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Abstract

Completely random signed measures are defined, characterized and related to Lévy random measures and Lévy bases.

1 Introduction

Completely random measures were defined in Kingman (1993). As described in Kingman (1993); Karr (1991) and more recently in Daley and Vere-Jones (2003, 2008) completely random measures are related to point process models, in particular Poisson cluster point processes. We make a natural extension of completely random measures to completely random signed measures and give a characterization of this class of signed random measures. It is shown that the class of Lévy random measures, introduced and used in Lévy adaptive regression kernel models Tu et al. (2006), and the class of Lévy bases, defined in Barndorff-Nielsen and Schmiegel (2004) and used in spatio-temporal modeling in Barndorff-Nielsen and Schmiegel (2004); Hellmund et al. (2008); Jónsdóttir et al. (2008), are natural extensions of completely random signed measures. Furthermore we show the assumption of infinitely divisibility in the definition of Lévy random measures and Lévy bases can be replaced by other very mild assumptions. The most basic concept involved in the definition of Lévy random measures and Lévy bases is thus independence.

2 Signed random measures

We let $\mathcal{X}$ denote a Borel subset of $\mathbb{R}^d$ for some $d \geq 1$ and $\mathcal{B} = \mathcal{B}(\mathcal{X})$ denote the trace of the Borel sigma algebra on $\mathcal{X}$. By $\mathcal{B}_b$ we denote the set of bounded Borel subsets of $\mathcal{X}$. A subset of $\mathcal{X}$ is bounded, if the closure of the set is compact.

Let $\mathcal{M}$ denote the set of signed Radon measures on $\mathcal{B}$, i.e. an element in $\mathcal{M}$ is a σ-additive set function, which takes finite values on compact sets, in particular on all bounded Borel subsets of $\mathcal{X}$. We let $\mathcal{M}^+$ denote the subset of $\mathcal{M}$ consisting

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of positive Radon measures. There are several different definitions of signed Radon measures in the literature, we use, what we believe is the most general, see Ash (1972).

We define a random signed measure M as a measurable mapping from a probability space $(\Omega, \mathcal{E}, P)$ into $(\mathcal{M}, \mathcal{F})$ where

$$\mathcal{F} = \sigma \left\{ \pi_f | f \in C_c(\mathcal{X}) \right\}, \quad \pi_f : \mathcal{M} \to \mathbb{R} : \mu \to \int_{\mathcal{X}} f(x)\, \mu(dx),$$

and $C_c(\mathcal{X})$ is the set of continuous functions on $\mathcal{X}$ with compact support.

Let $\mathcal{F}^+$ denote the trace of $\mathcal{F}$ on $\mathcal{M}^+$, then $\mathcal{F}^+ = \mathcal{B}(\mathcal{M}^+)$. A random measure is defined as a measurable mapping from a probability space into $(\mathcal{M}^+, \mathcal{F}^+)$.

The lemma below, used in the sequel, concerns the fixed atoms of a signed random measure. We say $x \in \mathcal{X}$ is a fixed atom of M if and only if

$$P(|M(\{x\})| > 0) > 0.$$

Lemma 2.1. *A signed random measure has at most countable many fixed atoms.*

Proof. Assume there are more than countable many fixed atoms, then there exist $\{x_n | n \geq 1\}$ contained in a bounded set, such that

$$\exists b > 0, a > 0 \forall n \geq 1 : P(|M(x_n)| > b) > a.$$

Thus $P(\limsup_n \{|M(x_n)| > b\}) \geq a$ and $\sum_n M(\{x_n\})$ cannot be convergent, which is a contradiction. □

3 A result on infinitely divisibility

Lemma 3.1. *Let M denote a stochastic process with index set $\mathcal{B}_b$ such that $(M(B_n))_{n\geq 1}$ are independent and*

$$M(\cup B_n) = \sum M(B_n)$$

P-a.s. for disjoint sets $(B_n)_{n\geq 1} \subset \mathcal{B}_b$ and $\cup B_n \in \mathcal{B}_b$.

Then $M(B)$ is infinitely divisible for any $B \in \mathcal{B}_b$, if the cumulant function of $M(A), A \in \mathcal{B}_B$ is of the form

$$\mathcal{C}\{M(A) \ddagger t\} = \log E\left[e^{itM(A)}\right] = \int_A f_{t,B}(x)\, \lambda_B(dx) \tag{1}$$

for some measurable function $f_{t,B} : \mathcal{X} \to \mathbb{C}$ for all $t \in \mathbb{R}\backslash\{0\}$ and an atom-less finite measure λ_B on $(B, \mathcal{B}_B)$, where $\mathcal{B}_B = \mathcal{B}(B)$.

Remark 3.2. For a given $t \in \mathbb{R}$ and $B \in \mathcal{B}_b$, because of independence, the cumulant transform defines a complex measure on $(B, \mathcal{B}_B)$. If $M(A)$ is zero with probability one on all sets in $\mathcal{B}_B$ with Lebesgue measure zero, then Lebesgue measure dominates the complex measure generated by the cumulant transform for all $t \in \mathbb{R}$ and thus, by Radon-Nikodym, condition (3.1) in the above lemma is fulfilled.

Proof. Let $B \in \mathcal{B}_b$ be given. Since $\lambda_B(B)$ is finite using Lemma 12.2, p.268 in Karlin and Studdun (1966) we can choose $(B_s)_{0\leq s\leq 1}$ such that $B_0 = \emptyset$, $B_{s'} \subseteq B_s, s' \leq s$, $B_1 = B$ and $\lambda_B(B_s) = s \cdot \lambda_B(B)$ for $s \in [0,1]$. It is clear the stochastic process $(M_s) = M(B_{s\wedge 1})_{s\geq 0}$ has independent increments and P-a.s. $M(B_0) = 0$. Furthermore (M_s) is stochastic continuous: Let $0 \leq s < 1$ be given, then for any $s' \in (s,1)$:

$$M_{s'} - M_s = M(B_{s'} \backslash B_s)$$

Since

$$\lambda_B(B_{s'} \backslash B_s) = (s' - s)\,\lambda_B(B) \to 0, s' \downarrow s,$$

we see

$$M(B_{s'} \backslash B_s) \tilde{\to} 0.$$

Thus in probability $M(B_{s'} \backslash B_s) \to 0$ as s' goes to s. Left continuity for $0 < s \leq 1$ is proved similar.

By definition 1.6 in Sato (2005) (M_s) is an additive process in law. Therefore $M_1 = M(B)$ is infinitely divisible, see Theorem 9.1 Sato (2005). □

4 Completely random signed measures

Definition 4.1. A completely random (signed) measure is a random (signed) measure M with independent values on disjoint sets, i.e. $\{M(A_n)\}$ are independent, if $\{A_n\}$ is a family of disjoint sets.

Corollary 4.2. *A completely random signed measure with cumulant transform satisfying condition (1) in Lemma 3.1 has infinitely divisible values.*

In the sequel we use the definition of a Poisson point process found in Kingman (1993).

Definition 4.3. A Poisson point process Φ on $\mathcal{Y}$, a Borel subset of $\mathbb{R}^l$ for some $l \geq 1$, is a random countable subset of $\mathcal{Y}$, such that

- The number of points $N(A)$ in a Borel subset A of $\mathcal{Y}$ is Poisson distributed with mean value $\mu(A)$, where μ is a measure on $\mathcal{B}(\mathcal{Y})$ (μ may be infinite on bounded sets!).
- Given disjoint sets $A_1, \ldots, A_n$ the random variables $N(A_1), \ldots, N(A_n)$ are independent.

The theorem below is stated in Kingman (1993), which also provides a sketch of a proof. We give a detailed proof, since we use important elements of the proof in the sequel.

Theorem 4.4. *Given a completely random measure M fulfilling condition (1) in Lemma 4.2 there exists a Radon measure m, an at most countable set of fixed atoms $\{x_i\}_{i\in I} \subset \mathcal{X}$, independent positive random variables $\{W_i\}$ and a Poisson point process Φ on $\mathcal{X} \times \mathbb{R}_+$, such for any $B \in \mathcal{B}_b$*

$$M(B) \sim m(B) + \sum_i W_i \cdot 1_B(x_i) + \sum_{(X_j,U_j)\in\Phi} U_j \cdot 1_B(X_j)\,. \tag{2}$$

Proof. Using Lemma 2.1 we note the set of fixed atoms is at most countable. In the remaining part of the proof we assume M has no fixed atoms.

By Lemma 3.1 $M(B)$ is infinitely divisible for any $B \in \mathcal{B}_b$.

For any B in $\mathcal{B}_b$ there exists a constant $m(B) \in \mathbb{R}_+$ and a measure ν_B on $\mathbb{R}_+$, such that $\int (|x| \wedge 1) \nu_B(dx)$ is finite and for any $t \in \mathbb{R}_+$:

$$\lambda_t(B) = \log E\left[e^{itM(B)}\right] = m(B) \cdot it + \int_{(0,\infty]} (e^{itz} - 1)\nu_B(dt),$$

see Exc. 11, Chap. 15 in Kallenberg (2002).

By the properties of λ_t, $(A, B) \to \nu_B(A)$ is a bi-measure. There exists a unique σ-finite measure ν on $(\mathcal{X} \times \mathbb{R}_+, \mathcal{B} \otimes B([0,\infty)))$ satisfying

$$\nu(B \times C) = \nu_B(C)$$

for all $B \in \mathcal{B}_b$ and $C \in B((0,\infty])$, see (9.17) in Sato (2005). We see m defines a Radon measure on $\mathcal{B}$. Assume without loss of generality, that $m \equiv 0$.

Let Φ denote a Poisson point process on $\mathcal{X} \times \mathbb{R}_+$ with mean measure ν. Notice the number of points from Φ in a bounded set need not be finite. Define

$$M'(B) = \sum_{(X_j,U_j)\in\Phi} U_j \cdot 1_B(X_j)$$

for any $B \in \mathcal{B}_b$. Then for any $B \in \mathcal{B}_b$, using ν is σ-finite and applying Campbell's Theorem Kingman (1993) we get

$$\begin{aligned} E[\exp(itM'(B))] &= \exp\left(\int_{B\times(0,\infty]} \left(e^{itz} - 1\right) \nu(dx \times dz)\right) \\ &= \exp\left(\int_{(0,\infty]} \left(e^{itz} - 1\right) \nu_B(dz)\right) \end{aligned}$$

Assuming without loss of generality, that M' is a random measure, M and M' are equal in distribution. □

We were not able to find a reference to the lemma below, concerning additive processes, thus a short proof is provided.

Lemma 4.5. *Given a continuous additive process (X_s), $s \geq 0$ with paths of bounded variation, $X_0 = 0$ and with characteristic triplet on the form $(A_s, 0, \gamma(s))$ we have $A_s \equiv 0$. i.e. the process has no Gaussian part and is deterministic, see Sato (2005) for notation.*

Proof. The total variation process $\left(V_s^X\right)$ of (X_s) is an increasing, continuous additive process. Set

$$Y_s = \frac{e^{-V_s^X}}{E\left[e^{-V_s^X}\right]}, s \geq 0$$

Then Y_s is a continuous martingale of bounded variation and thus constant, therefore V_s^X is deterministic, implying (X_s) is integrable and $\gamma(s)$ is of bounded variation.

$X_s - \gamma(s)$ therefore defines a continuous martingale of bounded variation, thus $X_s = \gamma(s)$ □

Lemma 4.6. *Let M denote a completely random signed measure and suppose the condition on the cumulant transform in Lemma 3.1 is fulfilled, then M has no Gaussian part.*

Proof. Let $B \in \mathcal{B}_b$ be given and let (M_s) denote the additive process in law constructed in the proof of Lemma 3.1 In the proof below, all references are to Sato (2005).

By Theorem 11.5 we can choose a *cadlag* modification of (M_s). Given a *cadlag* modification $\tilde{M}$ of M, $n \geq 1$ and the partition $0 = s_0 < s_1 = 1/n < \cdots < s_{n-1} = (n-1)/n < s_n = 1$ of the interval $[0,1]$ we have P-a.s.

$$\sum_i \left| \tilde{M}_{s_i} - \tilde{M}_{s_{i-1}} \right| = \sum_i \left| M_{s_i} - M_{s_{i-1}} \right| \leq |M|(B) < \infty,$$

since M is a random signed measure. Without loss of generality we assume M is an additive process of bounded variation (see Lemma 21.8 (i)).

Using Theorem 9.8 the law of M is uniquely determined by a characteristic triplet $(A_s, \nu_s, \gamma(s))$ each component satisfying some conditions given in the theorem. For every $s \geq 0$, ν_s is a Lévy measure on $\mathbb{R}$. Define $H = (0, \infty) \times \mathbb{R} \backslash \{0\}$ and let $\mathcal{B}(H)$ denote the Borel subsets of H.

By (19.1) we define

$$J(D, \omega) = \# \{ s > 0 | (s, M_s - M_{s-}) \in D \in \mathcal{B}(H) \}$$

Because of bounded variation we have (Lemma 21.8) for any $s > 0$

$$\int_{(0,s] \times \mathbb{R} \backslash \{0\}} |x| \, J(d(t,x), \omega) < \infty$$

as shown page 141_3-142^5 $\int_{\{|x| \leq 1\}} |x| \, \nu_s(dx) < \infty$ for all $s > 0$.

Using Theorem 19.3 and Lemma 21.8 there exist processes M^J and M^G, such that $M = M^J + M^G$ and M^G is P-a.s. an additive process, continuous in s with characteristic triplet $\left(A_s, 0, \gamma(s) - \int_{\{|x| \leq 1\}} x \nu_s(dx)\right)$ and of bounded variation, thus $A_s \equiv 0$ (see 4.5). $\square$

Theorem 4.7. *Given a completely random signed measure M fulfilling condition (1) in Lemma 3.1. Then for all $B \in \mathcal{B}_b$:*

$$M(B) \sim \mu^+(B) - \mu^-(B) + \sum_{i=1}^{k} W_i \cdot 1_B(x_i) + \sum_{(X_j, U_j) \in \Phi} U_j \cdot 1_B(X_j)$$

where Φ is a Poisson point process on $\mathcal{X} \times \mathbb{R}$, W_i are independent random variables, the $x_i, i \in I$ is an at most countable set of points in $\mathcal{X}$ and μ^+, μ^- are Radon measures.

Proof. Assume without loss of generality M has no fixed atoms. Applying Rajput and Rosinski (1989) Proposition 2.1 (see this reference for notation) for every $B \in \mathcal{B}_b$:

$$\mathcal{C}\{M(B) \ddagger t\} = it \cdot \left(a(B) - \int_{\{|z| \leq 1\}} z U_B(dz) \right) + \int_{\mathbb{R}} \left(e^{itz} - 1 \right) U_B(dz)$$

From the proof of the previous lemma, we have that

$$\int_{\{|z|\leq 1\}} (|z|)\, U_B\,(dz) < \infty. \tag{3}$$

Thus we can apply Campbell's Theorem Kingman (1993). In an argument, similar to the one found in the proof of Theorem 4.4, we can construct a Poisson point process Φ on $\mathcal{X} \times \mathbb{R}$, such that

$$\mathcal{C}\Bigg\{ \sum_{(X_j,U_j)\in\Phi} U_j \cdot 1_B\,(X_j) \ddagger t \Bigg\} = \int_{\mathbb{R}} \left(e^{itz} - 1\right) U_B\,(dz)$$

(see Lemma 2.3 in Rajput and Rosinski (1989) for existence of a (mean) measure on $\mathcal{X} \times \mathbb{R}$ with the acquired properties)

It remains to note that $a\,(B) - \int_{\{|z|\leq 1\}} zU_B\,(dz)$ is finite for all bounded sets B. $\square$

5 Lévy bases

Definition 5.1. A stochastic process L indexed by $\mathcal{B}_b$ is called a Lévy basis, if $L\,(B)$ is infinitely divisible for all B in $\mathcal{B}_b$ and $L\,(B_n)$,$n \geq 1$ are independent and

$$L\,(\cup_n B_n) = \sum_n L\,(B_n)$$

P-a.s. for any sequence of disjoint sets $(B_n)_{n\geq 1} \subseteq \mathcal{B}_b$, $\cup B_n \in \mathcal{B}_b$.

Remark 5.2. The condition of infinite divisibility can be left out, if condition (1) in Lemma 3.1 is fulfilled. As previously remarked the condition is fulfilled if $L\,(B) \in L^1\,(P)$ for all $B \in \mathcal{B}_b$ and $E\,[L\,(\{x\})] = 0$ for all $x \in \mathcal{X}$.

It is proved in Rajput and Rosinski (1989) Lemma 2.3 that the cumulant transform of a Lévy basis L can be written as

$$\begin{aligned}\mathcal{C}\{L\,(dx) \ddagger t\} = \Big\{ &ita\,(x) - \tfrac{1}{2}t^2 b\,(x) \\ &+ \int_{\mathbb{R}} \left(e^{itz} - 1 - itz \cdot 1_{[-1,1]}\,(z)\right) \rho\,(x, dz) \Big\} \lambda\,(dx)\end{aligned} \tag{4}$$

λ is called the control measure and is σ-finite, a is a Borel measurable mapping into the real numbers and b is a density wrt. λ of a measure, $\rho\,(x, \cdot)$ is a Lévy measure for given x.

A Lévy basis such that $a \equiv 0$ and $\rho \equiv 0$ is called a purely Gaussian Lévy basis.

Theorem 5.3. *Let L denote a Lévy basis. L has the same distribution as the sum of a purely Gaussian Lévy basis and a completely random signed measure restricted to $\mathcal{B}_b$ (all terms being independent) if and only if for any $B \in \mathcal{B}_b$:*

$$\int_B \int_{\{|z|\leq 1\}} |z|\, \rho\,(x, dz)\, \lambda\,(dx) < \infty$$

Proof. From the proof of Lemma 4.6 we see the condition is necessary.

Assume the condition is fulfilled. Using the representation of the cumulant transform of the Lévy we can make the following rearrangements:

$$\mathcal{C}\{dx \ddagger t\} = \Big\{it\Big(a(x) - \int_{\mathbb{R}} \big(z \cdot 1_{[-1,1]}(z)\big)\, \rho(x, dz)\Big) - \frac{1}{2}t^2 b(x) + \int_{\mathbb{R}} \big(e^{itz} - 1\big)\, \rho(x, dz)\Big\}\lambda(dx) \quad (5)$$

The non-Gaussian part of L has cumulant transform

$$\{it\left(a(x) - \int_{\mathbb{R}} \big(z \cdot 1_{[-1,1]}(z)\big)\, \rho(x, dz)\right) + \int_{\mathbb{R}} \big(e^{itz} - 1\big)\, \rho(x, dz)\}\lambda(dx) \quad (6)$$

Following the proof of Theorem 4.7 there is a completely random signed measure with cumulant transform (6). □

Definition 5.4. Lévy random measures are Lévy bases with no Gaussian part.

Corollary 5.5. *A Lévy random measure is a completely random signed measure if and only if for any $B \in \mathcal{B}_b$:*

$$\int_B \int_{\{|z| \le 1\}} |z|\, \rho(x, dz)\, \lambda(dx) < \infty.$$

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