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# Representation of Gaussian semimartingales with applications to the covariance function

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# Representation of Gaussian semimartingales with applications to the covariance function

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**Abstract**

The present paper is concerned with various aspects of Gaussian semimartingales. Firstly, generalizing a result of Stricker (1983), we provide a convenient representation of Gaussian semimartingales $X_t = X_0 + M_t + A_t$ as an $(\mathcal{F}_t^M)_{t\geq 0}$-semimartingale plus a process of bounded variation which is independent of $(M_t)_{t\geq 0}$. Secondly, we study stationary Gaussian semimartingales and their canonical decomposition. Thirdly, we give a new characterisation of the covariance function of Gaussian semimartingales which enable us to characterize the class of martingales and the processes of bounded variation among the Gaussian semimartingales. We conclude with two applications of the results.

*Keywords:* semimartingales; Gaussian processes; covariance function; stationary processes

*AMS Subject Classification:* 60G15; 60G10; 60G48

## 1 Introduction

Recently, there has been renewed interest in some of the fundamental properties of Gaussian processes, such as the semimartingale property and the existence of quadratic variation; see e.g. Barndorff-Nielsen and Schmiegel (2007).

Cheridito (2004), Cherny (2001) and Jeulin and Yor (1993) studied the semimartingale property of a certain class of Gaussian processes with stationary increments (or of a deterministic transformation of such processes). Jain and Monrad (1982) studied, among other topics, certain properties of Gaussian process of bounded variation. A good review of the literature about Gaussian semimartingales can be found in Liptser and Shiryayev (1989).

Stricker (1983, Théorème 2) showed the following. Let $(X_t)_{t\geq 0}$ be a Gaussian semimartingale with canonical decomposition $X_t = W_t + \int_0^t Z_s\, ds$, where $(W_t)_{t\geq 0}$ is a Brownian motion. Then there exists a Gaussian process $(Y_t)_{t\geq 0}$ which is independent of $(W_t)_{t\geq 0}$ and a deterministic function $(r, s) \mapsto \Psi_r(s)$ such that

$$X_t = W_t + \int_0^t \Big( \int_0^r \Psi_r(s)\, dW_s \Big) dr + \int_0^t Y_r\, dr.$$

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One of the purposes of the present paper is to generalize this result. Indeed, we show that a Gaussian process $(X_t)_{t\geq 0}$ is a semimartingale if and only if it can be decomposed as

$$X_t = X_0 + M_t + \int_0^t \Big(\int_0^r \Psi_r(s)\, dM_s\Big)\mu(dr) + \int_0^t Y_r\, \mu(dr). \tag{1.1}$$

where $(M_t)_{t\geq 0}$ is a Gaussian martingale, $(Y_t)_{t\geq 0}$ is a Gaussian process which is independent of $(M_t)_{t\geq 0}$, $\mu$ is a Radon measure on $\mathbb{R}_+$ and $(r,s)\mapsto \Psi_r(s)$ is a deterministic function. As a part of this we study Gaussian processes of bounded variation.

A second purpose of the paper is to study the canonical decomposition of stationary Gaussian semimartingales. Let $(X_t)_{t\in\mathbb{R}}$ be a stationary Gaussian process such that $(X_t)_{t\geq 0}$ is a semimartingale. We study the canonical decomposition of $(X_t)_{t\geq 0}$ and give a necessary and sufficient condition for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, where $\mathcal{F}_t^{X,\infty} := \sigma(X_s : s\in(-\infty,t])$ for $t\geq 0$.

In the last section of the paper we study the the covariance structure of Gaussian semimartingales. Let $(X_t)_{t\in\mathbb{R}}$ be a stationary Gaussian process. Then, Proposition 19 in Jeulin and Yor (1993) gives a necessary and sufficient condition on the spectral measure of $(X_t)_{t\in\mathbb{R}}$ for $(X_t)_{t\in\mathbb{R}}$ to be a semimartingale. Emery (1982) showed that a Gaussian process $(X_t)_{t\geq 0}$ is a semimartingale if and only if the mean-value function and the covariance function $\Gamma$ of $(X_t)_{t\geq 0}$ are of bounded variation and there exists an right-continuous increasing function $F$ such that for each $0\leq s<t$ and each elementary function $u\mapsto f_s(u)$ with $f_s(u)=0$ for $u>s$ we have

$$\frac{\Big|\int_s^t\int_0^s f_s(v)\,\Gamma(du,dv)\Big|}{\sqrt{\int_0^s\int_0^s f_s(u)f_s(v)\,\Gamma(du,dv)}} \leq F(t)-F(s).$$

However, based on the decomposition (1.1) we provide a new alternative characterisation of the covariance function (see Theorem 5.2). Some applications will be given as well. For example, we study the fractional Brownian motion.

The paper is organised as follows. Section 2 contains some preliminary results. We show that Gaussianity is preserved under various operations on a Gaussian semimartingale. Moreover, a suitable version of Fubini's Theorem is provided. Section 3 contains some representation results for Gaussian semimartingales. First, extending a result of Jeulin (1993), we characterize Gaussian process of bounded variation. Afterwards the decomposition (1.1) is provided. In section 4 the covariance function of Gaussian semimartingales is considered. We conclude with a few examples.

## 1.1 Notation

Let $(\Omega,\mathcal{F},P)$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of $\sigma$-algebras satisfying the usual conditions of right-continuity and completeness. If $(X_t)_{t\geq 0}$ is a stochastic process we denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the least filtration to which $(X_t)_{t\geq 0}$ is adapted.

A separable subspace $\mathbb{G}$ of $L^2(P)$ which contains all constants, is called a Gaussian space if $(X_1,\ldots,X_n)$ follows a multivariate Gaussian distribution whenever $n\geq 1$ and

$X_1, \dots, X_n \in \mathbb{G}$. Let $\mathbb{G}$ denote a Gaussian space and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Then we say that $\mathbb{G}$ is $(\mathcal{F}_t)_{t\geq 0}$-stable if $X \in \mathbb{G}$ implies $E[X|\mathcal{F}_t] \in \mathbb{G}$ for all $t \geq 0$. A typical example is $\mathbb{G} := \overline{\text{sp}}\{X_t : t \geq 0\}$ for a càdlàg Gaussian process $(X_t)_{t\geq 0}$ ($\overline{\text{sp}}$ denotes the $L^2(P)$-closure of the linear span) and $(\mathcal{F}_t)_{t\geq 0} = (\mathcal{F}_t^X)_{t\geq 0}$.

We say that a stochastic process $(X_t)_{t\geq 0}$ has stationary increments if for all $n \geq 1$, $0 \leq t_0 < \cdots < t_n$ and $0 < t$ we have

$$(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}) \overset{\mathcal{D}}{=} (X_{t_1+t} - X_{t_0+t}, \dots, X_{t_n+t} - X_{t_{n-1}+t}),$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution.

Let $\mu$ be a $\sigma$-finite measure on $\mathbb{R}$ and $f \colon \mathbb{R}_+ \to \mathbb{R}$ be a function. Then $f$ is said to be absolutely continuous w.r.t. $\mu$ if $f$ is of bounded variation and the total variation measure of $f$ is absolutely continuous w.r.t. $\mu$. A stochastic process $(X_t)_{t\geq 0}$ starting at 0 is said to be absolutely continuous w.r.t. $\mu$ if almost all sample paths of $(X_t)_{t\geq 0}$ are absolutely continuous w.r.t. $\mu$. Moreover for a locally $\mu$-integrable function $f$ we define $\int_a^b f\, d\mu := \int_{(a,b]} f\, d\mu$ for all $0 \leq a < b$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, if there exists a decomposition of $(X_t)_{t\geq 0}$ as

$$X_t = X_0 + M_t + A_t, \tag{1.2}$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale starting at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation starting at 0. We say that $(X_t)_{t\geq 0}$ is a semimartingale if it is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale. Moreover $(X_t)_{t\geq 0}$ is called a special $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale such that $(A_t)_{t\geq 0}$ in (1.2) can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable. In this case the representation (1.2) with $(A_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable is unique and is called the canonical decomposition of $(X_t)_{t\geq 0}$. From Liptser and Shiryayev (1989, Chapter 4, Section 9, Theorem 1) it follows that if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale then it is also an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale.

If $(A_t)_{t\geq 0}$ is a right-continuous Gaussian process of bounded variation then $(A_t)_{t\geq 0}$ is of integrable variation (see Stricker (1983, Proposition 4 and 5)) and we let $\mu_A$ denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto E[V_t(A)]$. For every Gaussian martingale $(M_t)_{t\geq 0}$ let $\mu_M$ denote the Lebesgue-Stieltjes measure induced by the mapping $t \mapsto E[M_t^2]$.

# 2 Preliminary results

In the following $\mu$ denotes a Radon measure on $\mathbb{R}_+$ and $(E, \mathcal{E}, \nu)$ is a $\sigma$-finite measure space.

**Lemma 2.1.** *Let $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and define $S := \overline{\text{sp}}\{\Psi_t : t \geq 0\}$. Assume $S$ is a separable subset of $L^2(\nu)$ and $t \mapsto \int \Psi_t(s) g(s)\, \nu(ds)$ is measurable for $g \in S$. Then, there exists a measurable mapping $\mathbb{R} \times E \ni (t,s) \mapsto \tilde{\Psi}_t(s) \in \mathbb{R}$ such that $\tilde{\Psi}_t = \Psi_t$ $\nu$-a.s. for $t \geq 0$.*

*Proof*. Since $S$ is a separable normed space, the Borel $\sigma$-algebra on $S$ induced by the norm-topology equals the $\sigma$-algebra induced by the mappings $S \ni f \mapsto \int fg\, d\nu \in \mathbb{R}$ for

$g \in S$. Therefore $t \mapsto \Psi_t$ is Bochner measurable, and thus a uniform limit of elements of the form $\Psi_t^n(s) = \sum_{k\geq 1} f_k^n(s) 1_{A_k^n}(t)$ where $f_k^n \in L^2(\nu)$ for $n, k \geq 1$ and $(A_k^n)_{k\geq 1}$ are disjoint $\mathcal{B}(\mathbb{R}_+)$-measurable sets for $n \geq 1$. Reducing if necessary to a subsequence we may assume that

$$\sup_{t\in\mathbb{R}_+} \|\Psi_t^n - \Psi_t\|_{L^2(\nu)} \leq 2^{-n}, \qquad n \geq 1. \tag{2.1}$$

Let $B := \{(t,s) \in \mathbb{R}_+ \times E : \limsup_{n\to\infty} |\Psi_t^n(s)| < \infty\}$ and define

$$\tilde{\Psi}_t(s) := \limsup_{k\to\infty} \Psi_t^n(s) 1_B((t,s)), \qquad (t,s) \in \mathbb{R}_+ \times E.$$

Then $(t,s) \mapsto \tilde{\Psi}_t(s)$ is measurable. Moreover by (2.1) it follows that $\tilde{\Psi}_t = \Psi_t$ $\nu$-a.s. for $t \in \mathbb{R}_+$, which completes the proof. □

Let $L^{2,1}(\nu,\mu)$ denote the space of all measurable mappings $\mathbb{R}_+ \times E \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ satisfying $\Psi_t \in L^2(\nu)$ for $t \geq 0$ and

$$\int_0^t \|\Psi_r\|_{L^2(\nu)} \, \mu(dr) < \infty, \qquad t > 0.$$

Furthermore $\mathcal{BV}(\nu)$ denotes the space of all measurable mappings $\mathbb{R}_+ \times E \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ for which $\Psi_t \in L^2(\nu)$ for all $t \geq 0$ and there exists a right-continuous increasing function $f$ such that $\|\Psi_t - \Psi_u\|_{L^2(\nu)} \leq f(t) - f(u)$ for $0 \leq u \leq t$.

**Lemma 2.2.** *Let $\Psi \in L^{2,1}(\nu,\mu)$. Then $r \mapsto \Psi_r(s)$ is locally $\mu$-integrable for $\nu$-a.a. $s \in E$ and by setting $\int_0^t \Psi_r(s)\, \mu(dr) = 0$ if $r \mapsto \Psi_r(s)$ is not locally $\mu$-integrable we have*

$$(t,s) \mapsto \int_0^t \Psi_r(s) \, \mu(dr) \in \mathcal{BV}(\nu). \tag{2.2}$$

*If in addition $S$ is a closed subspace of $L^2(\nu)$ such that $\Psi_r \in S$ for all $r \in [0,t]$, then*

$$s \mapsto \int_0^t \Psi_r(s) \, \mu(dr) \in S. \tag{2.3}$$

*Proof.* Let $t \geq 0$ be given. Tonelli's Theorem and Cauchy-Schwarz' inequality imply

$$\begin{aligned} &\int \Big( \int_0^t |\Psi_r(s)| \, \mu(dr) \Big)^2 \nu(ds) \qquad (2.4) \\ &\quad = \int_0^t \int_0^t \Big( \int |\Psi_r(s)\Psi_v(s)| \, \nu(ds) \Big) \mu(dr)\, \mu(dv) \leq \Big( \int_0^t \|\Psi_r\|_{L^2(\nu)} \, \mu(dr) \Big)^2 < \infty. \end{aligned}$$

This shows that $r \mapsto \Psi_r(s)$ is locally $\mu$-integrable for $\nu$-a.a. $s \in E$. By setting

$$\int_0^t \Psi_r(s) \, \mu(dr) = 0 \qquad \text{if } r \mapsto \Psi_r(s) \text{ is not locally } \mu\text{-integrable},$$

we have that $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(dr)$ is measurable and $s \mapsto \int_0^t \Psi_r(s)\,\mu(dr) \in L^2(\nu)$. Calculations as in (2.4) show that

$$\Big\|\int_0^t \Psi_r\,\mu(dr) - \int_0^u \Psi_r\,\mu(dr)\Big\|_{L^2(\nu)} \le \int_u^t \|\Psi_r\|_{L^2(\nu)}\,\mu(dr)$$
$$= \int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(dr) - \int_0^u \|\Psi_r\|_{L^2(\nu)}\,\mu(dr),$$

which yields (2.2). To show (2.3) fix $t \ge 0$. By the Projection Theorem it is enough to show

$$\Big\langle \int_0^t \Psi_r\,\mu(dr), g\Big\rangle_{L^2(\nu)} = 0 \qquad \text{for } g \in S^\perp.$$

Fix $g \in S^\perp$. Tonelli's Theorem and Cauchy-Schwarz' inequality shows that

$$\iint_0^t |\Psi_r(s)g(s)|\,\mu(dr)\,\nu(ds) \le \|g\|_{L^2(\nu)} \int_0^t \|\Psi_r\|_{L^2(\nu)}\,\mu(dr) < \infty.$$

Thus Fubini's Theorem shows that

$$\Big\langle \int_0^t \Psi_r\,\mu(dr), g\Big\rangle_{L^2(\nu)} = \int_0^t \langle \Psi_r, g\rangle_{L^2(\nu)}\,\mu(dr) = 0,$$

which completes the proof. □

For $\Psi \in L^{2,1}(\nu,\mu)$ we always define $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(dr)$ as in the above lemma.

**Lemma 2.3.** *For every $\Psi \in \mathcal{BV}(\nu)$ there exists a measurable mapping $(t,s) \mapsto \tilde\Psi_t(s)$ such that $t \mapsto \tilde\Psi_t(s)$ is right-continuous and of bounded variation for $s \in E$ and $\Psi_t = \tilde\Psi_t$ $\nu$-a.s. for $t \ge 0$.*

*Proof.* Define $\mathcal{D} := \{i2^{-n} : n \ge 1,\ i \ge 0\}$. We first show that $(A_t)_{t\in\mathcal{D}}$ has finite upcrossing over each finite interval $P$-a.s. by showing that $(\Psi_t)_{t\in\mathcal{D}\cap[0,N]}$ is of bounded variation $\nu$-a.s. for all $N \ge 1$. Fix $N \ge 1$. We have

$$\int \sup_{n\ge1} \sum_{i=1}^{N2^n} |\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}|\,d\nu = \int \liminf_{n\to\infty} \sum_{i=1}^{N2^n} |\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}|\,d\nu$$
$$\le \liminf_{n\to\infty} \sum_{i=1}^{N2^n} \int |\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}|\,d\nu \le \liminf_{n\to\infty} \sum_{i=1}^{N2^n} \|\Psi_{i2^{-n}} - \Psi_{(i-1)2^{-n}}\|_{L^2(\nu)} < \infty,$$

where the last inequality follows since $\Psi \in \mathcal{BV}(\nu)$. Since $(\Psi_t)_{t\in\mathcal{D}}$ has finite upcrossing over each finite interval $\nu$-a.s.

$$\tilde\Psi_t := \lim_{u\downarrow t,\ u\in\mathcal{D}} \Psi_u, \qquad t \ge 0,$$

is a well-defined càdlàg process. Moreover, since $\Psi \in \mathcal{BV}(\nu)$, $t \mapsto \Psi_t \in L^2(\nu)$ is right-continuous. This implies that $\tilde\Psi_t = \Psi_t$ $\nu$-a.s. for $t \ge 0$ and so $\tilde\Psi \in \mathcal{BV}(\nu)$. Thus it follows from calculations as above that $(\tilde\Psi_t)_{t\ge0}$ is of integrable variation. This completes the proof. □

# 3 General properties of Gaussian semimartingales

Our next result shows that the Gaussian property is preserved under various operations on a Gaussian semimartingale.

**Lemma 3.1.** *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ denote an $(\mathcal{F}_t)_{t\geq 0}$-stable Gaussian space. We have the following.*

*(i) Let $(X_t)_{t\geq 0} \subseteq \mathbb{G}$ be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale. Then $(X_t)_{t\geq 0}$ is a special $(\mathcal{F}_t)_{t\geq 0}$-semimartingale. Let $X_t = M_t + A_t + X_0$ be the $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$. Then, $(A_t)_{t\geq 0}, (M_t)_{t\geq 0} \subseteq \mathbb{G}$ and $(M_t)_{t\geq 0}$ is a (true) $(\mathcal{F}_t)_{t\geq 0}$-martingale which is independent of $X_0$.*

*(ii) Let $(M_t)_{t\geq 0} \subseteq \mathbb{G}$ be a Gaussian martingale starting at 0. Then*

$$\Big\{\int_0^t f(s)\,dM_s : f \in L^2(\mu_M)\Big\} = \overline{\text{sp}}\{M_u : u \leq t\Big\}, \qquad t \geq 0. \tag{3.1}$$

*In particular if $Y \in \mathbb{G}$ is an $\mathcal{F}_t^M$-measurable random variable with mean zero then there exists an $f \in L^2(\mu_M)$ such that*

$$Y = \int_0^t f(s)\,dM_s.$$

*Proof*. (i) follows by Stricker (1983, Proposition 4 and 5).
(ii): Fix $t \geq 0$. To show the inclusion '$\subseteq$' let $f \in L^2(\mu_M)$ be given. Since $\int_0^t f(s)\,dM_s$ is the $L^2(P)$-limit of $\int_0^t f_n(s)\,dM_s$ where the $f_n$'s are step functions such that $f_n \to f$ in $L^2(\mu_M)$, it follows that

$$\int_0^t f(s)\,dM_s \in \overline{\text{sp}}\{M_u : u \leq t\}.$$

Since $M_u = \int_0^t 1_{(0,u]}(s)\,dM_s$ for $u \in [0,t]$ and the left-hand side of (3.1) is closed the '$\supseteq$' inclusion follows and thus we have shown (3.1). Now assume that $Y \in \mathbb{G}$ is an $\mathcal{F}_t^M$-measurable random variable with mean zero. Let $(a_n)_{n\geq 1}$ be a dense subset of $[0,t]$ containing $t$. By Lévy's Theorem it follows that

$$E[Y|M_{a_1},\dots,M_{a_n}] \to E[Y|\mathcal{F}_t^M] = Y \qquad \text{in } L^2(P). \tag{3.2}$$

Since $(Y, M_{a_1},\dots,M_{a_n})$ is simultaneously Gaussian for every $n \geq 1$ the left-hand side of (3.2) belongs to the linear span of $\{M_{a_i} : 1 \leq i \leq n\}$. This shows that $Y \in \overline{\text{sp}}\{M_u : u \leq t\}$, which by (3.1) completes the proof of (ii). □

Let $(M_t)_{t\geq 0}$ denote a càdlàg Gaussian martingale and $(t,s) \mapsto \Psi_t(s)$ be a measurable mapping satisfying $\Psi_t \in L^2(\mu_M)$ for $t \geq 0$. Then we may and do choose $(\int \Psi_t(s)\,dM_s)_{t\geq 0}$ jointly measurable in $(t,\omega)$. To see this note that $S := \overline{\text{sp}}\{M_t : t \geq 0\}$ is a separable subspace of $L^2(P)$. Moreover Lemma 3.1 (ii) shows that each element in $S$ is on the form $\int f(s)\,dM_s$ for such $f \in L^2(\mu_M)$. Thus for $\int f(s)\,dM_s \in S$ we have

$$E\Big[\int \Psi_t(s)\,dM_s \int f(s)\,dM_s\Big] = \int \Psi_t(s) f(s)\,\mu_M(ds),$$

which shows that $t \mapsto E[\int \Psi_t(s)\,dM_s \int f(s)\,dM_s]$ is measurable. Hence by Lemma 2.1 there exists a measurable modification of $(\int \Psi_t(s)\,dM_s)_{t\geq 0}$.

**Lemma 3.2** (Stochastic Fubini result)**.** *Let $\mu$ be a $\sigma$-finite measure on $\mathbb{R}_+$, $(M_t)_{t\geq 0}$ be a càdlàg Gaussian martingale and $\Psi \in L^{2,1}(\mu_M, \mu)$. Then $t \mapsto \int \Psi_t(s)\, dM_s$ is locally $\mu$-integrable $P$-a.s. and*

$$\int_0^t \Big( \int \Psi_r(s)\, dM_s \Big) \mu(dr) = \int \Big( \int_0^t \Psi_r(s)\, \mu(dr) \Big) dM_s, \qquad t \geq 0. \tag{3.3}$$

*Proof.* We have

$$E\Big[ \int_0^t \Big| \int \Psi_r(s)\, dM_s \Big| \mu(dr) \Big] \leq \int_0^t \|\Psi_r\|_{L^2(\mu_M)}\, \mu(dt) < \infty,$$

which shows that $r \mapsto \int \Psi_r(s)\, dM_s$ is locally $\mu$-integrable $P$-a.s. Thus both sides of (3.3) are well-defined. The right-hand side belongs to $\overline{\text{sp}}\{M_t : t \geq 0\}$ and so does the left-hand side by Lemma 2.2. From Lemma 3.1 (ii) it follows that all elements in $\overline{\text{sp}}\{M_t : t \geq 0\}$ are on the form $\int g(s)\, dM_s$ for a $g \in L^2(\mu_M)$. Fix $\int g(s)\, dM_s \in \overline{\text{sp}}\{M_t : t \geq 0\}$. We have

$$E\Big[ \int g(s)\, dM_s \int \Big( \int_0^t \Psi_r(s)\, \mu(dr) \Big) dM_s \Big] = \int g(s) \int_0^t \Psi_r(s)\, \mu(dt)\, \mu_M(ds).$$

Moreover, it follows from Fubini's Theorem that

$$\begin{aligned} E\Big[ \int g(s)\, dM_s \int_0^t \Big( \int \Psi_r(s)\, dM_s \Big) \mu(dr) \Big] &= \int_0^t E\big[ \int g(s)\, dM_s \int \Psi_r(s)\, dM_s \big]\, \mu(dr) \\ = \int_0^t \int g(s) \Psi_r(s)\, \mu_M(ds)\, \mu(dr) &= \iint_0^t g(s) \Psi_t(s)\, \mu(dr)\, \mu_M(ds). \end{aligned}$$

Hence, the left- and the right-hand side of (3.3) have the same inner product with all elements of $\overline{\text{sp}}\{M_t : t \geq 0\}$ which means that they are equal. This completes the proof. □

# 4 Representation of Gaussian semimartingales

**Proposition 4.1.** *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ be a Gaussian space. Moreover let $(A_t)_{t\geq 0} \subseteq \mathbb{G}$ be $(\mathcal{F}_t)_{t\geq 0}$-adapted, right-continuous and of bounded variation. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-optional process $(Y_t)_{t\geq 0} \subseteq \mathbb{G}$ such that $\|Y_t\|_{L^2(P)} \leq 3$ for $t \geq 0$ and*

$$A_t = \int_0^t Y_s\, \mu_A(ds), \qquad t \geq 0. \tag{4.1}$$

*If $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)$-predictable then $(Y_t)_{t\geq 0}$ can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable and if $(A_t)_{t\geq 0}$ is a centered process we have $\|Y_r\|_{L^2(P)} = \sqrt{\pi/2}$ for $r \geq 0$.*

*Proof.* It follows from Jeulin (1993, Proposition 2) that $(A_t)_{t\geq 0}$ is absolutely continuous w.r.t. $\mu_A$. By Jacod and Shiryaev (2003, Proposition 3.13) there exists an $(\mathcal{F}_t)_{t\geq 0}$-optional process $(Z_t)_{t\geq 0}$ such that $A_t = \int_0^t Z_s\, \mu_A(ds)$ for $t \in \mathbb{R}_+$. Define

$$Z_s^n := \sum_{i=1}^{n2^n} \frac{A_{i2^{-n}} - A_{(i-1)2^{-n}}}{\mu_A(((i-1)2^{-n}, i2^{-n}])} 1_{((i-1)2^{-n}, i2^{-n}]}(s), \qquad s \geq 0,$$

where $0/0 := 0$. By reducing to probability measures we get from Dellacherie and Meyer (1982, page 50) that for almost all $\omega \in \Omega$, $Z^n_\cdot(\omega)$ converges to $Z_\cdot(\omega)$ $\mu_A$-a.s. Thus, Tonelli's Theorem shows that there exists a measurable $\mu_A$-null set $N$ such that for $t \notin N$, we have $Z^n_t$ converges to $Z_t$ $P$-a.s. For $t \geq 0$ define $Y_t := Z_t 1_{N^c}(t)$. Then $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-optional, $(Y_t)_{t\geq 0} \subseteq \mathbb{G}$ and $(Y_t)_{t\geq 0}$ satisfies (4.1). For all Gaussian random variables $X$ we have $\|X\|_{L^2(P)} \leq 3\|X\|_{L^1(P)}$. Now it follows

$$\mu_A((0,t]) = E\Big[\int_0^t |Y_s|\,\mu_A(ds)\Big] \geq 1/3 \int_0^t \|Y_s\|_{L^2(P)}\,\mu_A(ds),$$

by which we conclude that $\|Y_t\|_{L^2(P)} \leq 3$ for $\mu_A$-a.a. $t \geq 0$.

If $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable Jacod and Shiryaev (2003, Proposition 3.13) shows that the above $(Z_t)_{t\geq 0}$ can be chosen $(\mathcal{F}_t)_{t\geq 0}$-predictable and therefore $(Y_t)_{t\geq 0}$ will be $(\mathcal{F}_t)_{t\geq 0}$-predictable as well. □

The above result characterizes Gaussian processes of bounded variation. Indeed it follows from Proposition 4.1 and Lemma 2.2 that $(A_t)_{t\geq 0}$ is a Gaussian process which is right-continuous and of bounded variation if and only if

$$A_t = \int_0^t Y_r\,\mu(dr) \qquad t \geq 0,$$

for a Radon measure $\mu$ on $\mathbb{R}_+$ and a measurable Gaussian process $(Y_t)_{t\geq 0}$ which is bounded in $L^2(P)$.

Recall the definition of $\mu_M$ on page 3. Moreover, recall (e.g. from Rogers and Williams (1987)) the definition of the dual predictable projection of non-adapted processes.

**Proposition 4.2.** *Let $\mu$ be Radon measure on $\mathbb{R}_+$, $(M_t)_{t\geq 0}$ be a càdlàg Gaussian martingale and $\Psi \in L^{2,1}(\mu_M, \mu)$. Define*

$$A_t := \int_0^t \Big(\int \Psi_r(s)\,dM_s\Big)\mu(dr), \qquad t \geq 0.$$

*Then the dual $(\mathcal{F}_t)_{t\geq 0}$-predictable projection of $(A_t)_{t\geq 0}$ is for $t \geq 0$ given by*

$$A^p_t = \int_0^t \Big(\int_s^t \Psi_r(s)\,\mu(dr)\Big)dM_s = \int_0^t \Big(\int 1_{(0,r)}(s)\Psi_r(s)\,dM_s\Big)\mu(dr). \tag{4.2}$$

*In particular $(A_t)_{t\geq 0}$ is $(\mathcal{F}^M_t)_{t\geq 0}$-predictable if and only if $\Psi_t(s) = 0$ for $\mu_M \otimes \mu$-a.a. $(s,t)$ with $s \geq t$.*

*Proof.* Since $\Psi \in L^{2,1}(\mu_M, \mu)$ Lemma 2.2 shows that
$(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(dr) \in \mathcal{BV}(\mu)$. Now Lemma 3.2 and Lemma 4.3 below shows that

$$A^p_t = \int_0^t \Big(\int_s^t \Psi_r(s)\,\mu(dr)\Big)dM_s, \qquad t \geq 0.$$

The last identity in (4.2) follows from Lemma 3.2.

To conclude we note that $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable if and only if $A_t = A_t^{\mathrm{p}}$ for all $t \geq 0$. From (4.2) this is the case if and only if for $P \otimes \mu$-a.a. $(\omega, r)$ we have

$$\int 1_{(0,r)}(s)\Psi_r(s)\, dM_s(\omega) = \int \Psi_r(s)\, dM_s(\omega).$$

which by the isometric property of the integral corresponds to $1_{(0,r)}(s)\Psi_r(s) = \Psi_r(s)$ for $\mu_M \otimes \mu$-a.a. $(s, r)$. □

**Lemma 4.3.** *Let $(M_t)_{t\geq 0}$ be a càdlàg Gaussian martingale and let $\Psi \in \mathcal{BV}(\mu_M)$ satisfy that $t \mapsto \Psi_t(s)$ is càdlàg for $s \geq 0$. Then $s \mapsto \Psi_s(s)$ is locally $\mu_M$-square integrable. Let furthermore $(A_t)_{t\geq 0}$ be a modification of $(\int \Psi_t(s)\, dM_s)_{t\geq 0}$ which is right-continuous and of bounded variation. (Such a modification exists according to Lemma 2.3). Then the dual $(\mathcal{F}_t^M)_{t\geq 0}$-predictable projection of $(A_t)_{t\geq 0}$ exists and is given by*

$$A_t^p = \int_0^t \Big(\Psi_t(s) - \Psi_s(s)\Big) dM_s, \qquad t \geq 0. \tag{4.3}$$

*In particular, $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable if and only if for $t \geq 0$ we have $\Psi_t(s) = 0$ for $\mu_M$-a.a. $s \in [t, \infty)$.*

*Proof*. Fix $t \geq 0$. General theory shows that for $t \geq 0$ we have

$$\frac{1}{h}\int_0^t E[A_{u+h} - A_u | \mathcal{F}_u^M]\, du \to A_t^{\mathrm{p}} \text{ in the } \sigma(L^1, L^\infty)\text{-topology, as } h \downarrow 0, \tag{4.4}$$

see e.g. Dellacherie and Meyer (1982, Theorem 21.1). Thus from Gaussianity the convergence also takes place in the $\sigma(L^2, L^2)$-topology. We have

$$\begin{aligned}\frac{1}{h}\int_0^t E[A_{u+h} - A_u | \mathcal{F}_u^M]\, du &= \frac{1}{h}\int_0^t \Big(\int_0^u \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\, dM_s\Big) du \\ &= \int_0^t \Big(\frac{1}{h}\int_s^t \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\, du\Big) dM_s,\end{aligned}$$

where the second equality follows from Lemma 3.2 since $\Psi \in \mathcal{BV}(\mu_M) \subseteq L^{2,1}(\mu_M, \lambda)$ ($\lambda$ denotes the Lebesgue measure on $\mathbb{R}$). Thus (4.4) implies that there exists an $f_t \in L^2(\mu_M)$ such that

$$1_{[0,t]}(s)\frac{1}{h}\int_s^t \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\, du \longrightarrow_{h\downarrow 0} f_t(s) \qquad \text{in the } \sigma(L^2, L^2)$$

and $A_t^{\mathrm{p}} = \int_0^t f_t(s)\, dM_s$. Fix $s \in [0, t]$. The right-continuity of $t \mapsto \Psi_t(s)$ implies that

$$\begin{aligned}&\frac{1}{h}\int_s^t \big(\Psi_{u+h}(s) - \Psi_u(s)\big)\, du \\ &= \frac{1}{h}\int_t^{t+h} \Psi_u(s)\, du - \frac{1}{h}\int_s^{s+h} \Psi_u(s)\, du \to \Psi_t(s) - \Psi_s(s), \qquad \text{as } h \downarrow 0.\end{aligned}$$

This shows $f_t(s) = \Psi_t(s) - \Psi_s(s)$ for $\mu_M$-a.a. $s \in [0, t]$ and the proof of (4.3) is complete.

Since $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-predictable if and only if $A_t = A_t^{\mathrm{p}}$ for $t \geq 0$ the last part of the result is immediate. □

*Remark* 4.4. By writing $s \mapsto \Psi_s(s)$ as a telescoping sum of the functions $s \mapsto \Psi_t(s)$ it can also be seen directly that $s \mapsto \Psi_s(s)$ is locally $\mu_M$-square integrable.

We are now ready to state and prove one of the main results of the paper which describes the bounded variation component of a Gaussian semimartingale and generalizes a result of Stricker (1983).

**Theorem 4.5.** *$(X_t)_{t\geq 0}$ is a Gaussian semimartingale if and only if for $t \geq 0$ we have*

$$X_t = X + M_t + \Big( \int_0^t \Big( \int \Psi_r(s)\, dM_s \Big) \mu(dr) + \int_0^t Y_r\, \mu(dr) \Big), \tag{4.5}$$

*where $\mu$ is a Radon measure, $(M_t)_{t\geq 0}$ is a Gaussian martingale starting at 0, $\Psi$ is a measurable mapping such that $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\mu_M)$ and $\Psi_t(s) = 0$ for $\mu_M \otimes \mu$-a.a. $(s,t)$ with $s \geq t$, $(Y_t)_{t\geq 0}$ is a measurable process which is bounded in $L^2(P)$ and $X$ is a random variable such that $\{Y_t, X : t \geq 0\}$ is Gaussian and independent of $(M_t)_{t\geq 0}$.*

*In this case, $(X_t)_{t\geq 0}$ is (in addition) an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, where $\mathcal{F}_t := \mathcal{F}_t^M \vee \sigma(X, Y_s : s \geq 0)$ for $t \geq 0$ and (4.5) is the $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$.*

*Remark* 4.6. We actually prove the following. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ be an $(\mathcal{F}_t)_{t\geq 0}$-stable Gaussian space. Assume $(X_t)_{t\geq 0} \subseteq \mathbb{G}$ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale with $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition $X_t = X_0 + M_t + A_t$. Then $(X_t)_{t\geq 0}$ can be decomposed as in (4.5) with $\mu = \mu_A$, $(Y_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-predictable and $(M_t)_{t\geq 0}, (Y_t)_{t\geq 0} \subseteq \mathbb{G}$.

Theorem 4.5 also shows the following.

*Remark* 4.7. A Gaussian semimartingale $(X_t)_{t\geq 0}$ with martingale component $(M_t)_{t\geq 0}$ can be decomposed as $X_t = Z_t + B_t$, where $(Z_t)_{t\geq 0}$ is a Gaussian $(\mathcal{F}_t^M)_{t\geq 0}$-semimartingale and $(B_t)_{t\geq 0}$ is a Gaussian $(\mathcal{F}_t^X)_{t\geq 0}$-predictable process independent of $(M_t)_{t\geq 0}$ which is right-continuous and of bounded variation. In particular $\mathcal{F}_t^X = \mathcal{F}_t^M \vee \mathcal{F}_t^B$.

*Proof of Theorem 4.5. Only if*: We prove the more general result stated in Remark 4.6. Thus let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $\mathbb{G}$ be an $(\mathcal{F}_t)_{t\geq 0}$-stable Gaussian space. Assume $(X_t)_{t\geq 0} \subseteq \mathbb{G}$ and that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale with $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition $X_t = X_0 + M_t + A_t$. It follows from Lemma 3.1 (i) that $(A_t)_{t\geq 0}, (M_t)_{t\geq 0} \subseteq \mathbb{G}$, and since $(A_t)_{t\geq 0}$ is of bounded variation, Proposition 4.1 shows that there exists an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(Z_t)_{t\geq 0} \subseteq \mathbb{G}$ such that $\|Z_r\|_{L^2(P)} \leq 3$ for $r \geq 0$ and

$$A_t = \int_0^t Z_s\, \mu_A(ds), \qquad t \geq 0.$$

Let $({}^{\mathrm{p}}Z_t)_{t\geq 0}$ denote the $(\mathcal{F}_t^M)_{t\geq 0}$-predictable projection of $(Z_t)_{t\geq 0}$. The definition of $({}^{\mathrm{p}}Z_t)_{t\geq 0}$ shows that for $t \geq 0$ we have ${}^{\mathrm{p}}Z_t = E[Z_t|\mathcal{F}_{t-}^M]$. From Gaussianity it follows that ${}^{\mathrm{p}}Z_t$ is the projection of $Z_t$ on $\overline{\mathrm{sp}}\{M_s : s < t\}$ and thus ${}^{\mathrm{p}}Z_t \in \mathbb{G}$ for $t \geq 0$. This means that $\|Z_s\|_{L^2(P)} \geq \|{}^{\mathrm{p}}Z_s\|_{L^2(P)}$ for $r \geq 0$. Define $Y_t := Z_t - {}^{\mathrm{p}}Z_t$ for $t \geq 0$. Then $(Y_t)_{t\geq 0} \subseteq \mathbb{G}$ is bounded in $L^2(P)$. We claim that

$$E[Y_u M_t] = 0 \qquad \text{for } t, u \geq 0. \tag{4.6}$$

Since ${}^{\mathrm{p}}Z_u$ is the projection of $Z_u$ on $\overline{\mathrm{sp}}\{M_v : v < t\}$, (4.6) is obviously true for $0 \leq t < u$. Moreover since $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale it remains to be shown $E[Y_u M_u] = 0$ for $u \geq 0$. Fix $u \geq 0$. We have

$$E[Y_u M_u] = E[Y_u E[M_u|\mathcal{F}_{u-}]] = E[Y_u M_{u-}] = 0,$$

where the first equality follows since $Y_u$ is $\mathcal{F}_{u-}$ measurable and the second equality follows since $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale. This completes the proof of (4.6).

Since $(Y_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ both are subsets of $\mathbb{G}$ and $(M_t)_{t\geq 0}$ is a centered process, (4.6) implies that $(Y_t)_{t\geq 0}$ is independent of $(M_t)_{t\geq 0}$. It follows from Lemma 3.1 (ii) and Lemma 2.1 that ${}^{\mathrm{p}}Z_t = E[{}^{\mathrm{p}}Z_t] + \int \Psi_t(s)\, dM_s$ for $t \geq 0$ and some measurable mapping $\Psi$ such that $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\mu_M)$. Since $({}^{\mathrm{p}}Z_t)_{t\geq 0}$ is $(\mathcal{F}^M_t)_{t\geq 0}$-predictable, Proposition 4.2 shows $\Psi_t(s) = 0$ for $\mu_M \otimes \mu_A$-a.a. $(s,t)$ with $s \geq t$. This completes the proof of (4.5), by using $\tilde{Y}_t := Y_t + E[{}^{\mathrm{p}}Z_t]$ instead of $(Y_t)_{t\geq 0}$.

*If*: Assume conversely that (4.5) is satisfied. By Lemma 4.2

$$\int_0^t \Big( \int \Psi_r(s)\, dM_s \Big) \mu(dr), \qquad t \geq 0,$$

is $(\mathcal{F}^M_t)_{t\geq 0}$-predictable. Hence, $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale ($\mathcal{F}_t := \mathcal{F}^M_t \vee \sigma(X, Y_s : s \geq 0)$) and the $(\mathcal{F}_t)_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is (4.5). Since $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale, Stricker's Theorem (see Protter (2004, Chapter 2, Theorem 4)) shows that $(X_t)_{t\geq 0}$ in particular is a semimartingale, that is an $(\mathcal{F}^X_t)_{t\geq 0}$-semimartingale. This completes the proof. □

In the following we study the canonical decomposition of a Gaussian semimartingales. For a stochastic process $(X_t)_{t\in\mathbb{R}}$ we let $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$ denote the least filtration for which $X_s$ is $\mathcal{F}^{X,\infty}_t$-measurable for $t \geq 0$ and $s \in (-\infty, t]$.

**Theorem 4.8.** *Let $(X_t)_{t\in\mathbb{R}}$ be Gaussian process which either is stationary or has stationary increments and satisfies $X_0 = 0$. Assume $(X_t)_{t\geq 0}$ is a semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$. Then we have*

*(i) $(M_t)_{t\geq 0}$ is a Wiener process and hence $\mu_M$ equals the Lebesgue measure up to a scaling constant. Moreover $\mu_A$ is absolutely continuous with increasing density.*

*(ii) $(A_t)_{t\geq 0}$ has stationary increments if and only if $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$.*

*(iii) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale if and only if $\mu_A$ has a bounded density.*

*Proof*. The stationary increments of $(X_t)_{t\geq 0}$ imply that $(X_t)_{t\geq 0}$ has no fixed points of discontinuity. Since in addition $(X_t)_{t\geq 0}$ is a Gaussian semimartingale, it is continuous (see Stricker (1983, Proposition 3)). By continuity of $(X_t)_{t\geq 0}$ it follows that $(A_t)_{t\geq 0}$ is continuous as well.

(i): Since $(A_t)_{t\geq 0}$ is continuous we have $[M]_t = [X]_t$ for $t \geq 0$. (For a process $(Z_t)_{t\geq 0}$, $[Z]_t$ denotes the quadratic variation of $(Z_t)_{t\geq 0}$ on $[0,t]$.) By the stationary increments of $(X_t)_{t\geq 0}$, it follows that $[X]_t = Kt$ for all $t \geq 0$ and some constant $K \in \mathbb{R}_+$. Thus by Gaussianity it follows that $(M_t)_{t\geq 0}$ has stationary increments and therefore is a Wiener process with parameter $K$.

Let $v \geq 0$ be given and define $\mathcal{F}_t^{X,v} := \mathcal{F}_t^X \vee \sigma(X_s : s \in [-v, 0])$ for $t \geq 0$. In the following we shall use that for $0 \leq t_0 < t_1 < \cdots < t_n$ we have

$$(E[X_{t_i+v} - X_{t_{i-1}+v} | \mathcal{F}_{t_{i-1}+v}^X])_{i=1}^n \overset{\mathcal{D}}{=} (E[X_{t_i} - X_{t_{i-1}} | \mathcal{F}_{t_{i-1}}^{X,v}])_{i=1}^n. \tag{4.7}$$

In the case where $(X_t)_{t\in\mathbb{R}}$ has stationary increments and satisfies $X_0 = 0$ this is due to

$$\big(X_{t_i+v} - X_{t_{i-1}+v}, (X_s)_{s\in[0,t_{i-1}+v]}\big)_{i=1}^n \overset{\mathcal{D}}{=} \big(X_{t_i} - X_{t_{i-1}}, (X_s - X_{-v})_{s\in[-v,t_{i-1}]}\big)_{i=1}^n,$$

and $\sigma(X_s - X_{-v} : s \in [-v, t_{i-1}]) = \mathcal{F}_{t_{i-1}}^{X,v}$ for $i = 1, \ldots, n$. In the stationary case it follows since

$$\big(X_{t_i+v} - X_{t_{i-1}+v}, (X_s)_{s\in[0,t_{i-1}+v]}\big)_{i=1}^n \overset{\mathcal{D}}{=} \big(X_{t_i} - X_{t_{i-1}}, (X_s)_{s\in[-v,t_{i-1}]}\big)_{i=1}^n.$$

From (4.7) it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,v})_{t\geq 0}$-local quasimartingale and therefore also an $(\mathcal{F}_t^{X,v})_{t\geq 0}$-special semimartingale. Let $(A_t^v)_{t\geq 0}$ be the bounded variation component of $(X_t)_{t\geq 0}$ in the filtration $(\mathcal{F}_t^{X,v})_{t\geq 0}$. For $0 \leq u \leq t$ we have

$$\begin{aligned} A_t - A_u &= \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}_{(i-1)/2^n}^X] \\ &= \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[A_{i/2^n}^v - A_{(i-1)/2^n}^v | \mathcal{F}_{(i-1)/2^n}^X] \qquad \text{in } L^2(P), \end{aligned}$$

which shows

$$\|A_t - A_u\|_{L^1(P)} \leq \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[t2^n]} \|A_{i/2^n}^v - A_{(i-1)/2^n}^v\|_{L^1(P)} = \mu_{A^v}((u,t]). \tag{4.8}$$

From (4.7) it follows that (the limits are in $L^2(P)$)

$$\begin{aligned} A_t^v - A_u^v &= \lim_{n\to\infty} \sum_{[u2^n]+1}^{[t2^n]} E[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}_{(i-1)/2^n}^{X,v}] \\ &\overset{\mathcal{D}}{=} \lim_{n\to\infty} \sum_{[u2^n]+1}^{[t2^n]} E[X_{i/2^n+v} - X_{(i-1)/2^n+v} | \mathcal{F}_{(i-1)/2^n+v}^X] = A_{t+v} - A_{u+v}, \end{aligned}$$

and hence $\mu_{A^v}((u,t]) = \mu_A((u+v, t+v])$. Thus by (4.8) we conclude that

$$\mu_A((u,t]) \leq \mu_A((u+v, t+v]), \qquad 0 \leq u \leq t,\ 0 \leq v. \tag{4.9}$$

Define $f(t) := \mu_A((0,t])$ for $t \geq 0$ and let $T \geq 0$ be given. Choose a $t_0 \geq T$ such that $f$ is differentiable at $t_0$. Moreover let $t, h \geq 0$ satisfy $t + h \leq T$. Then

$$\begin{aligned} \mu_A((t, t+h]) &= f(t+h) - f(t) = \sum_{i=1}^n f(t + ih/n) - f(t + (i-1)h/n) \\ &\leq \sum_{i=1}^n f(t_0 + h/n) - f(t_0) = h \frac{f(t_0 + h/n) - f(t_0)}{h/n} \xrightarrow[n\to\infty]{} h f'(t_0), \end{aligned}$$

which shows $f$ is locally Lipschitz continuous and hence $\mu_A$ is absolutely continuous. From (4.9) it follows that $\mu_A$ has an increasing density.

(ii): Assume $(A_t)_{t\geq 0}$ has stationary increments. For $t \geq 0$ (4.7) shows

$$\begin{aligned} A_t &\overset{\mathcal{D}}{=} A_{t+v} - A_v = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[X_{i/2^n+v} - X_{(i-1)/2^n+v} | \mathcal{F}^X_{(i-1)/2^n+v}] \\ &\overset{\mathcal{D}}{=} \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] =: A_t^v \qquad \text{in } L^2(P). \end{aligned}$$

By the rules of successive conditioning it follows that $E[A_t A_t^v] = E[A_t^2]$. Since in addition $A_t \overset{\mathcal{D}}{=} A_t^v$ this shows that

$$E[(A_t - A_t^v)^2] = E[(A_t^v)^2] - E[A_t^2] = 0,$$

and hereby

$$A_t = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] \qquad \text{in } L^2(P).$$

This yields

$$M_t = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} X_{i/2^n} - X_{(i-1)/2^n} - E[X_{i/2^n} - X_{(i-1)/2^n} | \mathcal{F}^{X,v}_{(i-1)/2^n}] \qquad \text{in } L^2(P),$$

which implies that $M_t$ is independent of $X_u$ for $u \in [-v, 0]$. Since $v, t \in \mathbb{R}_+$ were arbitrarily chosen, we conclude that $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$.

Assume on the other hand that $(M_t)_{t\geq 0}$ is independent of $(X_t)_{t\leq 0}$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale with $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition given by $X_t = X_0 + A_t + M_t$. For $0 \leq u \leq t$ and $0 \leq v$ we have

$$A_{t+v} - A_{u+v} = \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[t2^n]} E[X_{i/2^n+v} - X_{(i-1)/2^n+v} | \mathcal{F}^{X,\infty}_{(i-1)/2^n+v}] \qquad \text{in } L^2(P)$$

from which we conclude that $(A_t)_{t\geq 0}$ has stationary increments.

(iii): Let $(X_t)_{t\geq 0}$ be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and let $(B_t)_{t\geq 0}$ denote the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-bounded variation component of $(X_t)_{t\geq 0}$. By arguments as above it follows that $(B_t)_{t\geq 0}$ has stationary increments and hence $E[|B_t - B_u|] \leq K(t-u)$ for $0 \leq u \leq t$ and some constant $K \in \mathbb{R}_+$. For $t \geq 0$ we have

$$A_t = \lim_{n\to\infty} \sum_{i=1}^{[t2^n]} E[B_{i/2^n} - B_{(i-1)/2^n} | \mathcal{F}^X_{(i-1)/2^n}] \qquad \text{in } L^2(P),$$

and hence

$$E[|A_t - A_u|] \leq \lim_{n\to\infty} \sum_{i=[u2^n]+1}^{[t2^n]} E[|E[B_{i/2^n} - B_{(i-1)/2^n} | \mathcal{F}^X_{(i-1)/2^n}]|] \leq K(t-u),$$

which shows $\mu_A$ has a bounded density.

Assume conversely that $\mu_A$ has a bounded density and let $K \in \mathbb{R}_+$ be a constant dominating the density. For $0 \leq u \leq t$ we have

$$\begin{aligned} E[|E[X_t - X_u|\mathcal{F}_u^{X,\infty}]|] &= \lim_{n\to\infty} E[|E[X_t - X_u|\mathcal{F}_u^{X,n}]|] \\ &= \lim_{n\to\infty} E[|E[X_{t+n} - X_{u+n}|\mathcal{F}_{u+n}^X]|] = \lim_{n\to\infty} E[|E[A_{t+n} - A_{u+n}|\mathcal{F}_{u+n}^X]|] \\ &\leq \lim_{n\to\infty} \mu_A((u+n, t+n]) \leq K(t-u). \end{aligned}$$

This shows $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-quasimartingale on bounded intervals and hence an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. □

Let $(X_t)_{t\geq 0}$ be a stationary Gaussian semimartingale with covariance function $\gamma(t) := \mathrm{Cov}(X_{u+t}, X_u) = E[X_t X_0] - E[X_0^2]$ for $t \geq 0$. Then $\gamma$ is locally Lipschitz continuous and Lipschitz continuous if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

To show this, let $(A_t)_{t\geq 0}$ be the bounded variation component of $(X_t)_{t\geq 0}$. For $0 \leq u, t$ we have

$$|\gamma(t+u) - \gamma(u)| = |E[(A_{t+u} - A_u)X_0]| \leq \|A_t - A_u\|_{L^2(P)} \|X_0\|_{L^2(P)},$$

and the statement thus follows from Theorem 4.8.

We believe that among the stationary Gaussian processes $(X_t)_{t\geq 0}$, the class of $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingales is strictly larger than the class of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales. However, we haven't found an example of an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale which isn't an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. This is equivalent (according to Theorem 4.8 (iii)) to finding a stationary Gaussian semimartingale $(X_t)_{t\geq 0}$ for which $\mu_A$ has an unbounded density ( $(A_t)_{t\geq 0}$ denotes the bounded variation component of $(X_t)_{t\geq 0}$).

# 5 The covariance function of Gaussian semimartingales

If $(X_t)_{t\geq 0}$ is a Gaussian process we let $\Gamma_X$ denote the corresponding covariance function, i.e. $\Gamma_X(t,s) := E[(X_t - E[X_t])(X_s - E[X_s])]$ for all $s, t \geq 0$. We need the following.

**Condition 5.1.** A function $G\colon \mathbb{R}_+^2 \to \mathbb{R}$ satisfies Condition 5.1, if $G$ is symmetric, positive semi-definite and there exists a right-continuous increasing function $f$ such that for all $0 \leq s \leq t$

$$\sqrt{G(t,t) + G(s,s) - 2G(s,t)} \leq f(t) - f(s).$$

Recall that $G$ is positive semi-definite if

$$\sum_{i,j=1}^n a_i G(t_i, t_j) a_j \geq 0$$

for all $n \geq 1$, $a_1, \ldots, a_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in \mathbb{R}_+$.

Assume that $G$ satisfies Condition 5.1 and denote by $(A_t)_{t\geq 0}$ a centered Gaussian process satisfying $\Gamma_A = G$. Then by Lemma 2.3 there exists a modification of $(A_t)_{t\geq 0}$ which is right-continuous and of bounded variation. Conversely, if $(A_t)_{t\geq 0}$ is a right-continuous Gaussian process of bounded variation, then $\Gamma_A$ satisfies Condition 5.1 with $f(t) = E[V_t(A)]$ for $t \geq 0$ since $(A_t)_{t\geq 0}$ is of integrable variation.

Thus $G$ satisfies Condition 5.1 if and only if $G = \Gamma_A$ for some right-continuous Gaussian process $(A_t)_{t\geq 0}$ of bounded variation.

A measurable mapping $\mathbb{R}_+^2 \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ is said to be a Volterra type kernel if $\Psi_t(s) = 0$ for all $s > t$. (A Volterra kernel is often assumed to be an $L^2(\lambda)$-kernel see e.g. Baudoin and Nualart (2003) and Smithies (1958). However, the latter assumption is not needed here.) Let $\mathbb{1}$ denote the Volterra type kernel given by $\mathbb{R}_+^2 \ni (t,s) \mapsto \mathbb{1}_t(s) = 1_{[0,t]}(s)$.

The next result is a new characterization of the covariance function of Gaussian semimartingales. The result is only formulated for centered Gaussian processes. This is no restriction since a Gaussian process $(X_t)_{t\geq 0}$ is a semimartingale if and only if $t \mapsto E[X_t]$ is right-continuous and of bounded variation and $(X_t - E[X_t])_{t\geq 0}$ is a semimartingale. To see this it is enough to show that the mean-value function of a Gaussian semimartingale is of bounded variation. Let $(X_t)_{t\geq 0}$ be a Gaussian semimartingale with bounded variation component $(A_t)_{t\geq 0}$. For $0 \leq u \leq t$ we have

$$|E[X_t] - E[X_u]| = |E[A_t] - E[A_u]| \leq E[V_t(A)] - E[V_u(A)],$$

by which we conclude that the mean-value function of $(X_t)_{t\geq 0}$ is of bounded variation.

**Theorem 5.2.** *Let $(X_t)_{t\geq 0}$ be a centered Gaussian process. Then the following conditions are equivalent:*

*(i) $(X_t)_{t\geq 0}$ is a semimartingale.*

*(ii) There exists a Radon measure $\mu$ on $\mathbb{R}_+$, a Volterra type kernel $\Phi$ such that $\Phi - \mathbb{1} \in \mathcal{BV}(\mu)$, and a function $G$ satisfying Condition 5.1 such that*

$$\Gamma_X(t,u) = G(t,u) + \int \Phi_t(s)\Phi_u(s)\,\mu(ds), \qquad u,t \geq 0.$$

*(iii) There exist Radon measures $\mu$ and $\nu$ on $\mathbb{R}_+$, a function $G$ satisfying Condition 5.1 and a Volterra type kernel $\Psi$ such that $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\nu)$ and such that for $0 \leq u,t$ we have*

$$\begin{aligned}\Gamma_X(t,u) = &G(t,u) + \nu((0,t\wedge u]) + \int_0^t \int_0^u \Psi_r(s)\,\nu(ds)\,\mu(dr) \\ &+ \int_0^t \int_0^u \Psi_r(s)\,\mu(dr)\nu(ds) + \int_0^t \int_0^u \langle \Psi_r, \Psi_v\rangle_{L^2(\nu)}\,\mu(dr)\,\mu(dv).\end{aligned} \tag{5.1}$$

*Proof.* We show (iii) ⇒ (ii) ⇒ (i) ⇒ (iii).

Assume (iii) is satisfied. Equation (5.1) can be written as

$$\Gamma_X(t,u) = G(t,u) + \int \Big(\mathbb{1}_t(s) + \int_0^t \Psi_r(s)\,\mu(dr)\Big)\Big(\mathbb{1}_u(s) + \int_0^u \Psi_r(s)\,\mu(dr)\Big)\,\nu(ds).$$

By Lemma 2.2, $(t,s) \mapsto \int_0^t \Psi_r(s)\,\mu(dr) \in \mathcal{BV}(\nu)$ which shows (ii).

Assume (ii) is satisfied. To show that $(X_t)_{t\geq 0}$ is a semimartingale it is enough to show that there exists a Gaussian semimartingale $(Z_t)_{t\geq 0}$ such that $(Z_t)_{t\geq 0}$ is distributed as $(X_t)_{t\geq 0}$. Indeed, assume that $(Z_t)_{t\geq 0}$ has been constructed. Then since $(Z_t)_{t\geq 0}$ is a càdlàg process, $(X_t)_{t\geq 0}$ is càdlàg through the rational numbers, and since $(X_t)_{t\geq 0}$ is right-continuous in $L^2(P)$, is it possible to choose a càdlàg modification of $(X_t)_{t\geq 0}$. For all $0 \leq s \leq t$ we have

$$E[|E[Z_t - Z_s|\mathcal{F}_s^Z]|] = E[|E[X_t - X_s|\mathcal{F}_s^X]|]. \tag{5.2}$$

Since a Gaussian process is a semimartingale if and only if it is quasimartingale on $[0,T]$ for all $T > 0$ according to Liptser and Shiryayev (1989) [Chapter 4, Section 9, Corollary of Theorem 1] and [Chapter 2, Section 1, Theorem 4], equation (5.2) shows that $(X_t)_{t\geq 0}$ is a semimartingale.

To construct $(Z_t)_{t\geq 0}$, note that since $G$ satisfies Condition 5.1 there exist two independent processes $(A_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$, with the properties that $(M_t)_{t\geq 0}$ is a càdlàg centered Gaussian martingale with $\mu_M = \mu$ for all $t \geq 0$ and $(A_t)_{t\geq 0}$ is a right-continuous centered Gaussian process of bounded variation such that $\Gamma_A = G$. Let $\Theta := \Phi - \mathbb{1}$ and

$$Z_t := M_t + \int \Theta_t(s)\,dM_s + A_t.$$

Then $(Z_t)_{t\geq 0}$ is a well-defined centered Gaussian process. Since $\Theta \in \mathcal{BV}(\mu)$, Lemma 2.3 implies that $(\int \Theta_t(s)\,dM_s)_{t\geq 0}$ can be chosen right-continuous and of bounded variation. Moreover, since $\Theta$ is a Volterra type kernel, $(\int \Theta_t(s)\,dM_s)_{t\geq 0}$ is $(\mathcal{F}_t^M)_{t\geq 0}$-adapted. Hence, since $(A_t)_{t\geq 0}$ is independent of $(M_t)_{t\geq 0}$, $(Z_t)_{t\geq 0}$ is a semimartingale. Since $\Gamma_X = \Gamma_Z$, Gaussianity implies that $(X_t)_{t\geq 0}$ is distributed as $(Z_t)_{t\geq 0}$, which completes the proof of (i).

Assume finally (i) is satisfied i.e. that $(X_t)_{t\geq 0}$ is a semimartingale. Choose, according to Remark 4.6, $(M_t)_{t\geq 0}, (Y_t)_{t\geq 0}, \Psi$ and $\mu_A$ such that for $t \geq 0$ we have

$$X_t = M_t + \int_0^t \Big(\int \Psi_r(s)\,dM_s\Big)\mu_A(dr) + \int_0^t Y_r\,\mu_A(dr) + X_0.$$

Since $\big(\int_0^t Y_r\,\mu_A(dr)\big)_{t\geq 0}$ is a Gaussian process of bounded variation, it follows that

$$G(t,u) := E\Big[\Big(\int_0^t Y_r\,\mu_A(dr) + X_0\Big)\Big(\int_0^u Y_r\,\mu_A(dr) + X_0\Big)\Big], \qquad t,u \geq 0$$

satisfies Condition 5.1. Since $\{(M_t)_{t\geq 0}, (Y_t)_{t\geq 0}, X_0\}$ are centered simultaneously Gaussian random variables and $(M_t)_{t\geq 0}$ is independent of $\{X_0, (Y_t)_{t\geq 0}\}$, it follows that (5.1) is satisfied. This completes the proof. □

The following definitions are taken from Jain and Monrad (1982). Let $f\colon \mathbb{R}_+^2 \to \mathbb{R}$. For $0 \leq s_1 \leq s_2$ and $0 \leq t_1 \leq t_2$ define

$$\Delta f((s_1,t_1);(s_2,t_2)) := f(s_2,t_2) - f(s_1,t_2) - f(s_2,t_1) + f(s_1,t_1)$$

and

$$V_{s,t}(f) := \sup \sum_{i,j} |\Delta f((s_{i-1},t_{j-1});(s_i,t_j))| + \sum_j |f(0,t_j) - f(0,t_{j-1})| \\ + \sum_i |f(s_i,0) - f(s_{i-1},0)| + |f(0,0)|,$$

where the sup is taken over all subdivisions $0 = s_0 < \cdots < s_p = s$ and $0 = t_0 < \cdots < t_q = t$ of $[0,s] \times [0,t]$.

We say that $f$ is of bounded variation if $V_{s,t}(f) < \infty$ for all $s,t > 0$ and in this case $f$ induces a signed Radon measure $\lambda_f$ on $\mathbb{R}_+^2$ given by

$$\lambda_f([0,t] \times [0,s]) = f(t,s), \qquad s,t \geq 0.$$

A function $f \colon \mathbb{R}_+^2 \to \mathbb{R}$ of bounded variation is said to be absolutely continuous if $(s,t) \mapsto V_{s,t}(f)$ is the restriction to $\mathbb{R}_+^2$ of the distribution function of a measure on $\mathbb{R}^2$ which is absolutely continuous w.r.t. $\lambda_2$ (the planar Lebesgue measure). This is equivalent to the existence of three locally integrable functions $h_1, h_2$ and $g$ such that

$$f(s,t) = \int_0^s \int_0^t g(u,v)\,du\,dv + \int_0^s h_1(u)\,du + \int_0^t h_2(v)\,dv + f(0,0).$$

If $\mu$ is a Radon measure on $\mathbb{R}_+$, let $\mu\Delta\mu$ denote the measure on $\mathbb{R}_+^2$ for which $(\mu\Delta\mu)(A \times B) = \mu(A \cap B)$ for all $A, B \in \mathcal{B}(\mathbb{R}_+)$. Let $\Delta := \{(x,y) \in (0,\infty)^2 : x = y\}$ denote the diagonal of $(0,\infty)^2$ and note that $\mu\Delta\mu$ is concentrated on $\Delta$ if $\mu$ has no mass at zero.

From the representation (5.1) it is easily seen that the covariance function of a Gaussian semimartingale is of bounded variation (a direct proof can be found e.g. in Liptser and Shiryayev (1989)). Thus if $(X_t)_{t\geq 0}$ is a Gaussian semimartingale then $\Gamma_X$ induces a signed Radon measure $\lambda_{\Gamma_X}$ on $\mathbb{R}_+^2$. The following result characterizes the martingales and the processes of bounded variation among the Gaussian semimartingales.

**Corollary 5.3.** *Let $(X_t)_{t\geq 0}$ be a continuous Gaussian semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$. Then $\lambda_{\Gamma_X}$ equals $\mu_M \Delta \mu_M$ on $\Delta$ and $\lambda_{\Gamma_X}$ is absolutely continuous with respect to $\mu \times \mu$ on $\mathbb{R}_+^2 \setminus \Delta$, where $\mu = \mu_M + \mu_A + \delta_0$ ($\delta_0$ denote the Dirac measure at 0). This shows that $(X_t)_{t\geq 0}$ is a martingale if and only if $\lambda_{\Gamma_X}$ is concentrated on $\Delta \cup \{(0,0)\}$ and is of bounded variation if and only if $\lambda_{\Gamma_X}$ is concentrated on $\mathbb{R}_+^2 \setminus \Delta$.*

*Proof.* Decompose $(A_t)_{t\geq 0}$ as in Remark 4.6 and let $u,t \geq 0$. By Fubini's Theorem,

$$\begin{aligned}
\Gamma_X(t,u) &= \operatorname{Cov}\Big[M_t + \int_0^t \int \Psi_r(s)\,dM_s\,\mu_A(dr), M_u + \int_0^u \int \Psi_r(s)\,dM_s\,\mu_A(dr)\Big] \\
&\quad + \operatorname{Cov}\Big[X_0 + \int_0^t Y_r\,\mu_A(dr), X_0 + \int_0^u Y_r\,\mu_A(dr)\Big] \\
&= \mu_M([0,t\wedge u]) + \int_0^t \int_0^u \Psi_r(s)\,\mu_M(ds)\,\mu_A(dr) + \int_0^t \int_0^u \Psi_r(s)\,\mu_A(dr)\,\mu_M(ds) \\
&\quad + \int_0^t \int_0^u \langle \Psi_r, \Psi_v \rangle_{L^2(\mu_M)}\,\mu_A(dr)\,\mu_A(dv) + \int_0^t \int_0^u E[Y_r Y_v]\,\mu_A(dr)\,\mu_A(dv) \\
&\quad + \int_0^t E[X_0 Y_r]\,\mu_A(dr) + \int_0^u E[X_0 Y_r]\,\mu_A(dr) + E[X_0^2],
\end{aligned}$$

which shows that there exists a measurable function $f\colon \mathbb{R}_+^2 \to \mathbb{R}$ such that with $\mu := \mu_M + \mu_A + \delta_0$ we have

$$\lambda_{\Gamma_X}([0,t]\times[0,u]) = \mu_M \Delta \mu_M([0,t]\times[0,u]) + \int_{[0,t]\times[0,u]} f \, d\mu \times \mu. \tag{5.3}$$

Furthermore, the continuity of $(X_t)_{t\geq 0}$ implies that $\mu$ is nonatomic on $(0,\infty)$ and by (5.3) we derive that $\lambda_{\Gamma_X}$ equals $\mu_M \Delta \mu_M$ on $\Delta$. From (5.3) we also derive that $\lambda_{\Gamma_X}$ is absolutely continuous with respect to $\mu \times \mu$ on $\mathbb{R}_+^2 \setminus \Delta$ and the proof is complete. □

Note that the distribution of a Gaussian martingale $(M_t)_{t\geq 0}$ is uniquely determined by $\mu_M$. Moreover Corollary 5.3 shows that for a continuous Gaussian semimartingale $(X_t)_{t\geq 0}$ with martingale component $(M_t)_{t\geq 0}$ we have

$$\mu_M((0,t]) = \lambda_{\Gamma_X}((s_1,s_2) \in \mathbb{R}_+^2 : s_1 = s_2 \leq t), \qquad t \geq 0.$$

Thus it is easy to find the distribution of the martingale component $(M_t)_{t\geq 0}$ from $\Gamma_X$.

The following two examples are applications of Corollary 5.3.

**Example 5.4.** The fractional Brownian Motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian processes $(X_t)_{t\geq 0}$ with covariance function

$$\Gamma_X(t,u) = \tfrac{1}{2}(t^{2H} + u^{2H} - |t-u|^{2H}). \tag{5.4}$$

Let $\varepsilon > 0$ be given. Below we prove that the fBm is not a semimartingale on $[0,\varepsilon]$ if $H \neq 1/2$. Let $H \neq 1/2$ and assume (for contradiction) that $(X_t)_{t\in[0,\varepsilon]}$ is a semimartingale. The Kolmogorov Continuity Theorem shows that $(X_t)_{t\geq 0}$ can be chosen continuous and using (5.4) it follows that

$$\int_0^t \int_0^u \frac{\partial^2 \Gamma_X}{\partial s \partial v} \, d\lambda_2 = \Gamma_X(t,u), \qquad t,u \geq 0, \tag{5.5}$$

which shows $\Gamma_X$ is absolutely continuous. Hereby we deduce that $(X_t)_{t\in[0,\varepsilon]}$ is of bounded variation on $[0,\varepsilon]$ (Corollary 5.3) and therefore also of integrable variation by Gaussianity. This contradicts that

$$\|X_t - X_u\|_{L^1(P)} = \sqrt{2/\pi}\, |t-u|^H, \qquad t,u \geq 0,$$

and we conclude that $(X_t)_{t\in[0,\varepsilon]}$ is not a semimartingale. For $H = 1/2$, we have $\frac{\partial^2 \Gamma_X}{\partial s \partial v} = 0$ $\lambda_2$-a.s. and hence (5.5) doesn't hold. ◊

**Example 5.5.** Let $(W_t)_{t\geq 0}$ be a Brownian Motion and define $(X_t)_{t\geq 0} := (W_{t+1} - W_t)_{t\geq 0}$. We want to show that $(X_t)_{t\in[0,\alpha]}$ is not a semimartingale for any $\alpha > 1$. Notice that

$$\Gamma_X(t,u) = (1 - |t-u|)^+, \qquad t,u \geq 0.$$

For contradiction assume that $(X_t)_{t\in[0,\alpha]}$ is a semimartingale. Theorem 4.8 shows that $\mu_M$ and $\mu_A$ are absolutely continuous and this implies by Corollary 5.3 that $\lambda_{\Gamma_X}$ is absolutely continuous on $[0,\alpha]^2 \setminus \Delta$. Since $\frac{\partial^2 \Gamma_X}{\partial u \partial t} = 0$ $\lambda_2$-a.s. there exist two functions $g_1$ and $g_2$ such that

$$\Gamma_X(u,t) = \int_0^t g_1(s)\,ds + \int_0^u g_2(s)\,ds + 1, \qquad 0 \le u < t \le \alpha.$$

For all $u \in [0, \alpha - 1]$ we have

$$0 = \Gamma_X(u,\alpha) = \int_0^\alpha g_1(s)\,ds + \int_0^u g_2(s)\,ds + 1,$$

which shows that $g_2(s) = 0$ for $\lambda$-a.a. $s \in [0, \alpha - 1]$. This contradicts

$$u = \Gamma_X(u,1) = \int_0^1 g_1(s)\,ds + 0 + 1, \qquad u \in [0, \alpha - 1],$$

and we have shown that $(X_t)_{t\in[0,\alpha]}$ is not a semimartingale.

Even though $(X_t)_{t\ge 0}$ is not a semimartingale on $\mathbb{R}_+$, we now show that on $[0,1]$ it is. By Yor (1997), $(W_t + W_1)_{t\in[0,1]}$ is a semimartingale with canonical decomposition

$$\Big(W_t - \int_0^t \frac{W_1 - W_s}{1-s}\,ds\Big) + \int_0^t \frac{W_1 - W_s}{1-s}\,ds + W_1. \tag{5.6}$$

Let

$$\mathcal{F}_t := \sigma(W_{s+1} - W_1 : s \in [0,t]) \vee \sigma(W_s : s \in [0,t]) \vee \sigma(W_1), \qquad t \ge 0.$$

Then (5.6) shows that $(X_t)_{t\in[0,1]}$ is a $(\mathcal{F}_t)_{t\in[0,1]}$-semimartingale with $(\mathcal{F}_t)_{t\in[0,1]}$-canonical decomposition given by

$$X_t = \Big[W_{t+1} - W_1 - W_t + \int_0^t \frac{W_1 - W_s}{1-s}\,ds\Big] - \int_0^t \frac{W_1 - W_s}{1-s}\,ds + X_0, \tag{5.7}$$

where the term in the first bracket is the martingale component. By forming the dual $(\mathcal{F}_t^X)_{t\in[0,1]}$-predictable projection on the bounded variation component of (5.7) it follows that the $(\mathcal{F}_t^X)_{t\in[0,1]}$-canonical decomposition of $(X_t)_{t\in[0,1]}$ is given by

$$X_t = \Big(W_{t+1} - W_1 - W_t + \int_0^t \frac{W_1 - E[W_s|\mathcal{F}_s^X]}{1-s}\,ds\Big) - \int_0^t \frac{W_1 - E[W_s|\mathcal{F}_s^X]}{1-s}\,ds + X_0.$$

Note that, even though $(X_t)_{t\ge 0}$ is not a semimartingale on $\mathbb{R}_+$ the quadratic variation of $(X_t)_{t\ge 0}$ does exist, and it is given by $[X]_t = 2t$ for all $t \ge 0$. $\Diamond$

It is known that the processes in Example 5.4 and 5.5 not are semimartingales (for the fBm case see Rogers (1997)). However, the proofs presented here are new and indicate the usefulness of the results in this paper.


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# References


Barndorff-Nielsen, O. and J. Schmiegel (2007). *Ambit processes; with applications to turbulence and cancer growth.* In Proceedings of the 2005 Abel Symposium on Stochastic Analysis and Applications. Heidelberg: Springer. (To appear).

Baudoin, F. and D. Nualart (2003). Equivalence of Volterra processes. *Stochastic Process. Appl. 107*(2), 327–350.

Cheridito, P. (2004). Gaussian moving averages, semimartingales and option pricing. *Stochastic Process. Appl. 109*(1), 47–68.

Cherny, A. (2001). When is a moving average a semimartingal? *MaPhySto Research Report 2001–28.*

Dellacherie, C. and P.-A. Meyer (1982). *Probabilities and Potential B: Theory of Martingales*, Volume 72 of *North-Holland Mathematics Studies.* Amsterdam: North-Holland Publishing Co.

Emery, M. (1982). Covariance des semimartingales gaussiennes. *C. R. Acad. Sci. Paris Sér. I Math. 295*(12), 703–705.

Jacod, J. and A. N. Shiryaev (2003). *Limit theorems for stochastic processes* (Second ed.), Volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Berlin: Springer-Verlag.

Jain, N. C. and D. Monrad (1982). Gaussian quasimartingales. *Z. Wahrsch. Verw. Gebiete 59*(2), 139–159.

Jeulin, T. (1993). Processus gaussiens á variation finie. *Ann. Inst. H. Poincaré Probab. Statist. 29*(1), 153–160.

Jeulin, T. and M. Yor (1993). Moyennes mobiles et semimartingales. *Séminaire de Probabilités XXVII*(1557), 53–77.

Liptser, R. S. and A. Shiryayev (1989). *Theory of Martingales*, Volume 49 of *Mathematics and its Applications (Soviet Series).* Dordrecht: Kluwer Academic Publishers Group.

Protter, P. E. (2004). *Stochastic integration and differential equations* (Second ed.), Volume 21 of *Applications of Mathematics (New York).* Berlin: Springer-Verlag. Stochastic Modelling and Applied Probability.

Rogers, L. and D. Williams (1987). *Diffusions, Markov Processes and Martingales, Volume 2: Itô Calculus.* Wiley Series in Probability and Mathematical Statistics. New York: John Wiley & Sons, Inc.

Rogers, L. C. G. (1997). Arbitrage with fractional brownian motion. *Math. Finance 7*(1), 95–105.

Smithies, F. (1958). *Integral equations.* Cambridge Tracts in Mathematics and Mathematical Physics, no. 49. New York: Cambridge University Press.

Stricker, C. (1983). Semimartingales gaussiennes – application au probleme de l'innovation. *Z. Wahrsch. Verw. Gebiete 64*(3), 303–312.

Yor, M. (1997). *Some aspects of Brownian Motion. Part II. Some recent martingale problems.* Lectures in Mathematics ETH Zürich. Basel: Birkhäuser Verlag.