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Gaussian Moving Averages and Semimartingales

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Abstract

In the present paper we study moving average processes $X_t = \int \varphi(t - s) - \psi(-s)\, dW_s$, where φ and ψ are deterministic functions and $(W_t)_{t\in\mathbb{R}}$ is a Wiener process. Necessary and sufficient condition on (φ, ψ) are provided for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, where $\mathcal{F}_t^{X,\infty} := \sigma(X_s : s \in (-\infty, t])$. Our results are constructive - meaning that they provide a simple method to obtain φ and ψ for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale or an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-Wiener process. Several examples are considered.

In the last part of the paper we study general Gaussian processes with stationary increments, $(X_t)_{t\in\mathbb{R}}$. We provide necessary and sufficient conditions on spectral measure for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Keywords: semimartingales; Gaussian processes; stationary processes; moving average processes

AMS Subject Classification: 60G15; 60G10; 60G48; 60G57

1 Introduction

In this paper we study moving average processes, that is processes $(X_t)_{t\in\mathbb{R}}$ on the form

$$X_t = \int \varphi(t - s) - \psi(-s)\, dW_s, \qquad t \in \mathbb{R}, \tag{1.1}$$

where φ and ψ are two deterministic function satisfying that $s \mapsto \varphi(t-s) - \psi(-s)$ is square integrable and $(W_t)_{t\in\mathbb{R}}$ is a Wiener process. We are concerned with the semimartingale property of $(X_t)_{t\geq 0}$ in the filtration $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$. The class of moving average processes includes all centered $L^2(P)$-continuous stationary Gaussian process with absolutely continuous spectral measure (see Doob (1990, Page 533)), the fractional Brownian motion and many Gaussian processes with stationary increments. It readily seen that all moving average processes are Gaussian processes with stationary increments.

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In the case where $\psi = 0$ and φ is 0 on $(-\infty, 0)$, Knight (1992, Theorem 6.5) proved that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$\varphi(t) = \alpha + \int_0^t h(s)\, ds, \qquad t \in \mathbb{R}_+,$$

for some $\alpha \in \mathbb{R}$ and a square integrable function h. This was reproved in Jeulin and Yor (1993) with a shorter proof. Moreover, their proof extends to processes where ψ is non-zero. In the case $\psi = 0$, Jeulin and Yor (1993) provided conditions on the Fourier transform of φ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Let $(X_t)_{t\geq 0}$ be given by (1.1) and note that it is easier for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale than an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale and harder than being an $(\mathcal{F}_t^{X})_{t\geq 0}$-semimartingale ($\mathcal{F}_t^X = \sigma(X_s : s \in [0,t])$). Assume ψ equals φ or 0 and that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X})_{t\geq 0}$-semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$. It follows from Basse (2007) that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if μ_A is absolutely continuous with a bounded density, where μ_A denotes the measure induced by the mapping $t \mapsto E[\mathrm{Var}_{[0,t]}(A)]$ and $\mathrm{Var}_{[0,t]}(A)$ denotes the total variation of $s \mapsto A_s$ on $[0,t]$.

Let S^1 denote the unit circle in the complex plane $\mathbb{C}$. For each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$, define $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t) := \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds,$$

where the limit is in λ-measure. We prove that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if φ can be decomposed as

$$\varphi(t) = \beta + \alpha\tilde{f}(t) + \int_0^t \widehat{f\hat{h}}(s)\, ds, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{R}$, $h \in L^2(\lambda)$ and $f\colon \mathbb{R} \to S^1$ is as above, and if $\alpha \neq 0$ then h and $\widehat{f(\widehat{\varphi - \psi})}$ are 0 on $\mathbb{R}_+$. In this case we can chose α, β, h and f such that the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by $X_t = X_0 + M_t + A_t$, where

$$M_t = \alpha \int \tilde{f}(t-s) - \tilde{f}(-s)\, dW_s \;\text{ and }\; A_t = \int_0^t \Big(\int \widehat{f\hat{h}}(s-u)\, dW_u \Big) ds.$$

As a special case we have that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale if and only if φ can be represented as in (1.2) with $h = 0$. Thus, to obtain examples of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-Wiener processes we have to calculate $\tilde{f}$ for some $f\colon \mathbb{R} \to S^1$ and examples of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales are obtained by also calculating $\widehat{f\hat{h}}$ and defining φ by (1.2). Examples are provided in Example 3.5.

In the last part of the paper we are concerned with the spectral measure of $(X_t)_{t\in\mathbb{R}}$, where $(X_t)_{t\in\mathbb{R}}$ is either a stationary Gaussian semimartingales or Gaussian semimartingales with stationary increments and $X_0 = 0$. In both cases we provide necessary and sufficient conditions on the spectral measure of $(X_t)_{t\in\mathbb{R}}$, for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

2 Notation and Hardy functions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras satisfying the usual conditions of right-continuity and completeness. For a stochastic process $(X_t)_{t\in\mathbb{R}}$ let $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ denote the least filtration subject to X_s is $\mathcal{F}_t^{X,\infty}$-measurable for $t \geq 0$ and $s \in (-\infty, t]$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if there exists a decomposition of $(X_t)_{t\geq 0}$ such that

$$X_t = X_0 + M_t + A_t,$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale which starts at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation which starts at 0.

A process $(W_t)_{t\in\mathbb{R}}$ is said to be a Wiener process if for all $n \geq 1$ and $t_0 < \cdots < t_n$

$$W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are independent, for $-\infty < s < t < \infty$ $W_t - W_s$ follows a centered Gaussian distributed with variance $\sigma^2(t-s)$ for some $\sigma^2 > 0$, and $W_0 = 0$. If $\sigma^2 = 1$ we say that $(W_t)_{t\in\mathbb{R}}$ is a standard Wiener process.

Let $f\colon \mathbb{R} \to \mathbb{R}$. Then (unless explicitly stated otherwise) all integrability matters of f are with respect to the Lebesgue measure λ on $\mathbb{R}$. For $t \in \mathbb{R}$, let $\tau_t f$ denote the function $s \mapsto f(t-s)$. We have:

Remark 2.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally square integrable function satisfying $\tau_t f - \tau_0 f \in L^2(\lambda)$ for $t \in \mathbb{R}$. Then $\mathbb{R} \ni t \mapsto \tau_t f - \tau_0 f \in L^2(\lambda)$ is continuous.

This follows by arguments as in Cheridito (2004, Lemma 3.4). Note that if f is square integrable then the result can be proved by approximation with continuous functions with compact support.

We now give a short survey of Fourier theory and Hardy functions. For a comprehensive survey see Dym and McKean (1976). The Hardy functions will become an important tool in the construction of the canonical decomposition of a moving average process. For square integrable functions $f, g\colon \mathbb{R} \to \mathbb{C}$ define their inner product as $\langle f, g\rangle := \int f\overline{g}\, d\lambda$, where $\overline{z}$ denotes the complex conjugate of the complex number z. For $f \in L^2(\lambda)$ define the Fourier transform of f as

$$\hat{f}(t) := \lim_{a\downarrow -\infty,\ b\uparrow\infty} \int_a^b f(x)e^{ixt}\, dx,$$

where the limit is in $L^2(\lambda)$. The Pancherel identity, shows that for all $f, g \in L^2(\lambda)$ we have $\langle \hat{f}, \hat{g}\rangle_{L^2(\lambda)} = 2\pi\langle f, g\rangle_{L^2(\lambda)}$. Moreover, for $f \in L^2(\lambda)$ we have that $\hat{\hat{f}} = 2\pi f(-\cdot)$. Thus, the mapping $f \mapsto \hat{f}$ is (up to the factor $\sqrt{2\pi}$) a linear isometry from $L^2(\lambda)$ onto $L^2(\lambda)$.

Let $\mathbb{C}_+$ denote the open upper half plane of the complex plane $\mathbb{C}$, i.e. $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$. An analytic function $H\colon \mathbb{C}_+ \to \mathbb{C}$ is a Hardy function if

$$\sup_{b>0} \int |H(a+ib)|^2\, da < \infty.$$

Let $\mathbb{H}^2_+$ denote the space of all Hardy functions. It can be shown that a function $H\colon \mathbb{C}_+ \to \mathbb{C}$ is a Hardy function if and only if there exists a function $h\colon \mathbb{R} \to \mathbb{C}$ which is 0 at $(-\infty, 0)$, belongs to $L^2(\lambda)$ and satisfies

$$H(z) = \int e^{izt} h(t)\, dt, \qquad z \in \mathbb{C}_+. \tag{2.1}$$

In this case $\lim_{b\downarrow 0} H(a+ib) = \hat{h}(a)$ for λ-a.a. $a \in \mathbb{R}$ and in $L^2(\lambda)$.

Let $H \in \mathbb{H}^2_+$ with h given by (2.1). Then H is called an outer function if it is non-trivial and for all $a + ib \in \mathbb{C}_+$ we have

$$\log(|H(a+ib)|) = \frac{b}{\pi} \int \frac{\log(|\hat{h}(u)|)}{(u-a)^2 + b^2}\, du.$$

An analytic function $J\colon \mathbb{C}_+ \to \mathbb{C}$ is called an inner function if $|J| \leq 1$ on $\mathbb{C}_+$ and with $j(a) := \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$ we have $|j| = 1$ λ-a.s. For $H \in \mathbb{H}^2_+$ (with h given by (2.1)) it is possible to factor H as a product of an outer function H^o and an inner function J. If h is a real function J can be chosen such that $\overline{J(z)} = J(-\overline{z})$ for all $z \in \mathbb{C}_+$.

For measurable functions f and g satisfying $\int |f(t-s)g(s)|\, ds < \infty$ for $t \in \mathbb{R}$, we let $f * g$ denote the convolution between f and g, that is $f * g$ is the mapping

$$t \mapsto \int f(t-s)g(s)\, ds.$$

A locally square integrable function f is said to have orthogonal increments if $\tau_t f - \tau_0 f \in L^2(\lambda)$ for all $t \in \mathbb{R}$ and for all $-\infty < t_0 < t_1 < t_2 < \infty$ we have that $\tau_{t_2} f - \tau_{t_1} f$ is orthogonal to $\tau_{t_1} f - \tau_{t_0} f$ in $L^2(\lambda)$.

3 Main results

By S^1 we shall denote the unit circle in the complex field $\mathbb{C}$, i.e. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ we define $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t) := \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds,$$

where the limit is in λ-measure. The limit exists since for $a \geq 1$ we have

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds = \int_{-1}^{1} \frac{e^{its} - 1}{is} f(s)\, ds + \int_{-a}^{a} e^{its} 1_{[-1,1]^c}(s) f(s)(is)^{-1}\, ds,$$

and the last term converges in $L^2(\lambda)$ to the Fourier transform of

$$s \mapsto 1_{[-1,1]^c}(s) f(s)(is)^{-1}.$$

Moreover, $\tilde{f}$ takes real values since $\overline{f} = f(-\cdot)$. Observe that for $u \leq t$, we have

$$\tilde{f}(t+\cdot) - \tilde{f}(u+\cdot) = \widehat{1_{[u,t]} f}, \qquad \lambda\text{-a.s.} \tag{3.1}$$

The function $\tilde{f}$ has orthogonal increments. To see this let $t_0 < t_1 < t_2 < t_3$ be given. By (3.1) we have

$$\begin{aligned}&\langle \tilde{f}(t_3-\cdot)-\tilde{f}(t_2-\cdot),\tilde{f}(t_1-\cdot)-\tilde{f}(t_0-\cdot)\rangle_{L^2(\lambda)}\\ &\qquad = 2\pi\langle \hat{1}_{[t_2,t_3]}f,\hat{1}_{[t_0,t_2]}f\rangle_{L^2(\lambda)} = \langle \hat{1}_{[t_2,t_3]},\hat{1}_{[t_0,t_2]}\rangle_{L^2(\lambda)} = \langle 1_{[t_2,t_3]},1_{[t_0,t_2]}\rangle_{L^2(\lambda)} = 0,\end{aligned}$$

which shows the result.

Let $(W_t)_{t\in\mathbb{R}}$ be a Wiener process and $\varphi,\psi\colon\mathbb{R}\to\mathbb{R}$ be measurable functions satisfying $\varphi(t-\cdot)+\psi(-\cdot)\in L^2(\lambda)$ for $t\in\mathbb{R}$. In the following we let $(X_t)_{t\in\mathbb{R}}$ be given by

$$X_t = \int \varphi(t-s)-\psi(-s)\,dW_s,\qquad t\in\mathbb{R}.$$

Now we are ready to characterize the class of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales.

Theorem 3.1. *$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if φ can be decomposed as*

$$\varphi(t) = \beta+\alpha\tilde{f}(t)+\int_0^t \widehat{f\hat{h}}(s)\,ds,\qquad \lambda\text{-a.a. } t\in\mathbb{R}, \tag{3.2}$$

where $\alpha,\beta\in\mathbb{R}$, $h\in L^2(\lambda)$, $f\colon\mathbb{R}\to S^1$ is a measurable function such that $\overline{f}=f(-\cdot)$, and if $\alpha\neq 0$ then h and $f\widehat{(\varphi-\psi)}$ are 0 on $\mathbb{R}_+$.

In this case we can chose α,β,h and f such that the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by $X_t=X_0+M_t+A_t$, where

$$M_t = \alpha\int \tilde{f}(t-s)-\tilde{f}(-s)\,dW_s \quad\text{and}\quad A_t=\int_0^t\Big(\int \widehat{f\hat{h}}(s-u)\,dW_u\Big)ds. \tag{3.3}$$

The proof is given is Section 5. The last term on the right-hand side of (3.2) is Lipschitz continuous of order 1/2 by Hölder's inequality.

Corollary 3.2. *$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale if and only if φ can be decomposed as in* (3.2) *with $h=0$.*

The corollary shows that the mapping $f\mapsto\tilde{f}$ (up to affine transformations) is onto the space of functions with orthogonal increments (recall the definition on page 4). Moreover, if $f,g\colon\mathbb{R}\to S^1$ are measurable functions satisfying $\overline{f}=f(-\cdot)$ and $\overline{g}=g(-\cdot)$ and $\tilde{f}=\tilde{g}$ λ-a.s. then (3.1) shows that for $u\leq t$ we have

$$\hat{1}_{[u,t]}f = \hat{1}_{[u,t]}g,\qquad \lambda\text{-a.s.}$$

which implies $f=g$ λ-a.s. Thus, we have shown:

Remark 3.3. The mapping $f\mapsto\tilde{f}$ is one to one and (up to affine transformations) onto the space of functions with orthogonal increments.

For each measurable function $f\colon\mathbb{R}\to S^1$ such that $\overline{f}=f(-\cdot)$ and for each $h\in L^2(\lambda)$ we have

$$\begin{aligned}\int_0^t \widehat{f\hat{h}}(s)\,ds &= \langle 1_{[0,t]},\widehat{f\hat{h}}\rangle_{L^2(\lambda)} = \langle \hat{1}_{[0,t]},(f\hat{h})(-\cdot)\rangle_{L^2(\lambda)} \\ &= \langle \hat{1}_{[0,t]}f,\hat{h}(-\cdot)\rangle = \langle \widehat{\hat{1}_{[0,t]}f},h\rangle_{L^2(\lambda)} = \int\big(\tilde{f}(t+s)-\tilde{f}(s)\big)h(s)\,ds,\end{aligned} \tag{3.4}$$

which gives an alternative way of writing the last term in (3.2).

In some cases it is of interest that $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. This situation is studied in the next result. We also study the case where $(X_t)_{t\geq 0}$ is a stationary process, which corresponds to $\psi = 0$.

Proposition 3.4. *We have*

(i) $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$*-semimartingale which is* $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$*-adapted if and only if* φ *is represented as* (3.2) *with* $f(a) = \lim_{b\downarrow 0} J(-a+ib)$ *for* λ*-a.a.* $a \in \mathbb{R}$*, for some inner function* J*. In this case* (3.2) *can be written as*

$$\varphi = \beta_1 + \alpha\tilde{f} + \tilde{f} * h_1, \qquad \lambda\text{-a.s.} \tag{3.5}$$

where $\beta_1 \in \mathbb{R}$ *and* $h_1 = h(-\cdot)$.

(ii) Assume $\psi = 0$*. Then* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$*-semimartingale if and only if*

$$\varphi(t) = \beta + \alpha\tilde{f}(t) + \int_0^t \widehat{f\hat{h}}(s)\, ds, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{3.6}$$

where $\alpha, \beta \in \mathbb{R}$, $h \in L^2(\lambda)$, $f\colon \mathbb{R} \to S^1$ *is a measurable function satisfying* $\overline{f} = f(-\cdot)$*, and if* $\alpha \neq 0$ *then* h *is 0 on* $\mathbb{R}_+$ *and* $t \mapsto \alpha + \int_0^t h(-s)\, ds$ *is square integrable on* $\mathbb{R}_+$.

According to Dym and McKean (1976, page 53), a function $J\colon \mathbb{C}_+ \to \mathbb{C}$ is an inner function if and only if J can be factorised as:

$$J(z) = Ce^{i\alpha z} \exp\Big(\frac{1}{\pi i}\int \frac{1+sz}{s-z}\, F(ds)\Big) \prod_{n\geq 1} \varepsilon_n \frac{z_n - z}{\overline{z}_n - z}, \tag{3.7}$$

where $C \in S^1$, $\alpha \geq 0$, $(z_n)_{n\geq 1} \subseteq \mathbb{C}_+$ satisfies $\sum_{n\geq 1} \Im(z_n)/(|z_n|^2+1) < \infty$ and $\varepsilon_n = z_n/\overline{z}_n$ or 1 according as $|z_n| \leq 1$ or not, and F is a nondecreasing bounded singular function. Thus, a measurable function $f\colon \mathbb{R} \to S^1$ with $\overline{f} = f(-\cdot)$ satisfies the condition in Proposition 3.4 (i) if and only if

$$f(a) = \lim_{b\downarrow 0} J(-a+b), \qquad \lambda\text{-a.a. } a \in \mathbb{R}, \tag{3.8}$$

for a function J given by (3.7). If $f\colon \mathbb{R} \to S^1$ is given by $f(t) = i\,\mathrm{sgn}(t)$ then obviously $\overline{f} = f(-\cdot)$ (sgn denotes the signum function defined by $\mathrm{sgn}(t) = -1_{(-\infty,0)}(t) + 1_{(0,\infty)}(t)$). Moreover, f does not satisfy the condition in Proposition 3.4 (i). This seen from Example 3.5 and Lemma 4.3.

The condition in Proposition 3.4 (i) is weaker than $(X_t)_{t\geq 0}$ being an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. The latter condition is satisfied if and only if φ can be represented as in (3.2) with f constant.

In the next example we illustrate the method to obtaining (φ, ψ) for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale or an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-Wiener process. The idea is simply to pick a function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ and calculate $\tilde{f}$. Moreover, if one wants $(X_t)_{t\geq 0}$ to be $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted one has to make sure that f is given by (3.8).

Example 3.5. *Let* $(X_t)_{t\in\mathbb{R}}$ *be given by*

$$X_t = \int \varphi(t-s) - \varphi(-s)\, dW_s, \qquad t \in \mathbb{R}.$$

(i) If φ *equals* $t \mapsto (e^{-t}-1/2)1_{\mathbb{R}_+}(t)$ *or* $t \mapsto \log(|t|)$ *then* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$*-Wiener process.*

(ii) If φ *equals*

$$t \mapsto \log(|t|) + \int_0^t \log\Big(\Big|\frac{s-1}{s+1}\frac{s^2-1}{s^2}\Big|\Big)\, ds,$$

then $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$*-semimartingale.*

We use Theorem 3.1 and Corollary 3.4 to show the above stated results. Notice that $\psi = \varphi$ and hence the condition that $f(\widehat{\varphi - \psi}) = 0$ on $\mathbb{R}_+$ is trivially satisfied.

To show the first part let f be the function $t \mapsto (t+i)(t-i)^{-1}$. For $t \geq 0$,

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds = 4\int_0^a \frac{\cos(ts) - 1_{[0,1]}(s)}{s^2+1}\, ds + 2\int_0^a \frac{\sin(ts)}{s}\frac{s^2-1}{s^2+1}\, ds$$
$$\to 4e^{-t} + \pi + 2(-1+2e^{-t}) = 8e^{-t} - 2 + \pi \qquad \text{as } a \to \infty,$$

and for $t \in (-\infty, 0)$,

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds \to 2 + \pi \qquad \text{as } a \to \infty,$$

which shows $\tilde{f}$ equals $t \mapsto (e^{-t} - 1/2)1_{\mathbb{R}_+}(t)$ up to an affine transformation.

Now let f be the function $t \mapsto i\,\mathrm{sgn}(t)$. For $t \geq 0$ and $a \geq 1$

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds = \int_{-a}^{a} \frac{\cos(ts) - 1_{[-1,1]}(s)}{is} f(s)\, ds$$
$$= 2\int_0^{at} \frac{\cos(s) - 1_{[0,t]}(s)}{is} f(s/t)\, ds = 2\Big(\int_0^{at} \frac{\cos(s) - 1_{[0,1]}(s)}{s}\, ds - \log(t)\Big).$$

and since $\tilde{f}(-t) = \tilde{f}(t)$ we conclude $\tilde{f}(t) = -2(\gamma + \log(|t|))$ for $t \in \mathbb{R}$, where γ denotes Euler's constant.

Let additionally $h(t) = 1_{[-1,0]}(t)$. Due to the fact that $\hat{h}(t) = \frac{1-\cos(t)}{it} + \frac{\sin(t)}{t}$ we obtain

$$\int_{-a}^{a} e^{its}\hat{h}(s) f(s)\, ds$$
$$= i\int_0^a \cos(ts)(\hat{h}(s) - \hat{h}(-s))\, ds - \int_0^a \sin(ts)(\hat{h}(s) + \hat{h}(-s))\, ds$$
$$= 2\Big(\int_0^a \cos(ts)\frac{1-\cos(s)}{s}\, ds - \int_0^a \sin(ts)\frac{\sin(s)}{s}\, ds\Big)$$
$$\to 2\Big(1/2\log\Big(\Big|\frac{t^2-1}{t^2}\Big|\Big) - 1/2\log\Big(\Big|\frac{t+1}{t-1}\Big|\Big)\Big)$$
$$= \log\Big(\Big|\frac{t-1}{t+1}\frac{t^2-1}{t^2}\Big|\Big) \qquad \text{as } a \to \infty,$$

which shows

$$\widehat{f\hat{h}}(t) = \log\Big(\Big|\frac{t-1}{t+1}\frac{t^2-1}{t^2}\Big|\Big), \qquad t \in \mathbb{R},$$

and by Theorem 3.1 completes the proof. ◊

As a consequence of Example 3.5 (i) we have the following: Let $(X_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck process given by

$$X_t = X_0 - \int_0^t X_s\, ds + W_t, \qquad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a Wiener process and $X_0 \overset{\mathcal{D}}{=} N(0,1/2)$ is independent of $(W_t)_{t\geq 0}$. Then $(B_t)_{t\geq 0}$, given by

$$B_t := W_t - 2\int_0^t X_s\, ds, \qquad t \geq 0,$$

is a Wiener process (in its own filtration). Representations of the Wiener process have been extensively studied by Lévy (1956), Cramér (1961), Hida (1961) and many others. One of the most famous examples of such a representation is

$$B_t = W_t - \int_0^t \frac{1}{s} W_s\, ds, \qquad t \geq 0.$$

Let $X_t = \int \varphi(t-s) - \varphi(-s)\, dW_s$ for $t \in \mathbb{R}$. Then φ has to be continuous on $[0,\infty)$ (in particular bounded on compacts of $\mathbb{R}$) for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingales. This is not the case for the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale property. Indeed, Example 3.5 shows that if $\varphi(t) = \log(|t|)$ then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale, but φ is unbounded on $[0,1]$.

4 Functions with orthogonal increments

In the following we collect some properties of functions with orthogonal increments.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with orthogonal increments. For $t \in \mathbb{R}$ we have

$$\begin{aligned} \|\tau_t f - \tau_0 f\|^2_{L^2(\lambda)} &= \|\tau_t f - \tau_{t/2} f\|^2_{L^2(\lambda)} + \|\tau_{t/2} f - \tau_0 f\|^2_{L^2(\lambda)} \\ &= 2\|\tau_{t/2} f - \tau_0 f\|^2_{L^2(\lambda)}. \end{aligned} \tag{4.1}$$

Moreover, since $t \mapsto \|\tau_t f - \tau_0 f\|^2_{L^2(\lambda)}$ is a continuous function by Remark 2.1, equation (4.1) shows that $\|\tau_t f - \tau_0 f\|^2_{L^2(\lambda)} = K|t|$, where $K := \|\tau_1 f - \tau_0 f\|^2_{L^2(\lambda)}$. This implies that $\|\tau_t f - \tau_u f\|^2_{L^2(\lambda)} = K|t-u|$ for $u,t \in \mathbb{R}$. For a step function $h = \sum_{j=1}^k a_j 1_{(t_{j-1},t_j]}$ define the mapping

$$\int h(u)\, d\tau_u f := \sum_{j=1}^k a_j(\tau_{t_j} f - \tau_{t_{j-1}} f).$$

Then $v \mapsto (\int h(u)\,d\tau_u f)(v)$ is square integrable and

$$\|h\|_{L^2(\lambda)} = \sqrt{K}\|\int h(u)\,d\tau_u f\|_{L^2(\lambda)}.$$

Hence, by standard arguments we can define $\int h(u)\,d\tau_u f$ through the above isometry for all $h \in L^2(\lambda)$ such that $L^2(\lambda) \ni h \mapsto \int h(u)\,d\tau_u f \in L^2(\lambda)$ is a linear isometry.

Assume in addition that $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a measurable function, and μ is a finite measure such that

$$\iint g(u,v)^2\,du\,\mu(dv) < \infty.$$

Then $(v,s) \mapsto (\int g(u,v)\,d\tau_u f)(s)$ can be chosen measurable and in this case we have

$$\int \Big(\int g(u,v)\,d\tau_u f\Big)\mu(dv) = \int \Big(\int g(u,v)\,\mu(dv)\Big) d\tau_u f. \tag{4.2}$$

Lemma 4.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by*

$$g(t) = \begin{cases} \alpha + \int_0^t h(v)\,dv & t \geq 0 \\ 0 & t < 0, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $h \in L^2(\lambda)$. Then, $g(t-\cdot) - g(-\cdot) \in L^2(\lambda)$ for all $t \in \mathbb{R}$.

Let f be a function with orthogonal increments.

(i) Let φ be a measurable function. Then there exists a constant $\beta \in \mathbb{R}$ such that

$$\varphi(t) = \beta + \alpha f(t) + \int_0^\infty \big(f(t-v) - f(-v)\big)h(v)\,dv, \quad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{4.3}$$

if and only if for all $t \in \mathbb{R}$ we have

$$\tau_t\varphi - \tau_0\varphi = \int g(t-u) - g(-u)\,d\tau_u f, \qquad \lambda\text{-a.s.} \tag{4.4}$$

(ii) Let g be a square integrable function. Then there exists a $\beta \in \mathbb{R}$ such that λ-a.s.

$$\int g(-u)\,d\tau_u f = \beta + \alpha f(-\cdot) + \int_0^\infty \big(f(-u-\cdot) - f(-u)\big)h(u)\,du. \tag{4.5}$$

Proof. To prove $g(t-\cdot)-g(-\cdot) \in L^2(\lambda)$ it is enough to show that $s \mapsto \int_{-s}^{t-s} h(u)\,du \in L^2(\lambda)$. But this follows since

$$\int \Big(\int_{-s}^{t-s} h(u)\,du\Big)^2 ds \leq t \int \int_{-s}^{t-s} h(u)^2\,du\,ds = t \int \int_0^t h(u-s)^2\,du\,ds$$
$$= t \int_0^t \int h(u-s)^2\,ds\,du = t^2 \int h(s)^2\,du < \infty.$$

(i): We may and do assume that h is 0 on $(-\infty,0)$. For $t,u\in\mathbb{R}$ we have that

$$g(t-u)-g(-u)=\begin{cases}\alpha 1_{(0,t]}(u)+\int_{-u}^{t-u}h(v)\,dv, & t\geq 0\\ -\alpha 1_{(t,0]}(u)-\int_{t-u}^{-u}h(v)\,dv. & t<0,\end{cases}$$

which by (4.2) implies that for $t\in\mathbb{R}$ we have λ-a.s.

$$\int g(t-u)-g(-u)\,d\tau_u f=\alpha(\tau_t f-\tau_0 f)+\int\big(\tau_{t-v}f-\tau_{-v}f\big)h(v)\,dv. \quad (4.6)$$

First assume (4.4) is satisfied. For $t\in\mathbb{R}$ it follows from (4.6) that

$$\tau_t\varphi-\tau_0\varphi=\alpha(\tau_t f-\tau_0 f)+\int[\tau_{t-v}f-\tau_{-v}f]h(v)\,dv, \qquad \lambda\text{-a.s.}$$

Hence, by Tonelli's Theorem there exists a sequence $(s_n)_{n\geq 1}$ such that $s_n\to 0$ and such that

$$\begin{aligned}\varphi(t-s_n)&=\varphi(-s_n)-\alpha f(s_n)+\alpha f(t-s_n) \quad (4.7)\\ &\quad+\int\big(f(t-v-s_n)-f(-v-s_n)\big)h(v)\,dv, \quad \forall n\geq 1,\ \lambda\text{-a.a. } t\in\mathbb{R}.\end{aligned}$$

From Remark 2.1 it follows that $\varphi(\cdot-s_n)-\varphi(\cdot)$ and $f(\cdot-s_n)-f(\cdot)$ converge to 0 in $L^2(\lambda)$ and

$$\int\big(f(t-v-s_n)-f(-v-s_n)\big)h(v)\,dv\to\int[f(t-v)-f(-v)]h(v)\,dv, \quad t\in\mathbb{R}.$$

Thus we obtain (4.5) by letting n tend to infinity in (4.7).

Assume conversely (4.3) is satisfied. For $t\in\mathbb{R}$ we have

$$\tau_t\varphi-\tau_0\varphi=\alpha(\tau_t f-\tau_0 f)+\int[\tau_{t-v}f-\tau_{-v}f]h(v)\,dv, \quad \lambda\text{-a.s.}$$

and hence we obtain (4.4) from (4.6).

(ii): Assume $g\in L^2(\lambda)$. By approximation we may assume g has compact support. Hence assume that g is on the form

$$g(t)=1_{[0,T]}(t)\Big(\int_0^T -h(s)\,ds+\int_0^t h(s)\,ds\Big)=-1_{[0,T]}(t)\int_t^T h(s)\,ds$$

for some $T>0$ and an $h\in L^2(\lambda)$. From (4.2) it follows that

$$\begin{aligned}\int g(-u)\,d\tau_u f&=\int\Big(\int -1_{(-u,T]}(s)1_{[0,T]}(-u)h(s)\,ds\Big)d\tau_u f\\ &=\int\Big(\int -1_{(-u,T]}(s)1_{[0,T]}(-u)h(s)\,d\tau_u f\Big)ds=\int_0^T -h(s)\Big(\int_{-s}^0 d\tau_u f\Big)ds\\ &=\int_0^T -h(s)\big(\tau_0 f-\tau_{-s}f\big)\,ds=g(0)\tau_0 f+\int_0^T h(s)\tau_{-s}f\,ds.\end{aligned}$$

Thus with $\beta:=\int_0^T h(s)f(-s)\,ds$ we have that

$$\int g(-u)\,d\tau_u f=\beta+g(0)f(-\cdot)+\int h(s)[f(-s-\cdot)-f(-s)]\,ds,$$

which completes the proof. □

Lemma 4.2. *Let $h \in L^2(\lambda)$, $f\colon \mathbb{R} \to \mathbb{R}$ be a function with orthogonal increments, $(W_t)_{t\in\mathbb{R}}$ be a Wiener process and let $(B_t)_{t\in\mathbb{R}}$ be given by*

$$B_t = \int f(t-s) - f(-s)\, dW_s = \int \tau_t f(s) - \tau_0 f(s)\, dW_s, \qquad t \in \mathbb{R}.$$

Then $(B_t)_{t\in\mathbb{R}}$ is a Wiener process and

$$\int h(s)\, dB_s = \int \Big(\int h(u)\, d\tau_u f \Big)(s)\, dW_s.$$

The proof is simple. For details see Basse (2007, Lemma 3.1 (ii)).

Lemma 4.3. *Let $f\colon \mathbb{R} \to S^1$ be a measurable function such that $\overline{f} = f(-\cdot)$. Then $\tilde{f}$ is constant on $(-\infty, 0)$ if and only if*

$$f(a) = \lim_{b\downarrow 0} J(-a+ib), \qquad \lambda\text{-a.a. } a \in \mathbb{R}, \tag{4.8}$$

for an inner function J.

Proof of Lemma 4.3. Assume $\tilde{f}$ is constant on $(-\infty,0)$ and let $t \geq 0$ be given. We have $\widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = 0$ for λ-a.a. $s \in (-\infty, 0)$ due to the fact that $\widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = \tilde{f}(s) - \tilde{f}(-t+s)$ for λ-a.a. $s \in \mathbb{R}$ and hence $\hat{1}_{[0,t]}\overline{f} \in \mathbb{H}^2_+$. Moreover, since $\hat{1}_{[0,t]}\overline{f}$ has outer part $\hat{1}_{[0,t]}$ we conclude that $\overline{f}(a) = \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$ and an inner function $J\colon \mathbb{C}_+ \to \mathbb{C}$.

Assume conversely (4.8) is satisfied and fix $t \geq 0$. Let $G \in \mathbb{H}^2_+$ be the Hardy function induced by $1_{[0,t]}$. Since J is an inner function, we obtain $GJ \in \mathbb{H}^2_+$ and thus

$$G(z)J(z) = \int e^{itz}\kappa(t)\, dt, \qquad z \in \mathbb{C}_+,$$

for some $\kappa \in L^2(\lambda)$ which is 0 on $(-\infty,0)$. The remark just below (2.1) shows

$$\widehat{1_{[0,t]}}(a)\overline{f}(a) = \lim_{b\downarrow 0} G(a+ib)J(a+ib) = \hat{\kappa}(a), \qquad \lambda\text{-a.a. } a \in \mathbb{R},$$

which implies

$$\tilde{f}(s) - \tilde{f}(-t+s) = \tilde{f}(s) - \tilde{f}(-t+s) = \widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = \hat{\hat{\kappa}}(-s) = 2\pi k(s),$$

for λ-a.a. $s \in \mathbb{R}$. Hence, we conclude that $\tilde{f}$ is constant on $(-\infty, 0)$ λ-a.s. □

We note that if f is the constant 1, then a simple calculation shows that $\tilde{f}(t) = \pi\,\mathrm{sgn}(t)$.

5 Proofs of main results

Let $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\,dW_s$ for $t \in \mathbb{R}$. Doob (1990, Chapter XII, Theorem 5.3) showed that if $(X_t)_{t\in\mathbb{R}}$ is a regular process then

$$X_t = \int_{-\infty}^{t} g(t-s)\,dB_s,\ t\in\mathbb{R} \qquad \text{and} \qquad (\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0},$$

for some Wiener process $(B_t)_{t\in\mathbb{R}}$ and some $g \in L^2(\lambda)$. However, we need the following explicit construction of $(B_t)_{t\in\mathbb{R}}$.

Lemma 5.1 (Main Lemma)**.** *Let $\varphi \in L^2(\lambda)$ and $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\,dW_s$ for $t \in \mathbb{R}$, where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process.*

(i) If

$$\int \frac{\log|\hat{\varphi}|(u)}{1+u^2}\,du > -\infty, \tag{5.1}$$

then there exist a measurable function $f\colon \mathbb{R}\to S^1$ with $\overline{f} = f(-\cdot)$, a function $g \in L^2(\lambda)$ which is 0 on $(-\infty,0)$ such that we have the following: First of all $(B_t)_{t\in\mathbb{R}}$ defined by

$$B_t = \int \tilde{f}(t-s) - \tilde{f}(-s)\,dW_s, \qquad t\in\mathbb{R},$$

is a Wiener process. Moreover,

$$X_t = \int_{-\infty}^{t} g(t-s)\,dB_s, \qquad t\in\mathbb{R},$$

and finally $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$.

(ii) If φ is 0 on $(-\infty,0)$, then φ satisfies (5.1) *and the above f is given by $f(a) = \lim_{b\downarrow 0} J(-a+b)$ for λ-a.a. $a\in\mathbb{R}$, where J is an inner function.*

Proof. (i): Due to the fact that $|\hat{\varphi}|^2$ is a positive integrable function which satisfies (5.1), Dym and McKean (1976, Chapter 2, Section 8, Exercise 4) shows there is an outer Hardy function $H^o \in \mathbb{H}^2_+$ such that $|\hat{\varphi}|^2 = |\hat{h}^0|^2$ and $\overline{\hat{h}^o} = \hat{h}^o(-\cdot)$, where h^0 is given by (2.1). Additionally, H^o is given by

$$H^o(z) = \exp\Big(\frac{1}{\pi i}\int \frac{uz+1}{u-z}\frac{\log|\hat{\varphi}|(u)}{u^2+1}\,du\Big), \qquad z\in\mathbb{C}_+.$$

Define $f\colon \mathbb{R}\to S^1$ by $\overline{f} = \hat{\varphi}/\hat{h}^o$ and note that $\overline{f} = f(-\cdot)$. According to Lemma 4.2, $(B_t)_{t\in\mathbb{R}}$ is a Wiener process due to the fact that $\tilde{f}$ has orthogonal increments.

We claim that

$$X_t = (2\pi)^{-1}\int h^o(t-s)\,dB_s = (2\pi)^{-1}\int \tau_t h^o(s)\,dB_s, \qquad t\in\mathbb{R}. \tag{5.2}$$

Fix $t \in \mathbb{R}$ and choose a sequence of step functions $(h_n)_{n\geq 1}$ such that h_n converges to $\tau_t h^o$ in $L^2(\lambda)$. Let $n \geq 1$ and assume $h_n = \sum_{i=1}^k a_i 1_{(t_{i-1},t_i]}$. We have

$$\int h_n(s)\,dB_s = \sum_{i=1}^k a_i(B_{t_i} - B_{t_{i-1}}) = \int \sum_{i=1}^k a_i\big(f(t_i - s) - f(t_{i-1} - s)\big)\,dW_s,$$

and

$$\begin{aligned}\sum_{i=1}^k a_i\big(f(t_i - s) - f(t_{i-1} - s)\big) &= \int_{\mathbb{R}} \sum_{i=1}^k a_i \frac{e^{it_i u} - e^{it_{i-1}u}}{iu} f(u)e^{-isu}\,du \\ &= \int_{\mathbb{R}} \hat{h}_n(u)f(u)e^{-isu}\,du = \widehat{\hat{h}_n f}(-s),\end{aligned}$$

by which we conclude

$$\int h_n(s)\,dB_s = \int \widehat{\hat{h}_n f}(-s)\,dW_s. \tag{5.3}$$

Since h_n converges to $\tau_t h^o$ in $L^2(\lambda)$, $\hat{h}_n f$ converges to $\widehat{\tau_t h^o} f$ in $L^2(\lambda)$. Moreover, for λ-a.a. $u \in \mathbb{R}$ we have

$$\widehat{\tau_t h^o}(u)f(u) = e^{itu}\hat{h^o}(-u)f(u) = e^{itu}\hat{\varphi}(-u) = \widehat{\tau_t\varphi}(u).$$

Thus, $\hat{h}_n f$ converges to $\widehat{\tau_t\varphi}$ in $L^2(\lambda)$ which implies $\widehat{\hat{h}_n f}$ converges to $\widehat{\widehat{\tau_t\varphi}} = 2\pi\varphi(t+\cdot)$ in $L^2(\lambda)$. In conclusion, $\widehat{\hat{h}_n f}(-\cdot)$ converges to $2\pi\tau_t\varphi$ in $L^2(\lambda)$. We obtain (5.2) by letting n tend to infinity in (5.3). Moreover since H^o is an outer function it follows from (5.2) and page 95 in Dym and McKean (1976) that $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$.

(ii): Assume $\varphi \in L^2(\lambda)$ is 0 on $(-\infty, 0)$. Equation (5.1) is satisfied according to Dym and McKean (1976, Section 2.6) since φ induces a Hardy function $H \in \mathbb{H}^2_+$ given by (2.1). Let h^o, f and $(B_t)_{t\in\mathbb{R}}$ be given as above (recall that $\overline{f} = f(-\cdot)$). It follows by Dym and McKean (1976, page 37) that $J := H/H^o$ is an inner function and the definition of J shows that $f(-a) = \lim_{b\downarrow 0} J(a + ib)$ for λ-a.a. $a \in \mathbb{R}$ which completes the proof. □

Lemma 5.2. *Let κ be a locally integrable function and let $\Delta_t\kappa$ denote the function*

$$s \mapsto t^{-1}(\kappa(t+s) - \kappa(s)), \qquad t > 0.$$

Then $(\Delta_t\kappa)_{t>0}$ is bounded in $L^2(\lambda)$ if and only if κ is absolutely continuous with square integrable density.

Proof. Assume $(\Delta_t\kappa)_{t>0}$ is bounded in $L^2(\lambda)$ and choose a sequence $(t_n)_{n\geq 1} \subseteq (0,\infty)$ converging to 0 such that $\Delta_{t_n}\kappa$ converges in the weak $L^2(\lambda)$-topology. Call the limit κ'. For $\lambda \otimes \lambda$-a.a. (u, v) with $u \leq v$ we have

$$\begin{aligned}\int_u^v \kappa'(s)\,ds &= \lim_{n\to\infty} \int_u^v \Delta_{t_n}\kappa(s)\,ds \\ &= \lim_{n\to\infty} t_n^{-1} \int_v^{v+t_n} \kappa(s)\,ds - \lim_{n\to\infty} t_n^{-1} \int_u^{u+t_n} \kappa(s)\,ds = \kappa(v) - \kappa(u),\end{aligned}$$

which shows that κ has density κ'.

Assume conversely that κ is absolutely continuous with density $\kappa' \in L^2(\lambda)$. For $0 < t$ we have

$$\int \big(\kappa(t+s) - \kappa(s)\big)^2 ds \leq t \iint_s^{s+t} \kappa'(v)^2 \, dv \, ds$$
$$= t \iint_0^t \kappa'(v+s)^2 \, ds \, dv \leq t^2 \int \kappa'(s)^2 \, ds,$$

by which we conclude $(\Delta_t \kappa)_{t>0}$ is bounded in $L^2(\lambda)$. □

Lemma 5.3. *Let $(W_t)_{t\in\mathbb{R}}$ be a Wiener process and Z be a random variable such that $\{Z, (W_t)_{t\in\mathbb{R}}\}$ is a Gaussian system. Define $(X_t)_{t\in\mathbb{R}}$ by*

$$X_t = \int_{-\infty}^t \varphi(t-s) \, dW_s, \qquad t \in \mathbb{R},$$

where $\varphi \in L^2(\lambda)$. Let $\mathcal{F}_t := \mathcal{F}_t^{W,\infty} \vee \sigma(Z)$ for $t \geq 0$. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale of unbounded variation then $(W_t)_{t\geq 0}$ is independent of Z.

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale and let $X_t = X_0 + M_t + A_t$ be the canonical $(\mathcal{F}_t)_{t\geq 0}$-decomposition of $(X_t)_{t\geq 0}$. By Basse (2007, Lemma 2.1 (iii) and (v)) there exists a family of square integrable functions $s \mapsto H_t(s)$ such that for all $t \geq 0$ we have $M_t = \int_{-\infty}^t H_t(s) \, dW_s$. The $(\mathcal{F}_t)_{t\geq 0}$-martingale property of $(M_t)_{t\geq 0}$ shows that

$$0 = E\big[\big(E[M_t | \mathcal{F}_0^{W,\infty}]\big)^2\big] = E\Big[\Big(\int_{-\infty}^0 H_t(s) \, dW_s\Big)^2\Big] = \int_{-\infty}^0 H_t(s)^2 \, ds, \qquad 0 \leq t,$$

and

$$0 = E\big[\big(E[M_t - M_u | \mathcal{F}_u^{W,\infty}]\big)^2\big] = \int_{-\infty}^u \left(H_t(s) - H_u(s)\right)^2 \, ds, \qquad 0 \leq u \leq t.$$

Thus, $H_t = \xi 1_{(0,t]}$ λ-a.s. for all $t \geq 0$ and some measurable function ξ. The stationary increments of $(X_t)_{t\in\mathbb{R}}$ implies that $[M]_t = [X]_t = kt$ for $t \geq 0$ and some $k \in \mathbb{R}_+$ and it follows that $E[M_t^2] = kt$ for $t \geq 0$. This yields $|\xi| = \sqrt{k}$ λ-a.s. and hence

$$\overline{\text{sp}}\{W_t : t \in [0,\infty)\} = \overline{\text{sp}}\{M_t : t \in [0,\infty)\}.$$

We have shown that $(W_t)_{t\geq 0}$ is independent of $\mathcal{F}_0$. In particular $(W_t)_{t\geq 0}$ is independent of Z. □

Remark 5.4. Let H denote a normed space and $\mathbb{R} \ni t \mapsto x(t) \in H$ be a continuous mapping satisfying $\|x_t - x_u\| = \|x_{t+v} - x_{u+v}\|$ for $t, v, u \in \mathbb{R}$. Then

$$\|x(t) - x(0)\| \leq \alpha + \beta|t|, \qquad t \in \mathbb{R},$$

for some $\alpha, \beta \in \mathbb{R}_+$.

The next lemma, which is inspired by Masani (1972), shows how one can transform a Gaussian process with stationary increments into a stationary Gaussian process.

Lemma 5.5. *Let $(X_t)_{t\in\mathbb{R}}$ be a continuous and centered Gaussian process with stationary increments. Then there exists a continuous, stationary and centered Gaussian process $(Y_t)_{t\in\mathbb{R}}$, satisfying*

$$X_t - X_0 = Y_t - Y_0 - \int_0^t Y_s \, ds, \qquad t \in \mathbb{R}. \tag{5.4}$$

Furthermore $\mathcal{F}_t^{X,\infty} = \sigma(X_0) \vee \mathcal{F}_t^{Y,\infty}$ for $t \geq 0$.

Assume in addition φ and ψ are measurable functions such that

$$\varphi(t-\cdot) + \psi(\cdot) \in L^2(\lambda) \text{ and } X_t = \int \varphi(t-s) + \psi(s) \, dW_s, \qquad t \in \mathbb{R}.$$

Then

$$\xi(t) := \int_{-\infty}^0 e^u \big(\varphi(t) - \varphi(u+t)\big) \, du, \qquad t \in \mathbb{R}, \tag{5.5}$$

is a well-defined square integrable function and $Y_t = \int \xi(t-s) \, dW_s$ for $t \in \mathbb{R}$.

Before proving the lemma we note that if

$$f(t) = \int_{-\infty}^0 e^s \big(g(t) - g(t+s)\big) \, ds = g(t) - e^{-t} \int_{-\infty}^t e^s g(s) \, ds, \qquad t \in \mathbb{R},$$

then

$$g(t) - g(0) = f(t) - f(0) - \int_0^t f(s) \, ds, \qquad t \in \mathbb{R}. \tag{5.6}$$

These identities are the main ingredients in the proof.

Proof of Lemma 5.5. Since $(X_t)_{t\in\mathbb{R}}$ is a Gaussian process with continuous sample paths, it is continuous in $L^2(P)$ as well. Due to the stationary increments of $(X_t)_{t\in\mathbb{R}}$ it hence follows from Remark 5.4 that there is a P-null set outside which

$$Y_t := \int_{-\infty}^0 e^u (X_t - X_{t+u}) \, du = X_t - e^{-t} \int_{-\infty}^t e^u X_u \, du, \qquad t \in \mathbb{R},$$

is well-defined and continuous. It is readily seen that $(Y_t)_{t\in\mathbb{R}}$ is a stationary centered Gaussian process. Moreover, (5.4) follows by (5.6). By definition of $(Y_t)_{t\in\mathbb{R}}$ we have $\mathcal{F}_t^{Y,\infty} \vee \sigma(X_0) \subseteq \mathcal{F}_t^{X,\infty}$ for $t \geq 0$ and (5.4) shows that $\mathcal{F}_t^{X,\infty} \subseteq \mathcal{F}_t^{Y,\infty} \vee \sigma(X_0)$ for $t \geq 0$. Thus, we conclude that $\mathcal{F}_t^{Y,\infty} \vee \sigma(X_0) = \mathcal{F}_t^{X,\infty}$ for $t \geq 0$.

Now assume that $X_t = \int \varphi(t-s) + \psi(s) \, dW_s$ for $t \in \mathbb{R}$. From Remark 5.4 we have $\|\varphi(t+\cdot) - \varphi(\cdot)\|_{L^2(\lambda)} \leq \alpha + \beta|t|$ for $t \in \mathbb{R}$ and some $\alpha, \beta \in \mathbb{R}_+$. This shows that

$$\xi(t) := \int_{-\infty}^0 e^u \big(\varphi(t) - \varphi(u+t)\big) \, du, \qquad t \in \mathbb{R}$$

is a well-defined function belonging to $L^2(\lambda)$. For $t \in \mathbb{R}$ we have

$$\begin{aligned} Y_t &= \int_{-\infty}^0 \Big(\int e^u \big(\varphi(t-s) - \varphi(t+u-s)\big)\, dW_s \Big) du \\ &= \int \Big(\int_{-\infty}^0 e^u \big(\varphi(t-s) - \varphi(t+u-s)\big)\, du \Big) dW_s = \int \xi(t-s)\, dW_s, \end{aligned}$$

where the second equality follows from Protter (2004, Chapter IV, Theorem 65). □

Proof of Theorem 3.1. If: Assume (3.2) is satisfied. We show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

(1) : The case $\alpha \neq 0$. Define $B_t := \int \tilde{f}(t-s) - \tilde{f}(-s)\, dW_s$ for $t \in \mathbb{R}$ and let $g \colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = \begin{cases} \alpha + \int_0^t h(u)\, du & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Since φ satisfies (3.2), Lemma 4.1–4.2 show

$$X_t - X_0 = \int \tau_t\varphi(s) - \tau_0\varphi(s)\, dW_s = \int g(t-s) - g(-s)\, dB_s, \qquad t \in \mathbb{R}.$$

Thus, from Cherny (2001, Theorem 3.1) it follows that $(X_t - X_0)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale with martingale component $(\alpha B_t)_{t\geq 0}$. For $t \geq 0$ we have

$$\begin{aligned} 0 &= \langle f(\widehat{\varphi - \psi}), 1_{[0,t]} \rangle_{L^2(\lambda)} = \langle (f(\widehat{\varphi - \psi}))(-\cdot), \widehat{1_{[0,t]}} \rangle_{L^2(\lambda)} \\ &= \langle (\widehat{\varphi - \psi})(-\cdot), f\widehat{1_{[0,t]}} \rangle_{L^2(\lambda)} = \langle \varphi(\cdot) - \psi(\cdot), \tilde{f}(t+\cdot) - \tilde{f}(\cdot) \rangle_{L^2(\lambda)}. \end{aligned} \tag{5.7}$$

That is $E[B_t X_0] = 0$ for all $t \geq 0$ and by Gaussianity it follows that X_0 is independent of $(B_t)_{t\geq 0}$. Hereby we conclude that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale due to $\mathcal{F}_t^{X,\infty} \subseteq \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$ for $t \geq 0$.

(2) : The case $\alpha = 0$. Since φ is absolutely continuous with square integrable density, Lemma 5.2 implies

$$E[(X_t - X_u)^2] = \int \big(\varphi(t-s) - \varphi(u-s)\big)^2 ds \leq K|t-u|^2, \qquad t, u \geq 0, \tag{5.8}$$

for some constant $K \in \mathbb{R}_+$. The Kolmogorov-Čentsov Theorem shows that $(X_t)_{t\geq 0}$ has a continuous modification and from (5.8) it follows that this modification is of integrable variation. Hence $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Only if: Assume conversely that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

(3) : First assume (in addition) that $(X_t)_{t\geq 0}$ is of unbounded variation. Let ξ and $(Y_t)_{t\in\mathbb{R}}$ be given as in Lemma 5.5. Since $(X_t)_{t\geq 0}$ is of unbounded variation it follows that $\mathcal{F}_0^{X,\infty} \neq \mathcal{F}_\infty^{X,\infty}$ and we conclude that $\mathcal{F}_0^{Y,\infty} \neq \mathcal{F}_\infty^{Y,\infty}$. Thus, Szegö's Alternative (see Dym and McKean (1976, page 84)) shows that

$$\int \frac{\log|\hat{\xi}|(u)}{1+u^2}\, du > -\infty.$$

Now choose f and g according to Lemma 5.1 (with (φ, X) replaced by (ξ, Y)) and let $(B_t)_{t\in\mathbb{R}}$ be given as in the lemma. The process $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale due to the fact that $(\mathcal{F}_t^{Y,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$. Lemma 5.3 shows that $(B_t)_{t\geq 0}$ is independent of X_0 which implies that $\tau_t \tilde{f} - \tau_0 \tilde{f}$ is orthogonal to $\varphi(-\cdot) + \psi(-\cdot)$ for $t \geq 0$. Therefore as in (5.7) it follows that $f\widehat{(\varphi - \psi)}$ is 0 on $\mathbb{R}_+$. Since $(Y_t)_{t\geq 0}$ (in particular) is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale, it follows from Knight (1992, Theorem 6.5) that

$$g(t) = \alpha + \int_0^t \zeta(u)\,du, \qquad t \geq 0,$$

for some $\alpha \in \mathbb{R}$ and some $\zeta \in L^2(\lambda)$. Let $\eta := \zeta + g$ and κ be given by

$$\kappa(t) = \alpha + \int_0^t \eta(u)\,du, \qquad t \geq 0,$$

and $\kappa(t) = 0$ for $t < 0$. For all $t \in \mathbb{R}$ we have

$$\begin{aligned} X_t - X_0 = Y_t - Y_0 - \int_0^t Y_u\,du &= Y_t - Y_0 - \int \Big(\int_0^t g(u-s)\,du\Big) dB_s \\ = \int \Big(g(t-s) - g(-s) + \int_{-s}^{t-s} g(u)\,du\Big) dB_s, &= \int \kappa(t-s) - \kappa(-s)\,dB_s, \end{aligned}$$

where the second equality follows from Protter (2004, Chapter IV, Theorem 65). By Lemma 4.2 we obtain

$$\tau_t\varphi - \tau_0\varphi = \int \kappa(t-u) - \kappa(-u)\,d\tau_u\tilde{f}, \qquad \lambda\text{-a.s. } \forall t \in \mathbb{R},$$

by which Lemma 4.1 (i) implies that

$$\varphi(t) = \beta + \alpha\tilde{f}(t) + \int_0^\infty \big(\tilde{f}(t-v) - \tilde{f}(-v)\big)\eta(v)\,dv, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\beta \in \mathbb{R}$. Thus, we obtain (3.2) (with $h = \eta(-\cdot)$) from (3.4). Moreover, for $t \geq 0$ we have

$$X_t - X_0 = \alpha B_t + \int \Big(\int_{-s}^{t-s} \widehat{f\hat{h}}(u)\,du\Big) dW_s = \alpha B_t + \int_0^t \Big(\int \widehat{f\hat{h}}(s-u)\,dW_u\Big) ds,$$

which shows that the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by (3.3) since $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale.

(4) : Assume $(X_t)_{t\geq 0}$ is of bounded variation and therefore of integrable variation (see Stricker (1983)). By Lemma 5.2 we conclude that φ is absolutely continuous with square integrable density and hereby on the form (3.2) with $\alpha = 0$ and $f = 1$. □

Proof of Proposition 3.4. (i) *Only if:* We have

$$\varphi(t-s)-\psi(-s)=0, \qquad \lambda\text{-a.a. } s\in(t,\infty)\ \forall t\geq 0,$$

since $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. By this it follows that φ is constant on $(-\infty,0)$ λ-a.s. Thus, we may and do assume $\varphi(s)=0$ for λ-a.a. $s\in(-\infty,0)$.

First assume $(X_t)_{t\geq 0}$ is of bounded variation. By arguing as in (4) it follows that φ is on the form (3.2) with f constant equal 1, and therefore f satisfies the additional condition in (i).

Second assume $(X_t)_{t\geq 0}$ is of unbounded variation. Proceed as in (3) in the proof of Theorem 3.1. Since φ is 0 on $(-\infty,0)$ it follows by (5.5) that ξ is 0 on $(-\infty,0)$ by which we can choose f such that the condition in (i) is satisfied according to Lemma 5.1.

If: According to Lemma 4.3, $\tilde{f}$ is constant on $(-\infty,0)$ λ-a.s. From (3.4) we obtain

$$\int \big(\tilde{f}(t-s)-\tilde{f}(-s)\big)h(-s)\,ds=\int_0^t \widehat{f\hat{h}}(s)\,ds,$$

which implies that φ is constant on $(-\infty,0)$ λ-a.s. Furthermore $f\widehat{(\varphi-\psi)}\in\mathbb{H}^2_+$ which shows $\mathbb{H}^2_+\ni\overline{f}f\widehat{(\varphi-\psi)}=\widehat{\varphi-\psi}$ and hereby $\varphi=\psi$ on $(-\infty,0)$. Thus, we conclude that $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted.

To prove (3.5) we assume that φ is represented as in equation (3.2) with $f(a)=\lim_{b\downarrow 0}J(-a+b)$ for λ-a.a. $a\in\mathbb{R}$ and some inner function J. By Lemma 4.3 it follows that $\tilde{f}$ is constant on $(-\infty,0)$ and hence we deduce (3.5) from (3.2) and (3.4).

(ii): Assume $\psi=0$. *Only if:* We may and do assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale of unbounded variation. We have to show that we can decompose φ as in (3.6) where $\alpha+\int_0^\cdot h(-s)\,ds$ is square integrable on $\mathbb{R}_+$. However, this follows as in (3) above (without referring to Lemma 5.5).

Assume conversely that (3.2) is satisfied, with α,β,f and h as stated. Let $g\in L^2(\lambda)$ be given by

$$g(t)=\begin{cases}\alpha+\int_0^t h(-v)\,dv & t\geq 0\\ 0 & t<0.\end{cases}$$

From Lemma 4.1 (ii) it follows that there exists a $\tilde{\beta}\in\mathbb{R}$ such that

$$\int g(-u)\,d\tau_u\tilde{f}=\tilde{\beta}+\alpha\tilde{f}(-\cdot)+\int\big(\tilde{f}(-v-\cdot)-\tilde{f}(-v)\big)h(-v)\,dv, \qquad \lambda\text{-a.s.}$$

which by (3.2) and (3.4) implies

$$\int g(-u)\,d\tau_u\tilde{f}=\tilde{\beta}-\beta+\varphi(-\cdot), \qquad \lambda\text{-a.s.}$$

The square integrability of φ shows $\tilde{\beta}=\beta$ and by setting

$$B_t:=\int \tilde{f}(t-s)-\tilde{f}(-s)\,dW_s, \qquad t\in\mathbb{R},$$

it follows from Lemma 4.2 that $X_0 = \int g(-s)\, dB_s$. Lemma 4.1 (i) and Lemma 4.2 shows that $X_t - X_0 = \int g(t-s) - g(-s)\, dB_s$ for $t \in \mathbb{R}$. Hence, we conclude that

$$X_t = \int g(t-s)\, dB_s, \qquad t \in \mathbb{R},$$

which show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale and thus an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. This completes the proof. □

6 The spectral measure of stationary semimartingales

For $t \in \mathbb{R}$, let $X_t = \int_{-\infty}^t \varphi(t-s)\, dW_s$ where $\varphi \in L^2(\lambda)$. In this section we use Knight (1992, Theorem 6.5) to give a condition on the Fourier transform of φ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. In the case where $(X_t)_{t\geq 0}$ is a Markov process we use this to provide a simple condition for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. In the last part of this section we study a general stationary Gaussian process $(X_t)_{t\in\mathbb{R}}$. Similar to Jeulin and Yor (1993), we provide a condition on the spectral measure of $(X_t)_{t\in\mathbb{R}}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Proposition 6.1. *Let $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\, dW_s$, where $\varphi \in L^2(\lambda)$ and $(W_t)_{t\in\mathbb{R}}$ is a Wiener process. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if*

$$\hat{\varphi}(t) = \frac{\alpha + \hat{h}(t)}{1 - it}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$.

The result follows directly from Knight (1992, Theorem 6.5), once we have shown the following technical result.

Lemma 6.2. *Let $\varphi \in L^2(\lambda)$. Then φ is on the form*

$$\varphi(t) = \begin{cases} \alpha + \int_0^t h(s)\, ds & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{6.1}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ if and only if

$$\hat{\varphi}(t) = \frac{\alpha + \hat{h}(t)}{1 - it}, \tag{6.2}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$.

Proof. Assume φ satisfies (6.1). The square integrability of φ shows that we can find a sequence $(a_n)_{n\geq 1}$ converging to infinity such that $\varphi(a_n)$ converges to 0. For all $n \geq 1$ we have

$$\begin{aligned}\int_0^{a_n} \varphi(s)e^{its}\,ds &= \int_0^{a_n} ce^{its}ds + \int_0^{a_n} \Big[\int_0^s h(u)\,du\Big]e^{its}\,ds \\ &= \frac{c(e^{ia_nt}-1)}{it} + \int_0^{a_n} h(u)\Big[\int_u^{a_n} e^{its}ds\Big]\,du \\ &= \frac{c(e^{ia_nt}-1)}{it} + \int_0^{a_n} h(u)\Big[\frac{e^{ia_nt}-e^{iut}}{it}\Big]\,du \\ &= \frac{1}{it}\Big[e^{ia_nt}\big(c+\int_0^{a_n} h(u)du\big) - c - \int_0^{a_n} h(u)e^{itu}du\Big] \\ &= \frac{1}{it}\Big[e^{ia_nt}\varphi(a_n) - c - \int_0^{a_n} h(u)e^{itu}du\Big].\end{aligned}$$

Hence by letting n tend to infinity, it follows that $\hat{\varphi}(t) = -(it)^{-1}(c+\hat{h}(t))$ and we obtain (6.2).

Assume conversely that (6.2) is satisfied and let $e(t) := e^{-t}1_{\mathbb{R}_+}(t)$ for $t \in \mathbb{R}$.

$$\hat{\varphi}(t) = \frac{\alpha+\hat{h}}{1-it} = \alpha\hat{e}(t) + \hat{h}(t)\hat{e}(t). \tag{6.3}$$

Note that $h * e$ is an square integrable function and $\widehat{h*e} = \hat{h}\hat{e}$. Thus from (6.3) it follows that $\varphi = \alpha e + h * e$ λ-a.s. This shows in particular that φ is 0 on $(-\infty, 0)$. It also yields that $h(t) - h * e(t) = \alpha e(t) + h(t) - \varphi(t) =: f(t)$, which implies that

$$h(t) - h(0) = f(t) - f(0) - \int_0^t f(s)\,ds,$$

and hence

$$\varphi(t) = \begin{cases} \varphi(0) + \int_0^t \varphi(s) - h(s)\,ds & t \geq 0 \\ 0 & t < 0. \end{cases}$$

This completes the proof of (6.1). □

Now we apply Proposition 6.1 to give conditions on φ for a Markov process $(X_t)_{t\in\mathbb{R}} = (\int_{-\infty}^t \varphi(t-s)\,dW_s)_{t\in\mathbb{R}}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. This process is always an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, but not always an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Recall that since $(X_t)_{t\in\mathbb{R}}$ is an stationary centered Gaussian Markov process which is continuous in probability it is actually an Ornstein-Uhlenbeck process, i.e. it is a centered Gaussian process with covariance function given by

$$E[X_tX_u] = e^{-\theta|t-u|}, \qquad t, u \in \mathbb{R},$$

for some $\theta \in (0, \infty)$.

Example 6.3. *Let $(X_t)_{t\in\mathbb{R}}$ be given by*

$$X_t = \int \varphi(t-s)\, dW_s, \qquad t \in \mathbb{R},$$

where $\varphi \in L^2(\lambda)$ is 0 on $(-\infty, 0)$. Assume $(X_t)_{t\geq 0}$ is an Markov process. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if $J - \alpha \in \mathbb{H}^2_+$ for some $\alpha \in \{-1, 1\}$, where J is the inner part of the Hardy function induced by φ. In particular if J is a singular inner function, that is on the form

$$J(z) = \exp\Big(\frac{-1}{\pi i}\int \frac{sz+1}{s-z}\frac{1}{1+s^2}\, F(ds)\Big), \qquad z \in \mathbb{C}_+,$$

where F is a singular measure which integrates $s \mapsto (1+s^2)^{-1}$, and we assume F is concentrated on $\mathbb{Z}$, $(F(\{k\}))_{k\in\mathbb{Z}}$ is bounded and $\sum_{k\in\mathbb{Z}} F(\{k\})^2 = \infty$, then $(X_t)_{t\geq 0}$ is not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale.

To prove the first part let J be the inner part of the Hardy function induced by φ. Since $(X_t)_{t\in\mathbb{R}}$ is an Ornstein-Uhlenbeck process we have (up to a scaling constant) that

$$|\hat{\varphi}(t)|^2 = (\theta + t^2)^{-1}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\theta \in (0, \infty)$, which implies

$$\hat{\varphi}(t) = \frac{j(t)}{\theta - it}, \qquad t \in \mathbb{R},$$

where $j(t) = \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$. Moreover, Proposition 6.1 shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$\hat{\varphi}(t) = \frac{\alpha + \hat{h}(t)}{\theta - it}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$. Thus, we conclude that $J - \alpha = H$, where H is the Hardy function induced by h and the proof of the first statement is complete.

To prove the last part, we note that

$$|J(a+ib)| = \exp\Big(\int \frac{-b}{\pi((s-a)^2+b^2)}\, F(ds)\Big).$$

Moreover, as a consequence of the Mean-Value Theorem, it follows that if $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded measurable function such that $f \notin L^2(\lambda)$ then $\exp(f) - 1 \notin L^2(\lambda)$. We will use this on

$$f(a) := \int \frac{-b}{\pi((s-a)^2+b^2)}\, F(ds), \qquad a \in \mathbb{R}.$$

The function f is bounded since $k \mapsto F(\{k\})$ is bounded. Moreover, $f \notin L^2(\lambda)$ since

$$\begin{aligned}\int |f(a)|^2\,da &= \Big(\frac{b}{\pi}\Big)^2 \int \Big(\sum_{j\in\mathbb{Z}} \frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da \\ &\geq \Big(\frac{b}{\pi}\Big)^2 \int \sum_{j\in\mathbb{Z}} \Big(\frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da = \Big(\frac{b}{\pi}\Big)^2 \sum_{j\in\mathbb{Z}} \int \Big(\frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da \\ &= \Big(\frac{b}{\pi}\Big)^2 \sum_{j\in\mathbb{Z}} \int \Big(\frac{F(\{j\})}{a^2+b^2}\Big)^2 da = \Big(\frac{b}{\pi}\Big)^2 \int \Big(\frac{1}{a^2+b^2}\Big)^2 da \sum_{j\in\mathbb{Z}} [F(\{j\})]^2 = \infty,\end{aligned}$$

where the first inequality follows from the fact that the terms in the sum is positive. Thus,

$$|J(a+ib)-\alpha| \geq ||J(a+ib)|-1| = \exp(f(a))-1,$$

which shows $J-\alpha \notin \mathbb{H}^2_+$ and hence $(X_t)_{t\geq 0}$ is not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. $\Diamond$

The proof of the next result is quite similar to the proof of Proposition 19 in Jeulin and Yor (1993).

Proposition 6.4. *Let $(X_t)_{t\in\mathbb{R}}$ be an $L^2(P)$-continuous stationary centered Gaussian process with spectral measure $\mu = \mu_s + f\,d\lambda$ (μ_s is the singular part of μ). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if $\int t^2\,\mu_s(dt) < \infty$ and*

$$f(t) = \frac{|\alpha + \hat{h}(t)|^2}{1+t^2}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$ if $\alpha \neq 0$. Moreover, $(X_t)_{t\geq 0}$ is of bounded variation if and only if $\alpha = 0$.

Proposition 6.4 extends the well-known fact that an $L^2(P)$-continuous stationary Gaussian process is of bounded variation if and only if $\int t^2\,\mu(dt) < \infty$ (μ denotes the spectral measure).

Proof of Proposition 6.4.
Only if: If $(X_t)_{t\geq 0}$ is of bounded variation then $\int t^2\,\mu(dt) < \infty$ and therefore μ is on the stated form. Thus, we may and do assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale of unbounded variation. Now it follows that $(X_t)_{t\in\mathbb{R}}$ is a regular process and can therefore be decomposed as (see e.g. Doob (1990))

$$X_t = V_t + \int_{-\infty}^t \varphi(t-s)\,dW_s, \qquad t \in \mathbb{R},$$

where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process which is independent of $(V_t)_{t\in\mathbb{R}}$ and $W_r - W_s$ is $\mathcal{F}_t^{X,\infty}$-measurable for $s \leq r \leq t$. The process $(V_t)_{t\in\mathbb{R}}$ is a stationary Gaussian and V_t is $\mathcal{F}_{-\infty}^{X,\infty}$-measurable for all $t \in \mathbb{R}$, where

$$\mathcal{F}_{-\infty}^{X,\infty} := \bigcap_{t\in\mathbb{R}} \mathcal{F}_t^{X,\infty}.$$

Moreover, $(V_t)_{t\in\mathbb{R}}$ and $(X_t - V_t)_{t\in\mathbb{R}}$ have spectral measure respectively μ_s and $f\,d\lambda$. For $0 \leq u \leq t$ we have

$$\begin{aligned} E[|V_t - V_u|] &= E[|E[V_t - V_u|\mathcal{F}_u^{V,\infty}]|] = E[|E[X_t - X_u|\mathcal{F}_u^{V,\infty}]|] \\ &\leq E[|E[X_t - X_u|\mathcal{F}_u^{X,\infty}]|], \end{aligned}$$

which shows that $(V_t)_{t\geq 0}$ is of integrable variation and hence $\int t^2\,\mu_s(dt) < \infty$. The fact that $(V_t)_{t\geq 0}$ is $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-adapted and of bounded variation implies that

$$\Big(\int_{-\infty}^t \varphi(t-s)\,dW_s\Big)_{t\geq 0}$$

is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and therefore also an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Thus, by Proposition 6.1 we conclude that

$$f(t) = |\hat{\varphi}(t)|^2 = \frac{|\alpha + \hat{h}(t)|^2}{1+t^2}\,, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$.

If: If $\int t^2\,\mu(dt) < \infty$, then $(X_t)_{t\geq 0}$ is of bounded variation and therefore an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Thus, we may and do assume $\int t^2 f(t)\,dt = \infty$. We will show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale by constructing a process $(Z_t)_{t\in\mathbb{R}}$ which equals $(X_t)_{t\in\mathbb{R}}$ in distribution and such that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. By Lemma 6.2 there exists a $\beta \in \mathbb{R}$ and a $g \in L^2(\lambda)$ such that with $\varphi(t) = \beta + \int_0^t g(s)\,ds$ for $t \geq 0$ and $\varphi(t) = 0$ for $t < 0$, we have $|\hat{\varphi}|^2 = f$. Define $(Z_t)_{t\in\mathbb{R}}$ by

$$Z_t = V_t + \int_{-\infty}^t \varphi(t-s)\,dW_s, \qquad t \in \mathbb{R},$$

where $(V_t)_{t\in\mathbb{R}}$ is a stationary Gaussian process with spectral measure μ_s and $(W_t)_{t\in\mathbb{R}}$ is a Wiener process which is independent of $(V_t)_{t\in\mathbb{R}}$. The processes $(X_t)_{t\in\mathbb{R}}$ and $(Z_t)_{t\in\mathbb{R}}$ are identical in distribution due to the fact that they are centered Gaussian processes with the same spectral measure. It readily seen that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. □

7 The spectral measure of semimartingales with stationary increments

Let $(X_t)_{t\in\mathbb{R}}$ be an $L^2(P)$-continuous Gaussian process with stationary increments such that $X_0 = 0$. Then there exists a unique positive symmetric measure μ on $\mathbb{R}$ which integrates $t \mapsto (1+t^2)^{-1}$ and satisfies

$$E[X_t X_u] = \int \frac{(e^{its}-1)(e^{-ius}-1)}{s^2}\,\mu(ds), \qquad t, u \in \mathbb{R}.$$

The measure μ is called the spectral measure for $(X_t)_{t\in\mathbb{R}}$. The spectral measure of the fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is

$$\mu(ds) = c_H |s|^{1-2H}\, ds,$$

where $c_H \in \mathbb{R}$ is a constant (see e.g. Yaglom (1987)). In particular the spectral measure of the Wiener process ($H = 1/2$) equals the Lebesgue measure up to a scaling constant. There should be no confusion with the spectral measure of stationary processes, since the intersection of the stationary processes and processes with stationary increments which starts at 0 is the 0 process.

Theorem 7.1. *Let $(X_t)_{t\in\mathbb{R}}$ be an $L^2(P)$-continuous, centered Gaussian process with stationary increments such that $X_0 = 0$. Moreover, let $\mu = \mu_s + f d\lambda$ be the spectral measure of $(X_t)_{t\in\mathbb{R}}$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if μ_s is a finite measure and*

$$f = |\alpha + \hat{h}|^2, \qquad \lambda\text{-a.s.}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$ if $\alpha \neq 0$. Moreover, $(X_t)_{t\geq 0}$ is of bounded variation if and only if $\alpha = 0$.

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Let $(Y_t)_{t\in\mathbb{R}}$ be the stationary centered Gaussian process given by Lemma 5.5 such that

$$X_t = Y_t - Y_0 + \int_0^t Y_s\, ds, \qquad t \in \mathbb{R}, \tag{7.1}$$

and let ν denote the spectral measure of $(Y_t)_{t\in\mathbb{R}}$, that is ν is a finite measure satisfying

$$E[Y_t Y_u] = \int e^{i(t-u)a}\, \nu(da), \qquad t, u \in \mathbb{R}.$$

By using Fubini's Theorem it follows that

$$E[X_t X_u] = \int \frac{\big(e^{its} - 1\big)\big(e^{-ius} - 1\big)}{s^2} (1 + s^2)\, \nu(ds), \qquad t, u \in \mathbb{R}. \tag{7.2}$$

Thus, by uniqueness of the spectral measure of $(X_t)_{t\in\mathbb{R}}$ we obtain $\mu(ds) = (1 + s^2)\, \nu(ds)$. Since $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale (7.1) implies that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale and hence Proposition 6.4 shows that the singular part ν_s of ν satisfies $\int t^2\, \nu(dt) < \infty$ and the absolute continuous part is on the form

$$|\alpha + \hat{h}(s)|^2 (1 + s^2)^{-1}\, ds,$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$ if $\alpha \neq 0$. Thus, we obtain the seeked decomposition of μ.

Conversely assume that μ_s is a bounded measure and $f = |\alpha + \hat{h}|^2$ for an $\alpha \in \mathbb{R}$ and an $h \in L^2(\lambda)$ which is 0 on $(-\infty, 0)$ if $\alpha \neq 0$. Let $(Y_t)_{t\in\mathbb{R}}$ be a centered Gaussian process such that

$$E[Y_t Y_u] = \int \frac{e^{i(t-u)a} f(a)}{1 + a^2}\, da, \qquad t, u \in \mathbb{R}.$$

By Proposition 6.4 it follows that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale. Thus, by defining $(Z_t)_{t\in\mathbb{R}}$ as

$$Z_t := Y_t - Y_0 + \int_0^t Y_s \, ds, \qquad t \in \mathbb{R},$$

we obtain that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale and hence also an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. Moreover, by calculations as in (7.2) it follows that $(Z_t)_{t\in\mathbb{R}}$ is distributed as $(X_t)_{t\in\mathbb{R}}$, which shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. This completes the proof. □

Let $(X_t)_{t\in\mathbb{R}}$ be a fBm with Hurst parameter $H \in (0,1)$. Then by use of the above theorem it is readily seen that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if $H = 1/2$ (this is already known by Rogers (1997)). As a consequence of the above theorem we also have:

Corollary 7.2. *Let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process with stationary increments. Then $(X_t)_{t\geq 0}$ is of bounded variation if and only if $(X_t - X_0)_{t\in\mathbb{R}}$ has finite spectral measure.*

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