x) - 1)(\mu_X - \nu)(dx, ds), \tag{5.8}$$

with

$$\psi_t^\flat = -\theta_t^\flat \sqrt{V_{t-}} \tag{5.9}$$

and

$$Y^\flat(t,x) = 1 - \theta_t^\flat(e^x - 1), \tag{5.10}$$

where

$$\theta_t^\flat = \frac{\mu + \lambda\kappa(\rho) + \tilde{\beta}V_{t-}}{V_{t-} + \lambda(\kappa(2\rho) - 2\kappa(\rho))}. \tag{5.11}$$

If

$$\Delta\tilde{N}_t^\flat > -1 \tag{5.12}$$

and

$$G_t^\flat = \mathcal{E}(\tilde{N}^\flat)_t \tag{5.13}$$

is a martingale, then the minimal martingale measure $P^\flat$ on $\mathcal{F}_T$ exists as a probability measure, $P^\flat \sim P$, and

$$\frac{dP^\flat}{dP} = \mathcal{E}(\tilde{N}^\flat)_T. \tag{5.14}$$

The process $(X_t)_{0\leq t\leq T}$ is a semimartingale under $P^\flat$. Its characteristics $(B^\flat, C^\flat, \nu^\flat)$ with respect to the zero truncation function are given by

$$dB_t^\flat = b_t^\flat dt, \quad dC_t^\flat = c_t^\flat dt, \quad \nu^\flat(dt, dx) = F^\flat(t, dx)dt, \tag{5.15}$$

where

$$b_t^\flat = \mu + \left(\beta - \theta_t^\flat\right)V_{t-}, \quad c_t^\flat = V_{t-}, \quad F^\flat(t,dx) = Y^\flat(t,x)\lambda U_\rho(dx). \tag{5.16}$$

Proof. We have Doob-Meyer decomposition $S = S_0 + A + M$ where

$$dA_t = (\mu + \lambda k(\rho) + \tilde{\beta}V_{t-})S_{t-}dt \tag{5.17}$$

and

$$dM_t = \sqrt{V_{t-}}S_{t-}dW_t + \int_0^\infty (e^x - 1)S_{t-}(\mu_X - \nu)(dx, dt). \tag{5.18}$$

The quadratic variation is

$$d[M]_t = S_{t-}^2 V_{t-}dt + \int_0^\infty S_{t-}^2(e^x - 1)^2\mu_X(dx, dt). \tag{5.19}$$

The predictable quadratic variation is thus

$$d\langle M\rangle_t = S_{t-}^2(V_{t-} + \lambda(k(2\rho) - 2k(\rho)))dt. \tag{5.20}$$

Let us define the process

$$\alpha_t = \frac{dA_t}{d\langle M\rangle_t}, \tag{5.21}$$

which is here

$$\alpha_t = \frac{\mu + \lambda k(\rho) + \tilde{\beta}V_{t-}}{V_{t-} + \lambda(k(2\rho) - 2k(\rho))} S_{t-}^{-1}. \tag{5.22}$$

Using these processes the density of the minimal martingale measure is given by $\mathcal{E}(-\int \alpha dM)$, see [Sch01, p.557]. □

In Appendix C we give sufficient conditions, granting that the process $G^\flat$ is positive and a proper martingale, and thus a density process.

Remark 5.6. *If we used the identity truncation function we had* $b_t^\flat = \mu + \left(\beta - \theta_t^\flat\right) V_{t-} + (1 - \theta_t^\flat)(k'(\rho) - \rho\lambda\zeta)$.

Remark 5.7. *We see from the above calculations that the mean-variance-tradeoff process for the BNS model with leverage is*

$$K_t = \int_0^t \frac{(\mu + \lambda k(\rho) + \tilde{\beta}V_{s-})^2}{V_{s-} + \lambda(k(2\rho) - 2k(\rho))} ds, \tag{5.23}$$

and so it is not deterministic.

Remark 5.8. *Related to the description of the minimal martingale measure is the minimal-variance strategy which has been explicitly calculated for European options in the BNS model in [CTV07].*

5.2 Structure preserving martingale measures

In this section we want to examine the behavior of the class of equivalent martingale measures for the BNS models which preserve the model structure, in order to compare them with the measures we obtained in the previous sections.

Under an arbitrary EMM Q it could be possible, that Z is not a Lévy process, that (W^Q, Z) are not independent, and thus under Q the log-price process is no longer described by a BNS model. We need a strong characterization of the subclass of EMMs which preserve the model structure. This class of measures has also been characterized in [NV03] with the following theorem.

Theorem 5.9. *Let* $y(x)$ *be a function* $y : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\int (\sqrt{y(x)} - 1)^2 U_\rho(dx) < \infty. \tag{5.24}$$

Then the process given by

$$\psi_t^y = -V_{t-}^{-1/2}\left(\mu + \tilde{\beta}V_{t-} + \lambda k^y(\rho)\right) \tag{5.25}$$

where

$$k^y(\theta) = \int_0^\infty \left(e^{\theta x} - 1\right) y(x) U(dx) \tag{5.26}$$

for $\Re(\theta) < 0$*, is such that*

$$\int_0^T \psi_s^2 ds < \infty \qquad P\text{-}a.s., \tag{5.27}$$

and

$$G_t^y = \mathcal{E}(\tilde{N}^y)_t, \tag{5.28}$$

where

$$\tilde{N}_t^y = \int_0^t \psi_s^y dW_s + \int_0^t \int_0^\infty (y(x) - 1)(\mu_X - \nu)(dx, ds), \tag{5.29}$$

is a density process. The probability measure defined by $dQ^y = G_T^y dP$ *is an EMM on* $\mathcal{F}_T$ *for the BNS model. The process* $(X_t)_{0 \le t \le T}$ *is a semimartingale under* Q *with semimartingale characteristics* (B^Q, C^Q, ν^Q) *given by*

$$dB_t^y = b_t^y dt, \quad dC_t^y = c_t^y dt, \quad \nu^y(dt, dx) = F^y(t, dx) dt, \tag{5.30}$$

where

$$b_t^y = -\frac{1}{2} V_{t-} - \int (e^x - 1) y(x) \lambda U_\rho(dx), \quad c_t^y = V_{t-}, \quad F^y(t, dx) = y(x) \lambda U_\rho(dx). \tag{5.31}$$

The process $W_t^y = W_t - \int_0^t \psi_s^y ds$ *is a* Q*-Brownian motion and* $Z_{\lambda t}$ *is a* Q^y*-Lévy process, such that* Z_1 *has Lévy measure* $U^y(dx) = y(x)U(dx)$ *and cumulant transform* $k^y(\theta)$*, and the processes* W^y *and* Z *are* Q^y*-independent.*

Conversely, for any Q *satisfying the requirements above, there exists a function* $y : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\int_0^\infty (\sqrt{y(x)} - 1)^2 U(dx) < \infty$*, such that* Q *coincides with* Q^y*.*

Remark 5.10. *If we used the identity truncation function we had* $b_t^y = -\frac{1}{2} V_{t-} - \int (e^x - 1 - x) y(x) \lambda U_\rho(dx)$ *for the first differential characteristic.*

Remark 5.11. *Structure Preserving Equivalent Martingale Measures (from now on SPEMM) are relevant since they allow to obtain some analytical results for option pricing, see [NV03]. Since the Laplace transform of log-prices has a simple expression, using a transform-based technique the authors can obtain some closed-form formulas for the price of European options in several relevant cases.*

Remark 5.12. *The structure preserving measures are in general not Esscher transforms with respect to* X *or* $\tilde{X}$*. In the special case, when only the law of the BDLP is changed such that* $y(x) = e^{\theta x}$*, the density process is given by*

$$L_t = e^{\theta Z_{\lambda t} - \lambda k^y(\theta) t} \tag{5.32}$$

and thus we have an Esscher transform with respect to the Lévy process $(Z_{\lambda t})$*.*

Whenever the distribution of the BDLP belongs to the same parametric class (such as gamma or inverse Gaussian, for example) under the original and under an equivalent martingale measure, we say the measure change is *distribution preserving*. The distribution preserving measures are obviously a subclass of the structure preserving measures.

Remark 5.13 (Uniqueness)**.** *It follows from the examples below, that neither the equivalent martingale measures, the structure preserving or the distribution preserving martingale measures are unique in general. But, the measure that does not change the law of the BDLP Z, is unique. In this case we have $Y = 1$, and equation* (5.25) *determines uniquely the change of drift for the Brownian motion W. This measure is trivially distribution preserving. Economically this choice of martingale measure corresponds to the (questionable) idea, that the jumps represent only* non-systematic *risk that is not reflected in derivatives prices, cf. [Mer76, p.133].*

5.3 The minimal entropy martingale measure

An important equivalent martingale measure, that can be defined for a wide class of general semimartingales is the minimal entropy martingale measure (MEMM). This measure is also relevant for its connection with utility maximization with respect to exponential utility. The definition of the MEMM and a systematic investigation on this connection is given in [Fri00, BF02]. For exponential Lévy models the MEMM coincides with the linear Esscher martingale transform, see [ES05, FM03, HS06]. A natural question is whether this property holds also more generally, and in particular for BNS models. We will see below by direct comparison, that in the non-leverage-case the answer is negative. The minimal entropy martingale measure for the BNS model in the leverage case, i.e., when $\rho \neq 0$, has been obtained by T. Rheinländer and G. Steiger. In [RS06, Corollary 4.5, p.1340f] they provide a representation formula in terms of the solution of a semi-linear integro-PDE, but from this representation formula it seems difficult to make a direct comparison in the general case. In the simple concrete example of the Poisson toy model it is possible to verify explicitly that the two measures do not coincide. This leads to the conclusion that in general, the MEMM and the (linear) Esscher transform for BNS models are different.

As Rheinländer and Steiger already remarked in [RS06, Remark 4.4.4, p.1339], the MEMM does not preserve the independence of increments of the BDLP, thus is not a structure preserving measure.

6 Simplifications for the BNS model without leverage

6.1 The Esscher martingale transform for BNS models without leverage

Let us now examine the simplification that occur in the simpler situation of without leverage, i.e., when $\rho = 0$. In this case it turns out, that both the Esscher martingale transforms for exponential and for linear processes as well as the minimal martingale measure coincide. In fact, this is true, for all Itô process, as we see from the following lemma.

Lemma 6.1. *Suppose the logarithmic return process X satisfies*

$$dX_t = \mu_t dt + \sigma_t dW_t \tag{6.1}$$

with W a standard Brownian motion, and μ and σ are adapted processes, such that (6.1) *is well-defined. Then the Esscher martingale transforms for the exponential process e^X, the Esscher martingale transform for the linear process $\tilde{X}$, and and the minimal martingale measure either exist and coincide, or neither of them exists.*

Proof. The modified Laplace cumulant process $K^X(\theta)$ of X in θ is given by

$$K^X(\theta)_t = \int_0^t \tilde{\kappa}^X(\theta)_s ds, \tag{6.2}$$

where

$$\tilde{\kappa}^X(\theta)_t = \mu_t\theta_t + \tfrac{1}{2}\sigma_t^2\theta_t^2. \tag{6.3}$$

Finding the parameter process θ that turns the exponential process e^X into a martingale requires to solve

$$\tilde{\kappa}^X(\theta+1) - \tilde{\kappa}^X(\theta) = 0, \tag{6.4}$$

i.e.,

$$\mu_t(\theta_t+1) + \tfrac{1}{2}\sigma_t^2(\theta_t+1)^2\mu_t\theta_t + \tfrac{1}{2}\sigma_t^2\theta_t^2 = 0. \tag{6.5}$$

That gives the solution

$$\theta_t^\sharp = -\frac{\mu_t + \frac{1}{2}\sigma_t^2}{\sigma_t^2}, \tag{6.6}$$

and we obtain

$$\frac{dP^\sharp}{dP} = \exp\left(\int_0^T \theta_t^\sharp dX_t - K^X(\theta^\sharp)_t\right), \tag{6.7}$$

and thus

$$\frac{dP^\sharp}{dP} = \exp\left(-\int_0^T \frac{\mu_t + \frac{1}{2}\sigma_t^2}{\sigma_t} dW_t - \frac{1}{2}\int_0^T \frac{(\mu_t + \frac{1}{2}\sigma_t^2)^2}{\sigma_t^2} dt\right), \tag{6.8}$$

provided that the density process is a proper martingale. Now let us compute the Esscher martingale transform for the linear process $\tilde{X}$. We have

$$d\tilde{X}_t = (\mu_t + \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t. \tag{6.9}$$

The modified Laplace cumulant process $K^{\tilde{X}}(\theta)$ of $\tilde{X}$ in θ is given by

$$K^{\tilde{X}}(\theta)t = \int_0^t \tilde{\kappa}^{\tilde{X}}(\theta)_s ds, \tag{6.10}$$

where

$$\tilde{\kappa}^{\tilde{X}}(\theta)_t = (\mu_t + \tfrac{1}{2}\sigma_t^2)\theta_t + \tfrac{1}{2}\sigma_t^2\theta_t^2. \tag{6.11}$$

We need the derivative

$$DK^{\tilde{X}}(\theta)t = \int_0^t D\tilde{\kappa}^{\tilde{X}}(\theta)_s ds, \tag{6.12}$$

where

$$D\tilde{\kappa}^{\tilde{X}}(\theta)_t = (\mu_t + \tfrac{1}{2}\sigma_t^2) + \sigma_t^2\theta_t. \tag{6.13}$$

We have to solve $DK^{\tilde{X}}(\theta) = 0$, which has in our case the solution

$$\theta_t^* = -\frac{\mu_t + \frac{1}{2}\sigma_t^2}{\sigma_t^2}. \tag{6.14}$$

Then the density for the Esscher martingale transform for the linear process $\tilde{X}$ is given by

$$\frac{dP^*}{dP} = \exp\left(\int_0^T \theta_t^* d\tilde{X}_t - K^{\tilde{X}}(\theta^*)_t,\right) \tag{6.15}$$

and thus

$$\frac{dP^*}{dP} = \exp\left(-\int_0^T \frac{\mu_t + \frac{1}{2}\sigma_t^2}{\sigma_t} dW_t - \frac{1}{2}\int_0^T \frac{(\mu_t + \frac{1}{2}\sigma_t^2)^2}{\sigma_t^2} dt\right), \tag{6.16}$$

provided that the density process is a proper martingale. We see that the expressions (6.8) and (6.16) coincide, and thus $P^\sharp = P^*$. By comparing our result with the expression for the density of the minimal martingale measure, see for example [Sch99, (1.1), p.28] we see that the Esscher martingale transforms agree with the minimal martingale measure. □

Remark 6.2. *As it is apparent from the proof the reason by which $P^\sharp$ and P^* coincide is the fact, that*

$$\tilde{X} - X = K^{\tilde{X}}(\theta) - K^X(\theta) \tag{6.17}$$

for any parameter process θ.

6.2 Minimal entropy for BNS without leverage

We want to recall in this section some results available for the minimal entropy martingale measure in the framework of the BNS model without leverage and we want to compare them with the measures we have obtained in order to show that for BNS, the MEMM and the Esscher martingale transform for the linear process in general do not coincide. F.E. Benth and T. Meyer-Brandis have obtained in [BMB05, Proposition 5.2, p.13] an explicit expression for the MEMM in the particular case of the BNS model without leverage, i.e., when the coefficient $\rho = 0$. The measure is obtained as the zero risk aversion limit of the martingale measure corresponding to the indifference price with respect to the exponential utility function.

Under some integrability conditions they have proved, that the MEMM is given by

$$\frac{dP^e}{dP} = \frac{\exp\left[-\int_0^T \frac{\mu+\tilde{\beta}V_{t-}}{\sqrt{V_{t-}}} dW_t - \int_0^T \frac{(\mu+\tilde{\beta}V_{t-})^2}{V_{t-}} dt\right]}{E\left[\exp(-\int_0^T \frac{(\mu+\tilde{\beta}V_{t-})^2}{2V_{t-}} dt)\right]}. \tag{6.18}$$

Actually, [BMB05] write V_t instead of V_{t-} but this does not make a difference in the present case. It is not difficult to see that this measure does not preserve the Lévy property, and thus the model structure; in order to have the Lévy property preservation, in fact, the measure should be of the form (5.1–5.2) in which $y(x)$ must be deterministic and time independent. Moreover a direct comparison of (6.18) with (6.8) shows that this measure does not coincide neither with the Esscher martingale transforms nor the minimal martingale measure. This remark allows to conclude that BNS models have a quite different behavior in comparison with exponential Lévy models with respect to these classes of measures. In the exponential Lévy case, in fact, it has been proved [ES05] that the MEMM coincides with the Esscher martingale transform for the linear process and that this measure has the special property of preserving the Lévy structure of the model.

Remark 6.3. *In [ES05] the MEMM for a particular stochastic volatility model with Lévy jumps has been investigated, for which it turns out that MEMM has the same properties of Lévy structure preservation and it coincides with the Esscher martingale transform for the linear process. This analogy with the exponential Lévy models breaks down for more complex models like BNS.*

Remark 6.4. *In contrast to the exponential Lévy model the minimal entropy measure is for the BNS models not time-independent. This means, for given $0 < T_1 < T_2$, the minimal entropy martingale measure for horizon T_1 is not obtained as restriction of the minimal entropy martingale measure for horizon T_2 to $\mathcal{F}_{T_1}$. This was also observed for the Stein and Stein / Heston model in [Rhe05].*

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A On existence and integrability conditions for the exponential Esscher martingale transforms

Lemma A.1. *Let*

$$\xi_1 = \sup\{\xi \geq 0 : E[e^{\xi Z_1}] < \infty\}. \tag{A.1}$$

and

$$\ell_0 = \inf_{\theta > \xi_1/\rho} \left[k(\rho(\theta+1)) - k(\rho\theta)\right]. \tag{A.2}$$

If one of the four conditions

1. $\xi_1 = +\infty$, *or*

2. $\xi_1 < +\infty$ *and* $\ell_0 = -\infty$,

3. $\xi_1 < +\infty$ *and* $\ell_0 > -\infty$, $\beta + 1/2 + \xi_1/\rho = 0$, *and* $\mu + \lambda\ell_0 \le 0$, *or*

4. $\xi_1 < +\infty$ *and* $\ell_0 > -\infty$, $\beta + 1/2 + \xi_1/\rho < 0$, *and* $V_0 e^{-\lambda T} \ge -\dfrac{\mu + \lambda\ell_0}{\beta + 1/2 + \xi_1/\rho}$,

holds, then there is a measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}$, *such that* $\vartheta_t^\sharp = \phi(V_{t-})$ *is a solution to* (3.1).

Proof. Let The function $k(\xi)$ is well-defined and analytic for $\xi < \xi_1$. Let us study now the behavior of

$$\ell(\theta) = k(\rho(\theta + 1)) - k(\rho\theta). \tag{A.3}$$

This function is well-defined for $\theta > \theta_0$, where

$$\theta_0 = \frac{\xi_1}{\rho}, \tag{A.4}$$

and we have

$$\ell(\theta) = \int_0^\infty e^{\theta\rho x}(e^{\rho x} - 1)U(dx). \tag{A.5}$$

This is a Laplace transform and we can differentiate under the integral to obtain

$$\ell'(\theta) = \int_0^\infty e^{\theta\rho x}(\rho x)(e^{\rho x} - 1)U(dx). \tag{A.6}$$

From (A.5) we infer that $\ell(\theta) < 0$, and, by monotone convergence, that

$$\lim_{\theta\to+\infty} \ell(\theta) = 0. \tag{A.7}$$

From (A.6) we see that $\ell(\theta)$ is increasing. We have to solve $f(\theta, v) = 0$, where

$$f(\theta, v) = \mu + (\beta + \tfrac{1}{2})v + v\theta + \lambda\ell(\theta) \tag{A.8}$$

for all $v > 0$. Under the conditions 1. and 2. we have

$$\inf_{\theta>\theta_0} f(\theta, v) = -\infty. \tag{A.9}$$

Under condition 3. and 4. we have

$$\inf_{\theta>\theta_0} f(\theta, v) = \mu + (\beta + 1/2)v + v\theta_0 + \lambda\ell_0 \tag{A.10}$$

Under conditions 3. and 4. this is less or equal to zero for all $v > 0$, and thus we have a solution. □

Lemma A.2. *Suppose* $\theta^\sharp$ *is a solution to* (3.1)*. Let*

$$N_t^\sharp = \int_0^t \theta_s^\sharp dX_s - K^X(\theta^\sharp)_t, \tag{A.11}$$

and

$$G_t^\sharp = e^{N_t^\sharp}. \tag{A.12}$$

If

$$E[Z_1 e^{\rho\Theta_0^\sharp Z_1}] < \infty, \qquad E[e^{\frac{1}{2}(\Theta_1^\sharp)^2 Z_1}] < \infty \tag{A.13}$$

where

$$\begin{aligned} \Theta_0^\sharp &= -\left[\frac{(\mu+\lambda k(\rho))_+}{V_0 e^{-\lambda T}} + \tilde\beta\right]_+, \\ \Theta_1^\sharp &= \max\left\{\left[\frac{(\mu+\lambda k(\rho))_+}{V_0 e^{-\lambda T}} + \tilde\beta\right]_+, \left[\frac{-(\mu+\lambda k(\rho))_-}{V_0 e^{-\lambda T}} + \tilde\beta\right]_-\right\} \end{aligned} \tag{A.14}$$

then $(G_t^\sharp)_{0\le t\le T}$ *is a martingale.*

Proof. The equation $f(\theta, v) = 0$ implies that $\theta_t^\sharp = \phi^\sharp(V_{t-})$ is bounded. Let us provide now concrete bounds for $\theta^\sharp$: We have

$$\theta_t^\sharp = -\left[\frac{\mu + \lambda\ell(\theta_t^\sharp)}{V_{t-}} + \tilde\beta\right]. \tag{A.15}$$

We have already observed that $\ell(\theta)$ is negative and increasing in θ, and $V_{t-} > V_0 e^{-\lambda T}$. Distinguishing the cases $\theta_t^\sharp \le 0$ and $\theta_t^\sharp > 0$ we obtain

$$-\left[\frac{(\mu+\lambda\ell(0))_+}{V_0 e^{-\lambda T}} + \tilde\beta\right] \le \theta_t^\sharp \le 0 \tag{A.16}$$

respectively

$$0 \le \theta_t^\sharp \le -\left[-\frac{(\mu+\lambda\ell(0))_-}{V_0 e^{-\lambda T}} + \tilde\beta\right] \tag{A.17}$$

The subscripts plus and minus in those inequalities denote the positive and negative part. We note, that $\ell(0) = k(\rho)$. Let us next consider the process $G^\sharp$. We can rewrite (A.12) as

$$G_t^\sharp = \mathcal{E}(\tilde N^\sharp)_t, \tag{A.18}$$

where $\tilde N^\sharp$ is the exponential transform of the process $N^\sharp$. Since $K^X(\theta)$ is continuous we have

$$\Delta N_t^\sharp = \theta_t^\sharp \rho \Delta Z_{\lambda t} \tag{A.19}$$

and thus

$$\Delta \tilde N_t^\sharp = e^{\theta_t^\sharp \rho \Delta Z_{\lambda t}} - 1. \tag{A.20}$$

To prove the lemma we use the integrability condition from [LM78, Theorem III.1, p.185f]. We have to show, that

$$A_t = \tfrac{1}{2}\langle \tilde N^{\sharp c}\rangle_t + \sum_{s\le t}(1+\Delta\tilde N_s^\sharp)\log(1+\Delta\tilde N_s^\sharp) - \Delta\tilde N_s^\sharp \tag{A.21}$$

admits a predictable compensator B such that $E[e^{B_T}] < \infty$. Using (A.20) we obtain, that A admits indeed a predictable compensator, which is given by

$$B_t = \tfrac{1}{2}\int_0^t (\theta_s^\sharp)^2 V_{s-}ds + \int_0^t \int_0^\infty \left[e^{\theta_s^\sharp \rho x}\theta_s^\sharp \rho x - \left(e^{\theta_s^\sharp \rho x} - 1\right)\right] U(dx)\lambda ds. \tag{A.22}$$

Using the boundedness for $\theta^\sharp$, a Taylor expansion at $x = 0$, and the first integrability condition in (A.13) we see that the second integral in (A.22) exists, and is, in fact, uniformly bounded by a constant. So we have $E[e^{B_T}] < \infty$ if

$$E\left[\exp\left(\frac{1}{2}\int_0^T (\theta_s^\sharp)^2 V_{s-}ds\right)\right] < \infty. \tag{A.23}$$

Using the bounds on $\theta^\sharp$ from above, the last inequality is implied by

$$E\left[\exp\left(\frac{1}{2}(\Theta_1^\sharp)^2 \int_0^T V_{s-}ds\right)\right] < \infty. \tag{A.24}$$

Finally using the inequality

$$\int_0^T V_{s-}ds \leq V_0 + Z_{\lambda T} \tag{A.25}$$

we see, that a sufficient condition for (A.24) is the second integrability condition in (A.13). □

B On existence and integrability conditions for the linear Esscher martingale transforms

Lemma B.1. *There exists always a measurable function $\phi^* : \mathbb{R}_+ \to \mathbb{R}$, such that $\vartheta_t^* = \phi^*(V_{t-})$ is a solution to* (3.11).

Proof. Let us study the behavior of $\tilde{\ell}(z) = \tilde{k}'_\rho(z)$: First we observe that $\tilde{M}$, which was given in (2.25), has bounded jumps and so $\tilde{k}_\rho(z)$ and $\tilde{k}'_\rho(z)$ exist and are entire functions, thus in particular continuous on $\mathbb{R}$. Differentiating (2.32) yields

$$\tilde{\ell}(z) = \int_0^\infty e^{z(e^{\rho x}-1)}(e^{\rho x} - 1)U(dx). \tag{B.1}$$

Using the integrability properties of $U(dx)$ we get by dominated convergence,

$$\lim_{z\to+\infty} \tilde{\ell}(z) = 0. \tag{B.2}$$

Let consider $b > a > 0$ such that $U([a, b]) > 0$. Then we have for $z < 0$

$$\tilde{\ell}(z) \leq e^{z(e^{\rho a}-1)}(e^{\rho a} - 1)U([a, b]) \tag{B.3}$$

and

$$\lim_{z\to-\infty} \tilde{\ell}(z) = -\infty. \tag{B.4}$$

We have to solve $\tilde{f}(\theta, v) = 0$, where

$$\tilde{f}(\theta, v) = \mu + \tilde{\beta} v + v\theta + \lambda \tilde{\ell}(\theta). \tag{B.5}$$

As $\tilde{\ell}(z)$ increases from $-\infty$ at $z \to -\infty$ to zero as $z \to +\infty$, there is a (unique) real zero of (3.11) for every $v > 0$. □

Lemma B.2. *Let*

$$N_t^* = \int_0^t \theta_s^* d\tilde{X}_s - K^{\tilde{X}}(\theta^*)_t, \tag{B.6}$$

and

$$G_t^* = e^{N_t^*} \tag{B.7}$$

where θ^* *is as above. If*

$$E[e^{\frac{1}{2}(\Theta_1^*)^2 Z_1}] < \infty \tag{B.8}$$

with

$$\Theta_1^* = \max\left\{ \left[\frac{(\mu + \lambda k(\rho))_+}{V_0 e^{-\lambda T}} + \tilde{\beta}\right]_+, \left[\frac{-(\mu + \lambda k(\rho))_-}{V_0 e^{-\lambda T}} + \tilde{\beta}\right]_- \right\} \tag{B.9}$$

then $(G_t^*)_{t\in[0,T]}$ *is a martingale.*

Proof. The equation $\tilde{f}(\theta, v) = 0$ implies that $\theta_t^* = \phi^*(V_{t-})$ is bounded. Let us provide now concrete bounds for θ^*: We have

$$\theta_t^* = -\left[\frac{\mu + \lambda \tilde{\ell}(\theta_t^*)}{V_{t-}} + \tilde{\beta}\right]. \tag{B.10}$$

Let us observe that $\tilde{\ell}(\theta)$ is negative and increasing in θ, and $V_{t-} > V_0 e^{-\lambda T}$. Distinguishing the cases $\theta_t^* \leq 0$ and $\theta_t^* > 0$ we obtain

$$-\left[\frac{(\mu + \lambda \tilde{\ell}(0))_+}{V_0 e^{-\lambda T}} + \tilde{\beta}\right] \leq \theta_t^* \leq 0 \tag{B.11}$$

respectively

$$0 \leq \theta_t^* \leq -\left[-\frac{(\mu + \lambda \tilde{\ell}(0))_-}{V_0 e^{-\lambda T}} + \tilde{\beta}\right] \tag{B.12}$$

This can be summarized by $|\theta_t^*| \leq \Theta_1^*$. We note that $\tilde{\ell}(0) = k(\rho)$. Let us next consider the process G^*. We can rewrite (B.7) as

$$G_t^* = \mathcal{E}(\tilde{N}^*)_t, \tag{B.13}$$

where $\tilde{N}^*$ is the exponential transform of the process N^*. Since $K^{\tilde{X}}(\theta)$ is continuous we have

$$\Delta N_t^* = \theta_t^*(e^{\rho \Delta Z_{\lambda t}} - 1) \tag{B.14}$$

and thus

$$\Delta \tilde{N}_t^* = \exp\left(\theta_t^*(e^{\rho \Delta Z_{\lambda t}} - 1)\right) - 1. \tag{B.15}$$

To prove the lemma we use the integrability condition from [LM78]. We have to show, that

$$A_t = \tfrac{1}{2}\langle \tilde{N}^{*c}\rangle_t + \sum_{s\le t}(1+\Delta\tilde{N}^*_s)\log(1+\Delta\tilde{N}^*_s) - \Delta\tilde{N}^*_s \tag{B.16}$$

admits a predictable compensator B such that $E[e^{B_T}] < \infty$. Using (B.15) we obtain, that A admits indeed a predictable compensator, which is given by

$$B_t = \tfrac{1}{2}\int_0^t (\theta^*_s)^2 V_{s-}ds + \int_0^t\int_0^\infty \left[e^{\theta^*_s(e^{\rho x}-1)}\theta^*_s(e^{\rho x}-1) - \left(e^{\theta^*_s(e^{\rho x}-1)}-1\right)\right]\lambda U(dx)ds. \tag{B.17}$$

Using the boundedness for θ^* and a Taylor expansion at $x = 0$ we see that the second integral in (B.17) is bounded by a constant. So we have $E[e^{B_T}] < \infty$ iff

$$E\left[\exp\left(\frac{1}{2}\int_0^T (\theta^*_s)^2 V_{s-}ds\right)\right] < \infty. \tag{B.18}$$

Using the bounds on θ^* from above, the last inequality is implied by

$$E\left[\exp\left(\frac{1}{2}(\Theta^*_1)^2\int_0^T V_{s-}ds\right)\right] < \infty. \tag{B.19}$$

Finally using the inequality

$$\int_0^T V_{s-}ds \le V_0 + Z_{\lambda T} \tag{B.20}$$

we see, that a sufficient condition for (B.19) is (B.8). □

Remark B.3. *We see that $\Theta^\sharp_1 = \Theta^*_1$ since $\ell(0) = \tilde{\ell}(0) = k(\rho)$. This is related to the observation, that one can use the same bound for the continuous quadratic variation parts in both the exponential and the linear Esscher martingale transforms.*

C On existence and integrability conditions for the minimal martingale measure

Lemma C.1. *A sufficient condition for 5.12 on $0 \le t \le T$ is*

$$\rho \le 0, \quad (\beta + 3/2)V_0 e^{-\lambda T} \ge -\mu + \lambda k(\rho) - \lambda k(2\rho), \quad \beta \ge -\tfrac{3}{2}. \tag{C.1}$$

When the jumps of Z are unbounded, this is also necessary.

Proof. From (5.8) we see, that

$$\Delta\tilde{N}^\flat_t = -\theta^\flat_t(e^{\rho\Delta Z_{\lambda t}} - 1). \tag{C.2}$$

As $\mathbb{V}[e^{\rho Z_1}] = e^{k(2\rho)} - e^{2k(\rho)} > 0$ we have

$$k(2\rho) - 2k(\rho) > 0. \tag{C.3}$$

Suppose $\Delta Z_{\lambda t} > 0$. Then we have

$$\theta_t^\flat \geq -1 > -\frac{1}{1 - e^{\rho \Delta Z_{\lambda t}}}. \tag{C.4}$$

By the assumptions of the lemma, this inequality holds, and thus $\Delta \tilde{N}_t^\flat > -1$ and $\mathcal{E}(\tilde{N}^\flat) > 0$. □

Lemma C.2. *If*

$$E[e^{\frac{1}{2}K_0^2 Z_1}] < \infty, \tag{C.5}$$

where

$$K_0 = \max\left(\left|\beta + \frac{1}{2}\right|, \left|\frac{\mu + \lambda k(\rho)}{\lambda(k(2\rho) - 2k(\rho))}\right|\right) \tag{C.6}$$

then $G^\flat$ is a martingale.

Proof. We use [LM78, Theorem III.1] to show, that $\mathcal{E}(\tilde{N}^\flat) > 0$ is a proper martingale. To apply that theorem, we consider

$$A_t = \tfrac{1}{2}\langle \tilde{N}^{\flat c}\rangle_t + \sum_{s \leq t}(1 + \Delta \tilde{N}_s^\flat)\log(1 + \Delta \tilde{N}_s^\flat) - \Delta \tilde{N}_s^\flat \tag{C.7}$$

and show it admits the predictable compensator B with $E[e^{B_T}] < \infty$. Let us observe that

$$\theta_t^\flat = \frac{a + bV_{t-}}{c + V_{t-}}, \tag{C.8}$$

with the constants $a = \mu + \lambda k(\rho)$, $b = \beta + 1/2$, $c = \lambda(k(2\rho) - 2k(\rho)) > 0$. Looking at the rational function $v \mapsto (a + bv)/(c + v)$ we get the bound

$$|\theta_t^\flat| \leq K_0. \tag{C.9}$$

Now we have the inequality

$$0 < 1 - \theta_t^\flat(e^{\rho x} - 1) < 1 + K_0. \tag{C.10}$$

The predictable quadratic variation of $(\tilde{N}^\flat)^c$ is

$$\langle (\tilde{N}^\flat)^c \rangle_t = \int_0^t \theta_s^{\flat 2} V_{s-} ds, \tag{C.11}$$

and we have the bound

$$\langle (\tilde{N}^\flat)^c \rangle_t \leq K_0^2 \int_0^t V_{s-} ds. \tag{C.12}$$

As $\theta_t^\flat$ is bounded we have for $f(x) = x\log x + 1 - x$ that $f\left(1 - \theta_t^\flat(e^{\rho x} - 1)\right) = \mathcal{O}(x^2)$ as $x \to 0$ and $f\left(1 - \theta_t^\flat(e^{\rho x} - 1)\right) = \mathcal{O}(1)$ as $x \to \infty$ and consequently there is a constant $K_1 > 0$ such that

$$0 \leq \int_0^\infty f\left(1 - \theta_t^\flat(e^{\rho x} - 1)\right) \lambda U(dx) \leq K_1. \tag{C.13}$$

This implies that

$$B_t = \int_0^t \left[\frac{1}{2}\theta_s^{\flat 2} V_{s-} + \int f\left(1 - \theta_t^{\flat}(e^{\rho x} - 1)\right) \lambda U(dx) \right] ds. \tag{C.14}$$

is well-defined, bounded, and we can conclude, that B is the predictable compensator of A. The inequalities (C.9) and

$$\int_0^t V_{s-} ds \leq V_0 + Z_{\lambda t} \tag{C.15}$$

imply there is a constant $K_2 > 0$ such that

$$B_T \leq \frac{1}{2} K_0^2 Z_{\lambda T} + K_2. \tag{C.16}$$

Since Z is a Lévy process we have $E[\exp(\frac{1}{2}K_0^2 Z_{\lambda T})] < \infty$ iff $E[\exp(\frac{1}{2}K_0^2 Z_1)] < \infty$. □

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