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Andreas Basse

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Abstract

The aim of the present paper is to characterize the spectral representation of Gaussian semimartingales. That is, we provide necessary and sufficient conditions on the kernel K for $(X_t)_{t\geq 0} = (\int K_t(s)\, dN_s)_{t\geq 0}$ to be a semimartingale. Here, N denotes an independently scattered Gaussian random measure on a general space S. We study the semimartingale property of $(X_t)_{t\geq 0}$ in three different filtrations. First the $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale property is considered and afterwards the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale property is treated in the case where $(X_t)_{t\in\mathbb{R}}$ is a moving average process and $\mathcal{F}_t^{X,\infty} = \sigma(X_s : s \in (-\infty, t])$. Finally we study a generalization of Gaussian Volterra processes. In particular we provide necessary and sufficient conditions on K for the Gaussian Volterra process $(\int_{-\infty}^t K_t(s)\, dW_s)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale ($(W_t)_{t\in\mathbb{R}}$ denotes a Wiener process). Hereby we generalize a result of Knight (1992), Cherny (2001) and Cheridito (2004) to the non-stationary case.

Keywords: semimartingales; Gaussian processes; Volterra processes; stationary processes; moving average processes

AMS Subject Classification: 60G15; 60G10; 60G48; 60G57

1 Introduction

Recently there has been major interest in Gaussian Volterra processes. That is, processes $(X_t)_{t\geq 0}$ given by

$$X_t = \int_{-\infty}^t K_t(s)\, dW_s, \qquad t \geq 0, \tag{1.1}$$

where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process with parameter space $\mathbb{R}$ and $s \mapsto K_t(s)$ is a square integrable function for $t \geq 0$. Knight (1992, Theorem 6.5), Cherny (2001), Cheridito (2004) and Jeulin and Yor (1993) studied Gaussian Volterra processes with K on the form $K_t(s) = k(t-s) + f(s)$ (such processes are called moving average processes). They characterized the set of K's for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale, where $\mathcal{F}_t^{W,\infty} := \sigma(W_s : s \in (-\infty, t])$. In the case where $K_t(s) = k(t-s)$ Jeulin and

*Department of Mathematical Sciences, University of Aarhus,
Ny Munkegade, DK-8000 Århus C, Denmark. E-mail: basse@imf.au.dk

Yor (1993, Proposition 19) gave a condition on the Fourier transform of k for $(X_t)_{t\geq 0}$ to an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale by using complex function theory (in particular Hardy theory).

A fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is an example of a Gaussian Volterra process (it is in fact a moving average process). In this case K is given by

$$K_t(s) = ((t-s)^+)^{H-1/2} - ((-s)^+)^{H-1/2}.$$

It is well-known (see Rogers (1997)) that the fBm is a semimartingale if and only if $H = 1/2$, i.e. it is a Brownian motion. Inspired by the fBm there has been developed (using Malliavin calculus) an integral for some Gaussian Volterra processes which are not semimartingales, see Alòs et al. (2001), Decreusefond (2005) and Marquardt (2006). This integral lacks some of the usual properties of the semimartingale integral by the characterization of semimartingales as stochastic integrators (the Bichteler-Dellacherie Theorem), see Protter (2004, Chapter 3, Theorem 43). Hence it is important to characterize the set of K's for which $(X_t)_{t\geq 0}$ is a semimartingale.

According to Kuelbs (1973) every centered Gaussian process $(X_t)_{t\geq 0}$, which is right-continuous in probability, has a spectral representation in distribution, i.e. $(X_t)_{t\geq 0}$ is distributed as $(\int K_t(s)\,dN_s)_{t\geq 0}$, where N is an independently scattered centered Gaussian random measure and $(t,s) \mapsto K_t(s)$ is a deterministic function. The semimartingale property of Gaussian processes is determined by the distribution of the process. Hence, $(X_t)_{t\geq 0}$ is a semimartingale if and only if $(\int K_t(s)\,dN_s)_{t\geq 0}$ has this property. The purpose of this paper is to characterize the spectral representation of Gaussian semimartingales, that is we characterize the family of kernels K for which

$$\Big(\int K_t(s)\,dN_s\Big)_{t\geq 0} \tag{1.2}$$

is a semimartingale. Note that the processes on the form (1.2) constitute a generalization of the Gaussian Volterra processes. We study the semimartingale property with respect to the natural filtration and with respect to two larger filtrations. In particular we characterize the K's for which a Gaussian Volterra process $(X_t)_{t\geq 0}$ given by (1.1) is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale or an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale (the latter condition is strongest). Hereby we generalize results of Cheridito (2004), Knight (1992, Theorem 6.5) and Cherny (2001). Our setting also covers Ambit processes with deterministic volatility, see Barndorff-Nielsen and Schmiegel (2007). Moreover, we characterize the functions k for which $(X_t)_{t\in\mathbb{R}} = (\int k(t-s)\,dW_s)_{t\in\mathbb{R}}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

The paper is organised as follows. Section 2 contains notation and preliminary results about Gaussian random measures. Section 3 contains measure-theoretic and Gaussian results. In section 4 we characterize the spectral representation of Gaussian semimartingales.

2 Notation and random measures

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras satisfying the usual conditions of right-continuity and completeness. If $(X_t)_{t\geq 0}$ is a stochastic process we denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the least filtration to which $(X_t)_{t\geq 0}$ is adapted. Let T equal $\mathbb{R}_+$ or $\mathbb{R}$. Then $(X_t)_{t\in T}$ is said to have stationary increments if for all $n \geq 1$, $t_0 < \cdots < t_n$ and $0 < t$ we have

$$(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}) \stackrel{\mathcal{D}}{=} (X_{t_1+t} - X_{t_0+t}, \ldots, X_{t_n+t} - X_{t_{n-1}+t}),$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, if there exists a decomposition of $(X_t)_{t\geq 0}$ as

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0, \tag{2.1}$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale starting at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation starting at 0. We say that $(X_t)_{t\geq 0}$ is a semimartingale if it is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale. Moreover $(X_t)_{t\geq 0}$ is called a special $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale such that $(A_t)_{t\geq 0}$ in (2.1) can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable. In this case the representation (2.1) with $(A_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable is unique and is called the canonical decomposition of $(X_t)_{t\geq 0}$. From Stricker's Theorem (see Protter (2004, Chapter 2, Theorem 4)) it follows that if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale then it is also an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale.

For each function $f\colon \mathbb{R}_+ \to \mathbb{R}$ of bounded variation, $V_t(f)$ denotes the total variation of f on $[0,t]$ for $t \geq 0$. If $(A_t)_{t\geq 0}$ is a right-continuous Gaussian process of bounded variation then $(A_t)_{t\geq 0}$ is of integrable variation (see Stricker (1983)) and we let μ_A denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto E[V_t(A)]$. For every Gaussian martingale $(M_t)_{t\geq 0}$ let μ_M denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto E[M_t^2]$.

A process $(W_t)_{t\in\mathbb{R}}$ is said to be a Wiener process if for all $n \geq 1$ and $t_0 < \cdots < t_n$,

$$W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are independent, for $-\infty < s < t < \infty$ $W_t - W_s$ follows a centered Gaussian distribution with variance $t - s$, and $W_0 = 0$.

We now give a short survey of properties of independently scattered centered Gaussian random measures. Let S denote a non-empty set and $\mathcal{A}$ be a family of subsets of S. Then $\mathcal{A}$ is called a ring if for every pair of sets in $\mathcal{A}$ the union, intersection and set difference are also in $\mathcal{A}$. A ring $\mathcal{A}$ is called a δ-ring if $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ implies $\bigcap A_n \in \mathcal{A}$. If $\mathcal{A}$ is a δ-ring and there exists a sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ satisfying $\bigcup A_n = S$ then $\mathcal{A}$ is said to be σ-finite. Throughout the paper let $\mathcal{A}$ denote a σ-finite δ-ring on a nonempty set S.

A family $N = \{N(A) : A \in \mathcal{A}\}$ of random variables is said to be an independently scattered centered Gaussian random measure if

1. For every sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ of pairwise disjoint sets with $\bigcup A_n \in \mathcal{A}$, $\sum_{i=1}^n N(A_i)$ converges to $N(\bigcup A_i)$ in probability as n tends to infinity.
2. For all $n \geq 1$ and all disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$, $N(A_1), \ldots, N(A_n)$ are independent centered Gaussian random variables.

For a general treatment of independently scattered random measures, see Rajput and Rosiński (1989). Let N denote an independently scattered centered Gaussian random measure. It is readily seen that there is a σ-finite measure ν on $(S, \sigma(\mathcal{A}))$ such that $N(A)$ has a centered Gaussian distribution with variance $\nu(A)$ for all $A \in \mathcal{A}$. Following Rajput and Rosiński (1989), ν is called the control measure of N. Throughout the paper N denotes a independently scattered centered Gaussian random measure with control measure ν. We shall assume in addition that $L^2(\nu)$ is separable.

Let $f = \sum_{i=1}^n \alpha_i 1_{A_i}$ be a simple function. That is, $n \geq 1$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathcal{A}$. Define $\int f(s)\, dN_s := \sum_{i=1}^n \alpha_i N(A_i)$. By a standard argument the integral $\int f(s)\, dN_s$ can be defined through the isometry

$$\Big\| \int f(s)\, dN_s \Big\|_{L^2(P)} = \|f\|_{L^2(\nu)}$$

for all $f \in L^2(S, \sigma(\mathcal{A}), \nu)$.

If $S = \mathbb{R}_+$, N could be the independently scattered random measure induced by a Brownian motion. More generally, if $S = \mathbb{R}_+^d$, N could be the independently scattered random measure induced by a d-parameter Brownian sheet. In this case ν is the Lebesgue measure on $\mathbb{R}_+^d$ and we can choose $\mathcal{A}$ to be $\{(s_1, t_1] \times \cdots \times (s_d, t_d] : s_i \leq t_i \ i = 1, \dots, d\}$. Another example is when $S = \mathbb{R}$ and N is the independently scattered random measure induced by a Brownian motion $(W_t)_{t \in \mathbb{R}}$ with parameter space $\mathbb{R}$.

3 Preliminary results

In this section we collect some measure-theoretical and Gaussian results. We let $(E, \mathcal{E}, m)$ be a σ-finite measure space and μ be a Radon measure on $\mathbb{R}_+$. If H is a normed space and $A \subseteq H$, then $\overline{\mathrm{sp}}\, A$ denotes the closure of the linear span of A. For each mapping $\mathbb{R}_+ \times E \ni (t, s) \mapsto \Psi_t(s) \in \mathbb{R}$ we denote by Ψ_t the mapping $s \mapsto \Psi_t(s)$ for $t \geq 0$. The following Lemma 3.1 – 3.2 are taken from Basse (2007).

Lemma 3.1. *Let $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and define $V := \overline{\mathrm{sp}}\{\Psi_t : t \geq 0\}$. Assume V is a separable subset of $L^2(m)$ and $t \mapsto \int \Psi_t(s) g(s)\, m(ds)$ is measurable for $g \in V$. Then there exists a measurable mapping $\mathbb{R}_+ \times E \ni (t, s) \mapsto \tilde{\Psi}_t(s) \in \mathbb{R}$ such that $\tilde{\Psi}_t = \Psi_t$ m-a.s. for $t \geq 0$.*

For a locally μ-integrable function f we define $\int_a^b f\, d\mu := \int_{(a,b]} f\, d\mu$ for $0 \leq a < b$. Let $\mathcal{BV}(m)$ denote the space of all measurable mappings $\mathbb{R}_+ \times S \ni (r, s) \mapsto \Psi_r(s) \in \mathbb{R}$ for which $\Psi_r \in L^2(m)$ for $r \geq 0$ and there exists a right-continuous increasing function f such that $\|\Psi_t - \Psi_u\|_{L^2(m)} \leq f(t) - f(u)$ for $0 \leq u \leq t$.

Lemma 3.2. *Let $(r, s) \mapsto \Psi_r(s)$ be a measurable mapping for which $(\Psi_r)_{r \geq 0}$ is bounded in $L^2(m)$. Then $r \mapsto \Psi_r(s)$ is locally μ-integrable for m-a.a. $s \in E$ and by setting $\int_0^t \Psi_r(s)\, \mu(dr) := 0$ for $t \geq 0$ if $r \mapsto \Psi_r(s)$ is not locally m-integrable we have*

$$(t, s) \mapsto \int_0^t \Psi_r(s)\, \mu(dr) \in \mathcal{BV}(m).$$

If in addition V is a closed subspace of $L^2(m)$ such that $\Psi_r \in V$ for all $r \in [0,t]$ then

$$s \mapsto \int_0^t \Psi_r(s)\,\mu(dr) \in V.$$

For a measurable mapping $(r,s) \mapsto \Psi_r(s)$ for which $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(m)$ we always define the mapping $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(dr)$ as in the above lemma.

Lemma 3.3. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(Y_t)_{t\geq 0} \subseteq L^1(P)$ be a measurable process with locally μ-integrable sample paths. Define*

$$A_t := \int_0^t Y_r\,\mu(dr), \qquad t \geq 0.$$

Then $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable if and only if Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$.

Proof. Assume $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(Z_t)_{t\geq 0}$ with locally μ-integrable sample paths such that $A_t = \int_0^t Z_r\,\mu(dr)$ for $t \geq 0$, see Jacod and Shiryaev (2003, Proposition 3.13). Hence $Y_t = Z_t$ P-a.s. for μ-a.a. $t \geq 0$ and we conclude that Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$.

Assume conversely that Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$ and let $({}^{\mathrm{p}}Y_t)_{t\geq 0}$ denote the $(\mathcal{F}_t)_{t\geq 0}$-predictable projection of $(Y_t)_{t\geq 0}$. Since Y_t is $\mathcal{F}_{t-}$-measurable for μ-a.a. $t \geq 0$ it follows that ${}^{\mathrm{p}}Y_t = Y_t$ P-a.s. for μ-a.a. $t \geq 0$. Thus

$$A_t = \int_0^t {}^{\mathrm{p}}Y_s\,\mu(ds), \qquad t \geq 0,$$

and it follows that $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable. This completes the proof. $\square$

Recall that N denotes an independently scattered centered Gaussian random measure with control measure ν. Let $\mathbb{R}_+ \times S \ni (r,s) \mapsto \Psi_r(s)$ be a measurable mapping for which $\Psi_r \in L^2(\nu)$ for $r \geq 0$. Then we may and do choose $(\int \Psi_t(s)\,dN_s)_{t\geq 0}$ jointly measurable in (t,ω). To see this note that $V := \overline{\mathrm{sp}}\{N(A) : A \in \mathcal{A}\}$ is a separable subspace of $L^2(P)$ and

$$V = \Big\{ \int f(s)\,dN_s : f \in L^2(\nu) \Big\}. \tag{3.1}$$

Hence for each element $\int f(s)\,dN_s \in V$ we have

$$E\Big[\int \Psi_t(s)\,dN_s \int f(s)\,dN_s\Big] = \int \Psi_t(s) f(s)\,\nu(ds),$$

which shows $t \mapsto E[\int \Psi_t(s)\,dN_s \int f(s)\,dN_s]$ is measurable. The existence of a measurable modification of $(\int \Psi_t(s)\,dN_s)_{t\geq 0}$ now follows from Lemma 3.1.

Lemma 3.4. *We have the following.*

(i) Let $(Y_t)_{t\geq 0}$ be a measurable process such that $(Y_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{N(A) : A \in \mathcal{A}\}$. Then there exists a measurable mapping $\mathbb{R}_+ \times S \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ with $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and such that $Y_t = \int \Psi_t(s)\, dN_s$ for $t \geq 0$.

(ii) Let $(r,s) \mapsto \Psi_r(s)$ be a measurable mapping for which $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\nu)$. Then $r \mapsto \int \Psi_r(s)\, dN_s$ is locally μ-integrable P-a.s. and for $t \geq 0$ we have

$$\int_0^t \Big(\int \Psi_r(s)\, dN_s\Big)\mu(dr) = \int \Big(\int_0^t \Psi_r(s)\, \mu(dr)\Big)\, dN_s. \tag{3.2}$$

(iii) Let $K_t \in L^2(\nu)$ for $t \geq 0$ and $(X_t)_{t\geq 0}$ be a right-continuous process satisfying $X_t = \int K_t(s)\, dN_s$ for $t \geq 0$. Then for $0 \leq u \leq t$ we have

$$E[X_t|\mathcal{F}_u^X] = \int \big(\mathfrak{P}_u K_t\big)(s)\, dN_s,$$

where $\mathfrak{P}_u K_t$ denotes the $L^2(\nu)$-projection of K_t on $\overline{\mathrm{sp}}\{K_v : v \in [0,u]\}$.

(iv) Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(A_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable centered Gaussian process which is right-continuous and of bounded variation. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(Y_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{A_t : t \geq 0\}$ satisfying $\|Y_t\|_{L^2(P)} = 1$ for $t \geq 0$ and

$$A_t = \int_0^t Y_r\, \mu(dr), \qquad t \geq 0,$$

where $\mu := \sqrt{2/\pi}\mu_A$.

Proof. (i): For $t \geq 0$ there exists, by (3.1), a $\Phi_t \in L^2(\nu)$ such that $Y_t = \int \Phi_t(s)\, dN_s$. Moreover for $f \in L^2(\nu)$, $t \mapsto \int \Phi_t(s) f(s)\, \nu(ds)$ is measurable since

$$E\Big[Y_t \int f(s)\, dN_s\Big] = \int \Phi_t(s) f(s)\, \nu(ds).$$

Hence it follows from Lemma 3.1 that there exists a Ψ as stated in (i).

(ii): Since for $t \geq 0$ we have

$$E\Big[\int_0^t \Big|\int \Psi_r(s)\, dN_s\Big|\mu(dr)\Big] \leq \int_0^t \|\Psi_r\|_{L^2(\nu)}\, \mu(ds) < \infty,$$

the mapping $r \mapsto \int \Psi_r(s)\, dN_s$ is locally μ-integrable P-a.s. Thus both sides of (3.2) are well-defined. The right-hand side belongs to $\overline{\mathrm{sp}}\{N(A) : A \in \mathcal{A}\}$ and so does the left-hand side by Lemma 3.2. Fix $Y = \int g(s)\, dN_s$ in $\overline{\mathrm{sp}}\{N(A) : A \in \mathcal{A}\}$. We have

$$E\Big[Y \int \Big(\int f(t,s)\, \mu(dt)\Big) dN_s\Big] = \int g(s) \int f(t,s)\, \mu(dt)\, \nu(ds).$$

Moreover from Fubini's Theorem we have

$$\begin{aligned}E\Big[Y\int\Big(\int f(t,s)\,dN_s\Big)\mu(dt)\Big] &= \int E[Y\int f(t,s)\,dN_s]\,\mu(dt)\\ &= \iint g(s)f(t,s)\,\nu(ds)\,\mu(dt) = \iint g(s)f(t,s)\,\mu(dt)\,\nu(dt).\end{aligned}$$

Hence, the left- and right-hand side of (3.2) have the same inner product with all elements of $\overline{\mathrm{sp}}\{N(A) : A \in \mathcal{A}\}$, from which equality follows.

(iii): From Gaussianity it follows that $E[X_t|\mathcal{F}^X_u]$ is the $L^2(P)$-projection of X_t on $\overline{\mathrm{sp}}\{X_v : v \le u\}$ and therefore (3.1) shows

$$E[X_t|\mathcal{F}^X_u] = \int f(s)\,dN_s,$$

for some $f \in L^2(\nu)$. Since $L^2(\nu) \ni g \mapsto \int g(s)\,dN_s \in L^2(P)$ is an isometry it is readily seen that $f = \mathfrak{P}_u K_t$.

(iv) is an immediate consequence of Basse (2007, Proposition 4.1). $\square$

4 Main results

In this section we characterize the spectral representation of Gaussian semimartingales $(X_t)_{t\ge0}$. We study three different filtrations. First we consider the natural filtration of $(X_t)_{t\ge0}$. Then we assume $(X_t)_{t\in\mathbb{R}}$ is a moving average process and the filtration is $(\mathcal{F}^{X,\infty}_t)_{t\ge0}$, where $(\mathcal{F}^{X,\infty}_t)_{t\ge0}$ is the least filtration for which X_s is $\mathcal{F}^{X,\infty}_t$-measurable for $t \ge 0$ and $s \in (-\infty, t]$. Finally the filtration is generated by the background driving random measure N. Recall that ν is the control measure of N.

Theorem 4.1. *Let $\mathbb{R}_+ \ni t \mapsto K_t \in L^2(\nu)$ be a right-continuous mapping and $(X_t)_{t\ge0}$ be given by $X_t = \int K_t(s)\,dN_s$ for $t \ge 0$. Then the following three conditions are equivalent:*

(i) $(X_t)_{t\ge0}$ is a semimartingale (in its natural filtration).

(ii) For $t \ge 0$ we have

$$K_t(s) = K_0(s) + H_t(s) + \int_0^t \Psi_r(s)\,\mu(dr), \qquad \nu\text{-a.a. } s \in S, \tag{4.1}$$

where $\mathbb{R}_+ \ni t \mapsto H_t \in L^2(\nu)$ is a right-continuous mapping satisfying $H_0 = 0$ and

$$\int \big(H_t(s) - H_u(s)\big)K_v(s)\,\nu(ds) = 0, \qquad 0 \le v \le u \le t, \tag{4.2}$$

$\mathbb{R}_+ \times S \ni (r, s) \mapsto \Psi_r(s) \in \mathbb{R}$ is a measurable mapping such that $\|\Psi_r\|_{L^2(\nu)} = 1$ and $\Psi_r \in \overline{\mathrm{sp}}\{K_v : v < r\}$ for $r \ge 0$, and μ is a Radon measure.

(iii) There exists a right-continuous increasing function $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|\mathfrak{P}_u K_t - K_u\|_{L^2(\nu)} \le f(t) - f(u), \qquad 0 \le u \le t,$$

where $\mathfrak{P}_u K_t$ denotes the $L^2(\nu)$-projection of K_t on $\overline{\mathrm{sp}}\{K_v : v \le u\}$.

The decomposition (4.1) *is unique and if* K *is represented as in* (4.1) *then the canonical decomposition of* $(X_t)_{t\geq 0}$ *is given by*

$$X_t = X_0 + \int H_t(s)\, dN_s + \int_0^t \Big(\int \Psi_r(s)\, dN_s \Big) \mu(dr). \tag{4.3}$$

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii): Assume $(X_t)_{t\geq 0}$ is a semimartingale. By Stricker (1983, Théorème 1) $(X_t)_{t\geq 0}$ is a special semimartingale with bounded variation component $(A_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{X_t : t \geq 0\}$. Hence by Lemma 3.4 (iv) there exists an $(\mathcal{F}_t^X)_{t\geq 0}$-predictable process $(Z_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{X_t : t \geq 0\}$ with $\|Z_r\|_{L^2(P)} = 1$ such that $A_t = \int_0^t Z_r\, \mu(dr)$ for $t \geq 0$, where $\mu = \sqrt{2/\pi}\mu_A$. Moreover Lemma 3.4 (i) shows that there exists a measurable mapping $(r, s) \mapsto \Psi_r(s)$ satisfying $\Psi_r \in L^2(\nu)$ and $Z_r = \int \Psi_r(s)\, dN_s$ for $r \geq 0$. Since Z_r is $\mathcal{F}_{r-}^X$-measurable, it follows from Gaussianity that $\Psi_r \in \overline{\mathrm{sp}}\{K_v : v < r\}$ for $r \geq 0$. From Lemma 3.4 (ii) we have

$$A_t = \int \Big(\int_0^t \Psi_r(s)\, \mu(dr) \Big) dN_s, \qquad t \geq 0.$$

Due to the fact that $(M_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{X_t : t \geq 0\}$, Lemma 3.4 (i) shows that for all $t \geq 0$, $M_t = \int H_t(s)\, dN_s$ for some $H_t \in L^2(\nu)$. The mapping $t \mapsto H_t \in L^2(\nu)$ is right-continuous since $(M_t)_{t\geq 0}$ is right-continuous. Stricker (1983, Théorème 1) shows that $(M_t)_{t\geq 0}$ is a true $(\mathcal{F}_t^X)_{t\geq 0}$-martingale and hence

$$0 = E[(M_t - M_u)X_v] = \int \big(H_t(s) - H_u(s) \big) K_v(s)\, \nu(ds), \qquad 0 \leq v \leq u \leq t.$$

This completes the proof of (4.1).

(ii) $\Rightarrow$ (i): Assume (4.1) is satisfied. We show that $(X_t)_{t\geq 0}$ is a semimartingale with canonical decomposition given by (4.3). For $t \geq 0$ define

$$M_t := \int H_t(s)\, dN_s \qquad \text{and} \qquad A_t := \int \Big(\int_0^t \Psi_r(s)\, \mu(dr) \Big) dN_s.$$

Note $X_t = X_0 + M_t + A_t$. Lemma 3.4 (ii) shows that

$$A_t = \int_0^t \Big(\int \Psi_r(s)\, dN_s \Big) \mu(dr), \qquad t \geq 0,$$

which implies that $(A_t)_{t\geq 0}$ is right-continuous and of bounded variation. Let $r \geq 0$. Since $\Psi_r \in \overline{\mathrm{sp}}\{K_v : v < r\}$, $\int \Psi_r(s)\, dN_s$ is $\mathcal{F}_{r-}^X$-measurable and hence it follows from Lemma 3.3 that $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^X)_{t\geq 0}$-predictable.

The only thing left to show is that $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t^X)_{t\geq 0}$-martingale. Since $M_t = X_t - X_0 - A_t$, $(M_t)_{t\geq 0}$ is $(\mathcal{F}_t^X)_{t\geq 0}$-adapted. Equation (4.2) shows that $E[(M_t - M_u)X_v] = 0$ for $0 \leq v \leq u \leq t$ and hence from Gaussianity it follows that $M_t - M_u$ is independent of X_v. The $(\mathcal{F}_t^X)_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ therefore follows by the $L^2(P)$ right-continuity of $(M_t)_{t\geq 0}$. Since $(\mathcal{F}_t^X)_{t\geq 0}$ satisfies the usual conditions we can choose a càdlàg modification of $(M_t)_{t\geq 0}$. Thus $(X_t)_{t\geq 0}$ is a semimartingale with canonical decomposition given by (4.3).

(i) $\Leftrightarrow$ (iii): From Stricker (1983, Théorème 1) it follows that $(X_t)_{t\geq 0}$ is a semimartingale if and only if it is a quasimartingale on each bounded interval. That is, for $t \geq 0$ we have

$$\sup \sum_{i=1}^{n} E[|E[X_{t_i} - X_{t_{i-1}} | \mathcal{F}^X_{t_{i-1}}]|] < \infty, \tag{4.4}$$

where the sup is taken over all finite partitions $0 = t_0 < \cdots < t_n = t$ of $[0,t]$. This is equivalent to the existence of a right-continuous and increasing function f satisfying

$$E[|E[X_t - X_u | \mathcal{F}^X_u]|] \leq f(t) - f(u), \qquad 0 \leq u \leq t.$$

The function f can now be chosen to be the left-hand side of (4.4). Moreover Lemma 3.4 (iii) shows that

$$\|\mathfrak{P}_u K_t - K_u\|_{L^2(\nu)} = \|E[X_t - X_u | \mathcal{F}^X_u]\|_{L^2(P)} = \sqrt{\frac{\pi}{2}}\, E[|E[X_t - X_u|\mathcal{F}^X_u]|],$$

which implies that (i) and (iii) are equivalent.

Decompose K as in (4.1). We show that this decomposition is unique. In the proof of "(ii) $\Rightarrow$ (i)" we showed that (4.3) is the canonical decomposition of $(X_t)_{t\geq 0}$ and since this is unique we have that $\mathbb{R}_+ \ni t \mapsto H_t \in L^2(\nu)$ is unique. Let $(A_t)_{t\geq 0}$ be the bounded variation component of the semimartingale $(X_t)_{t\geq 0}$. We have

$$\begin{aligned} E[V_t(A)] &= E\Big[\int_0^t \Big|\int \Psi_r(s)\, dN_s\Big|\, \mu(dr)\Big] = \int_0^t E\Big[\Big|\int \Psi_r(s)\, dN_s\Big|\Big]\, \mu(dr) \\ &= \sqrt{\frac{2}{\pi}} \int_0^t \|\Psi_r\|_{L^2(\nu)}\, \mu(dr) = \sqrt{\frac{2}{\pi}}\, \mu((0,t]), \end{aligned}$$

and hence μ is uniquely determined and it follows that $(t,s) \mapsto \Psi_t(s)$ is uniquely determined $\mu \otimes \nu$-a.s. This completes the proof. $\square$

The functions $t \mapsto H_t(s)$ can behave very differently for different H in the above theorem. An example of such an H is $H_t(s) = 1_{(0,t]}(s)$. In this case $t \mapsto H_t(s)$ is constant except at s where it has a jump of size one. But there are also examples of H for which $t \mapsto H_t(s)$ is continuous and nowhere differentiable (and hence of unbounded variation).

We now apply Theorem 4.1 on an example.

Example 4.2. Let $g, h \in C^1(\mathbb{R})$ be two strictly increasing functions such that $0 \leq g < h$ and $g(\infty) = \infty$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f > 0$. Define $K_t(s) = 1_{[g(t),h(t)]}(s) f(s)$ and let $(W_t)_{t\geq 0}$ be a Wiener process. We show that $(X_t)_{t\geq 0}$ given by

$$X_t = \int K_t(s)\, dW_s = \int_{g(t)}^{h(t)} f(s)\, dW_s, \qquad t \geq 0,$$

is not a semimartingale.

Choose $(a,b) \subseteq \mathbb{R}_+$ such that $h(0) \leq g(x) \leq h(a)$ for $x \in (a,b)$ and let $u,t \in (a,b)$ with $u \leq t$ be given. Moreover choose $c,d \geq 0$ satisfying $c \leq d \leq u$, $h(c) =$

$g(u)$ and $h(d) = g(t)$ and define $\psi := K_d - K_c = (1_{[g(u),g(t)]} - 1_{[g(c),g(d)]})f$. Let $\mathfrak{P}_u$ respectively $\mathfrak{P}_\psi$ denote the projection on $\overline{\text{sp}}\{K_v : v \in [0,u]\}$ respectively $\overline{\text{sp}}\{\psi\}$, where the closure is in $L^2(\lambda)$ (λ denotes the Lebesgue measure). We have that

$$\|\mathfrak{P}_u K_t - K_u\|_{L^2(\lambda)} = \|\mathfrak{P}_u f 1_{[g(u),g(t)]}\|_{L^2(\lambda)} \geq \|\mathfrak{P}_\psi f 1_{[g(u),g(t)]}\|_{L^2(\lambda)},$$

and by choosing $K_1, K_2 \in (0,\infty)$ such that $K_1 \leq f^2(s) \leq K_2$ for $s \in [0, g(t)]$, we get

$$|\mathfrak{P}_\psi f 1_{[g(u),g(t)]}| = \Big|\frac{\langle \psi, f 1_{[g(u),g(t)]}\rangle}{\langle \psi, \psi\rangle}\psi\Big| \geq \frac{K_1(g(t)-g(u))}{K_2(g(t)-g(u)+g(d)-g(c))}|\psi|.$$

Thus, by setting $\varphi = g \circ h^{-1} \circ g$, it follows that

$$\begin{aligned}
\|\mathfrak{P}_u K_t - K_u\|_{L^2(\lambda)} &\geq K_1 K_2^{-1} \frac{g(t)-g(u)}{g(t)-g(u)+g(d)-g(c)} \|\psi\|_{L^2(\lambda)} \\
&\geq K_1^{3/2} K_2^{-1} \frac{g(t)-g(u)}{g(t)-g(u)+g(d)-g(c)} \sqrt{g(t)-g(u)+g(d)-g(c)} \\
&= K_1^{3/2} K_2^{-1} \frac{g(t)-g(u)}{\sqrt{g(t)-g(u)+\varphi(t)-\varphi(u)}} \geq K\sqrt{t-u},
\end{aligned}$$

for some $K > 0$. Hence we conclude, by Theorem 4.1, that $(X_t)_{t\geq 0}$ is not a semimartingale. $\Diamond$

Let $(W_t)_{t\in\mathbb{R}}$ be a given Wiener process and k and f be measurable functions satisfying $k(t-\cdot) - f(-\cdot) \in L^2(\lambda)$ for $t \in \mathbb{R}$ (λ denotes the Lebesgue measure on $\mathbb{R}$). Then $(X_t)_{t\in\mathbb{R}}$ is said to be a (W_t)-moving average process with parameter (k, f) if

$$X_t = \int_{\mathbb{R}} k(t-s) - f(-s)\, dW_s, \qquad t \in \mathbb{R}.$$

For short we say $(X_t)_{t\in\mathbb{R}}$ is a (W_t)-moving average process. Note that we do not assume k and f are 0 on $(-\infty, 0)$. It is readily seen that all (W_t)-moving average processes have stationary increments. By Doob (1990, page 533) it follows that an $L^2(P)$-continuous, stationary and centered Gaussian process has absolutely continuous spectral measure if and only if it is a (W_t)-moving average process with parameter $(k, 0)$, for some Wiener process $(W_t)_{t\in\mathbb{R}}$ and function k.

Recall the definition of the filtration $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ on page 7.

Lemma 4.3. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(X_t)_{t\in\mathbb{R}}$ be a (W_t)-moving average. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale and either the martingale component or the bounded variation component of $(X_t)_{t\geq 0}$ is a (W_t)-moving average process, then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.*

Proof. Let $(X_t)_{t\in\mathbb{R}}$ be a process given by $X_t = \int k(t-s) - f(-s)\, dW_s$ and assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale where the martingale or the bounded variation component is a (W_t)-moving average. In either case the martingale component of $(X_t)_{t\geq 0}$ is given by $M_t = \int h(t-s) - h(-s)\, dW_s$ for $t \geq 0$ for some measurable function h. For $t, v \in \mathbb{R}_+$ we have

$$\begin{aligned}
E[M_t X_{-v}] = E[M_t(X_{-v} - X_0)] &= \int \big(h(t-s) - h(-s)\big)\big(k(-v-s) - k(-s)\big)\, ds \\
&= \int \big(h(t+v-s) - h(v-s)\big)\big(k(-s) - k(v-s)\big)\, ds \\
&= E[(M_{t+v} - M_v)(X_0 - X_v)] = 0,
\end{aligned}$$

and it follows from Gaussianity that $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$. This shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t \vee \mathcal{G})_{t\geq 0}$-semimartingale, where $\mathcal{G} := \sigma(X_s : s \in (-\infty, 0))$, and hence in particular an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. □

Theorem 4.4. *Let $(X_t)_{t\in\mathbb{R}}$ be a (W_t)-moving average process with parameters $(k, 0)$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if*

$$k(t) = h(t) + \int_0^t \psi(r)\, dr, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{4.5}$$

where h and ψ are measurable functions satisfying $h(t-\cdot) - h(-\cdot) \in L^2(\lambda)$ for $t \geq 0$,

$$\int \big(h(t-s) - h(u-s)\big) k(v-s)\, ds = 0, \qquad 0 \leq v \leq u \leq t, \tag{4.6}$$

and

$$\psi(t-\cdot) \in \overline{\mathrm{sp}}\{k(v-\cdot) : v \in (-\infty, t]\} \subseteq L^2(\lambda), \qquad 0 \leq t.$$

The above k and h are uniquely determined and the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by

$$X_t = X_0 + \int h(t-s) - h(-s)\, dW_s + \int_0^t \Big(\int \psi(r-s)\, dW_s\Big) dr, \tag{4.7}$$

and the martingale and the bounded variation component of $(X_t)_{t\geq 0}$ are (W_t)-moving average processes.

For each function $g\colon \mathbb{R} \to \mathbb{R}$ and $u \in \mathbb{R}$, we let $\theta_u g$ denote the function $s \mapsto g(s-u)$.

Proof of Theorem 4.4. Let $K_t(s) := k(t-s)$ for $t, s \in \mathbb{R}$.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. By the stationary increments, $(X_t)_{t\geq 0}$ has no fixed points of discontinuity. Moreover since $(X_t)_{t\geq 0}$ is a Gaussian semimartingale it follows from Stricker (1983, Proposition 3) that $(X_t)_{t\geq 0}$ is a continuous process. Let $X_t = X_0 + M_t + A_t$ be the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$. For $u \in \mathbb{R}_+$, let $\mathfrak{P}_u\colon L^2(\lambda) \to L^2(\lambda)$ denote the projection on $\overline{\mathrm{sp}}\{K_v : v \in (-\infty, u]\}$ and note that $\mathfrak{P}_{v+u} K_{t+u} = \theta_u \mathfrak{P}_v K_t$ for $v \leq t$ and $0 \leq u$. Standard theory shows that for $t \geq 0$ we have

$$\begin{aligned}
A_t &= \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,\infty}_{(i-1)/2^n}] \\
&= \lim_{n\to\infty} \int \sum_{i=1}^{[t2^n]} \big(\mathfrak{P}_{(i-1)/2^n} K_{i/2^n}(s) - K_{(i-1)/2^n}(s)\big)\, dW_s \qquad \text{in } L^2(P),
\end{aligned}$$

where the second equality follows from Lemma 3.4 (iii). Thus with

$$G_t := \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \big(\mathfrak{P}_{(i-1)/2^n} K_{i/2^n} - K_{(i-1)/2^n}\big) \qquad \text{in } L^2(\lambda),$$

we have $A_t = \int G_t(s)\, dW_s$. For $t, u \in \mathbb{R}_+$ it follows that

$$\begin{aligned} G_{t+u} - G_u &= \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[(t+u)2^n]} \mathfrak{P}_{(i-1)/2^n}\big(K_{i/2^n} - K_{(i-1)/2^n}\big) \\ &= \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \mathfrak{P}_{(i-1)/2^n+u}\big(K_{i/2^n+u} - K_{(i-1)/2^n+u}\big) \\ &= \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} \theta_u \mathfrak{P}_{(i-1)/2^n}\big(K_{i/2^n} - K_{(i-1)/2^n}\big) = \theta_u G_t \qquad \text{in } L^2(\lambda). \end{aligned} \tag{4.8}$$

Which shows $(A_t)_{t\geq 0}$ has stationary increments and therefore μ_A equals the Lebesgue measure up to a scaling constant. Arguments as in the prove of '(i) $\Rightarrow$ (ii)' in Theorem 4.1 shows that

$$A_t = \int \Big(\int_0^t \Psi_r(s)\, dr\Big) dW_s, \qquad t \geq 0,$$

for some measurable mapping $(t,s) \mapsto \Psi_t(s)$ satisfying that $t \mapsto \|\Psi_t\|_{L^2(\lambda)}$ is constant and $\Psi_t \in \overline{\text{sp}}\{K_u : u \in (-\infty, t]\}$ for $t \geq 0$. Hence $G_t(s) = \int_0^t \Psi_r(s)\, dr$ for λ-a.a. $s \in \mathbb{R}$ for $t \geq 0$. For $t, u \in \mathbb{R}_+$, (4.8) yields

$$\int_0^t \Psi_{r+u}(s)\, dr = \int_u^{t+u} \Psi_r(s)\, dr = \theta_u \int_0^t \Psi_r(s)\, dr = \int_0^t \theta_u \Psi_r(s)\, dr,$$

for λ-a.a. $s \in \mathbb{R}$, which implies that $\Psi_{r+u} = \theta_u \Psi_r$ λ-a.s. Thus there exists a $\psi \in L^2(\lambda)$ such that for $r \geq 0$, $\Psi_r(s) = \psi(r-s)$ for λ-a.a. $s \in \mathbb{R}$. By setting $h(t) = k(t) - \int_0^t \psi(r)\, dr$ for $t \in \mathbb{R}$, it follows that $h(t-\cdot) - h(-\cdot) \in L^2(\lambda)$ and $M_t = \int h(t-s) - h(-s)\, dW_s$ for $t \geq 0$. The $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ shows that h satisfies (4.6). This completes the proof of the *only if* statement.

Assume conversely k is on the form (4.5). By approximating k with continuous functions with compact support it is readily seen that

$$\lim_{t\to 0} \int \big(k(t-s) - k(-s)\big)^2\, ds = 0. \tag{4.9}$$

Since $(X_t)_{t\geq 0}$ is a stationary process, (4.9) shows that it is $L^2(P)$-continuous. For $t \geq 0$ define

$$M_t := \int h(t-s) - h(-s)\, dW_s \qquad \text{and} \qquad A_t := \int_0^t \Big(\int \psi(r-s)\, dW_s\Big) dr.$$

By Lemma 3.4 (ii) we have that

$$A_t = \int \Big(\int_0^t \psi(r-s)\, dr\Big) dW_s, \qquad t \geq 0,$$

which shows $X_t = X_0 + M_t + A_t$ for $t \geq 0$. Since $\psi(r-\cdot) \in \overline{\text{sp}}\{K_v : v \in (-\infty, r]\}$ for $r \geq 0$ it follows that $\int \psi(r-s)\,dW_s$ is $\mathcal{F}_r^{X,\infty}$-measurable for $r \geq 0$ and therefore $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-adapted and hence by continuity $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-predictable.

Equation (4.6) and the translation invariancy of the Lebesgue measure shows

$$\int \big(h(t-s) - h(u-s)\big)k(v-s)\,ds = 0, \qquad -\infty < v \leq u \leq t.$$

This yields $E[(M_t - M_u)X_v] = 0$ for $-\infty < v \leq u \leq t$ where $0 \leq u$ and it follows by Gaussianity that $M_t - M_u$ is independent of X_v. Since $M_t = X_t - X_0 - A_t$, $(M_t)_{t\geq 0}$ is continuous in $L^2(P)$. Moreover since $(M_t)_{t\geq 0}$ is a centered process we conclude that $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale. Since $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ satisfies the usual conditions, $(M_t)_{t\geq 0}$ has a càdlàg modification. Hence $(X_t)_{t\geq 0}$ is an semimartingale with canonical decomposition given by (4.7).

We finally show that h and k are uniquely determined. Thus assume (4.5) is satisfied for k, h and $\tilde{k}, \tilde{h}$. By the uniqueness of the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-decomposition of $(X_t)_{t\geq 0}$ is follows from (4.7) and Lemma 3.4 (ii) that

$$\int_0^t \psi(r-s)\,dr = \int_0^t \tilde{\psi}(r-s)\,dr, \qquad \lambda\text{-a.a. } s \in \mathbb{R},\ \text{all } t \geq 0,$$

which shows $\psi(r-s) = \tilde{\psi}(r-s)$ for λ-a.a. $r \geq 0$ and λ-a.a. $s \in \mathbb{R}$ and hence $\psi = \tilde{\psi}$ λ-a.s. Hereby it follows from (4.5) that $h = \tilde{h}$ λ-a.s. and the proof is complete. $\square$

As a consequence of Lemma 4.3 and Theorem 4.4 we have the following.

Corollary 4.5. *Let $(X_t)_{t\in\mathbb{R}}$ be a (W_t)-moving average process with parameter $(k, 0)$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if there exists a filtration in which $(X_t)_{t\geq 0}$ is a semimartingale with a martingale component which is a (W_t)-moving average process.*

For a (W_t)-moving average process on the form

$$X_t = \int_{-\infty}^t k(t-s)\,dW_s, \qquad t \in \mathbb{R},$$

Knight (1992, Theorem 6.5) proved that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if $k(t) = \alpha + \int_0^t g(s)\,ds$ for λ-a.a. $t \geq 0$, where $\alpha \in \mathbb{R}$ and $g \in L^2(\lambda)$. After proving this result he wrote "an interesting project for further research might be to test the present methods in the non-stationary Gaussian case". The following result generalizes his theorem to the non-stationary Gaussian case, but uses a different approach.

Let $(C_t)_{t\geq 0}$ be a family of increasing $\sigma(\mathcal{A})$-measurable sets satisfying

$$\bigcap_{u\in(t,\infty)} C_u = C_t, \qquad t \geq 0.$$

Let $(\mathcal{F}_t^N)_{t\geq 0}$ be the smallest filtration satisfying $N(A)$ is $\mathcal{F}_t^N$-measurable for $A \in \mathcal{A}$ with $A \subseteq C_t$, and let $(X_t)_{t\geq 0}$ be given by $X_t = \int_{C_t} K_t(s)\,dN_s$ for $t \geq 0$.

Theorem 4.6. *Let $(X_t)_{t\geq 0}$ and $(\mathcal{F}_t^N)_{t\geq 0}$ be given as above. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale if and only if for $t \geq 0$ we have*

$$K_t(s) = g(s) + \int_0^t \Psi_r(s)\,\mu(dr), \qquad \nu\text{-a.a. } s \in C_t, \tag{4.10}$$

where $g\colon S \to \mathbb{R}$ is square integrable w.r.t. ν on C_t for $t \geq 0$, μ is a Radon measure on $\mathbb{R}_+$ and $\mathbb{R}_+ \times S \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ is a measurable mapping satisfying $\|\Psi_r\|_{L^2(\nu)} = 1$ and $\Psi_r(s) = 0$ for ν-a.a. $s \notin \bigcup_{u<r} C_u$.

The decomposition (4.10) is unique and if K is represented as in (4.10), then the $(\mathcal{F}_t^N)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by

$$X_t = X_0 + \int_{C_t\setminus C_0} g(s)\,dN_s + \int_0^t \Big(\int \Psi_r(s)\,dN_s\Big)\mu(dr). \tag{4.11}$$

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale with $(\mathcal{F}_t^N)_{t\geq 0}$-canonical decomposition $X_t = X_0 + M_t + A_t$. From Stricker (1983, Proposition 4 and 5) it follows that $(M_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{X_t : t \geq 0\}$. Thus for each $t \geq 0$ there exists an $H_t \in L^2(\nu)$ such that $M_t = \int_{C_t} H_t(s)\,dN_s$. Let $0 \leq u \leq t$ be given. The $(\mathcal{F}_t^N)_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ implies

$$\begin{aligned}0 &= E[\big(E[M_t - M_u|\mathcal{F}_u^N]\big)^2] = E[\big(\int_{C_u} H_t(s) - H_u(s)\,dN_s\big)^2] \\ &= \int_{C_u} \big(H_t(s) - H_u(s)\big)^2\,\nu(ds),\end{aligned}$$

which shows $H_t(s) = H_u(s)$ for ν-a.a. $s \in C_u$. Thus there exists a measurable function $g\colon S \to \mathbb{R}$ which equals H_t ν-a.s. on C_t for $t \geq 0$. By Lemma 3.4 (iv) there exists a Radon measure μ and an $(\mathcal{F}_t^N)_{t\geq 0}$-predictable process $(Y_t)_{t\geq 0} \subseteq \overline{\mathrm{sp}}\{A_t : t \geq 0\}$ satisfying $\|Y_r\|_{L^2(P)} = 1$ for $r \geq 0$ and

$$A_t = \int_0^t Y_r\,\mu_A(dr), \qquad t \geq 0.$$

In particular Y_r is $\mathcal{F}_{r-}^N$ measurable for $r \geq 0$. Thus by Lemma 3.4 (i) there exists a measurable mapping $(r,s) \mapsto \Psi_r(s)$ satisfying $\Psi_r(s) = 0$ for ν-a.a. $s \notin \bigcup_{u<r} C_u$ and $Y_r = \int \Psi_r(s)\,dN_s$. From Lemma 3.4 (ii) it follows that

$$X_t = \int_{C_t} \Big(g(s) + K_0(s)\Big)\,dN_s + \int \Big(\int_0^t \Psi_r(s)\,\mu(dr)\Big)dN_s,$$

which shows (4.10).

Assume conversely (4.10) is satisfied. We show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale with canonical decomposition given by (4.11). From Lemma 3.4 (ii) it follows that

$$\begin{aligned}X_t &= X_0 + \int_{C_t\setminus C_0} g(s)\,dN_s + \int \Big(\int_0^t \Psi_r(s)\,\mu(dr)\Big)dN_s \\ &= X_0 + \int_{C_t\setminus C_0} g(s)\,dN_s + \int_0^t \Big(\int \Psi_r(s)\,dN_s\Big)\mu(dr).\end{aligned}$$

And since $(\int_{C_t \setminus C_0} g(s)\, dN_s)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-martingale it suffices to show that $\int_0^t \big(\int \Psi_r(s)\, dN_s\big)\mu(dr)$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-predictable process. But this follows from Lemma 3.3 since $\int \Psi_r(s)\, dN_s$ is $\mathcal{F}_{r-}^N$-measurable for $r \geq 0$.

To conclude the proof assume that K is decomposed as in (4.10). By uniqueness of the martingale component of $(X_t)_{t\geq 0}$ it follows that g is determined uniquely ν-a.s. on $\bigcup_{t\geq 0} C_t$. Using once more that $\|\Psi_r\|_{L^2(\nu)} = 1$ for $r \geq 0$, we have that $\mu = (2/\pi)^{1/2}\mu_A$ where $(A_t)_{t\geq 0}$ is the bounded variation component of $(X_t)_{t\geq 0}$, and hence μ is uniquely determined and it follows from (4.10) that Ψ is uniquely determined up to $\mu \otimes \nu$-null sets. This completes the proof. □

Let the setting be as in Theorem 4.6 and assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^N)_{t\geq 0}$-semimartingale. Then Theorem 4.6 in particular shows that $K_t = \tilde{K}_t$ ν-a.s. on C_t, where $(t,s) \mapsto \tilde{K}_t(s)$ is a measurable mapping satisfying that $t \mapsto \tilde{K}_t(s)$ is right-continuous and of bounded variation for $s \in S$.

If $(X_t)_{t\in\mathbb{R}}$ is given by

$$X_t = \int_{-\infty}^t k(t-s)\, dW_s, \qquad t \in \mathbb{R},$$

then $(X_t)_{t\geq 0}$ satisfies the following relations

$$\begin{aligned}(\mathcal{F}_t^{W,\infty})_{t\geq 0}\text{-semimartingale} &\Rightarrow (\mathcal{F}_t^{X,\infty})_{t\geq 0}\text{-semimartingale}\\ &\Rightarrow (\mathcal{F}_t^{X})_{t\geq 0}\text{-semimartingale.}\end{aligned}$$

Hence the assumptions on $(X_t)_{t\geq 0}$ are strongest in Theorem 4.6, weaker in Theorem 4.4 and weakest in Theorem 4.1.

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